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LIMIT CYCLES BIFURCATING FROM A NON-ISOLATED ZERO-HOPF EQUILIBRIUM OF THREE-DIMENSIONAL DIFFERENTIAL SYSTEMS

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Abstract. In this paper we study the limit cycles bifurcating from a non-isolated zero-Hopf equilibrium of a differential system in $\mathbb{R}^3$. The unfolding of the vector fields with a non-isolated zero-Hopf equilibrium is a family with at least three parameters. By using the averaging theory of the second order, explicit conditions are given for the existence of one or two limit cycles bifurcating from such a zero-Hopf equilibrium. To our knowledge, this is the first result on bifurcations from a non-isolated zero-Hopf equilibrium. This result is applied to study three-dimensional generalized Lotka-Volterra systems in [3]. The necessary and sufficient conditions for the existence of a non-isolated zero-Hopf equilibrium of this system are given, and it is shown that two limit cycles can be bifurcated from the non-isolated zero-Hopf equilibrium under a general small perturbation of three-dimensional generalized Lotka-Volterra systems.

1. Introduction

Zero-Hopf equilibrium is an equilibrium point of three-dimensional autonomous differential systems, which has a zero eigenvalue and a pair of purely imaginary eigenvalues. Usually the zero-Hopf bifurcation is a two-parameter unfolding (or family) of three-dimensional autonomous differential systems with a zero-Hopf equilibrium. The unfolding has an isolated equilibrium with a zero eigenvalue and a pair of purely imaginary eigenvalues if the two parameters take zero values, and the unfolding has different topological type of dynamics in the small neighborhood of this isolated equilibrium as the two parameters vary in a small neighborhood of the origin. This zero-Hopf bifurcation has been studied by Guckenheimer, Holmes, Scheurle, Marsden, Han and Kuznetsov in [8, 9, 24, 10, 15], and it has been shown that some complicated invariant sets of the unfolding could be bifurcated from the isolated zero-Hopf equilibrium under some conditions. Hence, zero-Hopf bifurcation could imply a local birth of "chaos" (cf. [5, 24]). Recently there are some theoretical analysis and numerical simulations which showed that three-dimensional or four-dimensional generalized Lotka-Volterra systems allow complicated dynamics such as chaotic behavior (cf. [1, 6, 26, 28] and references therein). The question naturally asked if a generalized Lotka-Volterra system can undergo the zero-Hopf bifurcation. However, we shall see such differential systems can not have an isolated zero-Hopf equilibrium in the set of all equilibria, but they may have non-isolated zero-Hopf equilibrium points. In other words, if generalized Lotka-Volterra systems

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have a zero-Hopf equilibrium then it is not isolated in the set of equilibria points of the generalized Lotka-Volterra system. Under a general small perturbation of a generalized Lotka-Volterra system with non-isolated zero-Hopf equilibrium, what dynamics it will happen is a challenge problem since the unfolding of the vector fields with a non-isolated zero-Hopf equilibrium is at least three parameters family. In 1973, Arnold in [2] proposed to investigate bifurcations of a three-parameter family with a zero eigenvalue and a pair of purely imaginary eigenvalues. As far as we know there are no works on the topic.

In this paper we first consider a three-dimensional polynomial differential system of degree two with a non-isolated zero-Hopf equilibrium at the origin. Then we study bifurcations of the non-isolated zero-Hopf equilibrium under a small perturbation of the polynomial differential system of degree two which keeps the equilibrium at origin. In the neighborhood of the origin, we reduce this perturbation system to a 2π-periodic differential system in a kind of cylindrical coordinates and re-scales variables. Applying the averaging theory of second order to this periodic differential system, we obtain the explicit conditions for the existence of one or two limit cycles bifurcating from the non-isolated zero-Hopf equilibrium. To our knowledge, this is the first result on bifurcations from a non-isolated zero-Hopf equilibrium.

As an application of the main theorem, we consider generalized Lotka- Volterra systems. It is well-known that n-dimensional generalized Lotka- Volterra systems are widely used as the first approximation for a community of n interacting species, each of which would exhibit logistic growth in the absence of other species in population dynamics. And this system is of wide interest in different branches of science, such as physics, chemistry, biology, evolutionary game theory, economics and etc.. We refer the reader to the book of Hofbauer and Sigmand [13] for its applications. The existence of limit cycles (isolated periodic orbits) for these models is interesting and significant both in mathematics and applications since the existence of stable limit cycle provided a satisfactory explanation for those species communities in which populations are observed to oscillate in a rather reproducible periodic manner (cf. [16, 17, 25] and references therein). To study the bifurcation of non-isolated zero-Hopf equilibrium in the Lotka-Volterra class, we consider three-dimensional generalized Lotka-Volterra systems, which describes the interaction of three species in a constant and homogeneous environment.

$$\frac{dx_i(t)}{dt} = x_i(t)(b_i + \sum_{j=1}^{3} a_{ij}x_j(t)), \;\; i = 1, 2, 3, \tag{1}$$

where $x_i(t)$ is the number of individuals in the i-th population at time t and $x_i(t) \geq 0$, b_i is the intrinsic growth rate of the i-th population, and the a_{ij} are interaction coefficients measuring the extent to which the j-th species affects the growth rate of the i-th, b_i and a_{ij} are parameters and the values of these parameters are not very small usually.

Over the last several decades, many researchers have devoted their effort to study the existence and number of isolated periodic solutions for system (1). There have been a series of achievements and unprecedented challenges on the theme even if system (1) is competitive system (cf. [7, 11, 12, 14, 21, 29, 30, 31]). In [3], Bobieński and Żoladek gave four components of center variety in the three-dimensional Lotka-Volterra class and studied the existence and number of isolated periodic solutions by certain Poincaré-Melnikov integrals of a new type. As far as we know there is no any results on periodic orbits bifurcating from a non-isolated zero-Hopf equilibrium of three-dimensional Lotka-Volterra systems. Here we make an analysis on the whole twelve dimensional parameters space of system (1), and give the necessary and sufficient conditions for the existence of a non-isolated zero-Hopf equilibrium of system (1). Hence, the three-dimensional generalized Lotka-Volterra systems with a non-isolated zero-Hopf equilibrium forms a subspace. We perturbs the subspace in the space of three-dimensional Lotka-Volterra systems with a positive equilibrium point, and look for the system arising isolated periodic solutions (i.e. limit cycles). Because the zero-Hopf equilibrium is not isolated and these parameters are not very small for such systems, the approach to deal with zero-Hopf bifurcation with two parameters does not work for system (1). By using our main theorem, we obtain two limit cycles bifurcating from a three-parameters family of three-dimensional Lotka-Volterra systems with a non-isolated zero-Hopf equilibrium .

This paper is organized as follows. In section 2 we study bifurcations of polynomial differential systems of degree two in $\mathbb{R}^3$ with a small positive parameter ε. When $\varepsilon = 0$, this system has a continuum of equilibria which fill a segment, or half-straight line, in which there exists a unique non-isolated zero-Hopf equilibrium of this system at the origin. When $\varepsilon \neq 0$, this system has a unique equilibrium at the origin. In a small neighborhood of the origin, we reduce this system to a 2π-periodic differential system in a kind of cylindrical coordinates and re-scales variables. Applying the averaging theory of second order to this periodic differential system, we obtain the explicit conditions for the existence of one or two limit cycles bifurcating from the non-isolated zero-Hopf equilibrium, see Theorem 4. In section 3 we do a preliminary analysis on the conditions for the existence of a positive zero-Hopf equilibrium for the Lotka-Volterra system (1) which will be non-isolated in the set of all equilibria, and we further reduce a normal form for a such non-isolated zero-Hopf equilibrium under a general perturbation in the Lotka-Volterra class. A three-parameters example inside the class of the Lotka-Volterra systems is provided to illustrate these results in the last section.

2. Limit cycles bifurcating from a non-isolated zero-Hopf equilibrium

In the section we study polynomial differential systems of degree two in $\mathbb{R}^3$ with a small positive parameter ε and other bounded parameters. When $\varepsilon = 0$, this system has a continuum of equilibria which fill a segment, or half-straight line, in which there exists a unique non-isolated zero-Hopf equilibrium at the origin. When $\varepsilon \neq 0$, this system has a unique equilibrium at the origin and some bifurcations occur. Limit cycles can be bifurcated from the non-isolated zero-Hopf equilibrium for this polynomial differential systems of degree two in $\mathbb{R}^3$, here a *limit cycle* means an isolated non-constant periodic orbit (or closed orbit) in phase space having the property that its neighboring trajectories are not periodic orbits and they spiral into it either as time approaches infinity or as time approaches negative infinity. To our knowledge there is no a general theory for studying the existence and the number of limit cycles which born from this zero-Hopf equilibrium under a small perturbation. We will use the second order averaging method to study this problem. It is well known that the averaging method has been widely used to look for periodic orbits of differential systems (see [4], [9], [18],[19] and references therein). For reader's convenience, we recall some basic results on the second order averaging method. The reader is referred to Theorem 3.1 in [4] for details about the following theorem 1.

Consider the differential system

$$\dot{x}(t) = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 Q(t, x, \varepsilon), \tag{2}$$

where F_1, F_2: $\mathbb{R} \times D \to \mathbb{R}^n$, and $Q: \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are continuous vector functions and T -periodic in the first variable t, here D is an open subset of $\mathbb{R}^n$ and $0 < \varepsilon_0 \ll 1$.

We further assume that $F_1(t, \cdot) \in C^1(D)$ for all $t \in \mathbb{R}$, $F_2(t, x)$, $Q(t, x, \varepsilon)$ and $D_x F_1(t, x)$ are locally Lipschitz with respect to x, and $Q(t, x, \varepsilon)$ is differentiable with respect to ε. We define

$$\begin{aligned} F_{10}(z) =& \frac{1}{T} \int_0^T F_1(s, z) ds, \\ F_{20}(z) =& \frac{1}{T} \int_0^T (D_z F_1(s, z) y_1(s, z) + F_2(s, z)) ds, \end{aligned} \tag{3}$$

where $D_z F_1(s, z)$ is the Jacobian matrix of the derivatives of the components of F_1 with respect to the components of z, and $y_1(s, z) = \displaystyle\int_0^s F_1(t, z) dt$. Then

$$\dot{z}(t) = \varepsilon F_{10}(z) + \varepsilon^2 F_{20}(z), \tag{4}$$

is called the *averaging differential system of second order* of (2). The following lemma shows the relation between the existence of non-degenerated equilibrium of the averaging differential system (4) and the existence of T-periodic solution of system (2).

Theorem 1. *Suppose that for $D_0 \subset D$ an open and bounded set and for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$, there exists $a_\varepsilon \in D_0$ such that*

$$F_{10}(a_\varepsilon) + \varepsilon F_{20}(a_\varepsilon) = 0,$$

and the Brouwer degree of the function $F_{10} + \varepsilon F_{20}: D_0 \to \mathbb{R}^n$ at the fixed point a_ε is not zero. Then for sufficiently small ε, $0 < |\varepsilon| < \varepsilon_0 \ll 1$, there exists a T-periodic solution $\phi(t, \varepsilon)$ of system (2) such that $\phi(0, \varepsilon) = a_\varepsilon$.

In Theorem 1, the non-zero Brouwer degree of the function $F_{10} + \varepsilon F_{20}$ at the fixed point a_ε can usually be verified by a sufficient condition, the Jacobian of the function $F_{10} + \varepsilon F_{20}$ at a_ε is not zero, which implies that a_ε is a non-degenerated equilibrium of system (4). Moreover, if F_{10} is not identically zero in D_0, then the zeros of $F_{10} + \varepsilon F_{20}$ are mainly determined by the zeros of F_{10} for ε sufficiently small. In this case Theorem 1 provides the averaging theory of first order. if F_{10} is identically zero and F_{20} is not identically zero in D_0, then the zeros of $F_{10} + \varepsilon F_{20}$ are determined by the zeros of F_{20}. In this case Theorem 1 provides the averaging theory of second order.

We now consider the following differential system of degree two in $\mathbb{R}^3$ with an equilibrium at the origin

$$
\begin{aligned}
\frac{dU}{dt} =& u\varepsilon U - vV + a_{200}(\varepsilon)U^2 + a_{110}(\varepsilon)UV + a_{101}(\varepsilon)UW + a_{020}(\varepsilon)V^2 \\
& + a_{011}(\varepsilon)VW + a_{002}(\varepsilon)W^2, \\
\frac{dV}{dt} =& vU + u\varepsilon V + b_{200}(\varepsilon)U^2 + b_{110}(\varepsilon)UV + b_{101}(\varepsilon)UW + b_{020}(\varepsilon)V^2 \\
& + b_{011}(\varepsilon)VW + b_{002}(\varepsilon)W^2, \\
\frac{dW}{dt} =& \varepsilon W + c_{200}(\varepsilon)U^2 + c_{110}(\varepsilon)UV + c_{101}(\varepsilon)UW + c_{020}(\varepsilon)V^2 \\
& + c_{011}(\varepsilon)VW + c_{002}(\varepsilon)W^2,
\end{aligned}
\tag{5}
$$

where $a_{ijk}(\varepsilon)$, $b_{ijk}(\varepsilon)$ and $c_{ijk}(\varepsilon)$ for $i, j, k = 0, 1, 2$ are smooth functions with respect to ε at $\varepsilon = 0$, u and $v > 0$ are bounded parameters. Without loss of generality, we assume that $\varepsilon > 0$. By linear algebra we obtain the following result.

Proposition 2. *Suppose that $\varepsilon = 0$. Then system* (5) *has a continuum of equilibria which fill a segment, or half-straight line passing through the origin if and only if $a_{002}(0) = b_{002}(0) = c_{002}(0) = 0$.*

For the sake of convenience we write

$$
\begin{aligned}
a_{ijk}(\varepsilon) =& a_{ijk0} + a_{ijk1}\varepsilon + O(\varepsilon^2), \\
b_{ijk}(\varepsilon) =& b_{ijk0} + b_{ijk1}\varepsilon + O(\varepsilon^2), \\
c_{ijk}(\varepsilon) =& c_{ijk0} + c_{ijk1}\varepsilon + O(\varepsilon^2),
\end{aligned}
$$

where $O(\varepsilon^2)$ denotes some continuous function with order at least two in ε. Assume that

$$(H_0): \quad a_{0020} = b_{0020} = c_{0020} = 0.$$

Then, by proposition 2, we have:

Lemma 3. *Assume that $\varepsilon = 0$ and (H_0) holds. Then system* (5) *has a continuum of equilibria which fill a segment, or half-straight line passing through the origin, in which the origin is a unique non-isolated zero-Hopf equilibrium of system* (5).

To study the periodic orbits of system (5) when $0 < \varepsilon \ll 1$, we introduce a class of cylindrical coordinates in a small neighborhood of the origin for system (5). Let

$$U = R\cos\theta, \quad V = R\sin\theta, \quad W = RZ, \quad R > 0. \tag{6}$$

This is a topological change of variables in a neighborhood of origin except at the origin.

Doing the transformation (6), system (5) becomes

$$
\begin{aligned}
\frac{dR}{dt} =& u\varepsilon R + R^2 \big(a_{200}(\varepsilon)\cos^3\theta + (a_{110}(\varepsilon) + b_{200}(\varepsilon))\sin\theta\cos^2\theta \\
& + (a_{020}(\varepsilon) + b_{110}(\varepsilon))\sin^2\theta\cos\theta + b_{020}(\varepsilon)\sin^3\theta \\
& + (a_{101}(\varepsilon)\cos^2\theta + (a_{011}(\varepsilon) + b_{101}(\varepsilon))\sin\theta\cos\theta + b_{011}(\varepsilon)\sin^2\theta)Z \\
& + (a_{002}(\varepsilon)\cos\theta + b_{002}(\varepsilon)\sin\theta)Z^2\big) \triangleq \mathbf{R}(\theta, R, Z),
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\theta}{dt} =& v + R\big(b_{200}(\varepsilon)\cos^3\theta + (b_{110}(\varepsilon) - a_{200}(\varepsilon))\sin\theta\cos^2\theta \\
&+ (b_{020}(\varepsilon) - a_{110}(\varepsilon))\sin^2\theta\cos\theta - a_{020}(\varepsilon)\sin^3\theta \\
&+ (b_{101}(\varepsilon)\cos^2\theta + (b_{011}(\varepsilon) - a_{101}(\varepsilon))\sin\theta\cos\theta - a_{011}(\varepsilon)\sin^2\theta)Z \\
&+ (b_{002}(\varepsilon)\cos\theta - a_{002}(\varepsilon)\sin\theta)Z^2\big) \triangleq \Theta(\theta, R, Z), \\
\frac{dZ}{dt} =& (1-u)\varepsilon Z + R\big(c_{200}(\varepsilon)\cos^2\theta + c_{110}(\varepsilon)\sin\theta\cos\theta + c_{020}(\varepsilon)\sin^2\theta \\
&+ (c_{011}(\varepsilon)\sin\theta + c_{101}(\varepsilon)\cos\theta - b_{020}(\varepsilon)\sin^3\theta - a_{200}(\varepsilon)\cos^3\theta \\
&- (a_{110}(\varepsilon) + b_{200}(\varepsilon))\sin\theta\cos^2\theta - (a_{020}(\varepsilon) + b_{110}(\varepsilon))\sin^2\theta\cos\theta)Z \\
&+ (c_{002}(\varepsilon) - a_{101}(\varepsilon)\cos^2\theta - (a_{011}(\varepsilon) + b_{101}(\varepsilon))\sin\theta\cos\theta \\
&- b_{011}(\varepsilon)\sin^2\theta)Z^2 - (a_{002}(\varepsilon)\cos\theta + b_{002}(\varepsilon)\sin\theta)Z^3\big) \triangleq \mathbf{Z}(\theta, R, Z).
\end{aligned} \tag{7}
$$

Note that $v > 0$, which implies that there exists a small neighborhood of $(R, Z) = (0, 0)$ such that $\Theta(\theta, R, Z) \neq 0$ for all θ in this neighborhood. Therefore, we consider an equivalent system of (7) in this neighborhood of $(R, Z) = (0, 0)$ taking θ as the new independent variable. Thus we get the system

$$
\begin{aligned}
\frac{dR}{d\theta} =& \frac{\mathbf{R}(\theta, R, Z)}{\Theta(\theta, R, Z)} \triangleq \mathcal{R}(\theta, R, Z), \\
\frac{dZ}{d\theta} =& \frac{\mathbf{Z}(\theta, R, Z)}{\Theta(\theta, R, Z)} \triangleq \mathcal{Z}(\theta, R, Z),
\end{aligned} \tag{8}
$$

where $\mathcal{R}(\theta, R, Z)$ and $\mathcal{Z}(\theta, R, Z)$ are smooth 2π-periodic functions in the variable θ in a neighborhood of $(R, Z) = (0, 0)$.

In order to use the averaging method, we re-scale the variables (R, Z) of system (8) as follows

$$R = \sqrt{\varepsilon}\, r, \quad Z = \sqrt{\varepsilon}\, z.$$

Then system (8) can be written as in power series of $\sqrt{\varepsilon}$, i.e.

$$
\begin{aligned}
\frac{dr}{d\theta} =& \sqrt{\varepsilon}\, R_1(\theta, r, z) + \varepsilon\, R_2(\theta, r, z) + O(\varepsilon^{3/2}), \\
\frac{dz}{d\theta} =& \frac{r}{v}\left(c_{2000}\cos^2\theta + c_{1100}\cos\theta\sin\theta + c_{0200}\sin^2\theta\right) + \sqrt{\varepsilon}\, Z_1(\theta, r, z) \\
&+ \varepsilon\, Z_2(\theta, r, z) + O(\varepsilon^{3/2}),
\end{aligned} \tag{9}
$$

where $R_i(\theta, r, z)$ and $Z_i(\theta, r, z)$ are polynomials in the variables (r, z) with coefficients 2π-periodic functions in the variable θ for $i = 1, 2$.

To apply the averaging method for studying system (9), we do the assumption

$$(H_1): \quad c_{2000} = c_{1100} = c_{0200} = 0.$$

Under the assumption (H_1), we can obtain the existence of limit cycles from a non-isolated zero-Hopf equilibrium as follows.

Theorem 4. *Assume that (H_0) holds. Then system (5) has a non-isolated zero-Hopf equilibrium at the origin when $\varepsilon = 0$. Moreover, if assumption (H_1) is satisfied, then there exists ε^*, $0 < \varepsilon^* \ll 1$ such that for any ε and $0 < \varepsilon < \varepsilon^*$, system (5) has ε-families of limit cycles bifurcating from the origin. More precisely:*

(i) *System (5) has one family of limit cycles bifurcating from the origin if one of the following conditions holds.*

(a1) $u(B_1^2+B_1B_2)<0$;
(b1) $u=0$ *and* $B_3(B_1^2+B_1B_2)>0$;
(c1) $c_{0201}+c_{2001}=0$, $uB_1<0$ *and* $B_2r_0^2+8v(1-u)\neq 0$;
(d1) $a_{1010}+b_{0110}=0$, $c_{0201}+c_{2001}\neq 0$, $uB_1<0$ *and* $B_2r_0^2+8v(1-u)\neq 0$.

(ii) *System* (5) *has two families of limit cycles bifurcating from the origin if one of the following conditions holds.*
(a2) $c_{0201}+c_{2001}=0$, $a_{1010}+b_{0110}\neq 0$, $uB_1<0$, $B_1+B_2<0$, $B_2r_0^2+8v(1-u)\neq 0$ *and* $-B_1+(B_1+B_2)u\neq 0$;
(b2) $B_3^2-256uv^2(B_1^2+B_1B_2)>0$, $u(B_1^2+B_1B_2)>0$ *and* $B_3(B_1^2+B_1B_2)>0$.

Here $r_0=\sqrt{-8uv/B_1}$, *and*

$$
\begin{aligned}
B_1 =&a_{0200}a_{1100}+a_{1100}a_{2000}+2a_{0200}b_{0200}-b_{0200}b_{1100}-2a_{2000}b_{2000}-b_{1100}b_{2000},\\
B_2 =&b_{1100}b_{2000}-a_{0200}a_{1100}-a_{1100}a_{2000}-2a_{0200}b_{0200}+b_{0200}b_{1100}+2a_{2000}b_{2000}\\
&+4a_{0200}c_{0110}+4a_{2000}c_{0110}-4b_{0200}c_{1010}-4b_{2000}c_{1010},\\
B_3 =&16v^2(a_{1010}c_{0201}+b_{0110}c_{0201}+a_{1010}c_{2001}+b_{0110}c_{2001})-8v(B_1+B_1u+B_2u).
\end{aligned}
$$

Proof. From Lemma 3 we know that the origin is a non-isolated zero-Hopf equilibrium of system (5) when $\varepsilon=0$.

If $0<\varepsilon\ll 1$, then we consider system (9) under assumptions (H_0) and (H_1), which becomes

$$
\begin{aligned}
\frac{dr}{d\theta} =&\sqrt{\varepsilon}\ r_1(\theta,r,z)+\varepsilon\ r_2(\theta,r,z)+O(\varepsilon^{\frac{3}{2}}),\\
\frac{dz}{d\theta} =&\sqrt{\varepsilon}\ z_1(\theta,r,z)+\varepsilon\ z_2(\theta,r,z)+O(\varepsilon^{\frac{3}{2}}),
\end{aligned}
\tag{10}
$$

where

$$
\begin{aligned}
r_1(\theta,r,z) =&\frac{r^2}{v}\left(a_{2000}\cos^3\theta+(a_{1100}+b_{2000})\cos^2\theta\sin\theta+(a_{0200}+b_{1100})\cos\theta\sin^2\theta\right.\\
&\left.+b_{0200}\sin^3\theta\right),\\
r_2(\theta,r,z) =&\frac{r}{v^2}\left(uv+vrz(a_{1010}\cos^2\theta+a_{0110}\cos\theta\sin\theta+b_{1010}\cos\theta\sin\theta+b_{0110}\sin^2\theta)\right.\\
&-r^2(b_{2000}\cos^3\theta-(a_{2000}-b_{1100})\cos^2\theta\sin\theta-(a_{1100}-b_{0200})\cos\theta\sin^2\theta\\
&-a_{0200}\sin^3\theta)(a_{2000}\cos^3\theta+(a_{1100}+b_{2000})\cos^2\theta\sin\theta+(a_{0200}+b_{1100})\\
&\left.\cos\theta\sin^2\theta+b_{0200}\sin^3\theta)\right),\\
z_1(\theta,r,z) =&-\frac{rz}{v}\left(a_{2000}\cos^3\theta-c_{1010}\cos\theta-c_{0110}\sin\theta+(a_{1100}+b_{2000})\cos^2\theta\sin\theta\right.\\
&\left.+(a_{0200}+b_{1100})\cos\theta\sin^2\theta+b_{0200}\sin^3\theta\right),\\
z_2(\theta,r,z) =&-\frac{1}{v^2}\left((-v+uv)z-rz^2(b_{2000}\cos^3\theta-a_{2000}\cos^2\theta\sin\theta\right.\\
&+b_{1100}\cos^2\theta\sin\theta-a_{1100}\cos\theta\sin^2\theta+b_{0200}\cos\theta\sin^2\theta-a_{0200}\sin^3\theta)\\
&(-c_{1010}\cos\theta+a_{2000}\cos^3\theta-c_{0110}\sin\theta+a_{1100}\cos^2\theta\sin\theta\\
&+b_{2000}\cos^2\theta\sin\theta+a_{0200}\cos\theta\sin^2\theta+b_{1100}\cos\theta\sin^2\theta+b_{0200}\sin^3\theta)\\
&-vr(c_{2001}\cos^2\theta+c_{1101}\cos\theta\sin\theta+c_{0201}\sin^2\theta-z^2(a_{1010}\cos^2\theta\\
&\left.+a_{0110}\cos\theta\sin\theta+b_{1010}\cos\theta\sin\theta+b_{0110}\sin^2\theta))\right).
\end{aligned}
$$

Hence system (10) is changed into the normal form with respect to the parameter $\sqrt{\varepsilon}$ for applying the averaging theory. We first consider the averaging differential system of first order for system (10). According to Theorem 1, by direct computation we have

$$F_{110}(r,z)=\int_0^{2\pi} r_1(\theta,r,z)d\theta\equiv 0,$$
$$F_{120}(r,z)=\int_0^{2\pi} z_1(\theta,r,z)d\theta\equiv 0.$$

It is clear that the first order averaging theory does not provide any information about system (10). Therefore, we further consider the averaging differential system of second order for system (10). From formula (3), we obtain that

$$\begin{aligned}
y_{11}(\theta,r,z)=&\int_0^{\theta} r_1(s,r,z)ds\\
=&\frac{r^2}{12v}\big(-4(a_{1100}+b_{2000})(-1+\cos^3\theta)+b_{0200}(8-9\cos\theta+\cos 3\theta)\\
&+4(a_{0200}+b_{1100})\sin^3\theta+a_{2000}(9\sin\theta+\sin 3\theta)\big),\\
y_{12}(\theta,r,z)=&\int_0^{\theta} z_1(s,r,z)ds\\
=&-\frac{rz}{12v}\big(12c_{0110}(-1+\cos\theta)-4(a_{1100}+b_{2000})(-1+\cos^3\theta)\\
&+b_{0200}(8-9\cos\theta+\cos 3\theta)-12c_{1010}\sin\theta\\
&+4(a_{0200}+b_{1100})\sin^3\theta+a_{2000}(9\sin\theta+\sin 3\theta)\big).
\end{aligned}$$

And

$$\begin{aligned}
F_{210}(r,z)=&\int_0^{2\pi}(y_{11}(\theta,r,z)\frac{\partial r_1(\theta,r,z)}{\partial r}+y_{12}(\theta,r,z)\frac{\partial r_1(\theta,r,z)}{\partial z}+r_2(\theta,r,z))d\theta\\
=&\frac{r}{8v^2}\big(B_1r^2+8uv+4v(a_{1010}+b_{0110})rz\big),\\
F_{220}(r,z)=&\int_0^{2\pi}(y_{11}(\theta,r,z)\frac{\partial z_1(\theta,r,z)}{\partial r}+y_{12}(\theta,r,z)\frac{\partial z_1(\theta,r,z)}{\partial z}+z_2(\theta,r,z))d\theta\\
=&\frac{1}{8v^2}\big(B_2r^2z+8v(1-u)z+4vr(c_{0201}+c_{2001}-(a_{1010}+b_{0110})z^2)\big).
\end{aligned}$$

Thus the averaged system of second order of system (10) is

$$\begin{aligned}
\frac{dr}{d\theta}&=\frac{r}{8v^2}\big(B_1r^2+8uv+4v(a_{1010}+b_{0110})rz\big),\\
\frac{dz}{d\theta}&=\frac{1}{8v^2}\big(B_2r^2z+8v(1-u)z+4vr(c_{0201}+c_{2001})\\
&\qquad -4vr(a_{1010}+b_{0110})z^2\big).
\end{aligned}\tag{11}$$

We divide the study of the zeros of system (11) into three cases.

Case I: $c_{0201}+c_{2001}=0$. It can be checked that the following conclusions are true.

(1) System (11) has a non-degenerated equilibrium $(r_0,z_0)=(\sqrt{-8uv/B_1},0)$ if $uB_1<0$ and $B_2r_0^2+8v(1-u)\neq 0$.
(2) System (11) has two non-degenerated equilibria: $(r_0,z_0)=(\sqrt{-8uv/B_1},0)$ and $(r_1,z_1)=(\sqrt{-8v/(B_1+B_2)},-(B_1r_1^2+8uv)/(4vr_1(a_{1010}+b_{0110})))$ if

$a_{1010}+b_{0110}\neq 0$, $uB_1<0$, $B_1+B_2<0$, $B_2r_0^2+8v(1-u)\neq 0$ and $-B_1+(B_1+B_2)u\neq 0$.

Case II: $c_{0201}+c_{2001}\neq 0$ and $a_{1010}+b_{0110}=0$. Then we obtain that system (11) has a non-degenerated equilibrium $(r_0,\bar z_0)=(\sqrt{-8uv/B_1},-(4rv(c_{0201}+c_{2001})/(B_2r_0^2+8v(1-u)))$, if $uB_1<0$ and $B_2r_0^2+8v(1-u)\neq 0$.

Case III: $a_{1010}+b_{0110}\neq 0$ and $c_{0201}+c_{2001}\neq 0$. Then, from $F_{210}(r,z)=0$ and $r>0$, we obtain that $\tilde z=-(B_1r^2+8uv)/(4(a_{1010}+b_{0110})rv)$. And if the following equation $({B_1}^2+B_1B_2)r^4-B_3r^2+64uv^2=0$ has a positive root $\tilde r$, then system (11) has an equilibrium $(\tilde r,\tilde z)$. Hence we obtain the following conclusions.

System (11) has two non-degenerated equilibria: $(\tilde r_1,\tilde z_1)$ and $(\tilde r_2,\tilde z_2)$ if $B_3^2-256uv^2(B_1^2+B_1B_2)>0$, $u(B_1^2+B_1B_2)>0$ and $B_3(B_1^2+B_1B_2)>0$. Where

$$\tilde r_1=\left(\frac{B_3-\sqrt{B_3^2-256uv^2(B_1^2+B_1B_2)}}{2(B_1^2+B_1B_2)}\right)^{\frac12},\quad \tilde z_1=-\frac{B_1\tilde r_1^2+8uv}{4(a_{1010}+b_{0110})\tilde r_1v},$$

$$\tilde r_2=\left(\frac{B_3+\sqrt{B_3^2-256uv^2(B_1^2+B_1B_2)}}{2(B_1^2+B_1B_2)}\right)^{\frac12},\quad \tilde z_2=-\frac{B_1\tilde r_2^2+8uv}{4(a_{1010}+b_{0110})\tilde r_2v}.$$

And system (11) has a non-degenerated equilibria $(\tilde r_2,\tilde z_2)$ if one of the following conditions holds:

(a) $u(B_1^2+B_1B_2)<0$,
(b) $u=0$ and $B_3(B_1^2+B_1B_2)>0$.

Applying Theorem 1 to the three cases, the proof of theorem 4 is completed. □

Remark: From the proof of Theorem 4, we can see that the averaging method of second order is valid if the origin is an isolated zero-Hopf equilibrium of system (5) as $\varepsilon=0$. That is, under only the assumption (H_2), we can prove that system (5) bifurcates at most two limit cycles from the zero-Hopf equilibrium doing a similar proof to the proof of Theorem 4.

3. The existence of a non-isolated zero-Hopf equilibrium for Lotka-Volterra system (1)

In this section we discuss the existence of zero-Hopf equilibrium for the Lotka-Volterra system (1). It is clear that system (1) always has an equilibrium at $(0,0,0)$ which is not zero-Hopf equilibrium for all 12 real parameters b_i and a_{ij}. Because of the biological meaning of $x_i(t)$, we consider system (1) in the first octant $\mathbb{R}_+^3$, where $\mathbb{R}_+=\{x\in\mathbb{R}:x>0\}$. For the sake of convenience, we give the classification of equilibria for system (1). An equilibrium is called *axial* if only one of its coordinates is positive, and *planar* if precisely two of its coordinates are positive, and *positive* if all of its coordinates are positive. An axial equilibrium of system (1) cannot become the zero-Hopf equilibrium for all real parameters, because it leaves on an invariant straight-line contained in the intersection of two invariant planes. Here invariant means it is invariant by the flow of system (1).

We now look for the conditions for the existence of positive equilibria of system (1), which is equivalent to find the positive solutions of the following system

$$b_i+\sum_{j=1}^{3}a_{ij}x_j=0,\quad i=1,2,3. \tag{12}$$

Note that equations (12) have finitely many solutions if and only if the determinant of the matrix $A = (a_{ij})_{3\times 3}$ is not zero (i.e. $|A| \neq 0$). Hence system (1) has at most one positive isolated equilibrium for all parameters if $|A| \neq 0$. The following lemma follows easily using linear algebra.

Lemma 5. *Assume that $|A| \neq 0$. System (1) has a unique positive equilibrium (x_{10}, x_{20}, x_{30}) if and only if the determinant of matrix B_i is not zero (i.e. $|B_i| \neq 0$) and $|A||B_i| < 0$ for $i = 1, 2, 3$, where $x_{10} = -|B_1|/|A|$, $x_{20} = -|B_2|/|A|$, $x_{30} = -|B_3|/|A|$ and*

$$B_1 = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}, B_2 = \begin{pmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{pmatrix}, B_3 = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{pmatrix}.$$

Suppose that the determinant of the matrix $A = (a_{ij})_{3\times 3}$ is zero (i.e. $|A| = 0$). Then equations (12) has infinitely many positive solutions if there exists a positive solution of (12). It is not difficult to check that the following lemma is true.

Lemma 6. *Assume that $|A| = 0$ and let*

$$A_b = \begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{pmatrix}.$$

Then system (1) has infinitely many positive equilibria (x_{10}, x_{20}, x_{30}) and these equilibria are not isolated if and only if one of the following conditions holds.

(i) *The rank of both matrices A and A_b is two, and system (1) has at least a planar equilibrium. In this case there exists a unique either segment, or half-line, which is filled with positive equilibria of system (1).*
(ii) *The rank of both matrices A and A_b is one, and system (1) has at least an axial equilibrium. In this case there exists a unique either a simplicial 2-complex, or a quarter-plane, which is filled with positive equilibria of system (1).*

In the following we only consider the case that system (1) has at least one positive equilibrium. Without loss of generality we move the positive equilibrium (x_{10}, x_{20}, x_{30}) to the point $(1, 1, 1)$ doing the change of variables

$$y_i = \frac{x_i}{x_{i0}}, \quad i = 1, 2, 3.$$

Note that $b_i = -\sum_{j=1}^{3} a_{ij} x_{j0}$ for $i = 1, 2, 3$. Then system (1) can be written as

$$\frac{dy_i}{dt} = y_i \sum_{j=1}^{3} a_{ij} x_{j0} (y_j - 1), \quad i = 1, 2, 3. \tag{13}$$

In order to study the dynamics of system (13) in a small neighborhood of the positive equilibrium $(1, 1, 1)$, we compute its Jacobian matrix

$$M = \begin{pmatrix} a_{11}x_{10} & a_{12}x_{20} & a_{13}x_{30} \\ a_{21}x_{10} & a_{22}x_{20} & a_{23}x_{30} \\ a_{31}x_{10} & a_{32}x_{20} & a_{33}x_{30} \end{pmatrix}.$$

The characteristic equation of system (13) at the equilibrium point $(1,1,1)$ is

$$\lambda^3 - \mathrm{tr}(M)\lambda^2 + m\lambda - |M| = 0, \tag{14}$$

where

$$\begin{aligned} \mathrm{tr}(M) =& a_{11}x_{10} + a_{22}x_{20} + a_{33}x_{30}, \\ m =& (a_{11}a_{22} - a_{12}a_{21})x_{10}x_{20} + (a_{11}a_{33} - a_{13}a_{31})x_{30}x_{10} \\ & + (a_{22}a_{33} - a_{23}a_{32})x_{20}x_{30}, \\ |M| =& (a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} \\ & - a_{12}a_{21}a_{33} + a_{11}a_{22}a_{33})x_{10}x_{20}x_{30} = |A|x_{10}x_{20}x_{30}. \end{aligned} \tag{15}$$

Hence equation (14) does not have a zero root if the positive equilibrium of system (13) is isolated (i.e. $|A| \neq 0$). And equation (14) has at least one zero root if system (13) has a continuum of positive equilibria containing the equilibrium $(1,1,1)$. We now consider the case that equation (14) has a zero root and a pair of pure imaginary roots, i.e. when system (13) has a positive zero-Hopf equilibrium. Some computations allow to obtain the next result.

Lemma 7. *System* (1) *has a positive zero-Hopf equilibrium* (x_{10}, x_{20}, x_{30}) *if and only if* (x_{10}, x_{20}, x_{30}) *is a non-isolated positive equilibrium of system* (1)*, and* $\mathrm{tr}(M) = 0$ *and* $m > 0$*, where the expressions of* $\mathrm{tr}(M)$ *and* m *are given in* (15)*.*

When $\mathrm{tr}(M) = 0$ and $m > 0$, if system (1) has a positive zero-Hopf equilibrium, it has a continuum of positive equilibria. This continuum of positive equilibria fills a segment or a half-line. Moreover, after some easy computations we obtain the following result.

Theorem 8. *System* (1) *has a unique positive zero-Hopf equilibrium if and only if there exist real numbers* k_1 *and* k_2*, and at least a pair* (i,j)*,* $i \in \{1,2\}$*,* $j \in \{1,2,3\}$ *and* $i \neq j$ *such that system* (1) *can be written into the form*

$$\begin{aligned} \frac{dy_1}{dt} =& y_1\left(b_{11}(y_1-1) + b_{12}(y_2-1) + b_{13}(y_3-1)\right), \\ \frac{dy_2}{dt} =& y_2\left(b_{21}(y_1-1) + b_{22}(y_2-1) + b_{23}(y_3-1)\right), \\ \frac{dy_3}{dt} =& y_3\left((k_1b_{11} + k_2b_{21})(y_1-1) + (k_1b_{12} + k_2b_{22})\right. \\ & \left.(y_2-1) + (k_1b_{13} + k_2b_{23})(y_3-1)\right), \end{aligned} \tag{16}$$

where

$$\begin{aligned} & b_{ii}b_{jj} - b_{ij}b_{ji} \neq 0, \\ & b_{11} + b_{22} + b_{13}k_1 + b_{23}k_2 = 0, \\ & b_{12}b_{21} - b_{11}b_{22} + k_1(b_{12}b_{23} - b_{13}b_{22}) + k_2(b_{13}b_{21} - b_{11}b_{23}) < 0. \end{aligned}$$

Now we shall investigate the normal form of the zero-Hopf equilibrium under a small perturbation of Lotka-Volterra type.

For doing this study we shall force that the point $(1,1,1)$ is also an equilibrium of the perturbed system (16) having eigenvalues ε, $\varepsilon u + vi$ and $\varepsilon u - vi$, where $v > 0$, $|\varepsilon| \ll 1$ and u are real parameters. Hence, when $\varepsilon = 0$, these eigenvalues are $0, vi$ and $-vi$, which implies that the perturbed system is a small perturbation of the zero-Hopf equilibrium $(1,1,1)$. Let

$$\Lambda \triangleq b_{13}(b_{21}b_{13} - b_{11}b_{23}) + b_{23}(b_{13}b_{22} - b_{12}b_{23}).$$

If $\Lambda \neq 0$, we consider a perturbed system (16) of the form

$$
\begin{aligned}
\frac{dy_1}{dt} &= y_1 \left(b_{11}(y_1-1) + b_{12}(y_2-1) + b_{13}(y_3-1) \right), \\
\frac{dy_2}{dt} &= y_2 \left(b_{21}(y_1-1) + b_{22}(y_2-1) + b_{23}(y_3-1) \right), \\
\frac{dy_3}{dt} &= y_3 \left(b_{31}(y_1-1) + b_{32}(y_2-1) + b_{33}(y_3-1) \right),
\end{aligned} \tag{17}
$$

where

$$
\begin{aligned}
b_{31} = &-\frac{1}{\Lambda}\big({b_{11}}^2 b_{13} b_{21} + b_{12} b_{13} {b_{21}}^2 + b_{11} b_{13} b_{21} b_{22} + b_{13} b_{21} {b_{22}}^2 \\
&- {b_{11}}^3 b_{23} - 2 b_{11} b_{12} b_{21} b_{23} - b_{12} b_{21} b_{22} b_{23} + b_{13} b_{21} v^2 \\
&- b_{11} b_{23} v^2 - b_{11} b_{13} b_{21} \varepsilon - b_{13} b_{21} b_{22} \varepsilon + {b_{11}}^2 b_{23} \varepsilon \\
&+ b_{12} b_{21} b_{23} \varepsilon - 2 b_{11} b_{13} b_{21} u \varepsilon - 2 b_{13} b_{21} b_{22} u \varepsilon + 2 {b_{11}}^2 b_{23} u \varepsilon \\
&+ 2 b_{12} b_{21} b_{23} u \varepsilon + b_{23} v^2 \varepsilon + 2 b_{13} b_{21} u \varepsilon^2 - 2 b_{11} b_{23} u \varepsilon^2 \\
&+ b_{13} b_{21} u^2 \varepsilon^2 - b_{11} b_{23} u^2 \varepsilon^2 + b_{23} u^2 \varepsilon^3 \big) \\
&\triangleq \alpha_0 + \alpha_1 \varepsilon + \alpha_2 \varepsilon^2 + \alpha_3 \varepsilon^3,
\end{aligned}
$$

$$
\begin{aligned}
b_{32} = &-\frac{1}{\Lambda}\big(b_{11} b_{12} b_{13} b_{21} + 2 b_{12} b_{13} b_{21} b_{22} + b_{13} {b_{22}}^3 - {b_{11}}^2 b_{12} b_{23} \\
&- {b_{12}}^2 b_{21} b_{23} - b_{11} b_{12} b_{22} b_{23} - b_{12} {b_{22}}^2 b_{23} + b_{13} b_{22} v^2 \\
&- b_{12} b_{23} v^2 - b_{12} b_{13} b_{21} \varepsilon - b_{13} {b_{22}}^2 \varepsilon + b_{11} b_{12} b_{23} \varepsilon \\
&+ b_{12} b_{22} b_{23} \varepsilon - 2 b_{12} b_{13} b_{21} u \varepsilon - 2 b_{13} {b_{22}}^2 u \varepsilon + 2 b_{11} b_{12} b_{23} u \varepsilon \\
&+ 2 b_{12} b_{22} b_{23} u \varepsilon - b_{13} v^2 \varepsilon + 2 b_{13} b_{22} u \varepsilon^2 - 2 b_{12} b_{23} u \varepsilon^2 \\
&+ b_{13} b_{22} u^2 \varepsilon^2 - b_{12} b_{23} u^2 \varepsilon^2 - b_{13} u^2 \varepsilon^3 \big) \\
&\triangleq \beta_0 + \beta_1 \varepsilon + \beta_2 \varepsilon^2 + \beta_3 \varepsilon^3, \\
b_{33} = &-b_{11} - b_{22} + \varepsilon + 2 u \varepsilon \\
&\triangleq \gamma_0 + \gamma_1 \varepsilon.
\end{aligned}
$$

It can be checked that the characteristic equation of system (17) at the equilibrium point $(1,1,1)$ has eigenvalues ε, $\varepsilon u + vi$ and $\varepsilon u - vi$ with $v > 0$.

We move the equilibrium $(1,1,1)$ to the origin, i.e. let $z_i = y_i - 1$, $i = 1,2,3$, then system (17) becomes

$$
\begin{aligned}
\frac{dz_1}{dt} &= (z_1+1)\left(b_{11} z_1 + b_{12} z_2 + b_{13} z_3 \right), \\
\frac{dz_2}{dt} &= (z_2+1)\left(b_{21} z_1 + b_{22} z_2 + b_{23} z_3 \right), \\
\frac{dz_3}{dt} &= (z_3+1)\left((\sum_{i=0}^{3} \alpha_i \varepsilon^i) z_1 + (\sum_{i=0}^{3} \beta_i \varepsilon^i) z_2 + (\sum_{i=0}^{1} \gamma_i \varepsilon^i) z_3 \right).
\end{aligned} \tag{18}
$$

The Jacobian matrix M of system (18) at $(0,0,0)$ has eigenvalues ε, $\varepsilon u+vi$ and $\varepsilon u-vi$ with $v>0$, where

$$M=\left(\begin{array}{ccc} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ \sum_{i=0}^{3}\alpha_i\varepsilon^i & \sum_{i=0}^{3}\beta_i\varepsilon^i & \sum_{i=0}^{1}\gamma_i\varepsilon^i \end{array}\right).$$

Doing a convenient linear change of variables, the Jacobian matrix M of system (18) can be written in its real Jordan normal form

$$J=\left(\begin{array}{ccc} u\varepsilon & -v & 0 \\ v & u\varepsilon & 0 \\ 0 & 0 & \varepsilon \end{array}\right).$$

Indeed, assume that $b_{12}b_{23}-b_{13}b_{22}\neq 0$. Then we consider the change of variables $(z_1,z_2,z_3)\to(U,V,W)$ given by

$$\left(\begin{array}{c} U \\ V \\ W \end{array}\right)=\left(\begin{array}{ccc} p_{11} & p_{12} & p_{13} \\ p_{21} & 1 & 0 \\ p_{31} & p_{32} & 1 \end{array}\right)\left(\begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array}\right), \tag{19}$$

where

$$\begin{aligned}
p_{11}=&\frac{1}{v(b_{12}b_{23}-b_{13}(b_{22}-\varepsilon))}\big(b_{23}u\varepsilon^2+(b_{13}b_{21}-b_{11}b_{23})(1+u)\varepsilon \\
&-b_{11}b_{13}b_{21}-b_{13}b_{21}b_{22}+{b_{11}}^2b_{23}+b_{12}b_{21}b_{23}\big),\\
p_{12}=&-\frac{1}{v(b_{12}b_{23}-b_{13}(b_{22}-\varepsilon))}\left(b_{13}(b_{22}-\varepsilon)(b_{22}-u\varepsilon)\right.\\
&\left.+b_{12}(b_{13}b_{21}-(b_{11}+b_{22})b_{23}+b_{23}(1+u)\varepsilon)\right),\\
p_{13}=&\frac{1}{v(b_{12}b_{23}-b_{13}(b_{22}-\varepsilon))}\big(b_{13}b_{23}(b_{11}-b_{22})+b_{12}{b_{23}}^2-{b_{13}}^2b_{21}\big),\\
p_{21}=&\frac{1}{v(b_{12}b_{23}-b_{13}(b_{22}+\varepsilon))}\left(b_{13}b_{21}-b_{11}b_{23}+b_{23}\varepsilon\right),\\
p_{31}=&-\frac{1}{\Lambda}\big(b_{13}b_{21}(b_{22}-2u\varepsilon)-{b_{11}}^2b_{23}+b_{11}(b_{13}b_{21}+2b_{23}u\varepsilon)\\
&-b_{23}(b_{12}b_{21}+v^2+u^2\varepsilon^2)\big),\\
p_{32}=&\frac{1}{\Lambda}\big(b_{12}(b_{13}b_{21}-b_{23}(b_{11}+b_{22}-2u\varepsilon))+b_{13}(v^2+(b_{22}-u\varepsilon)^2)\big).
\end{aligned}$$

When $b_{12}b_{23}-b_{13}b_{22}\neq 0$ and $|\varepsilon|\ll 1$, the linear transformation (19) is nonsingular. Thus, in the new variables (U,V,W) system (18) becomes

$$\begin{aligned}
\frac{dU}{dt}=&u\varepsilon U-vV+a_{200}U^2+a_{110}UV+a_{101}UW+a_{020}V^2\\
&+a_{011}VW+a_{002}W^2,\\
\frac{dV}{dt}=&vU+u\varepsilon V+b_{200}U^2+b_{110}UV+b_{101}UW+b_{020}V^2\\
&+b_{011}VW+b_{002}W^2,\\
\frac{dW}{dt}=&\varepsilon W+c_{200}U^2+c_{110}UV+c_{101}UW+c_{020}V^2\\
&+c_{011}VW+c_{002}W^2,
\end{aligned} \tag{20}$$

where the coefficients a_{ijk}, b_{ijk} and c_{ijk} for $i,j,k = 0,1,2$ are functions of the parameters b_{ij}, α_i, β_i and γ_i of system (18), and these functions are smooth at $\varepsilon = 0$, their explicit expressions are too long and we omit them.

From the foregoing we obtain the following result.

Theorem 9. *If system* (1) *has a non-isolated positive zero-Hopf equilibrium* (x_{10}, x_{20}, x_{30})*. Then system* (1) *is topologically equivalent to system* (16)*. Furthermore, if* $\Lambda \neq 0$ *and* $b_{13}b_{22} - b_{12}b_{23} \neq 0$*, then there exists a small perturbation of system* (16) *such that the perturbed system has a non-isolated positive zero-Hopf equilibrium at* $(1,1,1)$*, which has the normal form* (20) *with* $0 < |\varepsilon| \ll 1$.

Therefore, we can apply the main result in the previous section to system (20) and obtain the conclusions on non-isolated zero-Hopf bifurcation for three-dimensional Lotka-Volterra systems (1). In the next section, we will provide an example of three-dimensional Lotka-Volterra system to illustrate the conclusions.

4. An example of non-isolated zero-Hopf bifurcation for three-dimensional Lotka-Volterra systems

In this section we construct a concrete example of three-dimensional Lotka-Volterra systems according to Theorem 8 and Theorem 9. It is shown that this system undergoes non-isolated zero-Hopf bifurcation, and two limit cycles can be bifurcated from a non-isolated zero-Hopf equilibrium under some conditions.

We consider the following three-parameters Lotka-Volterra system in the first octant $\mathbb{R}^3_+$.

$$
\begin{aligned}
\frac{dx}{dt} =& x\left(2v - 2vx - vy + vz\right),\\
\frac{dy}{dt} =& y\left(-3v + vx + vy + vz\right),\\
\frac{dz}{dt} =& \frac{z}{5v^2}\big(10v^3 + 10uv\varepsilon^2 + 5u^2v\varepsilon^2 - x(10v^3 + 5v^2\varepsilon\\
&+8uv^2\varepsilon + 6uv\varepsilon^2 + 3u^2v\varepsilon^2 + u^2\varepsilon^3) + y(-5v^3 - 2uv^2\varepsilon\\
&-4uv\varepsilon^2 - 2u^2v\varepsilon^2 + u^2\varepsilon^3) + z(5v^3 + 5v^2\varepsilon + 10uv^2\varepsilon)\big),
\end{aligned}
\tag{21}
$$

where $0 \leq \varepsilon \ll 1$, $v > 0$ and u are bounded parameters.

When $\varepsilon = 0$, there exists a segment l with endpoints $(0, 5/2, 1/2)$ and $(5/3, 0, 4/3)$ such that each point in l is a positive equilibrium of system (21), where

$$
l = \left\{(x,y,z):\ x = 1+s,\ y = 1 - \frac{3}{2}s,\ z = 1 + \frac{1}{2}s,\ -1 \leq s \leq \frac{2}{3}\right\}.
$$

It can be checked that there is a unique point $(1,1,1)$ on the segment l, which is a zero-Hopf equilibrium of system (21). Hence, system (21) has a non-isolated zero-Hopf equilibrium at $(1,1,1)$ when $\varepsilon = 0$. We are interested in studying the number of limit cycles bifurcating from this non-isolated zero-Hopf equilibrium when $0 < \varepsilon \ll 1$.

Doing the change of variables

$$
X = x - 1,\ Y = y - 1,\ Z = z - 1,
$$

we move the equilibrium $(1,1,1)$ to the origin and we obtain the system

$$
\begin{aligned}
\frac{dX}{dt} =& -v(1+X)(2X+Y-Z),\\
\frac{dY}{dt} =& v(1+Y)(X+Y+Z),\\
\frac{dZ}{dt} =& \frac{1+Z}{5v^2}\big(Z(5v^3+5v^2\varepsilon+10uv^2\varepsilon)-X(10v^3+5v^2\varepsilon\\
&+8uv^2\varepsilon+6uv\varepsilon^2+3u^2v\varepsilon^2+u^2\varepsilon^3)+Y(-5v^3-2uv^2\varepsilon\\
&-4uv\varepsilon^2-2u^2v\varepsilon^2+u^2\varepsilon^3)\big).
\end{aligned} \tag{22}
$$

The Jacobian matrix of system (22) at $(0,0,0)$ has eigenvalues ε, $\varepsilon u+vi$ and $\varepsilon u-vi$ with $v>0$. To obtain the real Jordan normal form of system (22) at the origin, we do the linear transformation

$$
\begin{pmatrix} U_1\\ V_1\\ W_1 \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & p_{13}\\ p_{21} & 1 & 0\\ p_{31} & p_{32} & 1 \end{pmatrix} \begin{pmatrix} X\\ Y\\ Z \end{pmatrix},
$$

where

$$
\begin{aligned}
&p_{11}=3+5u+\frac{u\varepsilon}{v}+\frac{10(1+u)v}{-2v+\varepsilon},\; p_{12}=2-\frac{u\varepsilon}{v}+\frac{5v}{-2v+\varepsilon},\; p_{13}=-\frac{5v}{-2v+\varepsilon},\\
&p_{21}=-1-\frac{5v}{-2v+\varepsilon},\; p_{31}=-\frac{(v+u\varepsilon)(5v+u\varepsilon)}{5v^2},\; p_{32}=\frac{u\varepsilon(-4v+u\varepsilon)}{5v^2}.
\end{aligned}
$$

Then in the new variables (U_1,V_1,W_1) system (22) becomes

$$
\begin{aligned}
\frac{dU_1}{dt} =& u\varepsilon U_1 - vV_1 + a_{200}(\varepsilon)U_1^2 + a_{110}(\varepsilon)U_1V_1\\
&+a_{101}(\varepsilon)U_1W_1+a_{020}(\varepsilon)V_1^2+a_{011}(\varepsilon)V_1W_1+a_{002}(\varepsilon)W_1^2,\\
\frac{dV_1}{dt} =& vU_1 + u\varepsilon V_1 + b_{200}(\varepsilon)U_1^2 + b_{110}(\varepsilon)U_1V_1\\
&+b_{101}(\varepsilon)U_1W_1+b_{020}(\varepsilon)V_1^2+b_{011}(\varepsilon)V_1W_1+b_{002}(\varepsilon)W_1^2,\\
\frac{dW_1}{dt} =& \varepsilon W_1 + c_{200}(\varepsilon)U_1^2 + c_{110}(\varepsilon)U_1V_1 + c_{101}(\varepsilon)U_1W_1\\
&+c_{020}(\varepsilon)V_1^2+c_{011}(\varepsilon)V_1W_1+c_{002}(\varepsilon)W_1^2,
\end{aligned} \tag{23}
$$

where $a_{ijk}(\varepsilon)$, $b_{ijk}(\varepsilon)$ and $c_{ijk}(\varepsilon)$ have the following expressions

$$
\begin{aligned}
&a_{200}(\varepsilon) = \frac{2v}{5} + \frac{\varepsilon}{5}(-3+8u) + O(\varepsilon^2),\ a_{110}(\varepsilon) = \frac{2v}{5} + \frac{\varepsilon}{5}(17+18u) + O(\varepsilon^2),\\
&a_{101}(\varepsilon) = (6-2u)\varepsilon + O(\varepsilon^2),\ a_{020}(\varepsilon) = -10\varepsilon + O(\varepsilon^2),\\
&a_{011}(\varepsilon) = -3v + (-7-6u)\varepsilon + O(\varepsilon^2),\ a_{002}(\varepsilon) = -10\varepsilon + O(\varepsilon^2),\\
&b_{200}(\varepsilon) = \frac{12\varepsilon}{5} + O(\varepsilon^2),\ b_{110}(\varepsilon) = -2v + \frac{7\varepsilon}{5} + O(\varepsilon^2),\\
&b_{101}(\varepsilon) = -12\varepsilon + O(\varepsilon^2),\ b_{020}(\varepsilon) = \frac{2}{5}(-6+7u)\varepsilon + O(\varepsilon^2),\\
&b_{011}(\varepsilon) = 6v - \varepsilon + O(\varepsilon^2),\ b_{002}(\varepsilon) = 15\varepsilon + O(\varepsilon^2),\\
&c_{200}(\varepsilon) = \frac{16u\varepsilon}{25} + O(\varepsilon^2),\ c_{110}(\varepsilon) = \frac{56u\varepsilon}{25} + O(\varepsilon^2),\\
&c_{101}(\varepsilon) = \frac{2v}{5} + \frac{1}{5}(7-4u)\varepsilon + O(\varepsilon^2),\ c_{020}(\varepsilon) = -\frac{16u\varepsilon}{25} + O(\varepsilon^2),\\
&c_{011}(\varepsilon) = -\frac{4v}{5} + (\frac{6}{5} - \frac{22u}{5})\varepsilon + O(\varepsilon^2),\ c_{002}(\varepsilon) = -3\varepsilon + O(\varepsilon^2).
\end{aligned}
$$

It can be checked that system (23) satisfies hypothesis (H_1) and (H_2). Hence, by Theorem 4, we have the following conclusion.

Theorem 10. *Suppose that $u < 0$ and $u \neq -1/8$. Then there exists a small positive number ε^* such that for any ε, $0 < \varepsilon \leq \varepsilon^*$, system (23) has two limit cycles, that when $t = 0$ are near the points $(5\sqrt{-2u\varepsilon/v}\cos\theta, 5\sqrt{-2u\varepsilon/v}\sin\theta, 0)$ and $(5\sqrt{\varepsilon/(4v)}\cos\theta, 5\sqrt{\varepsilon/(4v)}\sin\theta, -(1+8u)\sqrt{\varepsilon}/(24v))$.*

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