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ALGEBRAIC INVARIANT CURVES AND ALGEBRAIC FIRST INTEGRALS FOR RICCATI POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. We characterize the algebraic invariant curves for the Riccati polynomial differential systems of the form $x' = 1$, $y' = a(x)y^2 + b(x)y + c(x)$, where $a(x)$, $b(x)$ and $c(x)$ are arbitrary polynomials. We also characterize their algebraic first integrals.

1. Introduction and statement of the main results

One of the more classical problems in the qualitative theory of planar differential equations depending on parameters is to characterize the existence or not of first integrals in function of these parameters.

Let x and y be complex variables. We consider the system

$$x' = 1, \qquad y' = a(x)y^2 + b(x)y + c(x), \tag{1}$$

where $a(x)$, $b(x)$ and $c(x)$ are C^1 functions on x and the prime denotes derivative with respect to the time t that can be either real or complex. In fact, if $a(x)c(x) \not\equiv 0$ these systems are called *Riccati differential systems*, if $a(x) \not\equiv 0$ and $c(x) \equiv 0$ they are a particular case of *Bernouilli differential systems*, and if $a(x) \equiv 0$ then they are *linear differential systems*.

Differential systems (1) are named after Count Jacobo Francesco Riccati (1676–1754). These equations have been intensively studied, and hundreds of applications have been found. Thus looking at MathSciNet there are more than 4000 papers having in their title the words “Riccati equations”. These equations have been studied in many books, see for instance [5, 6, 7, 11, 13, 17]. They are important since they can be used to solve second-order ordinary differential equations. An important application of the Riccati differential systems is to the 3rd order Schwarzian differential equation which appears in the theory of conformal mapping and univalent functions, see [12] for more details and references.

Our interest is on the *Riccati polynomial differential systems*, i.e. when the functions $a(x)$, $b(x)$ and $c(x)$ are polynomials. More precisely, we want to study the invariant algebraic curves of the Riccati polynomial differential systems and its algebraic integrability. But this problem has a long history. Thus when the functions $a(x)$, $b(x)$ and $c(x)$ are rational the irreducible invariant algebraic curves $f(x, y) = 0$

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(f is a polynomial) only can have as polynomial in the variable y degrees $1, 2, 4, 6$ and 12, see for more details Kovacic [8] (1986), who provided a constructive algorithm for finding invariant algebraic curves. In any case before arriving to this nice result there were many preliminary studies. The first known algorithm for finding invariant algebraic curves was due to Liouville [9] (1833). Later on Fuchs [3] (1878), and Pépin [14] (1881) provided also algorithms for finding invariant algebraic curves, see Ulmer and Weil [16] (1996). Singer in [15] (1979) states that all algorithms of these authors did not provide a complete description procedure, and the first complete algorithm was due to Baldassarri and Dwork [1] (1979).

The vector field associated to system (1) is

$$X = \frac{\partial}{\partial x} + (a(x)y^2 + b(x)y + c(x))\frac{\partial}{\partial y}.$$

The two main objectives of this paper are to characterize: (i) the algebraic first integrals, and (ii) the invariant algebraic curves of the Ricatti polynomial differential systems.

Partial results in this direction were obtained by Żołądek in [18] where the author characterized the possible algebraic solutions for the Riccati polynomial differential systems (1) with $a(x) = 1$ and $b(x) = 0$.

Let $U \subset \mathbb{C}^2$ be an open set. We say that the non-constant function $H\colon \mathbb{C}^2 \to \mathbb{C}$ is a *first integral* of the polynomial vector field X on U if $H(x(t), y(t))$ is constant for all values of t for which the solution $(x(t), y(t))$ of X is defined on U. Clearly H is a first integral of X on U if and only if $XH = 0$ on U.

An *algebraic first integral* is a first integral which is an algebraic function. An *algebraic function* $f(x) = C$ is a solution of

$$q_0(x) + q_1(x)C + q_2(x)C^2 + \cdots + q_{s-1}(x)C^{s-1} + C^s = 0,$$

where $q_j(x)$ are rational functions of x and s is the smallest positive integer for which such a relation holds. Of course, all rational functions are algebraic functions.

Theorem 1. *The Riccati polynomial differential system* (1) *has an algebraic first integral $H = H(x, y)$ if and only if $c(x) = \kappa^2 a(x)$, $b(x) = 2\kappa a(x)$ for some $\kappa \in \mathbb{C}$, and in this case the algebraic first integrals are algebraic functions in the variable $H = \int a(x)\,dx + 1/(y + \kappa)$.*

For proving Theorem 1 we need to characterize the Darboux polynomials of the Ricatti polynomial differential systems. Let $h = h(x, y) \in \mathbb{C}[x, y]\setminus\mathbb{C}$. As usual $\mathbb{C}[x, y]$ denotes the ring of all complex polynomials in the variables x and y. We say that $h = 0$ is an *invariant algebraic curve* of the vector field X associated to the Riccati polynomial differential system (1) if it satisfies

$$\frac{\partial h}{\partial x} + (a(x)y^2 + b(x)y + c(x))\frac{\partial h}{\partial y} = Kh,$$

the polynomial $K = K(x, y) \in \mathbb{C}[x, y]$ is called *the cofactor* of $h = 0$ and has degree at most

$$\max\{2 + \deg a(x), 1 + \deg b(x), \deg c(x)\} - 1.$$

When $h = 0$ is an invariant algebraic curve we also say that h is a *Darboux polynomial* of the Riccati polynomial differential system. Note that a *polynomial first integral* is a Darboux polynomial with zero cofactor.

Theorem 2. *The following statements hold for the Riccati polynomial differential systems.*

(a) *They have no polynomial first integrals.*

(b) *They have an irreducible Darboux polynomial f with nonzero cofactor K if and only if $c(x) = \kappa(b(x) - \kappa a(x))$ for some $\kappa \in \mathbb{C}$. In this case $f = y + \kappa$ and $K = b(x) + a(x)(y - \kappa)$. If additionally, $b(x) = \kappa_1 a(x)$ for some $\kappa_1 \in \mathbb{C}$ then:*

(b.1) *$f = y + \kappa_1 - \kappa$ is another irreducible Darboux polynomial with the nonzero cofactor $K = a(x)(y + \kappa)$, if $\kappa_1 \neq \kappa$.*

(b.2) *$f = 1 + (y + \kappa) \int a(x)\, dx$ is another irreducible Darboux polynomial with the nonzero cofactor $K = a(x)(y + \kappa)$, if $\kappa_1 = \kappa$.*

The proof of Theorems 1 and 2 are given in Section 2.

Let $\mathbb{R}[x, y]$ and $\mathbb{C}[x, y]$ be the ring of all real and complex polynomials in the variables x and y respectively. We note, from Theorem 2, that all complex irreducible Darboux polynomials in $\mathbb{C}[x, y]$ for the complex Riccati polynomial differential systems have degree 1 in the variable y. While also from Theorem 2 it follows that all real irreducible Darboux polynomials in $\mathbb{R}[x, y]$ for the real Riccati polynomial differential systems (i.e. when the polynomials $a(x)$, $b(x)$ and $c(x)$ are real) can have degree 1 or 2 in the variable y. So, there is a big difference between the Riccati rational differential systems and the Riccati polynomial differential systems, because as we have mentioned for the rational ones their invariant algebraic curves can have degrees $1, 2, 4, 6$ and 12 in the variable y.

2. Proof of Theorems 1 and 2

We separate the proof of Theorem 2 in several steps.

Lemma 3. *Riccati polynomial differential systems* (1) *have no polynomial first integrals.*

Proof. We proceed by contradiction. Let H be a polynomial first integral of system (1), that is

$$\frac{\partial H}{\partial x} + (a(x)y^2 + b(x)y + c(x))\frac{\partial H}{\partial y} = 0. \tag{2}$$

We write H as a polynomial in the variable y, i.e.

$$H(x, y) = \sum_{j=0}^{m} h_j(x)y^j, \quad \text{where } h_j(x) \text{ is a polynomial in the variable } x.$$

Without loss of generality we can assume that $h_m(x) \neq 0$.

Computing the coefficient of degree $m + 1$ in the variable y in (2) we get that

$$m\, a(x)h_m(x) = 0.$$

Since $a(x)h_m(x) \neq 0$, we must have $m = 0$. Then $H = h_0(x)$. In view of (2) we get that H satisfies

$$H'(x) = 0, \quad \text{that is} \quad H(x) \in \mathbb{C},$$

a contradiction with the fact that H is a polynomial first integral. □

The proof of the following proposition is well-known and can be found in [2].

Proposition 4. *We suppose that* $h \in \mathbb{C}[x, y]$ *and let* $h = h_1^{n_1} \cdots h_r^{n_r}$ *be its factorization in irreducible factors over* $\mathbb{C}[x, y]$. *Then for a polynomial system* (1) $h = 0$ *is an invariant algebraic curve with cofactor* K_h *if and only if* $h_i = 0$ *is an invariant algebraic curve for each* $i = 1, \ldots, r$ *with cofactor* K_{h_i}. *Moreover* $K = n_1K_{h_1} + \cdots + n_rK_{h_r}$.

In view of Proposition 4 to study the Darboux polynomials with non-zero cofactor it is enough to study the irreducible ones.

Lemma 5. *Let* $h = h(x, y)$ *be an irreducible Darboux polynomial of the Riccati polynomial differential system* (1) *with cofactor* $K \neq 0$. *Then* $K = K_0(x) + m\, a(x)y$ *with* m *a non-negative integer.*

Proof. We define the degree n of the Riccati polynomial differential system (1) as

$$n = \max\{2 + \deg a(x), 1 + \deg b(x), \deg c(x)\}.$$

We note that $n \geq 2$. Then the cofactor K has degree at most $n - 1$. We write it as $K(x, y) = \sum_{j=0}^{n-1} K_j(x)y^j$, where $K_j = K_j(x)$ is a polynomial in the variable x and has at most degree $n - 1 - j$. Since h is a Darboux polynomial of system (1) with cofactor K it satisfies

$$\frac{\partial h}{\partial x} + (a(x)y^2 + b(x)y + c(x))\frac{\partial h}{\partial y} = \left(\sum_{j=0}^{n-1} K_j(x)y^j\right)h. \tag{3}$$

We write h as a polynomial in the variable y, i.e. $h(x, y) = \sum_{j=0}^{m} h_j(x)y^j$, where each $h_j(x)$ is a polynomial in the variable x. Without loss of generality we can assume that $h_m(x) \neq 0$.

Assume $n \geq 3$. Computing the coefficient of y^{n+m-1} in (3) we get

$$0 = K_{n-1}(x)h_m(x) \quad \text{that is} \quad K_{n-1}(x) = 0.$$

Now proceeding inductively, we can show that for $\ell = n - 1, n - 2, \ldots, 3$ we have $K_\ell(x) = 0$. Therefore, $K = K_0(x) + yK_1(x)$. So, for $n \geq 2$ we have that always the cofactor of an irreducible Darboux polynomial is of the form $K = K_0(x) + yK_1(x)$.

Now computing the coefficient of y^{m+1} in (3) we get

$$m\, a(x)h_m(x) = h_m(x)K_1(x),$$

that is $(m\, a(x) - K_1(x))h_m(x) = 0$. Using that $h_m(x) \neq 0$ we must have $K_1(x) = m\, a(x)$ with m a non-negative integer. This completes the proof. □

Proposition 6. *The Riccati polynomial differential systems* (1) *have an irreducible Darboux polynomial* f *with nonzero cofactor* K *if and only if* $c(x) = \kappa(b(x) - \kappa a(x))$ *for some* $\kappa \in \mathbb{C}$. *In this case* $f = y + \kappa$ *and* $K = b(x) + a(x)(y - \kappa)$. *If additionally,* $b(x) = \kappa_1 a(x)$ *for some* $\kappa_1 \in \mathbb{C}$ *then:*

(b.1) *$f = y + \kappa_1 - \kappa$ is another irreducible Darboux polynomial with the nonzero cofactor $K = a(x)(y + \kappa)$, if $\kappa_1 \neq \kappa$.*

(b.2) *$f = 1 + (y + \kappa)\int a(x)\,dx$ is another irreducible Darboux polynomial with the nonzero cofactor $K = a(x)(y + \kappa)$, if $\kappa_1 = \kappa$.*

Proof. We write the Riccati polynomial differential system (1) as the differential equation

$$\frac{dy}{dx} = a(x)y^2 + b(x)y + c(x), \quad y = y(x). \tag{4}$$

Then, using Lemma 5 the Darboux polynomial $h = h(x, y) = h(x, y(x))$ satisfies

$$\frac{dh}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y}(a(x)y^2 + b(x)y + c(x)) = (K_0(x) + m\,a(x)y)h, \quad m \in \mathbb{N} \tag{5}$$

or equivalently

$$\log h = K + \int (K_0(x) + m\,a(x)y)\,dx, \quad \text{where} \quad K \in \mathbb{C}.$$

Hence

$$h = h(x, y(x)) = C\exp\left(\int (K_0(x) + m\,a(x)y)\,dx\right), \quad C \in \mathbb{C}\setminus\{0\}. \tag{6}$$

Now we write

$$a(x)y^2 + b(x)y + c(x) = (y + \Gamma_1(x))(a(x)y + \Gamma_2(x)) = (a(x)y + \tilde{\Gamma}_1(x))(y + \tilde{\Gamma}_2(x)), \tag{7}$$

where

$$\Gamma_1(x) = \frac{b(x)}{2a(x)} - \frac{\sqrt{b(x)^2 - 4a(x)c(x)}}{2a(x)}, \quad \Gamma_2(x) = \frac{b(x)}{2} + \frac{\sqrt{b(x)^2 - 4a(x)c(x)}}{2}, \tag{8}$$

and $\tilde{\Gamma}_1(x) = a(x)\Gamma_1(x)$, $\tilde{\Gamma}_2(x) = \Gamma_2(x)/a(x)$. We consider different cases.

Case 1*:* $\Gamma_1(x) = \kappa \in \mathbb{C}$. In this case, from (8) we have that

$$\frac{b(x)}{2a(x)} - \frac{\sqrt{b(x)^2 - 4a(x)c(x)}}{2a(x)} = \kappa, \quad \text{i.e.} \quad b(x) - \sqrt{b(x)^2 - 4a(x)c(x)} = 2\kappa a(x),$$

which yields

$$c(x) = \kappa(b(x) - \kappa a(x)). \tag{9}$$

Then, again from (8) we get

$$\Gamma_2(x) = b(x) - \kappa a(x).$$

Hence, it follows from (4) and (7) that

$$\frac{dy}{dx} = (y + \kappa)(a(x)y + \Gamma_2(x)), \quad \text{that is} \quad a(x)y + \Gamma_2(x) = \frac{dy/dx}{y + \kappa} = \frac{d\log(y + \kappa)}{dx},$$

which implies that (6) becomes

$$
\begin{aligned}
h &= C\exp\Big(\int (K_0(x) - m\Gamma_2(x) + m(a(x)y + \Gamma_2(x)))\,dx\Big)\\
&= C\exp\Big(\int \Big(K_0(x) - m\Gamma_2(x) + m\frac{d\log(y+\kappa)}{dx}\Big)\,dx\Big)\\
&= C(y+\kappa)^m \exp\Big(\int (K_0(x) - m\,b(x) + m\,a(x)\kappa)\,dx\Big)
\end{aligned}
$$

with $C \in \mathbb{C}\setminus\{0\}$. Since h must be a polynomial and $K_0(x)$ is a polynomial, we must have $K_0(x) = m\,b(x) - m\,a(x)\kappa$ and $h(x) = C(y+\kappa)^m$. Now using that h is irreducible we get that $m = 1$ and then $f = y+\kappa$ and $K = K_0(x) + m\,a(x)y = b(x) + a(x)(y-\kappa)$.

Now we consider two subcases.

Subcase 1.1*:* $\tilde\Gamma_2(x) = \Gamma_2(x)/a(x) \notin \mathbb{C}$. Now using that $\Gamma_1(x) = \kappa$ (and thus $\tilde\Gamma_1(x) = a(x)\kappa$), it follows from (4) and the second equation in (7) that

$$
a(x)(y+\kappa) = \frac{dy/dx}{y+\tilde\Gamma_2(x)} = \frac{d\log(y+\tilde\Gamma_2)}{dx} - \frac{d\tilde\Gamma_2/dx}{y+\tilde\Gamma_2(x)}.
$$

Therefore

$$
h = C(y+\tilde\Gamma_2(x))^m \exp\Big(\int (K_0(x) - m\,a(x)\kappa)\,dx\Big)\exp\Big(-m\int \frac{d\tilde\Gamma_2/dx}{y+\tilde\Gamma_2(x)}\,dx\Big),
$$

which is never a polynomial in the variables x and y. Hence in this case there is no more irreducible Darboux polynomials with non zero cofactor.

Subcase 1.2*:* $\tilde\Gamma_2(x) = \Gamma_2(x)/a(x) = \kappa_1 \in \mathbb{C}$. In this case $\Gamma_2(x) = \kappa_1 a(x) \in \mathbb{C}$, that is, $b(x) - \kappa a(x) = \kappa_1 a(x)$, i.e. $b(x) = (\kappa+\kappa_1)a(x)$, and from (9), we have that $c(x) = \kappa(b(x) - \kappa a(x)) = \kappa\kappa_1 a(x)$. Therefore since $\Gamma_1(x) = \kappa$ (and thus $\tilde\Gamma_1(x) = a(x)\kappa$), it follows from (4) and the second equation in (7) that the Riccati polynomial differential system is in this case

$$
\dot x = 1, \quad \dot y = a(x)(y+\kappa)(y+\kappa_1). \tag{10}
$$

We consider two different subcases.

Subcase 1.2.1*:* $\kappa_1 \neq \kappa$. In this case we have that $f = y+\kappa_1$ is an irreducible Darboux polynomial of the Riccati polynomial differential system (see (10)) with cofactor $K = a(x)(y+\kappa)$. No we shall prove that it is the only one. We proceed by contradiction. Assume $h = h(x,y)$ is another irreducible Darboux polynomial of system (10). Then if we denote by $\bar h = \bar h(x)$ the restriction of h to $y = -\kappa$, we get from (5) that $\bar h$ satisfies

$$
\frac{d\bar h}{dx} = (K_0 - ma(x)\kappa)\bar h.
$$

Solving this linear differential equation we obtain

$$
\bar h = Ce^{\int (K_0 - ma(x)\kappa)\,dx}, \quad C \in \mathbb{C}. \tag{11}
$$

It is clear that $\bar h \neq 0$ otherwise h would be reducible. Hence, since $\bar h$ must be a polynomial we get $K_0 = ma(x)\kappa$.

Furthermore, if we denote by $\tilde{h} = \tilde{h}(x)$ the restriction of h to $y = -\kappa_1$ and recalling that $K_0 = ma(x)\kappa$, we get from (5) that $\tilde{h}$ satisfies

$$\frac{d\tilde{h}}{dx} = ma(x)(\kappa - \kappa_1)\tilde{h}.$$

Solving this linear differential equation we obtain

$$\tilde{h} = Ce^{\int ma(x)(\kappa-\kappa_1)\,dx}, \quad C \in \mathbb{C}.$$

It is clear that $\tilde{h} \neq 0$ otherwise h would be reducible. Hence, since $\tilde{h}$ must be a polynomial we get $ma(x)(\kappa - \kappa_1) = 0$, which is a contradiction. Hence, the proof of this case is completed.

Subcase 1.2.2*:* $\kappa_1 = \kappa$. In this case we have that

$$f = 1 + (y + \kappa)\int a(x)\,dx$$

is an irreducible Darboux polynomial of the Riccati polynomial differential system (see (10)) with cofactor $K = a(x)(y + \kappa)$. No we shall prove that it is the only one. We proceed by contradiction. Assume $h = h(x, y)$ is another irreducible Darboux polynomial of system (10). Then if we denote by $\bar{h} = \bar{h}(x)$ the restriction of h to $y = -\kappa$, we get from (5) that $\bar{h}$ satisfies (11). Proceeding as in Subcase 1.2.1 we get that $K_0 = ma(x)\kappa$.

Furthermore, if we denote by $\tilde{h} = \tilde{h}(x)$ the restriction of h to $y = -\kappa - 1/\int a(x)\,dx$ and recalling that $K_0 = ma(x)\kappa$, we get from (5) that $\tilde{h}$ satisfies

$$\frac{d\tilde{h}}{dx} = -\frac{ma(x)}{\int a(x)\,dx}\tilde{h}.$$

Solving this linear differential equation we obtain

$$\tilde{h} = Ce^{\int -\frac{ma(x)}{\int a(u)\,du}\,dx}, \quad C \in \mathbb{C}.$$

It is clear that $\tilde{h} \neq 0$ otherwise h would be reducible. Hence, since $\tilde{h}$ must be a polynomial we get $\frac{ma(x)}{\int a(x)\,dx} = 0$, which is a contradiction. Hence, the proof of this case is completed.

Case 2*:* $\Gamma_1(x) \notin \mathbb{C}$ *and* $\tilde{\Gamma}_2 = \Gamma_2(x)/a(x) = \kappa \in \mathbb{C}$. In this case, from (8) we have that

$$\frac{b(x)}{2a(x)} + \frac{\sqrt{b(x)^2 - 4a(x)c(x)}}{2a(x)} = \kappa, \quad \text{i.e.} \quad b(x) + \sqrt{b(x)^2 - 4a(x)c(x)} = 2\kappa a(x),$$

which yields

$$c(x) = \kappa(b(x) - \kappa a(x)) \quad \text{and} \quad b(x) = 2\kappa a(x),$$

which is not possible because then we have $\Gamma_1(x) = \kappa \in \mathbb{C}$.

Case 3*:* $\Gamma_1(x) \notin \mathbb{C}$ *and* $\tilde{\Gamma}_2 = \Gamma_2(x)/a(x) \notin \mathbb{C}$. In this case, from (8) we can write

$$(a(x)y + \Gamma_2(x)) = \frac{dy/dx}{y + \Gamma_1(x)} = \frac{d\log(y + \Gamma_1)}{dx} - \frac{d\Gamma_1/dx}{y + \Gamma_1(x)}$$

and analogously,

$$(a(x)y + \tilde{\Gamma}_1(x)) = \frac{dy/dx}{y + \tilde{\Gamma}_2(x)} = \frac{d\log(y + \tilde{\Gamma}_2)}{dx} - \frac{d\tilde{\Gamma}_2/dx}{y + \tilde{\Gamma}_2(x)}.$$

In the first case, from (6), we have that

$$h = C(y + \Gamma_1(x))^m \exp\Big(\int (K_1(x) - m\Gamma_2(x))\,dx\Big) \exp\Big(-m\int \frac{d\Gamma_1/dx}{y + \Gamma_1(x)}\,dx\Big),$$

which is never a polynomial in the variables x and y. In the second case we have

$$h = C(y + \tilde{\Gamma}_2(x))^m \exp\Big(\int (K_1(x) - m\tilde{\Gamma}_1(x))\,dx\Big) \exp\Big(-m\int \frac{d\tilde{\Gamma}_2/dx}{y + \tilde{\Gamma}_2(x)}\,dx\Big),$$

which again is never a polynomial in the variables x and y. Hence this case is never possible. This completes the proof of the proposition. □

Proof of Theorem 2. The proof of Theorem 2 is an immediate consequence of Lemma 3 and Proposition 6. □

To prove Theorem 1 we need the following two auxiliary results. For a proof of the first result see [10] and for a proof of the second one, see [4].

Proposition 7. *The existence of a rational first integral for a polynomial differential system* (1) *implies either the existence of a polynomial first integral, or the existence of two Darboux polynomials having the same non–zero cofactor.*

Proposition 8. *If a polynomial differential system has an algebraic first integral, then it has a rational first integral.*

Proof of Theorem 1. From Proposition 8 to study the algebraic first integrals of the Riccati polynomial differential system (1) it is enough to study the rational ones. From Theorem 2 and Proposition 7, system (1) has a rational first integral if and only if it has two Darboux polynomials with the same non-zero cofactor. In view of Theorem 2 this is only possible if and only if $c(x) = \kappa(b(x) - \kappa a(x))$ and $b(x) = \kappa_1 a(x)$, i.e. $b(x) = \kappa_1 a(x)$ and $c(x) = \kappa(\kappa_1 - \kappa)a(x)$. We consider two different cases.

Case 1*:* $\kappa_1 \neq 2\kappa$. In this case the Darboux polynomials are of the form $(y+\kappa)^l(y + \kappa_1 - \kappa)^m$, with non-negative integers l and m. By Proposition 4, the cofactor is $a(x)[l(\kappa_1 - \kappa + y) + m(y+\kappa)]$. Then, using that $a(x) \neq 0$, in order to have a rational first integral we must have

$$l_1(\kappa_1 - \kappa + y) + m_1(y + \kappa) = l_2(\kappa_1 - \kappa + y) + m_2(y + \kappa), \tag{12}$$

with l_1, l_2, m_1, m_2 non-negative integers, $(l_1^2 + m_1^2)(l_2^2 + m_2^2) \neq 0$ and $(l_1, m_1) \neq (l_2, m_2)$.

Computing the coefficient of y in (12) we get that

$$l_1 + m_1 - l_2 - m_2 = 0, \quad \text{that is} \quad l_1 = l_2 + m_2 - m_1.$$

Introducing it in (12) and after simplifying by $a(x)$, we get

$$m_2(\kappa_1 - 2\kappa) = m_1(\kappa_1 - 2\kappa),$$

which is impossible because by assumptions $\kappa_1 \neq 2\kappa$, and $m_2 \neq m_1$ otherwise $(l_1, m_1) = (l_2, m_2)$ which is not possible. Hence in this case there are no algebraic first integrals.

Case 1*:* $\kappa_1 = 2\kappa$. In this case $b(x) = 2\kappa a(x)$ and $c(x) = \kappa^2 a(x)$, and thus system (1) becomes

$$x' = 1, \quad y' = a(x)(y^2 + 2\kappa y + \kappa^2) = a(x)(y+\kappa)^2,$$

whose rational first integral H is

$$H = \int a(x)\,dx + \frac{1}{y+\kappa} = \frac{1 + (y+\kappa)\int a(x)\,dx}{y+\kappa}.$$

This completes the proof of the theorem. □

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