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ON THE ANALYTIC INTEGRABILITY OF THE 5–DIMENSIONAL LORENZ SYSTEM FOR THE GRAVITY–WAVE ACTIVITY

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ABSTRACT. The 5-dimensional Lorenz system for the coupled Rosby and gravity waves has exactly two independent analytic first integrals.

1. Introduction

E.N. Lorenz constructed in [4] the following 5–dimensional differential system in $\mathbb{R}^5$

$$\begin{aligned} dU/dT &= -VW + bVZ, \\ dV/dT &= UW - bUZ, \\ dW/dT &= -UV, \\ dX/dT &= -Z, \\ dZ/dT &= bUV + X, \end{aligned} \qquad (L)$$

where $b \in \mathbb{R}$ is a parameter, describing coupled Rosby waves and gravity waves.

He was mainly interested in its slow manifolds. Here our interest will be in studying its analytic integrability, i.e. *what is the maximal number of independent analytic first integrals that the system* (L) *can exhibit*? This question has been considered for many other relevant differential systems and other classes of first integrals not necessarily analytic, see for instance [2], [5], [6], ...

Let U be an open subset of $\mathbb{R}^5$ invariant by the flow of the differential system (L), i.e. if a solution of system (L) has a point in U then all the points of this solution are contained in U. A *first integral* of the differential system (L) in U is a non–constant function $H = H(U, V, W, X, Z) : U \to \mathbb{R}$ satisfying that it is constant on every solution of system (L) contained in U. In other words, H is a first

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integral of system (L) in U if

$$(-VW + bVZ)\frac{\partial H}{\partial U} + (UW - bUZ)\frac{\partial H}{\partial V} - \\ UV\frac{\partial H}{\partial W} - Z\frac{\partial H}{\partial X} + (bUV + X)\frac{\partial H}{\partial z} = 0.$$

The maximal open set U for which $H : U \to \mathbb{R}$ is a first integral of system (L)is called the *domain of definition* of the first integral H.

Of course, when the first integral H is an analytic function we say that H is an *analytic first integral.*

Let $H_1 : U_1 \to \mathbb{R}$ and $H_2 : U_2 \to \mathbb{R}$ be two first integrals of the 5–dimensional Lorenz system (L). They are *independent* in $U_1 \cap U_2$ if their gradients are linearly independent over a full Lebesgue measure subset of $U_1 \cap U_2$.

It is well known (see [4]) and easy to check that the 5–dimensional Lorenz system (L) has the two independent analytic first integrals

$$U^2 + V^2$$

and

$$V^2 + W^2 + X^2 + Z^2,$$

having domains of definition equal to $\mathbb{R}^5$. We will show that system (L) has no independent analytic first integrals of $U^2 + V^2$ and $V^2 + W^2 + X^2 + Z^2$.

Theorem 1. *The* 5*–dimensional Lorenz system* (L) *has only two independent analytic first integrals.*

Theorem 1 is proved in section 2.

2. Proof of Theorem 1

In the proof we will use the two known results stated in the next two theorems.

Assume that we have an ordinary C^1 differential system $\dot{x} = f(x)$ on $\mathbb{R}^n$, i.e. $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function. Let ϕ_t be its flow; i.e. $\phi_t(x)$ is the solution of $\dot{x} = f(x)$ which pass through the point $x \in \mathbb{R}^n$ at time zero after the time t. Of course, assuming that the solution $\phi_t(x)$ exists for such a time t. The ordinary differential system $\dot{x} = f(x)$ on $\mathbb{R}^n$ is called *analytic* if the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is analytic.

A non–negative C^1 function $M : \mathbb{R}^n \to \mathbb{R}$ and non–identically vanishing on any open subset of $\mathbb{R}^n$, is called a *Jacobi multiplier* of the differential system $\dot{x} = f(x)$, if for any open set $D \subset \mathbb{R}^n$, and any $t \in \mathbb{R}$ for which $\phi_t(D)$ is defined, we have

$$\int_D M(x)dx = \int_{\phi_t(D)} M(x)dx.$$

Under good conditions Jacobi multipliers can be used for constructing an additional first integral, as it is stated in the following theorem (for a proof see for example Theorem 2.7 in [3]).

Theorem 2 (Jacobi)**.** *Assume that the analytic differential system $\dot{x} = f(x)$ on $\mathbb{R}^n$ admits an analytic Jacobi multiplier M and $n-2$ analytic first integrals. Then system $\dot{x} = f(x)$ admits an extra analytic first integral.*

In fact the analyticity is not strictly necessary in Theorem 2, but since we shall apply this result to analytic differential systems we state Theorem 2 for such systems.

Assume additionally that $f(0) = 0$ and the linear part at the origin 0 is $Df(0) = A$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A counted with their multiplicity. Let

$$\mathcal{G} = \left\{ (k_1, \ldots, k_n) : \sum_{i=1}^{n} k_i \lambda_i = 0, \ k_i \in \mathbb{N} \cup \{0\}, \ 1 \le i \le n \right\}.$$

The dimension of the smallest linear vector space $\bar{\mathcal{G}}$ containing $\mathcal{G}$ is an upper bound for the number of analytic first integrals of the analytic differential system $\dot{x} = f(x)$, more precisely:

Theorem 3 (Zhang [7])**.** *Let $\dot{x} = f(x)$ be an analytic differential system in $\mathbb{R}^n$ such that $f(0) = 0$, and let $m \le n$ be the dimension of $\bar{\mathcal{G}}$. Then the system $\dot{x} = f(x)$ admits at most m independent analytic first integrals.*

We recall that the *divergence* of a differential system $\dot{x} = f(x)$ of $\mathbb{R}^n$ with $f = (f_1, \ldots, f_n)$ is

$$\frac{\partial f_1}{\partial x_1} + \ldots + \frac{\partial f_n}{\partial x_n}.$$

Now we give the proof of Theorem 1.

Proof of Theorem 1. We first remark that the divergence of the 5–dimensional Lorenz system (L) is zero. Then, by the Liouville formula (see for instance [1]), the flow ϕ_t of system (L) preserves the volume; i.e. the volume of the subset D of $\mathbb{R}^n$ and the volume of $\phi_t(D)$ are equal, assuming that $\phi_t(D)$ is well defined. So, from the definition of Jacobi multiplier it follows that the constant function $M = 1$ is a Jacobi multiplier.

Now if we assume that the system (L) has a third independent analytic first integral with respect the two analytic first integrals $U^2 + V^2$ and $V^2 + W^2 + X^2 + Z^2$, Theorem 2 implies that there exists also a fourth independent analytic first integral.

It is easy to check that the singularities of the system (L) are the points of the U, V and W axes. For a point $(U, 0, 0, 0, 0)$ on the U–axis, computing the eigenvalues of the linear part we obtain that one of them is zero, and the other four are the roots of the polynomial equation

$$\lambda^4 + (1 + U^2 + b^2U^2)\lambda^2 + U^2 = 0,$$

in λ. The roots of this equation are of the form $\pm\alpha$ and $\pm\beta$, for some complex numbers α and β. Thus the five eigenvalues are $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (0, \alpha, -\alpha, \beta, -\beta)$. Given any parameter b, it is easy to find some U such that $\dfrac{\alpha}{\beta} \notin \mathbb{Q}$. Thus, for instance for $b = 1$ and $U = 2$ we get that

$$\alpha = \sqrt{\frac{1}{2}\left(9 + \sqrt{65}\right)}\, i, \quad \beta = \sqrt{\frac{1}{2}\left(9 - \sqrt{65}\right)}\, i,$$

and consequently

$$\frac{\alpha}{\beta} = \frac{1}{4}\left(9 + \sqrt{65}\right).$$

In short, a basis for the linear vector space $\bar{\mathcal{G}}$ when α/β is irrational is formed by the vectors $(1, 0, 0, 0, 0)$, $(0, 1, 1, 0, 0)$ and $(0, 0, 0, 1, 1)$, so the dimension of $\mathcal{G}$ is three. Applying Theorem 3 we obtain that system (L) can have at most three independent analytic first integrals, which contradicts the previous conclusion that there must exist a fourth one. Consequently the system (L) must have only two independent analytic first integrals. □

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