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INVERSE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS: APPLICATIONS TO MECHANICS

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Abstract. This paper is on the so called inverse problem of ordinary differential systems, i.e. the problem of determining the differential systems satisfying a set of given properties. More precisely, we characterize under very general assumptions the ordinary differential systems in $\mathbb{R}^N$ which have a given set of either $M \leq N$, or $M > N$ partial integrals, or $M < N$ first integrals, or $M \leq N$ partial and first integrals. Moreover, for such systems we determine the necessary and sufficient conditions for the existence of $N-1$ independent first integrals. For the systems with $M < N$ partial integrals we provide sufficient conditions for the existence of a first integral.

We give two relevant applications of the solutions of these inverse problems to constrained Lagrangian and constrained Hamiltonian systems. Additionally we provide a particular solution of the inverse problem in dynamics, and give a generalized solution of the problem of integration of the equation of motion in the classical Suslov problem on $SO(3)$.

1. Introduction and statement of the main results

In the theory of ordinary differential equations we can find two fundamental problems. The direct problem which consists in a broad sense in to find the solutions of a given ordinary differential equation, and the inverse problem. An *inverse problem of ordinary differential equations* as it was defined in [17] is to find the more general differential system satisfying a set of given properties. For instance what are the differential systems in $\mathbb{R}^N$ having a given set of invariant hypersurfaces, or of first integrals? The aim of the present paper is to provide an answer to these questions.

The first inverse problem in such sense was stated by Erugin in [15]. In this article the author stated and solved the problem of constructing a planar vector field for which a given curve is its invariant, i.e. formed by trajectories of the vector field. Erugin ideas were developed in particular in [17]. We observe that such kind of problem has recently been developed in $\mathbb{R}^2$ or $\mathbb{C}^2$ mainly restricted to polynomial differential equations (see for instance [5, 6, 7, 27, 39, 41, 42, 43]).

The aim of the present paper is to characterize under very general assumptions the ordinary differential systems in $\mathbb{R}^N$ which have a given set of either $M \leq N$, or $M > N$ partial integrals, or $M < N$ first integrals, or $M \leq N$ partial and first integrals.

By applying the obtained results we provide a solution of the following two inverse problems.

(i) For a given natural mechanical system with N degrees of freedom determine the most general field of force depending only on the position of the system and satisfying a given set of constraints, i.e. the *inverse problem for the constrained Lagrangian system.*

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One of the main objectives in this inverse problem is to study the behavior of the nonholonomic systems with linear constraints with respect to the velocity in a way different to the classical approach deduced from the d'Alembert-Lagrange principle. We shall explain this in more detail in Remark 18.

As a consequence of the solution of the inverse problem for the constrained Lagrangian system we obtain a solution for the *inverse problem in dynamics* (see for more details [16]). The first inverse problem in dynamics appeared in Celestial Mechanics, it was stated and solved by Newton (1687) [32] and concerns with the determination of the potential field of force that ensures the planetary motion in accordance to the observed properties, namely the Kepler's laws.

Bertrand (1877) in [2] proved that the expression for Newton's force of attraction can be obtained directly from the Kepler first law. He stated also a more general problem of determining a positional force, under which a particle describes a conic section under any initial conditions. Bertrand's ideas were developed in particular by [8, 46, 22, 14, 16, 36, 43].

In the modern scientific literature the importance of the inverse problem in Celestial Mechanics was already recognized by Szebehely (see [48]).

Clearly that in view of the second Newton law, acceleration is equal to force we obtain that the above inverse problems are equivalent to determine the second order differential equations from the given properties on the right hand side.

We give a generalized solution of the problem of integration of the equation of motion in the classical Suslov problem on $SO(3)$. This solution contains as a particular case the well known integrable cases of this problem.

(ii) For a given submanifold $\mathcal{M}$ of a symplectic manifold $\mathbb{M}$ we determine the differential systems having $\mathcal{M}$ invariant by their flow, i.e. the *inverse problem for constrained Hamiltonian system.*

We determine the equations of motion of a constrained Hamiltonian system in the following cases: (1) The given properties are l first integrals with $\dim \mathbb{M}/2 \leq l < \dim \mathbb{M}$. In particular we prove that these equations are Hamiltonian only if the first integrals are in involution, (2) the given properties are $M < \dim \mathbb{M}/2$ partial integrals. We deduce the differential equations which can be interpreted as a normal form of the equations of motion of nonholonomic system with in general nonlineal constraint with respect to the velocity.

Constrained Hamiltonian systems arise in many fields, for instance in multi-body dynamics or in molecular dynamics. The theory of such systems was mainly developed by Dirac (see for instance [9]). See general references for constrained dynamics in [45].

The statements of the inverse problem for constrained Hamiltonian and Lagrangian systems are new.

Now we shall provide the notations and definitions necessary for presenting our main results.

Let D be an open subset of $\mathbb{R}^N$. By definition an autonomous differential system is a system of the form

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}), \quad \mathbf{x} \in \mathrm{D}, \tag{1}$$

where the dependent variables $\mathbf{x} = (x_1, \ldots, x_N)$ are real, the independent variable (the time t) is real and the $\mathcal{C}^1$ functions $\mathbf{X}(\mathbf{x}) = (X_1(\mathbf{x}), ..., X_N(\mathbf{x}))$ are defined in the open set D.

The $\mathcal{C}^1$ function $g : \mathrm{D} \longrightarrow \mathbb{R}$ and the set $\{\mathbf{x} \in \mathrm{D} : g = g(\mathbf{x}) = 0\}$ are called *partial integral* and *invariant hypersurface* of the vector field $\mathbf{X}$ respectively, if $\mathbf{X}(g)|_{g=0} = 0$.

The function $H = H(\mathbf{x})$ defined in an open subset $\tilde{\mathrm{D}}_1$ of D such that its closure coincides with D is called a *first integral* if it is constant on the solutions of system (1) contained in $\tilde{\mathrm{D}}_1$, i.e. $\mathbf{X}(H)|_{\tilde{\mathrm{D}}_1} = 0$.

Let $h_j = h_j(\mathbf{x})$ for $j = 1, 2, \ldots, M$ with $M \le N$ be functions defined in an open subset $\tilde{\mathrm{D}}$ of D. We define the matrix

$$S_{M,N} = \begin{pmatrix} dh_1(\partial_1) & \ldots & dh_1(\partial_N) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ dh_M(\partial_1) & \ldots & dh_M(\partial_N) \end{pmatrix} = \begin{pmatrix} \partial_1 h_1 & \ldots & \partial_N h_1 \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \partial_1 h_M & \ldots & \partial_N h_M \end{pmatrix},$$

where $\partial_j h = \dfrac{\partial h}{\partial x_j}$ and $dh = \displaystyle\sum_{j=1}^{N} \partial_j h \, dx_j$.

We say that the functions h_j for $j = 1, \ldots, M \le N$ are *independent* if the rank of matrix $S_{M,N}$ is M for all $\mathbf{x} \in \tilde{\mathrm{D}}$, except perhaps in a subset of $\tilde{\mathrm{D}}$ of zero Lebesgue measure.

We shall say that the vector field $\mathbf{X}$ in $\mathrm{D} \in \mathbb{R}^N$ is *integrable* if it admits $N-1$ independent first integrals.

In this paper we present four different kind of results. First we characterize under very general assumptions the differential systems which have a given set of partial and first integrals. Second in $\mathbb{R}^N$ we provide some results on the integrability and on the existence of a first integral for the differential equations having $M < N$ partial integrals. Finally we solve the inverse problem in Lagrangian and Hamiltonian mechanics.

For simplicity we shall assume that all the functions which appear in this paper are of class $\mathcal{C}^\infty$, although most of the results remain valid under weaker hypotheses.

We define the matrix $S = S_{N,N}$. We note that S is the *Jacobian matrix* of the functions $h_1, \ldots, h_N$. The *Jacobian* of S, i.e. the determinant of S, is denoted by

$$|S| = \left| \frac{\partial(h_1, \ldots, h_N)}{\partial(x_1, \ldots, x_N)} \right| := \{h_1, \ldots, h_N\}.$$

This bracket is known in the literature as the *Nambu bracket* [29, 49, 21]. We provide new properties of the Nambu bracket in section 2. These properties will play a very important role in the proofs of the main results.

Our first result characterizes the differential systems (1) having a given set of M partial integrals with $M \le N$.

Theorem 1. *Let $g_j = g_j(\boldsymbol{x})$ for $j = 1, 2, \ldots, M$ with $M \le N$ be a given set of independent functions defined in an open set $\mathrm{D} \subset \mathbb{R}^N$. Then the most general differential systems in* D *which admit the set of partial integrals g_j for $j = 1, 2, \ldots, M$ are*

$$(2)\quad \dot{x}_j = \sum_{k=1}^{M} \Phi_k \frac{\{g_1, \ldots, g_{k-1}, x_j, g_{k+1}, \ldots, g_N\}}{\{g_1, g_2, \ldots, g_N\}} + \sum_{k=M+1}^{N} \lambda_k \frac{\{g_1, \ldots, g_{k-1}, x_j, g_{k+1}, \ldots, g_N\}}{\{g_1, g_2, \ldots, g_N\}}$$

where $g_{M+j} = g_{M+j}(\boldsymbol{x})$ for $j = 1, \ldots, N-M$, are arbitrary functions defined on D *which we choose in such a way that the Jacobian*

$$(3)\qquad |S| = \{g_1, \ldots, g_N\} \ne 0,$$

in the set D *and the functions $\Phi_j = \Phi_j(\boldsymbol{x})$, for $j = 1, 2, \ldots, M$ and $\lambda_{M+k} = \lambda_{M+k}(\boldsymbol{x})$ for $k = 1, 2, \ldots N-M$ are arbitrary functions such that*

$$(4)\qquad \Phi_j|_{g_j=0} = 0, \quad \text{for} \quad j = 1, 2, \ldots, M.$$

Theorem 1 is proved in section 2.

An immediate consequence of Theorem 1 is the next result.

Corollary 2. *Under the assumptions of Theorem 1 if $M = N$, then system* (2) *takes the form*

$$(5)\qquad \dot{x}_j = \Phi_1 \frac{\{x_j, g_2, \ldots, g_{N-1}, g_N\}}{\{g_1, g_2, \ldots, g_{N-1}, g_N\}} + \ldots + \Phi_N \frac{\{g_1, g_2, \ldots, g_{N-1}, x_j\}}{\{g_1, g_2, \ldots, g_{N-1}, g_N\}},$$

for $j = 1, 2, \ldots, N$.

Our second result determines the differential systems (1) having a given set of M partial integrals with $M > N$.

Theorem 3. *Let* $g_j = g_j(\boldsymbol{x})$ *for* $j = 1, 2, \ldots, M > N$ *be a set of functions defined in the open set* $\mathrm{D} \subset \mathbb{R}^N$ *such that at least* N *of them are independent at points of the set* D, *i.e. without loss of generality we can assume that* $\{g_1, \ldots, g_N\} \neq 0$ *in* D. *Then the most general differential systems in* D *which admit the partial integrals* g_j *for* $j = 1, 2, \ldots, M$ *are*

$$\dot{x}_j = \sum_{j_1,\ldots,j_{N-1}=1}^{M+N} G_{j_1,\ldots,j_{N-1}} \{g_{j_1}, \ldots, g_{j_{N-1}}, x_j\}, \tag{6}$$

for $j = 1, 2, \ldots, N$, *where* $G_{j_1,\ldots,j_{N-1}} = G_{j_1,\ldots,j_{N-1}}(\boldsymbol{x})$ *are arbitrary functions satisfying*

$$\dot{g}_j|_{g_j=0} = \left. \left(\sum_{j_1,\ldots,j_{N-1}=1}^{M+N} G_{j_1,\ldots,j_{N-1}} \{g_{j_1}, \ldots, g_{j_{N-1}}, g_j\} \right) \right|_{g_j=0} = 0, \tag{7}$$

for $j = 1, 2, \ldots, M$, *and* $g_{M+j} = x_j$ *for* $j = 1, 2, \ldots, N$.

Theorem 3 is proved in section 2.

Our third result characterizes the differential systems (1) having a given set of M_1 partial integrals and M_2 first integrals with $1 \leq M_2 < N$ and $M_1 + M_2 \leq N$.

Theorem 4. *Let* $g_l = g_l(\boldsymbol{x})$ *for* $l = 1, 2, \ldots, M_1$ *and* $f_k = f_k(\boldsymbol{x})$ *for* $k = 1, 2, \ldots, M_2 < N$ *with* $M_1 + M_2 = M \leq N$ *be independent functions defined in the open set* $\mathrm{D} \subset \mathbb{R}^N$. *Then the most general differential systems in* D *which admit the partial integrals* g_l *for* $j = 1, \ldots, M_1$ *and the first integrals* f_k *for* $k = 1, \ldots, M_2$ *are*

$$\begin{aligned} \dot{x}_j = & \sum_{k=1}^{M_1} \Phi_k \frac{\{g_1, \ldots, g_{k-1}, x_j, g_{k+1}, \ldots, g_{M_1}, f_1, \ldots, f_{M_2}, g_{M+1} \cdots g_N\}}{\{g_1, g_2, \cdots g_{M_1}, f_1, \ldots, f_{M_2}, g_{M+1}, \ldots, g_N\}} + \\ & \sum_{k=M+1}^{N} \lambda_k \frac{\{g_1, \ldots, g_{M_1}, f_1, \ldots, f_{M_2}, g_{M+1}, \ldots, g_{k-1}, x_j, g_{k+1}, \ldots, g_N\}}{\{g_1, \ldots, g_{M_1}, f_1, \ldots, f_{M_2}, g_{M+1}, g_{M+2}, \ldots, g_N\}}, \end{aligned} \tag{8}$$

for $j = 1, 2, \ldots, N$, *where* g_{M+j} *for* $j = 1, \ldots, N-M$ *are arbitrary functions satisfying that* $|S| = \{g_1, \ldots g_{M_1}, f_1, \ldots, f_{M_2}, g_{M+1}, \ldots, g_N\} \neq 0$ *in the set* D *where* c_j *for* $j = 1, \ldots, M_2$ *are arbitrary constants; the functions* $\Phi_l = \Phi_l(\boldsymbol{x})$, *for* $l = 1, 2, \ldots M_1$ *and* $\lambda_{M+k} = \lambda_{M+k}(\boldsymbol{x})$ *for* $k = 1, 2, \ldots N-M$ *are arbitrary functions such that* $\Phi_l|_{g_l=0} = 0$ *for* $l = 1, \ldots, M_1$.

Theorem 4 is proved in section 3.

Two results which follow easily from the proof of Theorem 4 are:

Corollary 5. *Under the assumptions of Theorem 4 but without partial integrals, i.e. if* $M_1 = 0$, *and* $M_2 = M < N$, *then the most general differential systems in* D *which admit the first integrals* f_k *for* $k = 1, \ldots, M_2$ *are*

$$\dot{x}_j = \sum_{k=M+1}^{N} \lambda_k \frac{\{f_1, \ldots, f_M, g_{M+1}, \ldots, g_{k-1}, x_j, g_{k+1}, \ldots, g_N\}}{\{g_1, \ldots, g_{M_1}, f_1, \ldots, f_{M_2}, g_{M+1}, g_{M+2}, \ldots, g_N\}}, \tag{9}$$

for $j = 1, 2, \ldots, N$, *where* g_{M+j} *for* $j = 1, \ldots, N-M$ *are arbitrary functions satisfying that* $|S| = \{f_1, \ldots, f_M, g_{M+1}, \ldots, g_{N-1}, g_N\} \neq 0$ *in the set* D.

Corollary 6. *Under the assumptions of Theorem 4 and if* $M_1 + M_2 = M = N$, *then the differential system* (8) *takes the form*

$$\dot{x}_j = \sum_{k=1}^{M_1} \Phi_k \frac{\{g_1, \ldots, g_{k-1}, x_j, g_{k+1}, \ldots, g_{M_1}, f_1, \ldots, f_{M_2}\}}{\{g_1, g_2, \cdots g_{M_1}, f_1, \ldots, f_{M_2}\}} \tag{10}$$

for $j = 1, 2, \dots, N$.

In the next result we provide a new proof of the classical result which states that a differential system in an open subset of $\mathbb{R}^N$ having $N-2$ first integrals and with zero divergence is integrable by quadratures. In fact this result goes back to Jacobi and Whittaker, see for more details on this result the book [20].

Theorem 7. *Under the assumptions of Corollary 5 for* $M_2 = N-2$ *and determining the functions* g_{N-1}, g_N, λ_{N-1}, λ_N *and* μ *satisfying*

$$\{f_1, \dots f_{N-2}, \mu\lambda_{N-1}, g_N\} + \{f_1, \dots f_{N-2}, g_{N-1}, \mu\lambda_N\} = 0, \tag{11}$$

where $\mu = \mu(\boldsymbol{x}) = \dfrac{U}{\{f_1, \dots, f_{N-2}, g_{N-1}, g_N\}}$ *for a convenient function* U*, then the solutions of the differential equation*

$$\dot{x}_j = \lambda_{N-2}\frac{\{f_1, \dots, f_{N-2}, x_j, g_N\}}{\{f_1, \dots, f_{N-2}, g_{N-1}, g_N\}} + \lambda_N\frac{\{f_1, \dots, f_{N-2}, g_{N-1}, x_j\}}{\{f_1, \dots, f_{N-2}, g_{N-1}, g_N\}} = X_j(\boldsymbol{x}), \tag{12}$$

for $j = 1, 2, \dots, N$*, can be computed by quadratures. Moreover* (11) *is the divergence of systems* (12).

Theorem 7 is proved in section 4.

In what follows we present five new results on the integrability of systems (2), (6), (8), (9) and (10).

Theorem 8. *Under the assumptions of Theorem 1 differential system* (2) *is integrable if and only if* $\Phi_l = \mu\{F_1, \dots, F_{N-1}, g_l\}$, $\quad \lambda_k = \mu\{F_1, \dots, F_{N-1}, g_k\}$ *for* $l = 1, \dots, M$ *and* $k = M+1, \dots, N$*, where* μ, $F_1, \dots, F_{N-1}$ *are arbitrary functions such that* $F_1, \dots, F_{N-1}$ *are independent in* D *and* $\mu\{F_1, \dots, F_{N-1}, g_l\}|_{g_l=0} = 0$.

The following results are proved in a similar way to the proof of Theorem 8.

Theorem 9. *Under the assumptions of Theorem 3 differential system* (6) *is integrable if and only if*

$$\Phi_l = \sum_{\alpha_1, \dots, \alpha_{N-1}=1}^{M+N} G_{\alpha_1, \dots, \alpha_{N-1}}\{g_{\alpha_1}, \dots, g_{\alpha_{N-1}}, g_l\} = \mu\{F_1, \dots, F_{N-1}, g_l\},$$

for $l = 1, \dots, M > N$*, where* μ, $F_1, \dots, F_{N-1}$ *are arbitrary functions such that* $F_1, \dots, F_{N-1}$ *are independent in* D *and* $\mu\{F_1, \dots, F_{N-1}, g_l\}|_{g_l=0} = 0$.

Theorem 10. *Under the assumptions of Theorem 4 differential system* (8) *is integrable if and only if* $\Phi_l = \mu\{F_1, \dots, F_{N-1}, g_l\}$, $\quad \lambda_k = \mu\{F_1, \dots, F_{N-1}, g_k\}$*, for* $l = 1, \dots, M_1$ *and* $k = M+1, \dots, N$*, where* μ, $F_1, \dots, F_{N-1}$ *are arbitrary functions such that* $F_1, \dots, F_{N-1}$ *are independent in* D *and* $\mu\{F_1, \dots, F_{N-1}, g_l\}|_{g_l=0} = 0$.

Corollary 11. *Under the assumptions of Corollary 5 differential system* (9) *is integrable if and only if* $\lambda_k = \mu\{F_1, \dots, F_{N-1}, g_k\}$*, for* $k = M+1, \dots, N$ *where* μ, $F_1, \dots, F_{N-1}$ *are arbitrary functions such that* $F_1, \dots, F_{N-1}$ *are independent in* D.

Corollary 12. *Under the assumptions of Corollary 6 differential system* (10) *is integrable if and only if* $\Phi_l = \mu\{F_1, \dots, F_{N-1}, g_l\}$*, where* μ, $F_1, \dots, F_{N-1}$ *are arbitrary functions,* $F_1, \dots, F_{N-1}$ *independent in* D *and* $\mu\{F_1, \dots, F_{N-1}, g_l\}|_{g_l=0} = 0$.

In the next result we provide sufficient conditions for the existence of a first integral of the differential system (2).

Theorem 13. *Suppose that we are under the assumptions of Theorem 1, and that in the differential system* (2) *we choose*

$$
\begin{aligned}
\lambda_{M+1} = & \quad L_0 g_{M+1} + L_1 g, \\
\lambda_{M+2} = & \quad L_0 g_{M+2} + L_1 g_{M+1} + L_2 g, \\
\vdots \quad & \quad \vdots \\
\lambda_N = & \quad L_0 g_N + L_1 g_{N-1} + \ldots + L_{N-M} g,
\end{aligned} \tag{13}
$$

with $L_0 = \sum_{j=1}^{M} \frac{\Phi_j \tau_j}{g_j}$, *where* $g = \prod_{j=1}^{M} |g_j|^{\tau_j}$,*and* τ_j *for* $j = 1, 2, \ldots M$ *are constants, and* $L_1, \ldots L_{N-M}$ *are functions satisfying that*

$$
\sum_{j=0}^{N-M} \nu_j L_j = 0, \tag{14}
$$

for convenient constants ν_j *for* $j = 0, 1, \ldots, M$.

Let $\boldsymbol{G} = (G_1, G_2, \ldots, G_{N-M})^T = \left(\frac{g_{M+1}}{g}, \frac{g_{M+2}}{g}, \ldots \frac{g_N}{g} \right)^T$ *and*

$$
B = \begin{pmatrix}
1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
G_1 & 1 & 0 & 0 & \ldots & 0 & 0 \\
G_2 & G_1 & 1 & 0 & \ldots & 0 & 0 \\
G_3 & G_2 & G_1 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
G_{N-M-2} & G_{N-M-3} & G_{N-M-4} & \vdots & \ldots & 1 & 0 \\
G_{N-M-1} & G_{N-M-2} & G_{N-M-3} & \vdots & \ldots & G_1 & 1
\end{pmatrix}
$$

be. Then there exists a function $\boldsymbol{R} = (R_1, \ldots, R_{N-M})^T$ *satisfying* $\boldsymbol{R} = \int B^{-1} d\boldsymbol{G}$, *being* $d\boldsymbol{G} = (dG_1, dG_2, \ldots, dG_{N-M})^T$ *where* dG_k *denotes the differential of* G_k *for* $k = 1, 2, \ldots N$. *Then*

$$
F = |g|^{\nu_0} \exp \left(\sum_{j=1}^{N-M} \nu_j R_j \right), \tag{15}
$$

is a first integral of system (2).

Theorem 13 is proved in section 5.

Such first integral already was obtained in [37]. We observe that these kind of first integrals appear also in the study of the invariant algebraic hypersurfaces with multiplicity of a polynomial vector field, see [4, 28], and for more details on the functions R_j see the proof of Theorem 13.

2. Applications to Lagrangian and Hamiltonian mechanics with constraints

As we observe from the previous section the solutions of the inverse problem in ordinary differential equations have a very hight arbitrariness due to the undetermined functions which appear. To obtain more exactly solutions need additional conditions to reduce this arbitrariness. In this section we shall obtain additional conditions for getting the equations of motion provided by the Lagrangian and Hamiltonian constrained mechanics. The aim of this section is to solve the inverse problem in Lagrangian and Hamiltonian system.

2.1. **Inverse problem for constrained Lagrangian systems.** Using Theorem 1 we will be able to provide an answer to the problems (i).

We shall introduce the notations and definitions that we need for presenting our applications.

We shall denote by Q an N-dimensional smooth manifold and by TQ the tangent bundle of Q with local coordinates $\mathbf{x} = (x_1, \ldots, x_N)$, and $(\mathbf{x}, \dot{\mathbf{x}}) = (x_1, \ldots, x_N,\ \dot{x}_1, \ldots, \dot{x}_N)$ respectively (see for instance [19]).

The following definitions can be found in [1].

A *Lagrangian system* is a pair $(\mathrm{Q}, \tilde{L})$ consisting of a smooth manifold Q, a function $\tilde{L} : T\mathrm{Q} \longrightarrow \mathbb{R}$. The point $\mathbf{x} \in \mathrm{Q}$ denotes the *position* of the system and we call each tangent vector $\dot{\mathbf{x}} \in T_{\mathbf{x}}\mathrm{Q}$ the *velocity* of the system at the point $\mathbf{x}$. A pair $(\mathbf{x}, \dot{\mathbf{x}})$ is called a state of the system. In Lagrangian mechanics it is usual to call Q, the *configuration space*, the tangent bundle TQ is called the *phase space*, $\tilde{L}$ is the *Lagrange function* or *Lagrangian* and the dimension N of Q is the number of *degrees of freedom*.

The equations

$$h_j = h_j(\mathbf{x}, \dot{\mathbf{x}}) = 0, \quad \text{for} \quad j = 1, \ldots, M \leq N, \tag{16}$$

with rank $\left(\dfrac{\partial(h_1, \ldots, h_M)}{\partial(\dot{x}_1, \ldots, \dot{x}_N)}\right) = M$, in all the points of Q, except in a zero Lebesgue measure set, define M independent *constraints* for the Lagrangian systems $(\mathrm{Q}, \tilde{L})$, i.e. we want that the orbits $(\mathbf{x}(t),\ \dot{\mathbf{x}}(t))$ of the mechanical system satisfy (16).

Let $\mathcal{M}^*$ be the submanifold of TQ defined by the equations (16), i.e.

$$\mathcal{M}^* = \{(\mathbf{x},\ \dot{\mathbf{x}}) \in T\mathrm{Q} : h_j(\mathbf{x}, \dot{\mathbf{x}}) = 0, \quad \text{for} \quad j = 1, \ldots, M \leq N\}$$

A *constrained Lagrangian system* is a triplet $(\mathrm{Q}, \tilde{L}, \mathcal{M}^*)$.

We call *the inverse problem for the constrained Lagrangian system* the problem of determining for a given constrained Lagrangian system $(\mathrm{Q}, \tilde{L}, \mathcal{M}^*)$, the field of force $\mathbf{F} = \mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), \ldots, F_N(\mathbf{x}))$ in such a way that the given submanifold $\mathcal{M}^*$ is invariant by the flow of the second order differential equations

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{x}_j}\right) - \frac{\partial \tilde{L}}{\partial x_j} = F_j(\mathbf{x}) \quad \text{for} \quad j = 1, \ldots, N.$$

We shall study the case when the constraints are linear in the velocities in $\mathcal{M}^*$, i.e.

$$h_j(\mathbf{x}, \dot{\mathbf{x}}) = \sum_{k=1}^{N} a_{jk}(\mathbf{x})\dot{x}_k + \alpha_j(\mathbf{x}) = 0, \quad \text{for} \quad j = 1, \ldots, M. \tag{17}$$

Our first main result provides the equations of motion of a constrained mechanical system with Lagrangian function

$$\tilde{L} = T = \frac{1}{2}\sum_{n,j=1}^{N} G_{jn}(\mathbf{x})\dot{x}_j\dot{x}_n := \frac{1}{2}\left\langle \dot{\mathbf{x}},\ \dot{\mathbf{x}}\right\rangle = \frac{1}{2}||\dot{\mathbf{x}}||^2, \tag{18}$$

where T is a Riemannian metric on Q (the *kinetic energy* of the system), and $M = N$ linear constraints given by

$$g_j = \sum_{n=1}^{N} G_{jn}(\mathbf{x})\left(\dot{x}_n - v_n(\mathbf{x})\right) = 0 \quad \text{for} \quad j = 1, \ldots, N, \tag{19}$$

where $\mathbf{v}(\mathbf{x}) = (v_1(\mathbf{x}), \ldots, v_N(\mathbf{x}))$ is a given vector field.

Theorem 14. *Let* Σ *be a constrained Lagrangian mechanical system with configuration space* Q, *kinetic energy* T *given in* (18), *and constraints given by* (19). *The equations of* *motion of Σ are the Lagrangian differential equations*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right)-\frac{\partial L}{\partial x_j}=0 \quad \textit{for} \quad j=1,\ldots,N, \tag{20}$$

with $L=\frac{1}{2}||\dot{\mathbf{x}}-\boldsymbol{v}||^2=T-\langle\dot{\mathbf{x}},\,\boldsymbol{v}\rangle+\frac{1}{2}||\boldsymbol{v}||^2$, *which are equivalently to*

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_j}\right)-\frac{\partial T}{\partial x_j} &= \frac{\partial}{\partial x_j}\left(\frac{1}{2}||\boldsymbol{v}||^2\right)+\sum_{n=1}^{N}\dot{x}_n\left(\frac{\partial p_j}{\partial x_n}-\frac{\partial p_n}{\partial x_j}\right)\\ &= \frac{\partial}{\partial x_j}\left(\frac{1}{2}||\boldsymbol{v}||^2\right)+\sum_{n=1}^{N}v_n\left(\frac{\partial p_j}{\partial x_n}-\frac{\partial p_n}{\partial x_j}\right),\end{aligned} \tag{21}$$

where

$$p_j=\sum_{n=1}^{N}G_{jn}v_n, \quad \textit{for} \quad j=1,2,\ldots,N. \tag{22}$$

In view of the second Newton law: *acceleration is equal to force* (see for instance [44]), we obtain that the right hand side of the equations of motion (21) are the generalized forces acting on the mechanical system which depends only on its position. Consequently the field of force $\mathbf{F}$ with components

$$F_j=\frac{\partial}{\partial x_j}\left(\frac{1}{2}||\mathbf{v}||^2\right)+\sum_{n=1}^{N}v_n\left(\frac{\partial p_j}{\partial x_n}-\frac{\partial p_n}{\partial x_j}\right) \tag{23}$$

is the most general field of force depending only on the position of the natural mechanical system which is constrained to move on the N dimensional subset of the phase space given by (19) In short the equations of motion (21) provide a complete answer to the inverse problem (i) when the constraints are given in the form (19).

Now we want to solve the inverse problem (i) for the classical constraints

$$\sum_{n=1}^{N}a_{jn}(\mathbf{x})\dot{x}_n=0 \quad \text{for} \quad j=1,\ldots,M. \tag{24}$$

We recall that the equations of motion of a constrained Lagrangian system with Lagrangian $\tilde{L}=\frac{1}{2}||\dot{\mathbf{x}}||^2-U(\mathbf{x})$, and constrains given by (24) but with a field of forces $\tilde{\mathbf{F}}=(\tilde{F}_1,\ldots,\tilde{F}_N)$ depending on positions and velocities are the *Lagrange equations with multipliers*

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_k}\right)-\frac{\partial T}{\partial x_k}=\tilde{F}_k(\mathbf{x},\dot{\mathbf{x}}) &= -\frac{\partial U}{\partial x_k}+\sum_{j=1}^{M}\mu_j a_{jk}, \quad \text{for} \quad k=1,\ldots,N,\\ \sum_{n=1}^{N}a_{jn}(\mathbf{x})\dot{x}_n &= 0, \quad \text{for} \quad j=1,\ldots,M,\end{aligned} \tag{25}$$

where $\mu_j=\mu_j(\mathbf{x},\dot{\mathbf{x}})$ are the *Lagrangian multipliers*. As we can observe the forces $\tilde{\mathbf{F}}$ are composed by the potential forces with components $-\frac{\partial U}{\partial x_k}$ and the reactive forces generated by constraints with components $\sum_{j=1}^{M}\mu_j a_{jk}$ for $k=1,\ldots,N$. For more details see [1].

In short we have two equations of motions: the ones given in (20), or what is the same (21) for constraints of type (19), and the classical ones given in (25) for the constraints (24). In order to solve the problem (i) for the constraints (24) we establish the relationship between these two sets of equations. For doing this we shall choose conveniently the vector field $\mathbf{v}$ which appear in (19).

In view of that the constraints (19) are equivalently to the constraints $\dot{x}_j = v_j(\mathbf{x})$ for $j = 1, \ldots, N$. On the other hand from (24) we obtain that $\langle a_j, \mathbf{v}\rangle = \sum_{n=1}^{N} a_{jn}v_n = 0$, thus $\mathbf{v}$ must be orthogonal to the independent vectors $a_j = (a_{j1}, \ldots, a_{jN})$ for $j = 1, \ldots, M$. So we introduce the N independent 1-forms, the first M of these 1-forms are associated to the M constraints (24), i.e.

$$\Omega_j = \sum_{n=1}^{N} a_{jn}(\mathbf{x})dx_n \quad \text{for} \quad j = 1, \ldots, M, \tag{26}$$

and we choose the 1-forms Ω_j for $j = M + 1, \ldots, N$ arbitrarily, but satisfying that the determinant $|\Upsilon|$ of the matrix $\Upsilon = (a_{jk})$:

$$\Upsilon = \begin{pmatrix} \Omega_1(\partial_1) & \ldots & \Omega_1(\partial_N) \\ \vdots & \vdots & \vdots \\ \Omega_N(\partial_1) & \ldots & \Omega_N(\partial_N) \end{pmatrix} = \begin{pmatrix} a_{11} & \ldots & a_{N1} \\ \vdots & \vdots & \vdots \\ a_{N1} & \ldots & a_{NN} \end{pmatrix}, \tag{27}$$

is nonzero. The ideal case would be when this determinant is constant. In other words the N 1-forms Ω_j for $j = 1, \ldots, N$ are independent. Now we define the vector field $\mathbf{v}$ as

$$\mathbf{v} = -\frac{1}{|\Upsilon|}\begin{vmatrix} \Omega_1(\partial_1) & \ldots & \Omega_1(\partial_N) & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \Omega_M(\partial_1) & \ldots & \Omega_M(\partial_N) & 0 \\ \Omega_{M+1}(\partial_1) & \ldots & \Omega_{M+1}(\partial_N) & \nu_{M+1} \\ \vdots & \vdots & \vdots & \vdots \\ \Omega_N(\partial_1) & \ldots & \Omega_N(\partial_N) & \nu_N \\ \partial_1 & \ldots & \partial_N & 0 \end{vmatrix} = \left\langle \Upsilon^{-1}\mathbf{P}, \partial_{\mathbf{x}} \right\rangle, \tag{28}$$

where $\mathbf{P} = (0, \ldots, 0, \nu_{M+1}, \ldots, \nu_N)^T$, the functions $\nu_j = \nu_j(\mathbf{x})$ are nonzero arbitrary functions due to the arbitrariness of Ω_j for $j = M + 1, \ldots, N$.

Proposition 15. *The vector field* (28) *is the most general vector field satisfying the constraints* (24)*, i.e.* $\Omega_j(\boldsymbol{v}) = 0$ *for* $j = 1, \ldots, M$*, where the* Ω_j *are given in* (26)*.*

We define

$$\Lambda = \Lambda(\mathbf{x}) = (\Lambda_1(\mathbf{x}), \ldots, \Lambda_N(\mathbf{x}))^T = (\Upsilon^T)^{-1}H\mathbf{v}(\mathbf{x}) = A\mathbf{P}, \tag{29}$$

where $A = (A_{jk})$ is an $N \times N$ antisymmetric matrix such that

$$A = \left(\Upsilon^T\right)^{-1} H\Upsilon^{-1}, \quad H = (H_{jn}) = \left(\frac{\partial p_n}{\partial x_j} - \frac{\partial p_j}{\partial x_n}\right). \tag{30}$$

Theorem 16. *Let* Σ *be a constrained Lagrangian mechanical system with configuration space* Q*, kinetic energy* T *given in* (18)*, and constraints given by* (19) *with* $\boldsymbol{v} = (v_1, \ldots, v_N)^T$: *given by* (28)*.*

The equations of motion of Σ *are*

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_j}\right) - \frac{\partial T}{\partial x_j} = F_j(\mathbf{x}) = \frac{\partial}{\partial x_j}\left(\frac{1}{2}||\boldsymbol{v}||^2\right) + \sum_{k=1}^{M} \Lambda_k a_{kj}, \tag{31}$$

for $j = 1, \ldots, N$*, where the* Λ_k*'s are defined in* (29) *with*

$$\Lambda_k = \sum_{j=1}^{M} A_{kj}\nu_j = 0 \quad \text{for} \quad k = M + 1, \ldots, N. \tag{32}$$

Remark 17. *Equations* (32) *define a system of the first order partial differential equations with unknown functions* $\nu_{M+1},\ldots,\nu_N$ *(see* (28), (30) *and* (32)*).*

We observe that equations (32) *can be rewritten as follows*

$$\tilde{A}\boldsymbol{b}=\boldsymbol{0} \quad \text{with} \quad \boldsymbol{b}=(\nu_{M+1},\ldots,\nu_N)^T, \tag{33}$$

where $\tilde{A}$ *is an* $(N-M)\times(N-M)$ *antisymmetric matrix. Thus if* $N-M$ *is even then, from* (28)*, it follows that the vector* $\boldsymbol{b}$ *is nonzero, consequently the determinant of the matrix* $|\tilde{A}|=\mu^2_{N,M}$ *must be zero, i.e.* $\mu_{N,M}=0$*. If* $N-M$ *is odd then* $|\tilde{A}|$ *is always zero. If in this case rank*$(\tilde{A})=r$*, then without loss of generality we can assume that* (32) *takes the form*

$$\sum_{j=M+1}^{N} A_{kj}\nu_j=0 \quad \text{for} \quad k=M+1,\ldots,M+r. \tag{34}$$

In particular for $M=1$, $N=3$, $M=2$, $N=4$ *we obtain respectively*

$$\begin{aligned}\mu_{3,1} &= a_1H_{23}+a_2H_{31}+a_3H_{12}=0,\\ \mu_{4,2} &= (\alpha_{42}\alpha_{31}-\alpha_{32}\alpha_{41})H_{12}+(\alpha_{41}\alpha_{22}-\alpha_{21}\alpha_{42})H_{13}+\\ &\quad(\alpha_{21}\alpha_{32}-\alpha_{31}\alpha_{22})H_{14}+(\alpha_{42}\alpha_{11}-\alpha_{12}\alpha_{41})H_{23}+\\ &\quad(\alpha_{12}\alpha_{31}-\alpha_{32}\alpha_{11})H_{24}+(\alpha_{22}\alpha_{11}-\alpha_{12}\alpha_{21})H_{34}=0.\end{aligned} \tag{35}$$

Remark 18. *Equations* (31) *can be interpreted as the equations of motion of the constrained Lagrangian system with Lagrangian* $\tilde{L}=T+\frac{1}{2}||\boldsymbol{v}||^2$ *and constraints* (24)*. The field of force with components*

$$F_j(\mathbf{x})=\frac{\partial}{\partial x_j}\left(\frac{1}{2}||\boldsymbol{v}||^2\right)+\sum_{k=1}^{M}\Lambda_k a_{kj}, \tag{36}$$

for $j=1,\ldots,N$*, has the same structure than the field of forces determine in* (25)*, but there are three important differences. First the potential and reactive components in* (36) *are related through the vector field* $\boldsymbol{v}$ *(which itself is determined by the constraints), while in* (25) *the potential* U *is completely independent of the reactive forces with components* $\sum_{k=1}^{M}\mu_k a_{kj}$*. Second the multipliers* $\Lambda_1,\ldots,\Lambda_M$ *in* (36) *depend only on the position of the mechanical system, while in* (25) *the Lagrangian multipliers* μ_j *depends on the position and velocity, and finally system* (31) *was deduced from Lagrangian differential system* (20)*, while system* (25) *in general has no relations with the Lagrangian equations.*

In the applications of Theorem 16 we will determine the functions $\nu_{M+1},\ldots,\nu_N$ *as solutions of* (32) *together with the condition*

$$U=-\frac{1}{2}||\boldsymbol{v}||^2+h, \tag{37}$$

where h *is a constant. Under the potential* (37) *we obtain that between the fields of force* $\tilde{\boldsymbol{F}}$ *given in* (25) *and* $\boldsymbol{F}$ *given in* (36) *the only difference consists in the coefficients which determine the reactive forces.*

The following two questions arises: There exist solutions of equations (32) *and* (37) *in such a way that the solutions of the differential system*

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_j}\right)-\frac{\partial T}{\partial x_j}=-\frac{\partial U}{\partial x_j}+\sum_{k=1}^{M}\Lambda_k a_{kj}, \tag{38}$$

for $j=1,\ldots,N$*, where the* Λ_k*'s are defined in* (29)*, coincide with the solutions of* (25)*?*

If the answer to the previous question is always positive, then there are equations of motion with field of forces only depending on the positions (31) *equivalent to the Lagrangian* *equations of motions with constraints* (25)*. In short, we would have a new model to describe the behavior of the mechanical systems with linear constraints with respect to the velocity.*

The second question is: What is the mechanical meaning of the differential equations generated by the vector field (28)*, i.e.*

$$\dot{\mathbf{x}} = \boldsymbol{v}(\mathbf{x}) = \Upsilon^{-1}\boldsymbol{P}, \tag{39}$$

under the conditions (32) *and of the differential equations*

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_j}\right) - \frac{\partial T}{\partial x_j} = \frac{\partial \frac{1}{2}||\boldsymbol{v}||^2}{\partial x_j} + \sum_{k=1}^{M} \mu_k a_{kj} \ ? \tag{40}$$

Partial answer to theses questions are given in the examples of section 8.

Now we consider a mechanical system with configuration space Q of the dimension N and kinetic energy T given by (18). The problem of determining the most general field of force depending only on the position of the system, for which the curves defined by

$$f_j = f_j(\mathbf{x}) = c_j \in \mathbb{R} \quad \text{for} \quad \mathrm{j} = 1, \ldots, \mathrm{N}-1, \tag{41}$$

are formed by orbits of the mechanical system, is called as the *generalized Dainelli's inverse problem in dynamics.* If we assume that the given family of curves (41) admits the family of orthogonal hypersurfaces $S = S(\mathbf{x}) = c_N$, then this problem is called the *generalized Dainelli Joukovski's inverse problem.*

If the field of force is potential in the generalized Dainelli inverse problems, then such problems coincide with the *Suslov's inverse problem,* or the *inverse problem in Celestial Mechanics* and generalized Dainelli Joukovski's inverse problem coincide with the *Joukovski problem* (for more details see [43]).

The solutions of the generalized Dainelli's problem for $N = 2$, and of the Joukovski's problems for $N = 2, 3$ can be found in [50, 8, 22, 16]. A complete solution of the Suslov problem can be found in [46], but this solution in general is complicate to implement.

The following result provides a solution of these inverse problems.

Theorem 19. *Under the assumptions of Theorem 16 if the given $M = N-1$ 1-forms* (26) *are closed, i.e. $\Omega_j = df_j$ for $j = 1, \ldots, N-1$, then the following statements hold.*

(a) *System* (31) *takes the form*

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_j}\right) - \frac{\partial T}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{1}{2}||\boldsymbol{v}||^2\right) + \nu_N \sum_{k=1}^{N-1} A_{Nk}\frac{\partial f_k}{\partial x_j} =: F_j, \tag{42}$$

for $j = 1, \ldots, N$, where $\nu_N = \nu_N(\mathbf{x})$ is an arbitrary function. Clearly F_j are the components of the most general field of force that depends only on the position under which a given $N-1$ parametric family of curves (41) *can be described as orbits of the mechanical system.*

(b) *If*

$$\nu_N \sum_{k=1}^{N-1} A_{Nk}\frac{\partial f_k}{\partial x_j} = -\frac{\partial h}{\partial x_j}, \tag{43}$$

for $j = 1, \ldots, N-1$, where $h = h(f_1, \ldots, f_{N-1})$, then the family of curves (41) *can be freely described by a mechanical system under the influence of forces derived from the potential function $V = -U = \frac{1}{2}||\boldsymbol{v}||^2 - h(f_1, \ldots, f_{N-1})$.*

(c) *If we assume that the given family of curves* (41) *admits the family of orthogonal hypersurface $S = S(\mathbf{x}) = c_N$ defined by*

$$\left\langle \frac{\partial S}{\partial \mathbf{x}}, \frac{\partial f_j}{\partial \mathbf{x}} \right\rangle = 0 \quad \text{for} \quad j = 1, \ldots, N-1, \tag{44}$$

then the most general field of force that depends only on the position of the system under which the given family of curves are formed by orbits of (42) *is*

$$\boldsymbol{F} = \frac{\partial}{\partial \mathbf{x}}\left(\frac{\nu}{\sqrt{2}}\left\|\frac{\partial S}{\partial \mathbf{x}}\right\|\right)^2 + \left\langle \frac{\partial}{\partial \mathbf{x}}\left(\frac{\nu^2}{2}\right), \frac{\partial S}{\partial \mathbf{x}}\right\rangle \frac{\partial S}{\partial \mathbf{x}} - \left\|\frac{\partial S}{\partial \mathbf{x}}\right\|^2 \frac{\partial}{\partial \mathbf{x}}\left(\frac{\nu^2}{2}\right), \tag{45}$$

where $\nu = \nu(\mathbf{x})$ *is an arbitrary function on* Q. *If we choose* ν *and* $h = h(f_1, \ldots, f_{N-1})$ *satisfying the first order partial differential equation*

$$\left\langle \frac{\partial}{\partial \mathbf{x}}\left(\frac{\nu^2}{2}\right), \frac{\partial S}{\partial \mathbf{x}}\right\rangle \frac{\partial S}{\partial \mathbf{x}} - \left\|\frac{\partial S}{\partial \mathbf{x}}\right\|^2 \frac{\partial}{\partial \mathbf{x}}\left(\frac{\nu^2}{2}\right) = -\frac{\partial h}{\partial \mathbf{x}}, \tag{46}$$

then the field of force $\boldsymbol{F}$ *is given by the potential*

$$V = \frac{\nu^2}{2}\left\|\frac{\partial S}{\partial \mathbf{x}}\right\|^2 - h(f_1, \ldots, f_{N-1}). \tag{47}$$

If (41) *is such that* $f_j = x_j = c_j$ *for* $j = 1, \ldots, N-1$ *then* (47) *takes the form*

$$V = \frac{\nu^2|\tilde{G}|}{2\Delta}\left(\frac{\partial S}{\partial x_N}\right)^2 - h(x_1, \ldots, x_{N-1}). \tag{48}$$

where $\tilde{G} = (\tilde{G}_{nm})$ *is the inverse matrix of the matrix* G *and*

$$\Delta = \begin{vmatrix} \tilde{G}_{11} & \ldots & \tilde{G}_{1,N-1} \\ \vdots & \ldots & \vdots \\ \tilde{G}_{1,N-1} & \ldots & \tilde{G}_{N-1,N-1} \end{vmatrix}.$$

Clearly (46) *holds in particular if* $\nu = \nu(S)$ *and* h *is a constant.*

(d) *Under the assumption* (*b*) *we have that* $\displaystyle\int_{g_v^t(\gamma)} \sigma = const.$, *where* $\sigma = \langle \boldsymbol{v}, d\mathbf{x}\rangle$ *is the 1-form associated to vector field* $\boldsymbol{v}$, g_v^t *is the flow of* $\boldsymbol{v}$, *and* γ *is an arbitrary closed curve on* Q.

We note that statement (a) of Theorem 19 provides the answer to the generalized Dainelli's inverse problem, which before was only solved for $N = 2$ by Dainelli. Statement (b) of Theorem 19 gives a simpler solution to the Suslov's inverse problem, already solved by the same Suslov. Statement (c) of Theorem 19 provides the answer to the generalized Dainelli-Joukovski's problem solved by Joukovski for the case when the field of force is potential and $N = 2, 3$. Finally statement (d) of Theorem 19 is the well known Thomson's Theorem (see [26]) in our context.

Theorems 14, 16 and Proposition 15 are proved in section 8. Theorem 19 is proved in section 10.

2.2. **Inverse problem for constrained Hamiltonian systems.** In this section we shall apply Theorems 1 and 4, Corollaries 5 and 6 to solve the problem (ii) of the introduction.

Now we consider $\mathbb{M}$ a $2N$-dimensional smooth manifold with local coordinates $(\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_N, y_1, \ldots, y_N)$, and let Ω^2 be a closed non-degenerate 2-form, i.e. $(\mathbb{M}, \Omega^2)$ is a *symplectic manifold,* $H : \mathbb{M} \longrightarrow \mathbb{R}$ *a smooth function, and* $\mathcal{M}$ *a submanifold* of $\mathbb{M}$. The quaternary $(\mathbb{M}, \Omega^2, \mathcal{M}, H)$ is called *constrained Hamiltonian system* (see [1]).

We call the *inverse problem for constrained Hamiltonian systems*, the problem of the determination of the vector field $\mathbf{W}$ with components $(W_1, \ldots, W_{2N})$ with $W_j = W_j(\mathbf{x}, \mathbf{y})$ in such a way that the submanifold $\mathcal{M}$ is invariant by the flow of the differential system

$$\dot{x}_k = \{H, x_k\}^* + W_k, \quad \dot{y}_k = \{H, y_k\}^* + W_{N+k}, \quad \text{for} \quad k = 1, \ldots, N, \tag{49}$$

where

$$\{H,\, G\}^* = \sum_{k=1}^{N} \left(\frac{\partial H}{\partial y_k}\frac{\partial G}{\partial x_k} - \frac{\partial H}{\partial x_k}\frac{\partial G}{\partial y_k} \right), \tag{50}$$

is the *Poisson bracket.*

In particular if $W_k = 0$ for $k = 1, \ldots, N$, then (49) coincides with the standard Hamiltonian equations for a mechanical system which are under the action of the external forces with components $W_{N+1}, \ldots, W_{2N}$.

Theorem 20. *Let $(\mathbb{M}, \Omega^2, \mathcal{M}_1, H)$ be a constrained Hamiltonian system and let $f_j = f_j(\mathbf{x}, \mathbf{y})$ for $j = 1, \ldots, N$ be a given set of independent functions defined in $\mathbb{M}$. Assume that $\{f_1, \ldots, f_N, x_1, \ldots, x_N\} \neq 0$, in $\mathbb{M}$, then the manifold*

$$\mathcal{M}_1 = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{M} : f_j(\mathbf{x}, \mathbf{y}) = c_j \in \mathbb{R} \quad \textit{for} \quad j = 1, \ldots, N\},$$

where c_j for $j = 1, \ldots, N$ are arbitrary constants, is invariant by the flow of differential system

$$\begin{aligned} \dot{x}_k =& \quad \{H,\, x_k\}^*, \\ \dot{y}_k =& \quad \{H,\, y_k\}^* - \sum_{j=1}^{N} \frac{\{H,\, f_j\}^* \{f_1, \ldots, f_{j-1}, y_k, f_{j+1}, \ldots f_N, x_1, \ldots, x_N\}}{\{f_1, \ldots, f_N, x_1, \ldots, x_N\}} \\ =& \quad \{H,\, y_k\}^* + W_{k+N}, \end{aligned} \tag{51}$$

for $k = 1, \ldots, N$.

Under the assumptions

$$\{f_1, \ldots, f_N, x_1, \ldots, x_N\} = 0 \quad \textit{and} \quad \{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_1\} \neq 0, \tag{52}$$

the submanifold $\mathcal{M}_1$ is invariant by the flow of the differential system

$$\begin{aligned} \dot{x}_k \quad &= \{H,\, x_k\}^*, \quad \textit{for} \quad k = 1, \ldots, N-1, \\ \dot{x}_N \quad &= \{H,\, x_N\}^* - \\ &\sum_{j=1}^{N} \frac{\{H,\, f_j\}^* \{f_1, \ldots, f_{j-1}, x_N, f_{j+1}, \ldots, f_N, x_1, \ldots, x_{N-1}, y_1\}}{\{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_1\}} \\ &= \{H,\, x_N\}^* + W_N, \\ \dot{y}_1 \quad &= \{H,\, y_1\}^* + \lambda\{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_1\} \\ &= \{H,\, y_1\}^* + W_{1+N}, \\ \dot{y}_k \quad &= \{H,\, y_k\}^* - \\ &\sum_{j=1}^{N} \frac{\{H,\, f_j\}^* \{f_1, \ldots, f_{j-1}, y_k, f_{j+1}, \ldots f_N, x_1, \ldots, x_{N-1}, y_1\}}{\{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_1\}} + \\ &\lambda\{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_k\} \\ &= \{H,\, y_k\}^* + W_{k+N}, \quad \textit{for} \quad k = 2, \ldots, N, \end{aligned} \tag{53}$$

where $\lambda = \lambda(\mathbf{x}, \mathbf{y})$ is an arbitrary function.

We observe that the solution (51) of the inverse problem in constrained Hamiltonian systems for the case when the first integrals are pairwise in involution, and $H = H(f_1, \ldots, f_N)$ becomes into the Hamiltonian system $\dot{x}_k = \{H,\, x_k\}^*, \quad \dot{y}_k = \{H,\, y_k\}^*$. Additionally system (53), when the first integrals are pairwise in involution satisfying (52) and $H = H(f_1, \ldots, f_N)$, becomes into the differential system

$$\dot{x}_k = \{H,\, x_k\}^*, \quad \dot{y}_k = \{H,\, y_k\}^* + \lambda\{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_k\},$$

for $k = 1, \ldots, N$. These equations are the equations of motion of the mechanical system with the constraints $\{f_1, \ldots, f_N, x_1, \ldots, x_N\} = 0$.

Theorem 21. *Let* $(\mathbb{M}, \Omega^2, \tilde{\mathcal{M}}_1, H)$ *be a constrained Hamiltonian system and let* $f_j = f_j(\mathbf{x}, \mathbf{y})$ *for* $j = 1, \ldots, N + r$, *with* $r < N$ *be a given set of independent functions defined in* $\mathbb{M}$ *and such that* $\{f_1, \ldots, f_{N+r}, x_1, \ldots, x_{N-r}\} \neq 0$. *Then the manifold*

$$\tilde{\mathcal{M}}_1 = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{M} : f_j(\mathbf{x}, \mathbf{y}) = c_j \in \mathbb{R} \quad for \quad j = 1, \ldots, N + r\},$$

where c_j *are arbitrary constants, is invariant by the flow of the differential system*

$$\begin{aligned} \dot{x}_k &= \{H, x_k\}^*, \\ \dot{x}_n &= \{H, x_n\}^* - \sum_{j=1}^{N+r} \frac{\{H, f_j\}^* \{f_1, \ldots, f_{j-1}, x_n, f_{j+1}, \ldots, f_{N+r}, x_1, \ldots, x_{N-r}\}}{\{f_1, \ldots, f_{N+r}, x_1, \ldots, x_{N-r}\}} \\ &= \{H, x_n\}^* + W_n, \\ \dot{y}_m &= \{H, y_m\}^* - \sum_{j=1}^{N+r} \frac{\{H, f_j\}^* \{f_1, \ldots, f_{j-1}, y_m, f_{j+1}, \ldots, f_{N+r}, x_1, \ldots, x_{N-r}\}}{\{f_1, \ldots, f_{N+r}, x_1, \ldots, x_{N-r}\}} \\ &= \{H, y_m\}^* + W_{m+N}, \end{aligned} \tag{54}$$

for $k = 1, \ldots, N - r$, $n = N - r + 1, \ldots, N$, $m = 1, \ldots, N$.

Remark 22. *With respect to Theorems 20 and 21 we observe the following. If we assume that* $\{f_1, \ldots, f_N, x_1, \ldots, x_N\} \neq 0$, ***in*** $\mathbb{M}$, *and* $H = H(f_1, \ldots, f_N)$ *then the system of equations* $f_j(\mathbf{x}, \mathbf{y}) = c_j$, *for* $j = 1, \ldots, N$ *can be solved locally with respect to* $\mathbf{y}$, *(momenta) i.e.* $y_j = u_j(\mathbf{x}, \mathbf{c})$, *for* $j = 1, \ldots, N$ *where* $\mathbf{c} = (c_1, \ldots, c_N)$. *If the given first integrals are pairwise in involution, i.e.* $\{f_j, f_k\} = 0$, *then* $\sum_{j=1}^{N} u_j(\mathbf{x}, \mathbf{c}) dx_j = dS(\mathbf{x})$. *Consequently from the Liouville theorem:*

Theorem 23. If a Hamiltonian system has N independent first integrals in involution, which can be solved with respect to the momenta, then its motion can be obtained with quadratures, that is, the equation of motion can be solved simply by evaluating integrals.

We obtain that the Hamiltonian system $\dot{x}_k = \{H, x_k\}^*$, $\dot{y}_k = \{H, y_k\}^*$, *for* $k = 1, \ldots, N$ *is integrable by quadratures (for more details see* [26].

In general the given set of first integrals is not necessarily in involution. The solution of the inverse problem in constrained Hamiltonian system shows that in this case the differential equations which have as invariant the submanifold $\mathcal{M}_1$ *is in general not Hamiltonian. The origin of the theory on noncommutative integration is the Nekhoroshevs Theorem (see* [30]*). The following result holds (see* [26]*).*

Theorem 24. If a Hamiltonian system with N degrees of freedom has $N + r$ independent first integrals f_j for $j = 1, \ldots, N + r$, such that the $f_1, \ldots, f_{N-r}$ first integrals are in involution with all integrals $f_1, \ldots, f_{N+r}$. Then the Hamiltonian system is integrable by quadratures.

If $f_1, f_2, \ldots, f_{N-r}$ *are the first integrals which are in involution with all the first integrals and* $H = H(f_1, f_2, \ldots, f_{N-r})$, *then the differential system* (54) *is Hamiltonian and is integrable by quadratures.*

Theorem 25. *Let* $(\mathbb{M}, \Omega^2, \mathcal{M}_2, H)$ *be a constrained Hamiltonian system and let* $g_j : \mathbb{M} \longrightarrow \mathbb{R}$ *for* $j = 1, \ldots, M < N$ *be given independent functions in* $\mathbb{M}$, *where*

$$\mathcal{M}_2 = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{M} : g_j(\mathbf{x}, \mathbf{y}) = 0 \quad for \quad j = 1, \ldots, M < N\},$$

We choose the arbitrary functions g_m *for* $m = M+1,\ldots,2N$ *in such a way that the determinant* $\{g_1,\ldots,g_M,g_{M+1},\ldots,g_{2N}\}\neq 0$ *in* $\mathbb{M}$.

We shall study only the case when $\{g_1,\ldots,g_M,g_{M+1},\ldots,g_N,x_1,\ldots,x_N\}\neq 0$. *Then the submanifold* $\mathcal{M}_2$ *is an invariant manifold by the flow of the differential system*

$$
\begin{aligned}
\dot{x}_k &= \{H,\,x_k\}^*,\\
\dot{y}_k &= \{H,\,y_k\}^*+\\
&\sum_{j=1}^{M}\frac{(\Phi_j-\{H,\,g_j\}^*)\,\{g_1,\ldots,g_{j-1},y_k,g_{j+1},g_{2M},g_{2M+1},\ldots,g_N,x_1,\ldots,x_N\}}{\{g_1,\ldots,g_N,x_1,\ldots,x_N\}}+\\
&\sum_{j=M+1}^{N}\frac{(\lambda_j-\{H,\,g_j\}^*)\{g_1,\ldots,g_{2M+1},\ldots,g_{j-1},y_k,g_{j+1},\ldots g_N,x_1,\ldots,x_N\}}{\{g_1,\ldots,g_N,x_1,\ldots,x_N\}}\\
&=\{H,\,y_k\}^*+W_{k+N},
\end{aligned}
\tag{55}
$$

for $k=1,\ldots,N$, *where* λ_j *for* $j=M+1,\ldots,N$, *and* Φ_j *are arbitrary functions satisfying* $\Phi_j|_{g_j=0}=0$ *for* $j=1,\ldots,M$.

We observe that equations (55) on the submanifold $\mathcal{M}_2$ when the arbitrary functions λ_k are $\lambda_k=\{H,g_k\}^*$ become

$$
\begin{aligned}
\dot{x}_j =&\ \{H,x_j\}^*,\\
\dot{y}_j =&\ \{H,y_j\}^*+\\
&-\sum_{k=1}^{M}(\{H,g_k\}^*)\frac{\{g_1,\ldots,g_{k-1},y_j,g_{k+1},\ldots,g_{N_1},\ldots,g_N,x_1,\ldots,x_N\}}{\{g_1,\ldots,g_N,x_1,\ldots,x_N\}},
\end{aligned}
\tag{56}
$$

for $j=1,\ldots,N$. This system can be interpreted as the equations of motion of the constrained mechanical system with Hamiltonian H under the action of the external forces with components

$$
W_{j+N}=-\sum_{k=1}^{M}\{H,g_k\}^*\frac{\{g_1,\ldots,g_{k-1},y_j,g_{k+1},\ldots,g_{N_1},\ldots,g_N,x_1,\ldots,x_N\}}{\{g_1,\ldots,g_N,x_1,\ldots,x_N\}},
$$

generated by the constraints $g_j=0$ for $j=1,\ldots,M$.

Theorem $20, 21$ and 25 are proved in section 11.

3. Preliminaries and new properties of the Nambu bracket

The Nambu bracket $\left|\dfrac{\partial(h_1,\ldots,h_N)}{\partial(x_1,\ldots,x_N)}\right| := \{h_1,\,\ldots,h_N\}$ was proposed by Nambu to generalize Hamiltonian mechanics. This skew symmetric bracket satisfies the Leibniz rule and the *fundamental identity*

$$
\begin{aligned}
&0=F\,(f_1\ldots,f_{N-1},g_1\ldots,g_N):=\\
&\sum_{n=1}^{N}\{g_1,\ldots,g_{n-1},\{f_1\ldots,f_{N-1},g_n\},g_{n+1},\ldots,g_N\}-\{f_1\ldots,f_{N-1},\{g_1\ldots,g_N\}\},
\end{aligned}
\tag{57}
$$

where $f_1,\,f_2,\,\ldots,f_{N-1},\,g_1,\,\ldots,g_N$ are arbitrary functions. For more details see [29, 49, 21].

In this section we show new properties of this bracket which we will use in the proofs and in the applications of the results stated in the two previous sections.

We shall need the next result:

Proposition 26. *The following identities hold*

$$\begin{aligned}
&\sum_{j=1}^{N}\frac{\partial f}{\partial x_j}\{g_1,\ldots,g_{k-1},x_j,g_{k+1},\ldots,g_N\}=\{g_1,\ldots,g_{k-1},f,g_{k+1}\ldots,g_N\},\\
&\frac{\partial f}{\partial x_k}=\{x_1,\ldots,x_{k-1},f,x_{k+1},\ldots,x_N\},\\
&K_j:=\sum_{j=1}^{N}\frac{\partial}{\partial x_j}\{g_1,\ldots,g_{k-1},x_j,g_{k+1}\ldots,g_N\}=0,
\end{aligned}\tag{58}$$

for $k=1,2,\ldots,N$ *and*

$$\begin{aligned}
&\frac{\partial f_1}{\partial x_N}\left|\frac{\partial\,(G,f_2,\ldots,f_N)}{\partial\,(y_1,\ldots,y_N)}\right|+\ldots+\frac{\partial f_N}{\partial x_N}\left|\frac{\partial\,(f_1,\ldots,f_{N-1},G)}{\partial\,(y_1,\ldots,y_N)}\right|\\
&=\frac{\partial G}{\partial y_1}\left|\frac{\partial\,(f_1,\ldots,f_N)}{\partial\,(x_N,y_2,\ldots,y_N)}\right|+\ldots+\frac{\partial G}{\partial y_N}\left|\frac{\partial\,(f_1,\ldots,f_N)}{\partial\,(y_1,\ldots,y_{N-1},x_N)}\right|.
\end{aligned}\tag{59}$$

Here the functions $g_1,\ldots,g_N,f_1,\ldots,f_N,G$ *and* f *are arbitrary.*

Proof. The proof of the first relation is the following

$$\{g_1,\ldots,g_{k-1},f,g_{k+1}\ldots,g_N\}=\begin{vmatrix}\partial_1 g_1 & \ldots & \partial_N g_1\\ \vdots & & \vdots\\ \partial_1 g_{k-1} & \ldots & \partial_N g_{k-1}\\ \partial_1 f & \ldots & \partial_N f\\ \partial_1 g_{k+1} & \ldots & \partial_N g_{k+1}\\ \vdots & & \vdots\\ \partial_1 g_N & \ldots & \partial_N g_N\end{vmatrix}$$

$$=\partial_1 f\begin{vmatrix}\partial_1 g_1 & \ldots & \partial_N g_1\\ \vdots & & \vdots\\ \partial_1 g_{k-1} & \ldots & \partial_N g_{k-1}\\ 1 & 0 & \ldots\\ \partial_1 g_{k+1} & \ldots & \partial_N g_{k+1}\\ \vdots & & \vdots\\ \partial_1 g_N & \ldots & \partial_N g_N\end{vmatrix}+\ldots+\partial_N f\begin{vmatrix}\partial_1 g_1 & \ldots & \partial_N g_1\\ \vdots & & \vdots\\ \partial_1 g_{k-1} & \ldots & \partial_N g_{k-1}\\ 0 & \ldots & 1\\ \partial_1 g_{k+1} & \ldots & \partial_N g_{k+1}\\ \vdots & & \vdots\\ \partial_1 g_N & \ldots & \partial_N g_N\end{vmatrix}$$

$$=\{g_1,\ldots,g_{k-1},x_1,g_{k+1},\ldots,g_N\}\partial_1 f+\ldots+\{g_1,\ldots,g_{k-1},x_N,g_{k+1},\ldots,g_N\}\partial_N f.$$

The proof of the second relation follows easily from the definition of the Nambu bracket. The proof of the third relation is the following. Taking in the second identity of (58) $f=\{g_1,\ldots,g_{k-1},x_j,g_{k+1}\ldots,g_N\}$ we obtain

$$K_j:=\sum_{j=1}^{N}\{x_1,\ldots,x_{j-1},\{g_1,\ldots,g_{k-1},x_j,g_{k+1},\ldots,g_N\},x_{j+1},\ldots,x_N\}.$$

By using the fundamental identity (57) we get

$$K_j=\{g_1,\ldots,g_{k-1},g_{k+1},\ldots,g_n,\ \{x_1,\ldots,x_{j-1},x_j,x_{j+1},\ldots,x_N\}\}=0,$$

because $\{x_1,\ldots,x_{j-1},x_j,x_{j+1},\ldots,x_N\}=1$. We observe that this identity can be proved by applying only the properties of the determinants, but this proof is long.

The proof of (59) is easy to obtain by considering that the value of determinant

$$\begin{vmatrix} \dfrac{\partial f_1}{\partial y_1} & \dots & \dfrac{\partial f_1}{\partial y_N} & \dfrac{\partial f_1}{\partial x_N} \\ \vdots & \dots & \vdots & \vdots \\ \dfrac{\partial f_N}{\partial y_1} & \dots & \dfrac{\partial f_N}{\partial y_N} & \dfrac{\partial f_N}{\partial x_N} \\ \dfrac{\partial G}{\partial y_1} & \dots & \dfrac{\partial G}{\partial y_N} & 0 \end{vmatrix}$$

can be obtained by developing by the last row or, what is the same by the last column. □

Proposition 27. *The Nambu bracket satisfy the identities*

(60)

$$0 = \Omega\left(f_1 \dots, f_{N-1}, g_1 \dots, g_N, G\right) :=$$

$$\sum_{n=1}^{N} \{f_1, \dots, f_{N-1}, g_n\}\{g_1, \dots, g_{n-1}, G, g_{n+1}, \dots, g_N\} - \{f_1 \dots, f_{N-1}, G\}\{g_1 \dots, g_N\},$$

$$0 = F_\lambda\left(f_1 \dots, f_{N-1}, g_1 \dots, g_N\right) :=$$

$$\sum_{n=1}^{N} \{g_1, \dots, g_{n-1}, \lambda\{f_1 \dots, f_{N-1}, g_n\}, g_{n+1}, \dots, g_N\} - \{f_1 \dots, f_{N-1}, \lambda\{g_1 \dots, g_N\}\},$$

for arbitrary functions $f_1, \dots, f_{N-1}$, G, $g_1, \dots, g_N, \lambda$.

Note that the second identity is a generalization of the fundamental identity (57) which is obtained when $\lambda = 1$.

Proof. Indeed, using the first property of (58) we obtain the first identity (see for instance [43])

$$\Omega\left(f_1 \dots, f_{N-1}, g_1 \dots, g_N, G\right) = \begin{vmatrix} dg_1(\partial_1) & \dots & dg_1(\partial_N) & \{f_1, \dots, f_{N-1}, g_1\} \\ \vdots & \dots & \vdots & \vdots \\ dg_N(\partial_1) & \dots & dg_N(\partial_N) & \{f_1, \dots, f_{N-1}, g_N\} \\ dG(\partial_1) & \dots & dG(\partial_N) & \{f_1, \dots, f_{N-1}, G\} \end{vmatrix}$$

$$= \begin{vmatrix} dg_1(\partial_1) & \dots & dg_1(\partial_N) & \sum_{j=1}^{N}\{f_1, \dots, f_{N-1}, x_j\}dg_1(\partial_j) \\ \vdots & \dots & \vdots & \vdots \\ dg_N(\partial_1) & \dots & dg_N(\partial_N) & \sum_{j=1}^{N}\{f_1, \dots, f_{N-1}, x_j\}dg_N(\partial_j) \\ dG(\partial_1) & \dots & dG(\partial_N) & \sum_{j=1}^{N}\{f_1, \dots, f_{N-1}, x_j\}dG(\partial_j) \end{vmatrix}$$

$$= \sum_{j=1}^{N}\{f_1, \dots, f_{N-1}, x_j\} \begin{vmatrix} dg_1(\partial_1) & \dots & dg_1(\partial_N) & dg_1(\partial_j) \\ \vdots & \dots & \vdots & \vdots \\ dg_N(\partial_1) & \dots & dg_N(\partial_N) & dg_N(\partial_j) \\ dG(\partial_1) & \dots & dG(\partial_N) & dG(\partial_j) \end{vmatrix} = 0.$$

This proves the first identity.

The proof of the second identity is as follows. Taking $G = x_j$ in the first identity of (60) we obtain

$$\lambda\Omega\left(f_1,\ldots,f_{N-1},g_1,\ldots,g_N,x_j\right) :=$$
$$\lambda\{f_1,\ldots,f_{N-1},g_1\}\{x_j,g_2,\ldots,g_N\}+\ldots+\lambda\{f_1,\ldots,f_{N-1},g_N\}\{g_1,\ldots,g_{N-1},x_j\}$$
$$\ldots-\lambda\{f_1,\,\ldots,f_{N-1},x_j\}\{g_1,\ldots,g_N\}=0.$$

Using the third identity of (58), from the last expression we have

$$0=\sum_{j=1}^{N}\frac{\partial}{\partial x_j}\left(\lambda\Omega(f_1,\ldots,f_{N-1},g_1,\ldots,g_N,x_j)\right)=$$
$$\sum_{j=1}^{N}\{x_j,g_2,\ldots,g_N\}\frac{\partial}{\partial x_j}\left(\lambda\{f_1,\ldots,f_{N-1},g_1\}\right)+\ldots$$
$$+\sum_{j=1}^{N}\Big(\{g_1,g_2,\ldots,g_{N-1},x_j\}\frac{\partial}{\partial x_j}\left(\lambda\{f_1,\,\ldots,f_{N-1},g_N\}\right)-$$
$$\{f_1,\,\ldots,f_{N-1},x_j\}\frac{\partial}{\partial x_j}\left(\lambda\{g_1,\ldots,g_N\}\right)\Big).$$

Now using the first identity of (58), the previous expression becomes

$$0=\{\lambda\{f_1,\,\ldots,f_{N-1},g_1\},g_2,\ldots,g_N\}+\ldots+\{g_1,g_2,\ldots,g_{N-1},\lambda\{f_1,\,\ldots,f_{N-1},g_N\}\}\ldots$$
$$-\{f_1,\,\ldots,f_{N-1},\lambda\{g_1,\ldots,g_N\}\}=F_\lambda\left(f_1,\ldots,f_{N-1},g_1,\ldots,g_N\right).$$

This complete the proof of the second identity. □

Remark 28. *We note that the second identity of* (60) *has obtained from the first identity of* (60)*. So, in some sense the first identity of* (60) *is more basic. In fact, from the proof of the second identity of* (60) *we obtain*

$$F_\lambda\left(f_1,\ldots,f_{N-1},g_1,\ldots,g_N\right)=\sum_{j=1}^{N}\frac{\partial}{\partial x_j}\left(\lambda\Omega\left(f_1,\ldots,f_{N-1},g_1,\ldots,g_N,x_j\right)\right).$$

Now we establish the relationship between the Nambu bracket and the classical Poisson bracket. We suppose that $N=2n,$ and $x_j=x_j$ and $x_{j+n}=y_j$ for $j=1,\ldots,n.$

Proposition 29. *Between the Poisson bracket and the Nambu bracket there are the following equalities*

$$\begin{aligned}&\sum_{j=1}^{n}\{H,f,\,x_1\ldots,x_{j-1},x_{j+1},\ldots,x_n,y_1,\ldots,y_{j-1},y_{j+1},\ldots,y_n\}=\{H,f\}^*,\\&\sum_{j=1}^{2n}\{H,f_j\}^*\{f_1,\ldots,f_{j-1},G,f_{j+1},\ldots,f_{2n}\}=\{H,G\}^*\{f_1,\ldots,f_{2n}\}.\end{aligned}\tag{61}$$

Proof. The first equality it is easy to obtain by using the definition of the Nambu bracket. The second equality follows in view of the identity $\Omega\left(f_1\ldots,f_{N-1},g_1\ldots,g_N,G\right)=0$. Indeed,

$$
\begin{aligned}
&\sum_{k=1}^{2n}\{H,f_k\}^*\{f_1,\ldots,f_{k-1},G,f_{k+1},\ldots,f_{2n}\}\\
=\;&\sum_{k=1}^{2n}\Big(\sum_{j=1}^{n}\{H,f_k,\,x_1\ldots,x_{j-1},x_{j+1},\ldots,x_n,y_1,\ldots,y_{j-1},y_{j+1},\ldots,y_n\}\cdot\\
&\{f_1,\ldots,f_{k-1},G,f_{k+1},\ldots,f_{2n}\}\Big)\\
=\;&\sum_{j=1}^{n}\Big(\sum_{k=1}^{2n}\{H,f_k,\,x_1\ldots,x_{j-1},x_{j+1},\ldots,x_n,y_1,\ldots,y_{j-1},y_{j+1},\ldots,y_n\}\cdot\\
&\{f_1,\ldots,f_{k-1},G,f_{k+1},\ldots,f_{2n}\}\Big)\\
=\;&\sum_{j=1}^{n}\{H,G,\,x_1\ldots,x_{j-1},x_{j+1},\ldots,x_n,y_1,\ldots,y_{j-1},y_{j+1},\ldots,y_n\}\{f_1,\ldots,f_{2n}\}\\
=\;&\{H,G\}^*\{f_1,\ldots,f_{2n}\}.
\end{aligned}
$$

□

4. Proof of Theorems 1 and 3

In this section we construct the most general autonomous differential system on $\mathrm{D}\subset\mathbb{R}^N$ having the set of invariant hypersurfaces $g_j=0$ for $j=1,2,\ldots,M$, with $M\le N$, and $M>N$.

Proof of Theorem 1. We consider the vector field

(62)
$$
\begin{aligned}
\mathbf{X}=\;& -\frac{1}{\{g_1,\ldots,\,g_N\}}\begin{vmatrix}
dg_1(\partial_1) & \ldots & dg_1(\partial_N) & \Phi_1\\
dg_2(\partial_1) & \ldots & dg_2(\partial_N) & \Phi_2\\
\vdots & \vdots & \vdots & \vdots\\
dg_M(\partial_1) & \ldots & dg_M(\partial_N) & \Phi_M\\
dg_{M+1}(\partial_1) & \ldots & dg_{M+1}(\partial_N) & \lambda_{M+1}\\
\vdots & \vdots & \vdots & \vdots\\
dg_N(\partial_1) & \ldots & dg_N(\partial_N) & \lambda_N\\
\partial_1 & \ldots & \partial_N & 0
\end{vmatrix}\\
=\;& \sum_{k,j=1}^{N}\frac{S_{jk}P_j}{|S|}\partial_k=\left\langle S^{-1}\mathbf{P},\partial_{\mathbf{x}}\right\rangle,
\end{aligned}
$$

where S_{jk} for $k,j=1,\ldots,N$ is the determinant of the adjoint of the matrix S after removing the row j and the column k (see (3)), S^{-1} is the inverse matrix of S, and $\mathbf{P}=(P_1,\ldots,P_N)^T=\left(\Phi_1,\ldots,\Phi_M,\,\lambda_{M+1},\ldots,\lambda_N\right)^T$. From (62) it is easy to obtain the relationship

$$
\begin{aligned}
\mathbf{X}(g_j) = \;& \Phi_1\frac{\{g_j,g_2,\ldots,g_N\}}{\{g_1,g_2,\ldots,g_N\}}+\ldots+\Phi_M\frac{\{g_1,\ldots,g_{M-1},\, g_j,g_{M+1},\ldots,g_N\}}{\{g_1,\ldots,g_{M-1},g_M,g_{M+1},\ldots,g_N\}}+\\
& \lambda_{M+1}\frac{\{g_1,\ldots,g_M,g_j,g_{M+2},\ldots,g_N\}}{\{g_1,\ldots,g_M,g_{M+1},g_{M+2},\ldots,g_N\}}+\ldots+\lambda_N\frac{\{g_1,\ldots,g_{N-1},g_j\}}{\{g_1,\ldots,g_{N-1},g_N\}}\\
= \;& \begin{cases}\Phi_j & \text{for} \quad 1\le j\le M\\ \lambda_j & \text{otherwise.}\end{cases}
\end{aligned} \tag{63}
$$

Thus

$$
\mathbf{X}(g_j)=\Phi_j, \quad \mathbf{X}(g_{M+k})=\lambda_{M+k}, \tag{64}
$$

for $j=1,2,\ldots M$, and $k=1,\ldots,N-M$. In view of assumption $\Phi_j|_{g_j=0}=0$ we obtain that the $g_j=0$ for $j=1,2,\ldots M$ are invariant hypersurfaces of the vector field $\mathbf{X}$.

The vector field $\mathbf{X}$ already was used in [43, 37]. Note that it is well defined in view of assumption (3).

Now we shall prove that system (2) is the most general differential system which admits the given set of independent partial integrals. Indeed let $\dot{\mathbf{x}}=\tilde{\mathbf{X}}(\mathbf{x})$ be another differential system having $g_1,\, g_2, ..., g_M$ as partial integrals, i.e. $\tilde{\mathbf{X}}(g_j)|_{g_j=0}=0$ for $j=1,2,\ldots,M$. Then taking

$$
\Phi_j=\tilde{\mathbf{X}}(g_j)=\sum_{l=1}^{N}\tilde{X}_l\partial_l g_j=\sum_{l=1}^{N}\tilde{X}_l\{x_1,\ldots,x_{l-1},g_j,x_{l+1},\ldots,x_N\},
$$

for $j=1,2,..,M$, and

$$
\lambda_{M+k}=\tilde{\mathbf{X}}(g_{M+k})=\sum_{l=1}^{N}\tilde{X}_l\partial_l g_{M+k}=\sum_{l=1}^{N}\tilde{X}_l\{x_1,\ldots,x_{l-1},g_{M+k},x_{l+1},\ldots,x_N\},
$$

for $k=1,\ldots,N-M$, (here we use the second identity of (58)) and substituting Φ_j and λ_{M+k} into formula (62) we get for arbitrary function F

$$
\begin{aligned}
\mathbf{X}(F) = \;& \sum_{l=1}^{N}\Phi_j\frac{\{g_1,\ldots,g_{j-1},F,g_{j+1}\ldots,g_M,\ldots,g_N\}}{\{g_1,g_2,\ldots,g_N\}}+\\
& \sum_{j=M+1}^{N}\lambda_{M+j}\frac{\{g_1,\ldots,g_M,g_{M+1},\ldots,g_{j-1},F,g_{j+1}\ldots,g_N\}}{\{g_1,g_2,\ldots,g_N\}}\\
= \;& \sum_{j=1}^{N}\sum_{l=1}^{N}\tilde{X}_l\{x_1,\ldots,x_{l-1},g_j,x_{l+1},\ldots,x_N\}\frac{\{g_1,\ldots,g_{j-1},F,g_{j+1},\ldots,g_N\}}{\{g_1,g_2,\ldots,g_N\}}\\
= \;& \sum_{l=1}^{N}\tilde{X}_l\sum_{j=1}^{N}\{x_1,\ldots,x_{l-1},g_j,x_{l+1},\ldots,x_N\}\frac{\{g_1,\ldots,g_{j-1},F,g_{j+1},\ldots,g_N\}}{\{g_1,g_2,\ldots,g_N\}}\\
= \;& \sum_{l=1}^{N}\tilde{X}_l\{x_1,\ldots,x_{l-1},F,x_{l+1},\ldots,x_N\}=\tilde{\mathbf{X}}(F),
\end{aligned}
$$

Here we have used the first identity of (60) and the second of (58). Hence, in view of arbitrariness of F the theorem has been proved. □

Proof of Theorem 3. First of all we determine the differential system by using the N independent functions $g_j=g_j(\mathbf{x})$ for $j=1,2,\ldots,N$. Thus we obtain system (5). Clearly this differential system admits additional partial integrals g_j for $j=N+1,\ldots,M$ if and only if $\mathbf{X}(g_\nu) = \Phi_\nu, \quad \Phi_\nu|_{g_\nu=0} = 0$, for $\nu = N+1, \ldots, M$. Equivalently, using (62) can be written as

$$\Phi_1\{g_\nu, \ldots, g_N\} + \ldots + \Phi_N\{g_1, \ldots, g_{N-1}, g_\nu\} - \Phi_\nu\{g_1, \ldots, g_{N-1}, g_N\} = 0. \tag{65}$$

Now we prove that

$$\Phi_\nu = \sum_{\alpha_1,\ldots,\alpha_{N-1}=1}^{M+N} G_{\alpha_1,\ldots,\alpha_{N-1}}\{g_{\alpha_1}, \ldots, g_{\alpha_{N-1}}, g_\nu\} \tag{66}$$

is a solution of (65) for $\nu = 1, 2, \ldots, M \geq N$, where $G_{\alpha_1,\ldots,\alpha_{N-1}} = G_{\alpha_1,\ldots\alpha_{N-1}}(\mathbf{x})$ are arbitrary functions satisfying (7).

Indeed, in view of (65) and (66) we obtain

$$\sum_{\alpha_1,\ldots,\alpha_{N-1}=1}^{M+N} G_{\alpha_1,\ldots,\alpha_{N-1}}(\{g_{\alpha_1}, \ldots, g_{\alpha_{N-1}}, g_1\}\{g_\nu, g_2, \ldots, g_{N-1}, g_N\} + \ldots$$
$$+\{g_{\alpha_1}, \ldots, g_{\alpha_{N-1}}, g_N\}\{g_1, g_2, \ldots, g_{N-1}, g_\nu\} - \{g_{\alpha_1}, \ldots, g_{\alpha_{N-1}}, g_\nu\}\{g_1, g_2, \ldots, g_{N-1}, g_N\})$$
$$= \sum_{\alpha_1,\ldots,\alpha_{N-1}=1}^{M+N} G_{\alpha_1,\ldots,\alpha_{N-1}}\Omega\left(\{g_{\alpha_1}, \ldots, g_{\alpha_{N-1}}, g_1, \ldots, g_N, g_\nu\right) = 0$$

which is identically zero by (60).

Inserting (66) into (5) and from the second identity of (60) we obtain from the equation

$$\dot{x}_\nu = \Phi_1\frac{\{x_\nu, g_2 \ldots, g_N\}}{\{g_1, \ldots, g_N\}} + \ldots + \Phi_N\frac{\{g_1, \ldots, g_{N-1}, x_\nu\}}{\{g_1, \ldots, g_N\}}$$
$$= \sum_{\alpha_1\ldots\alpha_{N-1}=1}^{M+N} \frac{G_{\alpha_1,\ldots,\alpha_{N-1}}}{\{g_1, \ldots, g_N\}}\sum_{n=1}^{N}\{g_{\alpha_1}, \ldots, g_{\alpha_{N-1}}, g_n\}\{g_1, \ldots, g_{n-1}, x_\nu, g_{n+1}, \ldots, g_N\}$$
$$= \sum_{\alpha_1,\ldots,\alpha_{N-1}=1}^{M+N} G_{\alpha_1,\ldots,\alpha_{N-1}}\left(\{g_{\alpha_1}, \ldots, g_{\alpha_{N-1}}, x_\nu\}\right)$$

for $j = 1, 2, \ldots, N$. Now we prove that this differential system which coincides with (6) is the most general. Indeed using that $g_{M+j} = x_j$ for $j = 1, \ldots, N$, system (6) admits the representation

$$\begin{aligned} \dot{x}_1 &= \left(\sum_{\substack{\alpha_1,\ldots,\alpha_{N-1}=1\\ \alpha_1,\ldots,\alpha_{N-1}\neq(M+2,\ldots,N)}}^{M+N} G_{\alpha_1,\ldots,\alpha_{N-1}}\{g_{\alpha_1}, \ldots, g_{\alpha_{N-1}}, x_1\}\right) + \\ &\qquad G_{M+2,\,M+3\ldots,M+N}\{x_2, \ldots, x_N, x_1\}, \\ &\vdots \qquad\qquad \vdots \\ \dot{x}_N &= \left(\sum_{\substack{\alpha_1,\ldots,\alpha_{N-1}=1\\ \alpha_1,\ldots,\alpha_{N-1}\neq(M+1,\ldots,N-1)}}^{M+N} G_{\alpha_1,\ldots,\alpha_{N-1}}\{g_{\alpha_1}, \ldots, g_{\alpha_{N-1}}, x_N\}\right) + \\ &\qquad G_{M+1,\,M+2\ldots,M+N-1}\{x_1, \ldots, x_{N-1}, x_N\}. \end{aligned} \tag{67}$$

Note that $\{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N, x_j\} \in \{-1, 1\}$. Therefore if $\dot{x}_j = \tilde{X}_j$ for $j = 1, \ldots, N$ is another differential system having the given set of partial integrals, then by choosing conveniently functions $G_{M+2,\,M+3\,\ldots,M+N},\ldots G_{M+1,\,M+2\,\ldots,M+N-1}$ we deduce that the constructed vector field (67) contain the vector field $\tilde{\mathbf{X}} = \left(\tilde{X}_1,\,\tilde{X}_2,\ldots,\tilde{X}_N\right)$. So the proof of Theorem 3 follows. □

Corollary 30. *Under the assumptions of Theorem 3 for $N=2$ system (6) takes the form*

$$
\begin{aligned}
\dot{x} = &\ \sum_{j=1}^{M} G_j\{g_j,\,x\} + G_{M+1}\{y,\,x\} = \sum_{j=1}^{M} G_j\{g_j,\,x\} - G_{M+1},\\
\dot{y} = &\ \sum_{j=1}^{M} G_j\{g_j,\,y\} + G_{M+2}\{x,\,y\} = \sum_{j=1}^{M} G_j\{g_j,\,y\} + G_{M+2},
\end{aligned}
\tag{68}
$$

where $G_j = G_j(x,y)$ for $j=1,2,\ldots M+2$ are arbitrary functions satisfying (7). Moreover (7) becomes

$$
\left.\left(\sum_{j=1}^{M} G_j\{g_j,\,g_k\} + G_{M+1}\{y,g_k\} + G_{M+2}\{x,\,g_k\}\right)\right|_{g_k=0} = 0,
\tag{69}
$$

for $k=1,2,\ldots M$.

Proof of Corollary 30. It follows immediately from Theorem 3. □

Remark 31. *We note that conditions (69) hold in particular if*

$$
G_j = \prod_{\substack{m=1\\ m\neq j}}^{M} g_m\,\tilde{G}_j
\tag{70}
$$

where $\tilde{G}_j = \tilde{G}_j(x,y)$ for $j=1,\ldots,M+2$ are arbitrary functions.

Inserting (70) into (68) we obtain the following differential system

$$
\dot{x} = \sum_{j=1}^{M}\prod_{\substack{m=1\\ m\neq j}}^{M} g_m\tilde{G}_j\{g_j,\,x\} - \prod_{m=1}^{M} g_m\tilde{G}_{M+1},\ \dot{y} = \sum_{j=1}^{M}\prod_{\substack{m=1\\ m\neq j}}^{M} g_m\tilde{G}_j\{g_j,\,y\} + \prod_{m=1}^{M} g_m\tilde{G}_{M+2}.
\tag{71}
$$

We observe that system (71) coincides with polynomial differential (5) of [5] when the partial integrals are polynomial and generic in the sense defined in [5].

5. Proof of Theorem 4 and Theorem 7

Proof of Theorem 4. Let $\mathbf{X}$ be the vector field

$$
\mathbf{X} = -\frac{1}{|S|}\begin{vmatrix}
dg_1(\partial_1) & \ldots & dg_1(\partial_N) & \Phi_1\\
\vdots & \vdots & \vdots & \\
dg_{M_1}(\partial_1) & \ldots & dg_{M_1}(\partial_N) & \Phi_{M_1}\\
df_1(\partial_1) & \ldots & df_1(\partial_N) & 0\\
\vdots & \vdots & \vdots & \vdots\\
df_{M_2}(\partial_1) & \ldots & df_{M_2}(\partial_N) & 0\\
dg_{M+1}(\partial_1) & \ldots & dg_{M+1}(\partial_N) & \lambda_{M+1}\\
\vdots & \vdots & \vdots & \vdots\\
dg_N(\partial_1) & \ldots & dg_N(\partial_N) & \lambda_N\\
\partial_1 & \ldots & \partial_N & 0
\end{vmatrix} = \left\langle S^{-1}\mathbf{P}, \partial_{\mathbf{x}}\right\rangle,
$$

where $\mathbf{P} = (P_1,\ldots,P_N)^T = (\Phi_1,\ldots,\Phi_{M_1},0,\ldots,0,\lambda_{M+1},\ldots,\lambda_N)^T$, which is the vector field associated to differential system (8) where $|S| = \{g_1,\ldots,g_{M_1},f_1,\ldots,f_{M_2},g_{M+1},\ldots,g_N\}$. Clearly this vector field is well defined in view of the assumptions.

From $\mathbf{X}(g_j) = \Phi_j, \quad \Phi|_{g_j=0} = 0$, for $j = 1, \ldots, M_1$ we deduce that g_j are partial integrals of vector field $\mathbf{X}$ and $\mathbf{X}(f_j) = 0$ for $j = 1, \ldots, M_2$ we obtain that f_j are first integrals of vector field $\mathbf{X}$.

Now we prove that system (8) is the most general differential system admitting the partial integrals g_j and the first integrals f_k. Indeed let $\dot{\mathbf{x}} = \tilde{\mathbf{X}}(\mathbf{x})$ be another differential system which admits g_j for $j = 1, \ldots, M_1$ partial integrals and f_k for $k = 1, \ldots, M_2$ first integrals with $M_1+M_2 \leq N$, i.e. $\tilde{\mathbf{X}}(g_j)|_{g_j=0} = 0$ for $j = 0, 1, \ldots, M_1$ and $\tilde{\mathbf{X}}(f_k) = 0$ for $k = 1, \ldots, M_2$. Then taking $\Phi_j = \tilde{\mathbf{X}}(g_j)$ and $\lambda_{M+k} = \tilde{\mathbf{X}}(f_k)$ and analogously to the proof of Theorem 1 we deduce that the vector field $\tilde{\mathbf{X}}$ is a particular case of the vector field $\mathbf{X}$. Thus the theorem is proved. □

Proof of Theorem 7. In view of Corollary 5 it follows (12), or equivalently

$$\dot{\mathbf{x}} = S^{-1}\mathbf{P}, \tag{72}$$

where $\mathbf{P} = (0, \ldots, 0, \lambda_{N-1}, \lambda_N)^T$ (see for more details the proof of Theorem 1, i.e. (62)). Hence Corollary 5 gives the most general differential system which admits first integrals f_j for $j = 1, \ldots, N-2$. After the change of variables $(x_1, \ldots, x_N) \longrightarrow (y_1, \ldots, y_N)$ where $y_j = f_j$ for $j = 1, \ldots, N-2$, and $y_{N-1} = x_{N-1}$, $y_N = x_N$ we obtain that the differential system (72) on the set

$$E_c = \{(y_1, y_2, \ldots y_N) \in \mathbb{R}^N : y_1 = c_1, \ldots, y_{N-2} = c_{N-2}\}$$

becomes $\dot{\mathbf{x}} = B^{-1}\dot{\mathbf{y}} = B^{-1}\hat{S}^{-1}\tilde{\mathbf{P}}$, where $\hat{S}$ and B are defined by

$$\begin{aligned} S = & \frac{\partial\left(f_1, \ldots, f_{N-2},\, g_{N-1},\, g_N\right)}{\partial\left(x_1, \ldots, x_N\right)} = \frac{\partial\left(f_1, \ldots, f_{N-2},\, g_{N-1},\, g_N\right)}{\partial(y_1, \ldots, y_N)} \frac{\partial\left(y_1, \ldots, y_N\right)}{\partial\left(x_1, \ldots, x_N\right)} = \hat{S}B \\ \dot{x}_j = & \sum_{k=1}^{N} \left(\frac{\partial\, x_j}{\partial y_k}\right) \dot{y}_k, \end{aligned}$$

and $\tilde{z}$ denotes the function $z(x_1, \ldots, x_N)$ expressed in the variables $\mathbf{y} = (y_1, \ldots, y_N)$.

Clearly we have that

$$\hat{S} = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 \\ dg_{N-1}(\partial_1) & dg_{N-1}(\partial_2) & \ldots & dg_{N-1}(\partial_{N-2}) & dg_{N-1}(\partial_{N-1}) & dg_{N-1}(\partial_N) \\ dg_N(\partial_1) & dg_N(\partial_2) & \ldots & dg_N(\partial_{N-2}) & dg_N(\partial_{N-1}) & dg_N(\partial_N) \end{pmatrix},$$

where $\partial_j = \frac{\partial}{\partial y_j}$ and consequently

$$|\hat{S}| = \begin{vmatrix} dg_{N-1}(\partial_{N-1}) & dg_{N-1}(\partial_N) \\ dg_N(\partial_{N-1}) & dg_N(\partial_N) \end{vmatrix} = \frac{\partial g_{N-1}}{\partial y_{N-1}} \frac{\partial g_N}{\partial y_N} - \frac{\partial g_{N-1}}{\partial y_N} \frac{\partial g_N}{\partial y_{N-1}} = \{g_{N-1},\, g_N\}.$$

After a change of variables $x_j = x_j(y_1, \ldots, y_N)$ for $j = 1, \ldots, N$ system $\dot{x}_j = X_j(\mathbf{x})$ can be rewritten as $\dot{\mathbf{y}} = \mathbf{Y}(\mathbf{y})$. A computation shows that

$$\begin{aligned}\mathbf{Y} &= \left\langle \hat{S}^{-1}\tilde{\mathbf{P}}, \partial_{\mathbf{y}} \right\rangle \\ &= -\frac{1}{|\hat{\Upsilon}|}\begin{vmatrix} 1 & \ldots & 0 & 0 & 0 & 0 \\ 0 & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 & 0 & 0 \\ dg_{N-1}(\partial_1) & \ldots & dg_{N-1}(\partial_{N-2}) & dg_{N-1}(\partial_{N-1}) & dg_{N-1}(\partial_N) & \lambda_{N-1} \\ dg_N(\partial_1) & \ldots & dg_N(\partial_{N-2}) & dg_N(\partial_{N-1}) & dg_N(\partial_N) & \lambda_N \\ \partial_1 & \ldots & \partial_{N-2} & \partial_{N-1} & \partial_N & 0 \end{vmatrix} \\ &= -\frac{1}{|\hat{S}|}\begin{vmatrix} dg_{N-1}(\partial_{N-1}) & dg_{N-1}(\partial_N) & \lambda_{N-1} \\ dg_N(\partial_{N-1}) & dg_N(\partial_N) & \lambda_N \\ \partial_{N-1} & \partial_N & 0 \end{vmatrix}.\end{aligned}$$

Thus

$$\begin{aligned} \dot{y}_{N-1} = \mathbf{Y}(y_{N-1}) = \quad & \tilde{\lambda}_{N-1}\frac{\{\, y_{N-1}, \tilde{g}_N, \}}{\{\tilde{g}_{N-1},\, \tilde{g}_N\}} + \tilde{\lambda}_N\frac{\{\tilde{g}_{N-1},\, y_{N-1},\, \}}{\{\tilde{g}_{N-1},\, \tilde{g}_N\}} = Y_{N-1}(\mathbf{y}), \\ \dot{y}_N = \mathbf{Y}(y_N) = \quad & \tilde{\lambda}_{N-1}\frac{\{\, y_N,\, \tilde{g}_N\}}{\{\tilde{g}_{N-1},\, \tilde{g}_N\}} + \tilde{\lambda}_N\frac{\{\tilde{g}_{N-1},\, y_N\,\}}{\{\tilde{g}_{N-1},\, \tilde{g}_N\}} = Y_N(\mathbf{y}), \\ \dot{y}_j = \mathbf{Y}(y_j) = \quad & 0, \quad \text{for} \quad j = 1, \ldots, N-2. \end{aligned} \tag{73}$$

On the other hand from (11) and (12) and Remark 26 it follows that

$$\sum_{j=1}^{N}\frac{\partial(UX_j)}{\partial x_j} = \{f_1, \ldots, f_{N-2}, \mu\lambda_{N-1}, g_N\} + \{f_1, \ldots, f_{N-2}, g_{N-1}, \mu\lambda_N\} = 0. \tag{74}$$

On the other hand from the relations

$$\sum_{j=1}^{N}\frac{\partial X_j}{\partial x_j} = \frac{1}{D}\sum_{m=1}^{N}\frac{\partial\,(DY_m)}{\partial y_m},$$

where $D = |S|$ (see (3)) is the Jacobian. Hence we obtain from (74) the well known relation

$$\sum_{j=1}^{N}\frac{\partial(UX_j)}{\partial x_j} = \frac{1}{D}\sum_{j=1}^{N}\frac{\partial\left(\tilde{U}DY_j\right)}{\partial y_j} = \frac{1}{D}\left(\frac{\partial\left(\tilde{U}DY_{N-1}\right)}{\partial\, y_{N-1}} + \frac{\partial\left(\tilde{U}DY_N\right)}{\partial y_N}\right) = 0,$$

Consequently the function $D\tilde{U}$ is an integrating factor of (73).

The equality (11) is obtained from Proposition 26 by considering the vector fields $\mathbf{X}$ determined by (12), hence

$$\begin{aligned} &\operatorname{div}\left(\tilde{\lambda}_{N-1}\{f_1, \ldots, f_{N-2}, x_j, g_N\} + \tilde{\lambda}_N\{f_1, \ldots, f_{N-2}, g_{N-1}, x_j\}\right) \\ &= \sum_{j=1}^{N}\frac{\partial\tilde{\lambda}_{N-1}}{\partial x_j}\{f_1, \ldots, f_{N-2}, x_j, g_N\} + \sum_{j=1}^{N}\frac{\partial\tilde{\lambda}_N}{\partial x_j}\{f_1, \ldots, f_{N-2}, g_{N-1}, x_j\} \\ &= \{f_1, \ldots, f_{N-2}, \tilde{\lambda}_{N-1}, g_N\} + \{f_1, \ldots, f_{N-2}, g_{N-1}, \tilde{\lambda}_N\}, \end{aligned}$$

where $\tilde{\lambda}_j = \dfrac{\lambda_j}{\{f_1, \ldots, f_{N-2}, g_{N-1}, g_N\}}$ for $j = N-1,\, N$. Thus the theorem is proved. □

6. Proof of Theorem 8

Proof of Theorem 8. Assume that the vector field $\mathbf{X}$ associated to differential system (2) is integrable, i.e. admit $N-1$ independent first integrals $F_1,\ldots,F_{N-1}$. Thus from the equations $\mathbf{X}(F_j)=0$ for $j=1,\ldots,N-1$ we obtain the representation $\mathbf{X}=\mu\{F_1,...,F_{N-1},*\}$, where μ is an arbitrary function. Thus $\mathbf{X}(g_l)=\Phi_l=\mu\{F_1,...,F_{N-1},g_l\}$, $\mathbf{X}(g_k)=\lambda_k=\mu\{F_1,...,F_{N-1},g_k\}$, for $l=1,\ldots,M$ and $k=M+1,\ldots,N$. So the "only if" part of the theorem follows. Now we shall prove the "if" part.

We suppose that $\Phi_l=\mu\{F_1,...,F_{N-1},g_l\}$, and $\lambda_k=\mu\{F_1,...,F_{N-1},g_k\}$. Thus the vector field associated to differential system (2) takes the form

$$\begin{aligned}\mathbf{X}(x_j)=&\sum_{n=1}^{M}\Phi_n\frac{\{g_1,\ldots,g_{n-1},x_j,g_{n+1},g_M,\ldots,g_N\}}{\{g_1,\ldots,g_N\}}+\\&\sum_{n=M+1}^{N}\lambda_n\frac{\{g_1,\ldots,g_M,g_{M+1},\ldots,g_{n-1},x_j,g_{n+1},\ldots,g_N\}}{\{g_1,\ldots,g_N\}}\\=&\ \mu\sum_{n=1}^{N}\{F_1,...,F_{N-1},g_n\}\frac{\{g_1,\ldots,g_{n-1},x_j,g_{n+1},\ldots,g_N\}}{\{g_1,\ldots,g_N\}}.\end{aligned}$$

In view of the first identity (60) we obtain that

$$\mathbf{X}(x_j)=\mu\{F_1,...,F_{N-1},x_j\}\frac{\{g_1,\ldots,g_N\}}{\{g_1,\ldots,g_N\}}=\mu\{F_1,...,F_{N-1},x_j\}.$$

Thus functions $F_1,\ldots,F_{N-1}$ are first integrals of $\mathbf{X}$. Hence the vector field is integrable. $\square$

7. Proof of Theorem 13

Proof of Theorem 13. Let $\mathbf{X}=(X_1,\ldots,X_N)$ be the vector field associated to system (2).

Since $g_{M+j}=gG_j$ for $j=1,2,\ldots N-M$, and using (13) and (64) we obtain

$$\begin{aligned}\mathbf{X}(g)=&\sum_{j=1}^{M}\frac{g\tau_j}{g_j}\mathbf{X}(g_j)=g\sum_{j=1}^{M}\frac{\tau_j\Phi_j}{g_j}=L_0g,\\ \mathbf{X}(g_{M+1})=&\ \lambda_{M+1}\text{ equivalently }\mathbf{X}(g)G_1+g\mathbf{X}(G_1)=L_0\,g\,G_1+L_1g,\end{aligned}$$

and similarly it follows that

$$\begin{aligned}L_0\,g\,G_2+g\mathbf{X}(G_2)=&\ L_0\,g\,G_2+L_1gG_1+L_2g,\\ &\vdots\\ L_0\,g\,G_{N-M}+g\mathbf{X}(G_{N-M})=&\ L_0\,g\,G_{M-N}+\ldots+L_{N-M}g.\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{X}(g)=&\ L_0g,\\ \mathbf{X}(G_1)=&\ L_1,\\ \mathbf{X}(G_2)=&\ L_1\,G_1+L_2,\\ &\vdots\\ \mathbf{X}(G_{N-M})=&\ L_1\,G_{N-M-1}+L_2\,G_{N-M-2}\ldots+L_{N-M},\end{aligned}\tag{75}$$

or, in matrix form $\mathbf{X}(\mathbf{G})=B\mathbf{L}$, where $\mathbf{L}=(L_1,\ldots,\,L_{N-M})^T$.

We introduce the 1-forms $\omega_1, \omega_2, ..., \omega_{N-M}$ as follows

$$\begin{aligned} dG_1 = & \quad \omega_1 \\ dG_2 = & \quad G_1\omega_1 + \omega_2 \\ & \quad \vdots \\ dG_{N-M} = & \quad G_{M-M-1}\omega_1 + ... + G_1\omega_{N-M-1} + \omega_{N-M}. \end{aligned}$$

or equivalently

$$dG = BW, \tag{76}$$

where $\mathbf{W} = (\omega_1, \ldots, \omega_{N-M})^T$. Consequently by considering (75) and relation $d\mathbf{G}(\mathbf{X}) = \mathbf{X}(\mathbf{G})$ we obtain that

$$\mathbf{W}(\mathbf{X}) = \mathbf{L}. \tag{77}$$

A one-form is said to be a *closed one-form* if its exterior derivative is everywhere equal to zero. Denoting by $\wedge$ the *wedge product* on the differential 1-forms, we obtain

$$\begin{aligned} 0 = d^2G_1 = & \quad d\omega_1, \\ 0 = d^2G_2 = & \quad dG_1 \wedge \omega_1 + G_1 d\omega_1 + d\omega_2 = \omega_1 \wedge \omega_1 + G_1 d\omega_1 + d\omega_2 = d\omega_2, \\ 0 = d^2G_3 = & \quad dG_2 \wedge \omega_1 + G_2 d\omega_1 + dG_1 \wedge \omega_2 + G_2 d\omega_2 + d\omega_3, \\ = & \quad G_1 (\omega_1 \wedge \omega_1) + \omega_2 \wedge \omega_1 + \omega_1 \wedge \omega_2 + G_2 d\omega_1 + G_2 d\omega_2 + d\omega_3 = d\omega_3, \end{aligned}$$

analogously we deduce that $d\omega_j = 0$ for $j = 4, \ldots, N-M$, thus the 1-forms ω_j are closed. Therefore $\omega_j = dR_j$, where R_j is a convenient function. Hence, by (77), we get

$$\omega_j(\mathbf{X}) = L_j, \quad \text{for} \quad j = 1, 2, \ldots, N-M.$$

Let $\mathbf{R} = (R_1, \ldots, R_{N-M})^T$ be the vector defined by $d\mathbf{R} = (\omega_1, \ldots, \omega_{N-M})^T = \mathbf{W} = B^{-1}d\mathbf{G}$, obtained from (76).

After the integration of the system $d\mathbf{R} = B^{-1}d\mathbf{G}$ we obtain $\mathbf{R} = \displaystyle\int B^{-1}d\mathbf{G}$. Hence

$$\begin{aligned} R_1 = & \quad G_1, \\ R_2 = & \quad G_2 - \frac{G_1^2}{2!}, \\ R_3 = & \quad G_3 - G_1G_2 + \frac{G_1^3}{3!}, \\ R_4 = & \quad G_4 - G_1G_3 + G_1^2G_2 - \frac{G_1^4}{4!} - \frac{G_2^2}{2!}, \\ R_5 = & \quad G_5 - G_1G_4 + G_1^2G_3 - G_1^3G_2 + \frac{G_1^5}{5!} + \frac{G_2^3}{3!}, \\ & \quad \vdots \end{aligned}$$

Therefore, since $G_j = \dfrac{g_{M+j}}{g}$ we deduce the representations

$$\begin{aligned} R_1 = & \quad \frac{g_{M+1}}{g} = \frac{A_1}{g}, \\ R_2 = & \quad \frac{g_{M+2}}{g} - \frac{1}{2!}\left(\frac{g_{M+1}}{g}\right)^2 = \frac{A_2}{g^2}, \\ R_3 = & \quad \frac{g_{M+3}}{g} - \frac{g_{M+1}g_{M+2}}{g^2} + \frac{1}{3!}\left(\frac{g_{M+1}}{g}\right)^3 = \frac{A_3}{g^3}, \\ & \quad \vdots \end{aligned}$$

So we have $R_j = \dfrac{A_j}{g^j}$, for $j = 1, 2, \ldots N-M$, where A_j are functions previously defined.

From the equalities $\mathbf{X}(\ln|g|) = L_0$, $\mathbf{X}(R_j) = dR_j(X) = \omega_j(X) = L_j$ for $j = 1, \ldots N-M$ and (14) we have that

$$0 = \sum_{j=0}^{N-M} \nu_j L_j = \nu_0 \mathbf{X}(\ln g) + \sum_{j=1}^{N-M} \nu_j \mathbf{X}(R_j) = \mathbf{X}\left(\ln\left[g^{\nu_0}\exp\left(\sum_{j=1}^{N-M}\nu_j R_j\right)\right]\right) = 0.$$

Thus $F = g^{\nu_0}\exp\left(\sum_{j=1}^{N-M}\nu_j R_j\right) = g^{\nu_0}\exp\left(\sum_{j=1}^{N-M}\nu_j \dfrac{A_j}{g^j}\right)$, is a first integral of differential system (2). We observe that the functions g_j for $j = 1, \ldots, M$ in general are not algebraic. □

8. Proof of Theorems 14, 16 and Proposition 19.

Proof of Theorem 14. We consider the differential system (2) with N as $2N$ and with invariant hypersurfaces $g_j(x_1, \ldots, x_{2N}) = 0$ for $j = 1, \ldots, N_1 \le N$. Taking the functions g_m for $m = N_1, \ldots, 2N$ as follows $g_\alpha = g_\alpha(x_1, \ldots, x_{2N})$, $\quad g_{N+j} = x_j$, for $\alpha = N_1 + 1, \ldots, N$ if $N_1 < N$ and $j = 1, \ldots, N$, . We assume that $\{g_1, g_2, \ldots, g_N, x_1, \ldots, x_N\} \ne 0$. Hence the system (2) takes the form

$$\begin{aligned} \dot{x}_j =\ & \lambda_{N+j}, \\ \dot{x}_{j+N} =\ & \sum_{k=1}^{N_1}\Phi_k \frac{\{g_1, \ldots, g_{k_1}, x_{j+N}, g_{k+1}, \ldots, g_N, x_1, \ldots, x_N\}}{\{g_1, g_2, \ldots, g_N, x_1, \ldots, x_N\}} + \ldots + \\ & \sum_{k=N_1+1}^{2N}\lambda_k \frac{\{g_1, \ldots, g_{N_1+1}, \ldots, g_{k-1}, x_{j+N}, g_{k+1}, \ldots, g_N, x_1, \ldots, x_N\}}{\{g_1, \ldots, g_N, x_1, \ldots, x_N\}}, \end{aligned} \tag{78}$$

for $j = 1, \ldots, K$.

In particular if we take $g_j = x_{N+j} - p_j(x_1, \ldots, x_N) = 0$, where $p_j = p_j(x_1, \ldots, x_N)$ are convenient functions for $j = 1, \ldots, N$, then from (78) we obtain

$$\dot{x}_j = \lambda_{N+j}, \quad \dot{x}_{N+j} = \Phi_j + \sum_{n=1}^{N}\lambda_{N+n}\frac{\partial p_j}{\partial x_n},$$

thus

$$\dot{x}_j = \lambda_{N+j}, \quad \frac{d}{dt}\left(x_{N+j} - p_j\right) = \Phi_j. \tag{79}$$

Taking the arbitrary functions λ_{N+j} and Φ_j as follows $\lambda_{N+j} = \sum_{n=1}^{N}\tilde{G}_{jn}x_{N+n}$, $\quad \Phi_j = \dfrac{\partial L}{\partial x_j}$, for $j = 1, \ldots, N$, where $\tilde{G}_{jn} = \tilde{G}_{jn}(x_1, \ldots, x_N)$ are elements of a symmetric definite positive matrix $\tilde{G}$, and

$$L = \frac{1}{2}\sum_{n,j=1}^{N} G_{jn}(\mathbf{x})(\dot{x}_j - v_j)(\dot{x}_n - v_n) = \frac{1}{2}||\dot{\mathbf{x}} - \mathbf{v}||^2 = \frac{1}{2}||\dot{\mathbf{x}}||^2 - \langle\mathbf{v}, \dot{\mathbf{x}}\rangle + \frac{1}{2}||\mathbf{v}||^2 = T - \langle\mathbf{v}, \dot{\mathbf{x}}\rangle + \frac{1}{2}||\mathbf{v}||^2,$$

where $G = (G_{jk})$ is the inverse matrix of $\tilde{G} = (\tilde{G}_{jk})$.

We can write g_j as $g_j = x_{j+N} - p_j = \sum_{n=1}^{N} G_{jn}(\dot{x}_n - v_n) = 0$ for $j = 1, \ldots, N$. Then, $g_j = 0$ if and only if $\dot{x}_1 - v_1 = \ldots = \dot{x}_N - v_N = 0$. Since $\Phi_j = \dfrac{\partial L}{\partial x_j} = -\left\langle \dot{\mathbf{x}} - \mathbf{v}, \dfrac{\partial \mathbf{v}}{\partial x_j} \right\rangle$, hence $\Phi_j|_{g_j=0} = \dfrac{\partial L}{\partial x_j}|_{\dot{\mathbf{x}}=\mathbf{v}} = 0$, for $j = 1, \ldots, N$.

On the other hand in view of the relations $g_j = x_{j+N} - p_j = \sum_{n=1}^{N} G_{jn}\left(\dot{x}_n - v_n\right) = \dfrac{\partial L}{\partial \dot{x}_j}$, we finally deduce that equations (79) can be written as the Lagrangian differential equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) - \frac{\partial L}{\partial x_j} = 0, \quad \text{for} \quad j = 1, \ldots, N. \tag{80}$$

After computation and in view of the constraints (19) we finally obtain differential system (21). This complete the proof of the theorem. □

Proof of Proposition 19. First we prove that the vector field (28) is such that

$$\begin{aligned} \sum_{n=1}^{N} \Omega_j(\partial_n) v_n = &\quad \Omega_j(\mathbf{v}) = 0 \quad \text{for} \quad j = 1, \ldots, M, \\ \sum_{n=1}^{N} \Omega_{M+k}(\partial_n) v_n = &\quad \Omega_{M+k}(\mathbf{v}) = \nu_{M+k} \quad \text{for} \quad k = M+1, \ldots, N. \end{aligned} \tag{81}$$

Indeed, from the relation $\mathbf{v}(\mathbf{x}) = S^{-1}\mathbf{P}$ we get that

$$\Upsilon \mathbf{v}(\mathbf{x}) = \left(\Omega_1(\mathbf{v}), \ldots, \Omega_M(\mathbf{v}), \Omega_{M+1}(\mathbf{v}), \ldots, \Omega_N(\mathbf{v})\right)^T = \mathbf{P} = \left(0, \ldots, 0, \nu_{M+1}, \ldots, \nu_N\right)^T.$$

Thus we obtain (81). Consequently the vector field $\mathbf{v}$ satisfies the constraints.

Now we show that vector field $\mathbf{v}$ is the most general satisfying these constraints. Let $\tilde{\mathbf{v}} = (\tilde{v}_1, \ldots, \tilde{v}_N)$ be another vector field satisfying the constraints, i.e. $\sum_{n=1}^{N} \Omega_j(\partial_n)\tilde{v}_n = \Omega_j(\tilde{\mathbf{v}}) = 0$ for $j = 1, \ldots, M$. Taking the arbitrary functions $\nu_{M+1}, \ldots, \nu_N$ as follows $\nu_{M+k} =$

$\sum_{n=1}^{N} \Omega_{M+j}(\partial_n)\tilde{v}_n$ we obtain from (28) the relations

$$\mathbf{v} = -\frac{1}{|\Upsilon|}\begin{vmatrix} \Omega_1(\partial_1) & \dots & \Omega_1(\partial_N) & \sum_{n=1}^{N}\Omega_1(\partial_n)\tilde{v}_n \\ \vdots & \dots & \vdots & \vdots \\ \Omega_M(\partial_1) & \dots & \Omega_M(\partial_N) & \sum_{n=1}^{N}\Omega_M(\partial_n)\tilde{v}_n \\ \Omega_{M+1}(\partial_1) & \dots & \Omega_{M+1}(\partial_N) & \sum_{n=1}^{N}\Omega_{M+1}(\partial_n)\tilde{v}_n \\ \vdots & \dots & \vdots & \vdots \\ \Omega_N(\partial_1) & \dots & \Omega_N(\partial_N) & \sum_{n=1}^{N}\Omega_N(\partial_n)\tilde{v}_n \\ \partial_1 & \dots & \partial_N & 0 \end{vmatrix}$$

$$= -\frac{1}{|\Upsilon|}\sum_{n=1}^{N}\tilde{v}_n\begin{vmatrix} \Omega_1(\partial_1) & \dots & \Omega_1(\partial_N) & \Omega_1(\partial_n) \\ \vdots & \dots & \vdots & \vdots \\ \Omega_M(\partial_1) & \dots & \Omega_M(\partial_N) & \Omega_M(\partial_n) \\ \Omega_{M+1}(\partial_1) & \dots & \Omega_{M+1}(\partial_N) & \Omega_{M+1}(\partial_n) \\ \vdots & \dots & \vdots & \vdots \\ \Omega_N(\partial_1) & \dots & \Omega_N(\partial_N) & \Omega_N(\partial_n) \\ \partial_1 & \dots & \partial_N & 0 \end{vmatrix}.$$

Thus

$$\mathbf{v} = -\frac{1}{|\Upsilon|}\sum_{n=1}^{N}\tilde{v}^n\begin{vmatrix} \Omega_1(\partial_1) & \dots & \Omega_1(\partial_N) & 0 \\ \vdots & \dots & \vdots & \vdots \\ \Omega_M(\partial_1) & \dots & \Omega_M(\partial_N) & 0 \\ \Omega_{M+1}(\partial_1) & \dots & \Omega_{M+1}(\partial_N) & 0 \\ \vdots & \dots & \vdots & \vdots \\ \Omega_N(\partial_1) & \dots & \Omega_N(\partial_N) & 0 \\ \partial_1 & \dots & \partial_N & -\partial_n \end{vmatrix} = \sum_{j,k=1}^{N}\tilde{v}_n\partial_n = \tilde{\mathbf{v}}.$$

Thus Proposition 15 is proved. □

Proof of Theorem 16. Let σ is the 1-form associated to the vector field $\mathbf{v}$, i.e.

$$\sigma = \langle \mathbf{v},\, d\mathbf{x}\rangle = \sum_{j,k=1}^{N} G_{jk}v_j dx_k = \sum_{n=1}^{N} p_n\, dx_n. \tag{82}$$

Then the 2-form $d\sigma$ admit the development

$$d\sigma = \sum_{n,j=1}^{N}\left(\frac{\partial p_n}{\partial x_j} - \frac{\partial p_j}{\partial x_n}\right) dx_j \wedge dx_n = \frac{1}{2}\sum_{n,j=1}^{N} A_{nj}\Omega_n \wedge \Omega_j. \tag{83}$$

Here we have used that the 1-forms $\Omega_1, \dots, \Omega_N$ are independent, and consequence they form a basis of the 1-form space. Hence $\Omega_k \wedge \Omega_n$ for $k, n = 1, \dots, N$ form a basis of the 2-form space.

From (83) we have that the inner product of vector field $\mathbf{v}$ and $d\sigma$ i.e. $\imath_{\mathbf{v}} d\sigma$ is such that

$$\imath_{\mathbf{v}} d\sigma = \sum_{n,j=1}^{N} v_n \left(\frac{\partial p_n}{\partial x_j} - \frac{\partial p_j}{\partial x_n} \right) dx_j = \langle H\mathbf{v}, d\mathbf{x} \rangle \, , \tag{84}$$

where the matrix H is $\left(\frac{\partial p_n}{\partial x_j} - \frac{\partial p_j}{\partial x_n} \right)$.

Again from (83) we have that

(85)

$$\begin{aligned} \imath_{\mathbf{v}} d\sigma(*) = & \quad d\sigma(\mathbf{v}, *) = \frac{1}{2} \sum_{n,j=1}^{N} b_{nj} \Omega_n \wedge \Omega_j(\mathbf{v}, *) \\ = & \quad \frac{1}{2} \sum_{n,j=1}^{N} A_{nj} \left(\Omega_n(\mathbf{v}) \Omega_j(*) - \Omega_j(\mathbf{v}) \Omega_n(*) \right) \\ = & \quad \frac{1}{2} \sum_{n,j=1}^{N} A_{nj} \Omega_n(\mathbf{v}) \Omega_j(*) - \frac{1}{2} \sum_{n,j=1}^{N} A_{jn} \Omega_n(\mathbf{v}) \Omega_j(*) \\ = & \quad \frac{1}{2} \sum_{n,j=1}^{N} (A_{nj} - A_{jn}) \Omega_n(\mathbf{v}) \Omega_j(*) = \sum_{n,j=1}^{N} A_{nj} \Omega_n(\mathbf{v}) \Omega_j(*) = \sum_{n=1}^{N} \Lambda_n \Omega_n(*). \end{aligned}$$

Now from the last equality and (83) we have

$$\imath_{\mathbf{v}} d\sigma(\partial_j) = \sum_{n=1}^{N} \Lambda_n \Omega_n(\partial_j) = \sum_{n,j=1}^{N} v_n \left(\frac{\partial p_n}{\partial x_j} - \frac{\partial p_j}{\partial x_n} \right). \tag{86}$$

Clearly, from these relations it follows that $H\mathbf{v}(\mathbf{x}) = \Upsilon^T \Lambda$, hence $\Lambda = (\Upsilon^T)^{-1} H \mathbf{v}(\mathbf{x}) = (\Upsilon^T)^{-1} H \Upsilon^{-1} \mathbf{P} = A\mathbf{P}$, here we used the equality $\mathbf{v}(\mathbf{x}) = \Upsilon^{-1} \mathbf{P}$.

From (86) and (21) we obtain

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_j} \right) - \frac{\partial T}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\frac{1}{2} ||\mathbf{v}||^2 \right) + \sum_{j=1}^{N} \Lambda_j \Omega_j(\partial_k), \tag{87}$$

for $k = 1, \ldots, N$. From (87), (26) and (32) we get (31). In short Theorem 16 is proved. □

9. Applications of Theorems 14, 16 and Proposition 19

In this section we illustrate in some particular cases the relation between three mathematical models:

(i) the classical model deduced from the d'Alembert-Lagrange principle (see (25)),
(ii) the model deduced from the Lagrangian equations (20) (see (31)), and
(iii) the model obtained from the first order differential equations (39) under the conditions (32).

9.1. **Suslov problem on $SO(3)$.** In this example we study the problem of integration of equations of motion in the classical problem of nonholonomic dynamics formulated by Suslov [47]. We consider the rotational motion of a rigid body around a fixed point and subject to the nonholonomic constraint $< \tilde{\mathbf{a}}, \omega >= 0$ where $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity of the body, $\tilde{\mathbf{a}}$ is a constant vector, and $<,>$ is the scalar product. Suppose that the body rotates in a force field with potential $U(\gamma) = U(\gamma_1, \gamma_2, \gamma_3)$. Applying the method of Lagrange multipliers we write the equations of motion (25) in the form

$$I\dot{\omega} = I\omega \wedge \omega + \gamma \wedge \frac{\partial U}{\partial \gamma} + \mu \tilde{\mathbf{a}}, \qquad \dot{\gamma} = \gamma \wedge \omega, \qquad < \tilde{\mathbf{a}}, \omega >= 0, \tag{88}$$

where

$$\gamma = (\gamma_1,\, \gamma_2,\, \gamma_3) = (\sin z \sin x,\, \sin z \cos x,\, \cos z), \tag{89}$$

$(x, y, z) = (\varphi, \psi, \theta)$ are the Euler angles, and I is the tensor of inertia.

Using the constraint equation $< \tilde{\mathbf{a}}, \omega >= 0$, the Lagrange multiplier μ can be expressed as a function of ω and γ as follows

$$\mu = -\frac{\left\langle \tilde{\mathbf{a}}, I\omega \wedge \omega + \gamma \wedge \dfrac{\partial U}{\partial \gamma} \right\rangle}{\langle \tilde{\mathbf{a}}, I^{-1}\tilde{\mathbf{a}} \rangle}.$$

System (88) always has three independent integrals

$$K_1 = \frac{1}{2}(I\omega,\, \omega) + U(\gamma), \quad K_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2, \quad K_3 = \langle \tilde{\mathbf{a}}, \omega \rangle .$$

Note that K_1 is the energy first integral.

In order to have real motions we must take $K_2 = 1$, $K_3 = 0$. In this case we can reduce the problem of integration of (88) to the problem of existence of an invariant measure and fourth independent integrals. Thus, if there exist a fourth first integral K_4 independent with K_1, K_2, K_3, then the Suslov problem is integrable [25]. It is well-known the following result, see [24].

Proposition 32. *If $\tilde{\mathbf{a}}$ is an eigenvector of operator I, i.e.*

$$I\tilde{\mathbf{a}} = \kappa\tilde{\mathbf{a}}, \tag{90}$$

then the phase flow of system (88) *preserves the "standard" measure in* $\mathbb{R}^6 = \mathbb{R}^3\{\omega\}\times\mathbb{R}^3\{\gamma\}$.

G.K. Suslov has considered a particular case when the body is in absence of external forces: $U \equiv 0$. If (90) holds, then the equations (88) have the additional first integral $K_4 = (I\omega, I\omega)$. E.I. Kharlamova in [23] studied the case when the body rotates in the homogenous force field with the potential $U = (\mathbf{b}, \gamma)$ where the vector $\mathbf{b}$ is orthogonal to the vector $\tilde{\mathbf{a}}$. Under these conditions the equations of motion have the first integral $K_4 = (I\omega, \mathbf{b})$. V.V. Kozlov in [24] consider the case when $\mathbf{b} = \varepsilon\tilde{\mathbf{a}}$, $\varepsilon \neq 0$. The integrability problem in this case was study in particular in [25, 33]. For the case $U = \varepsilon \det I(I^{-1}\gamma,\, \gamma)$ system (88) has the Clebsch-Tisserand first integral [24] $K_4 = \frac{1}{2}(I\omega, I\omega) - \dfrac{1}{2}\varepsilon \det I(I^{-1}\gamma,\, \gamma)$.

From now on we suppose that equality (90) is fulfilled. We assume that vector $\tilde{\mathbf{a}}$ coincides with one of the principal axes and without loss of generality we can choose it as the third axis, i.e., $\tilde{\mathbf{a}} = (0, 0, 1)$ and consequently the constrained becomes $\omega_3 = 0$. Equations of motion have the following form

$$\begin{aligned} &I_1\dot{\omega}_1 = \gamma_2\partial_{\gamma_3}U - \gamma_3\partial_{\gamma_2}U, \quad I_2\dot{\omega}_2 = \gamma_3\partial_{\gamma_1}U - \gamma_1\partial_{\gamma_3}U, \\ &\dot{\gamma}_1 = -\gamma_3\omega_2, \quad \dot{\gamma}_2 = \gamma_3\omega_1, \quad \dot{\gamma}_3 = \gamma_1\omega_2 - \gamma_2\omega_1, \end{aligned} \tag{91}$$

where I_k are the principal moments of inertia of the body with respect to the $k-$ axis, i.e., $I = \operatorname{diag}(I_1, I_2, I_3)$. The second group of differential system from (91) is well–known as *Poisson differential equations.* We observe that the above mentioned choice of $\tilde{\mathbf{a}}$ guarantees that the phase flow of system (91) preserves the standard measure in $\mathbb{R}^5\{\omega_1,\, \omega_2, \gamma\}$.

Now we illustrate the partial answer for the stated questions in Remark 18 . We study the integrability of the Suslov problem in the case of equations (91). We suppose that the manifold Q is the special orthogonal group of rotations of $\mathbb{R}^3$, i.e. $\mathrm{Q} = SO(3)$, with the Riemann metric G given by

$$\begin{pmatrix} I_3 & I_3\cos z & 0 \\ I_3\cos z & (I_1\sin^2 x + I_2\cos^2 x)\sin^2 z + I_3\cos^2 z & (I_1 - I_2)\sin x\cos x\sin z \\ 0 & (I_1 - I_2)\sin x\cos x\sin z & I_1\cos^2 x + I_2\sin^2 x \end{pmatrix},$$

with determinant $|G| = I_1 I_2 I_3 \sin^2 z$.

In this case we have that the constraint is $\omega_3 = \dot{x} + \dot{y}\cos z = 0$.

By choosing the 1-form Ω_j for $j = 1, 2, 3$ as follows $\Omega_1 = dx + \cos z\, dy, \quad \Omega_2 = dy, \quad \Omega_3 = dz$. we obtain $|\Upsilon| = 1$. Hence the differential system (39) can be written as

$$\dot{x} = \nu_2 \cos z \quad \dot{y} = -\nu_2, \quad \dot{z} = -\nu_3 \tag{92}$$

From (22) we compute

$$\begin{aligned} p_1 &= 0, \\ p_2 &= (I_1 \sin^2 x + I_2 \cos^2 x)\nu_2 \sin^2 z + (I_2 - I_1)\nu_3 \cos x \sin x \sin z, \\ p_3 &= -\nu_3(I_2 \sin^2 x + I_1 \cos^2 x) + (I_2 - I_1)\nu_2 \sin x \cos x \sin z. \end{aligned} \tag{93}$$

Changing ν_1 and ν_2 by μ_1 and μ_2 as

$$\mu_1 = I_2(\nu_3 \sin x - \nu_2 \sin z \cos x), \quad \mu_2 = I_1(\nu_3 \cos x + \nu_2 \sin z \sin x),$$

we obtain

$$p_1 = 0, \quad p_2 = \mu_1 \sin z \cos x - \mu_2 \sin z \sin x, \quad p_3 = \sin x \mu_1 + \cos x \mu_2.$$

Now the first of condition (35) takes the form

$$\mu_{3,1} = a_1 H_{23} + a_2 H_{31} + a_3 H_{12} = \partial_z p_2 - \partial_y p_3 + \cos z \partial_x p_3 = 0 \tag{94}$$

After the change $\gamma_1 = \sin z \sin x, \quad \gamma_2 = \sin z \cos x, \quad \gamma_3 = \cos z$, the system (92) by considering the constraints and condition (94) can be written as

$$\dot{\gamma_1} = \frac{1}{I_2}\mu_1\gamma_3, \quad \dot{\gamma_2} = \frac{1}{I_1}\mu_2\gamma_3, \quad \dot{\gamma_3} = -\frac{1}{I_1 I_2}\left(I_1\mu_1\gamma_1 + I_2\mu_2\gamma_2\right) \tag{95}$$

$$\sin z \left(\gamma_3 \left(\frac{\partial \mu_1}{\partial \gamma_2} - \frac{\partial \mu_2}{\partial \gamma_1}\right) - \gamma_2 \frac{\partial \mu_1}{\partial \gamma_3} + \gamma_1 \frac{\partial \mu_2}{\partial \gamma_3})\right) - \cos x\, \partial_y \mu_2 - \sin x\, \partial_y \mu_1 = 0, \tag{96}$$

respectively.

Clearly if $\mu_j = \mu_j(x, z, K_1, K_4)$ for $j = 1, 2$, then the equation (96) takes the form

$$\gamma_3 \left(\frac{\partial \mu_1}{\partial \gamma_2} - \frac{\partial \mu_2}{\partial \gamma_1}\right) - \gamma_2 \frac{\partial \mu_1}{\partial \gamma_3} + \gamma_1 \frac{\partial \mu_2}{\partial \gamma_3} = 0, \tag{97}$$

By comparing (91) with (95) we obtain that $\omega_1 = -\dfrac{\mu_2}{I_1}, \quad \omega_2 = \dfrac{\mu_1}{I_2}$.

We define F_1 and F_2 as

$$F_1 = I_1\omega_1 - \mu_2(\gamma_1, \gamma_2, \gamma_3, K_1, K_4), \quad F_2 = I_2\omega_2 + \mu_1(\gamma_1, \gamma_2, \gamma_3, K_1, K_4). \tag{98}$$

and we assume that

$$J = \frac{\partial F_1}{\partial K_1}\frac{\partial F_2}{\partial K_4} - \frac{\partial F_2}{\partial K_1}\frac{\partial F_1}{\partial K_4} \neq 0, \quad \text{in} \quad \text{for all } (\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^5, \tag{99}$$

Clearly if (99) holds then $F_1 = F_2 = 0$ can be solved with respect to K_1 and K_4 , i.e. $K_1 = K_1(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3), \quad K_4 = K_4(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$.

In order to give a partial answer to the question stated in Remark 18 we shall study system (91) with the potential

$$U = -||\mathbf{v}||^2 + h = -\frac{1}{2I_1 I_2}(I_1\mu_1^2 + I_2\mu_2^2) + h, \tag{100}$$

(see formula (37)).

The following result holds (see [40])

Theorem 33. *We suppose that a body in the Suslov problem rotates under the action of the force field defined by the potential* (100) *where* $\mu_1 = \mu_1(\gamma_1, \gamma_2, \gamma_3, K_1, K_4)$ *and* $\mu_2 = \mu_2(\gamma_1, \gamma_2, \gamma_3, K_1, K_4)$ *are solutions of the first order partial differential equation* (97) *for arbitrary constants* K_1 *and* K_4 *and such that* (99) *takes place. Then the following statements hold.*

(a) *The equations* (91) *have the first integrals* K_1 *and* K_4 *defined implicitly through the equations* $F_1 = F_2 = 0$ *given in* (98). *Consequently they are integrable by quadratures. In particular*

$$\mu_1 = \frac{\partial \tilde{S}(\gamma_1, \gamma_2, \gamma_3, K_1, K_4)}{\partial \gamma_1} + \Psi_1(\gamma_2^2 + \gamma_3^2,\ \gamma_1, K_1, K_4),$$

(101)

$$\mu_2 = \frac{\partial \tilde{S}(\gamma_1, \gamma_2, \gamma_3, K_1, K_4)}{\partial \gamma_2} + \Psi_2(\gamma_1^2 + \gamma_3^2,\ \gamma_2, K_1, K_4),$$

are solutions of (97), *where* $\tilde{S}, \Psi_1,\ \Psi_2,\ \Omega$ *are arbitrary smooth functions such that*

$$\tilde{S}(\gamma_1, \gamma_2, \gamma_3, K_1, K_4) = S(\gamma_1, \gamma_2, K_1, K_4) + \int \Omega(\gamma_1^2 + \gamma_2^2,\ \gamma_3, K_1, K_4) d(\gamma_1^2 + \gamma_2^2)$$

(b) *The Suslov's, Kharlamova-Zabelina's, Kozlov's, Clebsch-Tisserand's, Tisserand-Okunova's and Dragović-Gajić-Jovanović's first integrals can be obtained from* (98) *with* μ_1 *and* μ_2 *determined by* (101).

Proof. After some calculations we obtain that the derivative of F_1 along the solutions of (91) takes the form

$$\begin{aligned}
\dot{F}_1 = &\ I_1\dot{\omega}_1 + \dot{\mu}_2 \\
= &\ \gamma_2\frac{\partial U}{\partial \gamma_3} - \gamma_3\frac{\partial U}{\partial \gamma_2} - \frac{\partial \mu_2}{\partial \gamma_1}\gamma_3\omega_2 + \frac{\partial \mu_2}{\partial \gamma_2}\gamma_3\omega_1 + \frac{\partial \mu_2}{\partial \gamma_3}(\gamma_1\omega_2 - \gamma_2\omega_1) \\
= &\ \gamma_2\frac{\partial U}{\partial \gamma_3} - \gamma_3\frac{\partial U}{\partial \gamma_2} + \omega_2(\gamma_1\frac{\partial \mu_2}{\partial \gamma_3} - \gamma_3\frac{\partial \mu_2}{\partial \gamma_1}) + \omega_1(\gamma_3\frac{\partial \mu_2}{\partial \gamma_2} - \gamma_2\frac{\partial \mu_2}{\partial \gamma_3}) \\
= &\ \gamma_2\frac{\partial U}{\partial \gamma_3} - \gamma_3\frac{\partial U}{\partial \gamma_2} + \frac{F_2 + \mu_1}{I_2}(\gamma_1\frac{\partial \mu_2}{\partial \gamma_3} - \gamma_1\frac{\partial \mu_2}{\partial \gamma_3}) + \frac{F_1 - \mu_2}{I_1}(\gamma_3\frac{\partial \mu_2}{\partial \gamma_2} - \gamma_2\frac{\partial \mu_2}{\partial \gamma_3}) \\
= &\ \gamma_2\frac{\partial}{\partial \gamma_3}\left(U + \frac{1}{2I_1I_2}(I_1\mu_1^2 + I_2\mu_2^2)\right) - \gamma_3\frac{\partial}{\partial \gamma_2}\left(U + \frac{1}{2I_1I_2}(I_1\mu_1^2 + I_2\mu_2^2)\right) \\
&+\frac{\mu_1}{I_2}\left(\gamma_3\left(\frac{\partial \mu_1}{\partial \gamma_2} - \frac{\partial \mu_2}{\partial \gamma_1}\right) - \gamma_2\frac{\partial \mu_1}{\partial \gamma_3} + \gamma_1\frac{\partial \mu_2}{\partial \gamma_3}\right) \\
&+\frac{F_2}{I_2}\left(\gamma_1\frac{\partial \mu_2}{\partial \gamma_3} - \gamma_3\frac{\partial \mu_2}{\partial \gamma_1}\right) + \frac{F_1}{I_1}\left(\gamma_3\frac{\partial \mu_2}{\partial \gamma_2} - \gamma_2\frac{\partial \mu_2}{\partial \gamma_3}\right).
\end{aligned}$$

A similar relation can obtained for $\dot{F}_2$.

In view of (100), (97) and (98) we deduce that $\dot{F}_1 = \dot{F}_2 = 0$. By solving the equations $F_j = 0$ for $j = 1, 2$ with respect to K_1, K_4 we finally obtain the require first integrals. Hence the proof of the first part of statement (a) follows. The integrability by quadratures comes from the Euler-Jacobi Theorem (see for instance [3]).

Finally it is easy to check that the functions μ_1 and μ_2 defined in (101) satisfy the equation (97). This completes the proof of statement (a).

Now we prove the statement (b). First we consider the functions

$$\mu_1 = \frac{\partial \tilde{S}(\gamma_1, \gamma_2, \gamma_3, K_1, K_4)}{\partial \gamma_1} = \frac{\partial \tilde{S}}{\partial \gamma_1}, \quad \mu_2 = \frac{\partial \tilde{S}(\gamma_1, \gamma_2, \gamma_3, K_1, K_4)}{\partial \gamma_2} = \frac{\partial \tilde{S}}{\partial \gamma_2}.$$

Hence the implicit first integrals K_1 and K_4 defined implicitly by the equations

$$F_1 = I_1\omega_1 - \frac{\partial \tilde{S}}{\partial \gamma_2} = 0, \quad F_2 = I_2\omega_2 + \frac{\partial \tilde{S}}{\partial \gamma_1} = 0. \tag{102}$$

Now we show that the Suslov's, Kharlamova-Zabelina's and Kozlov's first integral can be obtained from (102).

For the *Suslov's integrable case* we have that $\tilde{S} = C_1\gamma_1 + C_2\gamma_2$, where C_1 and C_2 are arbitrary constants. Thus $\mu_1 = C_1, \quad \mu_2 = C_2$ and U =const. as a consequence the functions F_1 and F_2 of (98) become $F_1 = I_1\omega_1 - C_2 = 0, \quad F_2 = I_2\omega_2 + C_1 = 0$. Thus $K_1 = I_1\omega_1^2 + I_2\omega_2^2 = \dfrac{C_2^2}{I_2} + \dfrac{C_1^2}{I_1}, \quad K_4 = I_1^2\omega_1^2 + I_2^2\omega_2^2 = C_2^2 + C_1^2.$

For the *Kharlamova-Zabelina's integrable case* we have

$$\tilde{S} = \frac{2/3}{\sqrt{I_1 b_1^2 + I_2 b_2^2}} \left(\tilde{h} + b_1\gamma_1 + b_2\gamma_2\right)^{3/2} - \frac{K_4}{b_1^2 I_1 + b_2^2 I_2}\left(b_2 I_2\gamma_1 - b_1 I_1\gamma_2\right),$$

where $\tilde{h} = I_1 I_2 \left(\dfrac{K_4^2 I_1 I_2}{b_1^2 I_1 + b_2^2 I_2} - K_1\right)$, K_1 and K_4 are arbitrary constants, then

$$\mu_1 = \frac{b_1}{\sqrt{I_1 b_1^2 + I_2 b_2^2}}\sqrt{\tilde{h} + b_1\gamma_1 + b_2\gamma_2} - \frac{K_4 b_2 I_2}{b_1^2 I_1 + b_2^2 I_2},$$

$$\mu_2 = \frac{b_2}{\sqrt{I_1 b_1^2 + I_2 b_2^2}}\sqrt{\tilde{h} + b_1\gamma_1 + b_2\gamma_2} + \frac{K_4 b_1 I_1}{b_1^2 I_1 + b_2^2 I_2}.$$

Hence the the functions F_1 and F_2 of (98) are

$$\begin{aligned} F_1 &= I_1\omega_1 - \left(\frac{b_2}{\sqrt{I_1 b_1^2 + I_2 b_2^2}}\sqrt{\tilde{h} + b_1\gamma_1 + b_2\gamma_2} + \frac{K_4 b_1 I_1}{b_1^2 I_1 + b_2^2 I_2}\right) = 0, \\ F_2 &= I_2\omega_2 + \left(\frac{b_1}{\sqrt{I_1 b_1^2 + I_2 b_2^2}}\sqrt{\tilde{h} + b_1\gamma_1 + b_2\gamma_2} - \frac{K_4 b_2 I_2}{b_1^2 I_1 + b_2^2 I_2}\right) = 0. \end{aligned} \tag{103}$$

Thus

$$K_1 = I_1\omega_1^2 + I_2\omega_2^2 - \frac{1}{I_1 I_2}\left(b_1\gamma_1 + b_2\gamma_2\right), \quad K_4 = I_1\omega_1 b_1 + I_2\omega_2 b_2.$$

The first integral K_4 is the well–know Kharlamova-Zabelina's first integral [23].

For the *Kozlov's integrable case* we have $I_1 = I_2$ and

$$\tilde{S} = -K_4 \arctan\frac{\gamma_1}{\gamma_2} + \frac{1}{2}\int D(\gamma_1^2 + \gamma_2^2) d(\gamma_1^2 + \gamma_2^2),$$

where

$$D(u) = I_1\sqrt{\frac{K_1 + a\sqrt{1-u}}{u} - \frac{K_4^2}{u^2}},$$

a is a real constant, K_1 and K_4 are arbitrary real constants. Hence

$$\mu_1 = -\frac{\gamma_2 K_4}{\gamma_1^2 + \gamma_2^2} + \gamma_1 D(\gamma_1^2 + \gamma_2^2), \quad \mu_2 = \frac{\gamma_1 K_4}{\gamma_1^2 + \gamma_2^2} + \gamma_2 D(\gamma_1^2 + \gamma_2^2).$$

Consequently the functions F_1 and F_2 of (98) are

$$F_1 = \omega_1 - \left(\frac{\gamma_1 K_4}{\gamma_1^2 + \gamma_2^2} + \gamma_2 D(\gamma_1^2 + \gamma_2^2\right) = 0, \quad F_2 = \omega_2 + \left(-\frac{\gamma_2 K_4}{\gamma_1^2 + \gamma_2^2} + \gamma_1 D(\gamma_1^2 + \gamma_2^2\right) = 0.$$

Thus $K_1 = \omega_1^2 + \omega_2^2 - a\sqrt{1 - \gamma_1^2 - \gamma_2^2} = \omega_1^2 + \omega_2^2 - a\gamma_3 \quad K_4 = \omega_1\gamma_1 + \omega_2\gamma_2$. This case correspond to the well–known integrable "Lagrange case" of the Suslov problem [25].

Finally we analyze the case when the functions μ_1 and μ_2 are given by the formula

$$\mu_1 = \Psi_1(\gamma_1^2 + \gamma_3^2, \gamma_1, K_1, K_4), \quad \mu_2 = \Psi_2(\gamma_1^2 + \gamma_3^2, \gamma_2, K_1, K_4). \tag{104}$$

The potential function (100) in this case coincides with the potential obtained by Dragović-Gajić-Jovanović in [11]. We call this case the *Generalized Tisserand case.* In particular, if

$$\mu_1 = \sqrt{h_1 + (a_1 + a_3)(\gamma_3^2 + \gamma_2^2) + (b_1 + a_3)\gamma_1^2 + f_1(\gamma_1)},$$

$$\mu_2 = \sqrt{h_2 + (a_2 + a_4)(\gamma_3^2 + \gamma_1^2) + (b_2 + a_4)\gamma_2^2 + f_2(\gamma_2)},$$

where a_1, a_2, a_3, a_4, b_1, b_2, , h_1, h_2, are arbitrary real constants: $h_1 = \dfrac{I_2(I_1K_1 - K_4)}{I_1 - I_2}$, and $h_2 = \dfrac{I_1(I_2K_1 - K_4)}{I_1 - I_2}$, and $f_1 = f_1(\gamma_1)$ and $f_2 = f_2(\gamma_2)$ are arbitrary functions, then the functions F_1 and F_2 of (98) take the form

$$\begin{aligned} F_1 &= I_1\omega_1 - \sqrt{h_2 + (a_2 + a_4)(\gamma_3^2 + \gamma_1^2) + (b_2 + a_4)\gamma_2^2 + f_2(\gamma_2)} = 0, \\ F_2 &= I_2\omega_2 + \sqrt{h_1 + (a_1 + a_3)(\gamma_3^2 + \gamma_2^2) + (b_1 + a_3)\gamma_1^2 + f_1(\gamma_1)} = 0. \end{aligned}$$

The case when $f_j(\gamma_j) = \alpha_j\,\gamma_j$, for $j = 1, 2$ was studied in [35], where α_1 and α_2 are real constants. If $f_1 = f_2 = 0$, we obtain the Tisserand's case [24]. The first integrals in the Clebsch-Tisserand's case are

$$\begin{aligned} K_1 = \quad & I_1\omega_1^2 + I_2\omega_2^2 - \left(\frac{b_1 + a_3}{I_2} + \frac{a_2 + a_4}{I_1}\right)\gamma_1^2 \\ & - \left(\frac{a_1 + a_3}{I_2} + \frac{b_2 + a_4}{I_1}\right)\gamma_2^2 - \left(\frac{a_1 + a_3}{I_2} + \frac{a_2 + a_4}{I_1}\right)\gamma_3^2, \\ K_4 = \quad & I_1^2\omega_1^2 + I_2^2\omega_2^2 - (b_1 + a_3 + a_2 + a_4)\,\gamma_1^2 \\ & - (a_1 + a_3 + b_2 + a_4)\,\gamma_2^2 - (a_1 + a_3 + a_2 + a_4)\,\gamma_3^2, \end{aligned}$$

it is easy to obtain from $F_1 = 0$ and $F_2 = 0$. Thus statement (b) follows. In short, the theorem is proved. □

9.2. **Nonholonomic Chaplygin systems.** We illustrated Theorem 16 in the noholonomic Chaplygin systems.

It was pointed out by Chaplygin [4] that in many nonholonomic systems the generalized coordinates $x_1, \ldots, x_N$ can be chosen in such a way that the equations of the non–integrable constraints, can be written in the form

$$\dot{x}_j = \sum_{k=M+1}^{N} \hat{a}_{jk}(x_{M+1}, \ldots, x_N)\dot{x}_k, \quad \text{for} \quad j = 1, 2, \ldots, M, \tag{105}$$

A *constrained Chaplygin-Lagrangian mechanical system* is the mechanical system with Lagrangian $\tilde{L} = \tilde{L}(x_{M+1}, \ldots, x_N, \dot{x}_1, \ldots, \dot{x}_N)$, subject to M linear nonholonomic constraints (105).

We shall solve the inverse problem for this constrained system when the Lagrangian function is the following

$$\tilde{L} = T = \frac{1}{2}\sum_{n,j=1}^{N} G_{jn}(x_{M+1}, \ldots, x_N)\dot{x}_j\dot{x}_n \tag{106}$$

In this section we determine the vector field (28) and differential system (31) for constrained Chaplygin-Lagrangian mechanical system with Lagrangian (106).

First we determine the 1-forms Ω_j for $j = 1, \ldots, N$. Taking

$$\begin{aligned} \Omega_j = \quad & dx_j - \sum_{k=M+1}^{N} \hat{a}_{jk}(x_{M+1}, \ldots, x_N)dx_k, \quad \text{for} \quad j = 1, 2, \ldots, M, \\ \Omega_k = \quad & dx_k \quad \text{for} \quad k = M + 1, \ldots, N, \end{aligned}$$

we obtain that

$$\Upsilon=\begin{pmatrix}1&0&\dots&0&0&-\hat{a}_{1\,M+1}&\dots&-\hat{a}_{1\,N}\\0&1&\dots&0&0&-\hat{a}_{2\,M+1}&\dots&-\hat{a}_{2\,N}\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&\dots&1&-\hat{a}_{M\,M+1}&\dots&-\hat{a}_{M\,N}\\0&0&0&\dots&0&1&0&0\\\vdots&\vdots&\vdots&\dots&\vdots&\vdots&\vdots&\vdots\\0&0&0&\dots&0&0&0&1\end{pmatrix},\tag{107}$$

thus $|\Upsilon|=1$ and consequently

$$\Upsilon^{-1}=\begin{pmatrix}1&0&\dots&0&0&\hat{a}_{1\,M+1}&\dots&\hat{a}_{1\,N}\\0&1&\dots&0&0&\hat{a}_{2\,M+1}&\dots&\hat{a}_{2\,N}\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&\dots&1&\hat{a}_{M\,M+1}&\dots&\hat{a}_{M\,N}\\0&0&0&\dots&0&1&0&0\\\vdots&\vdots&\vdots&\dots&\vdots&\vdots&\vdots&\vdots\\0&0&0&\dots&0&0&0&1\end{pmatrix}.$$

Thus the vector field (28) in this case generate the following differential equations

$$\dot{x}_j=\sum_{n=M+1}^{N}\hat{a}_{jn}\nu_n\quad \dot{x}_k=\nu_k\quad\text{for}\quad j=1,\dots,M,\quad k=M+1,\dots,N.\tag{108}$$

Differential system (31) in this case admits the representation

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{x}_k}\right)&=\frac{\partial}{\partial x_k}\left(\frac{1}{2}||\mathbf{v}||^2\right)+\Lambda_k,\\\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{x}_j}\right)-\frac{\partial T}{\partial x_j}&=\frac{\partial}{\partial x_j}\left(\frac{1}{2}||\mathbf{v}||^2\right)-\sum_{k=1}^{M}\Lambda_k\hat{a}_{kj},\end{aligned}\tag{109}$$

for $j=M+1,\dots,N$, $k=1,\dots,M$, where $\Lambda_1,\dots,\Lambda_M$ are determine by the formula (29), (30) and (32).

We observe that system (109) coincide with the Chaplygin system. Indeed, excluding Λ_k from the first of the equations of (109) and denoting by L^* the expression in which the velocities $\dot{x}_1,\dots,\dot{x}_M$, have been eliminated by means of the constraints equations (105), i.e.

$$L^*=L|_{\dot{x}_j=\sum_{k=M+1}^{N}\hat{a}_{jk}\dot{x}_k}=(T+\frac{1}{2}||\mathbf{v}||^2)\Big|_{\dot{x}_j=\sum_{k=M+1}^{N}\hat{a}_{jk}\dot{x}_k},$$

Therefore, we obtain

$$\frac{\partial L^*}{\partial\dot{x}_j}=\frac{\partial L}{\partial\dot{x}_j}+\sum_{\alpha=1}^{M}\frac{\partial L}{\partial\dot{x}_\alpha}\hat{a}_{\alpha j},\qquad\frac{\partial L^*}{\partial x_j}=\frac{\partial L}{\partial x_j}+\sum_{\alpha=1}^{M}\sum_{m=M+1}^{N}\frac{\partial L}{\partial\dot{x}_\alpha}\dot{x}_m\frac{\partial\hat{a}_{\alpha m}}{\partial x_j},$$

for $j=M+1,\dots,N$.

From these relations, we have

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial\dot{x}_j}\right)-\frac{\partial L^*}{\partial x_j}=\sum_{m=M+1}^{N}\sum_{l=1}^{M}\left(\frac{\partial\hat{a}_{lj}}{\partial x_m}-\frac{\partial\hat{a}_{lm}}{\partial x_j}\right)\dot{x}_m\frac{\partial L}{\partial\dot{x}_l},$$

for $j=M+1,\dots,N$, $k=1,\dots,M$, which are the equations which Chaplygin published in the Proceeding of the Society of the Friends of Natural Science in 1897.

9.3. **The Chapliguin–Caratheodory sleigh.** We shall now analyze one of the most classical nonholonomic systems : The Chapliguin–Carathodory's sleigh (for more details see [31]). Hence, one has the constrained Lagrangian system with the configuration space $Q = \mathbb{S}^1 \times \mathbb{R}^2$, with the Lagrangian function

$$\tilde{L} = \frac{m}{2}\left(\dot{y}^2 + \dot{z}^2 + \frac{J_C}{2}\dot{x}^2\right) - U(x, y, z),$$

and with the constraint $\varepsilon\dot{x} + \sin x\dot{y} - \cos x\dot{z} = 0$, where m, J_c and ε are parameters related with the sleigh. Observe that the Chapliguin's skate is a particular case of this mechanical system and can be obtained when $\varepsilon = 0$

To determine the vector field (28) in this case we choose the 1-forms Ω_j for $j = 1, 2, 3$ as follows (see [43]) $\Omega_1 = \varepsilon dx + \sin x\, dy - \cos x\, dz$, $\quad \Omega_2 = \cos x\, dy + \sin x\, dz$, $\quad \Omega_3 = dx$, hence $|\Upsilon| = 1$.

Differential equations (39) and the first condition of (35) take the form respectively

$$\dot{x} = \nu_3, \quad \dot{y} = \nu_2 \cos x - \varepsilon\lambda_3 \sin x, \quad \dot{z} = \nu_2 \sin x + \varepsilon\nu_3 \cos x, \tag{110}$$

where $\nu_j = \nu_j(x, y, z, \varepsilon)$ for $j = 2, 3$ are solutions of the partial differential equation

$$\begin{aligned} &0 = \mu_{3,1} = a_1 H_{23} + a_2 H_{31} + a_3 H_{12} = \\ &\sin x(J\partial_z\nu_3 + \varepsilon m\partial_y\nu_2) + \cos x(J\partial_y\nu_3 - \varepsilon m\partial_z\nu_2) - m(\partial_x\nu_2 - \varepsilon\nu_3), \end{aligned} \tag{111}$$

where $J = J_C + \varepsilon^2 m$.

For the Chapliguin skate ($\varepsilon = 0$) we have

$$\dot{x} = \nu_3, \quad \dot{y} = \nu_2 \cos x, \quad \dot{z} = \nu_2 \sin x, \quad \dot{y}\cos x - \dot{x}\cos x = 0, \tag{112}$$

$$J_C(\sin x\partial_z\nu_3 + \cos x\partial_y\nu_3) - m\partial_x\nu_2 = 0, \tag{113}$$

where $\nu_j = \nu_j(x, y, z, 0)$ for $j = 2, 3$. Now we study the behavior of the Chapliguin skate by using the differential equations generated by the vector field $\mathbf{v}$ with ν_2 and ν_3 satisfying partial differential equation (113).

Proposition 34. *All the trajectories of the Chapliguin skate* ($\varepsilon = 0$) *under the action of the potential field of force with potential* $U = mgy$ *can be obtained from differential system* (112) *where* ν_2 *and* ν_3 *are solutions of* (113).

Proof. Indeed, for the case when $\varepsilon = 0$ the equation of motions of Chapliguin skate obtained from (25) are

$$\ddot{x} = 0, \quad \ddot{y} = mg + \sin x\mu, \quad \ddot{z} = -\cos x\mu, \quad \sin x\dot{y} - \cos x\dot{z} = 0.$$

Hence, we obtain $\dfrac{d}{dt}(\dfrac{\dot{z}}{\sin x}) = g\cos x$. We study only the case when $\dot{x}|_{t=t_0} = C_0 \neq 0$, as a consequence,

$$\dot{x} = C_0, \quad \dot{y} = \left(\frac{g\sin x}{C_0} + C_1\right)\cos x, \quad \dot{z} = \left(\frac{g\sin x}{C_0} + C_1\right)\sin x. \tag{114}$$

Clearly, the solutions of these equations coincide with the solutions of (112) and (113) under the condition $||\mathbf{v}||^2 = J_C\nu_3^2 + m\nu_2^2 = 2(-mgy + h)$. Indeed, taking

$$\nu_3 = C_0, \quad \nu_2 = \sqrt{\frac{2(-mgy + h) - J_C\, C_0^2}{m}},$$

where C_0 is an arbitrary constant. We obtain the differential system

$$\dot{x} = C_0, \quad \dot{y} = \sqrt{\frac{2(-mgy + h) - J_C\, C_0^2}{m}}\cos x, \quad \dot{z} = \sqrt{\frac{2(-mgy + h) - J_C\, C_0^2}{m}}\sin x.$$

The solutions of this system coincide with the solutions of (114). In short the proposition is proved. □

In what follows we study the motion of the Chapliguin–Carathodory's sleigh without action of the active forces.

Proposition 35. *All the trajectories of Chapliguin–Carathodory's sleigh in absence of active forces can be obtained from* (110) *with the condition* (111).

Proof. Indeed, taking in (111) $\nu_j = \nu_j(x, \varepsilon)$, $j = 1, 2$ such that $\partial_x \nu_2 = \varepsilon\, \nu_3$, then all the trajectories of equation (110) are given by

$$y = y_0 + \int \frac{(\nu_2 \cos x - \varepsilon\nu_3 \sin x)dx}{\nu_3}, \quad z = z_0 - \int \frac{(\nu_2 \sin x - \varepsilon\nu_3 \cos x)dx}{\nu_3}, \quad t = t_0 + \int \frac{dx}{\lambda_3(x, \varepsilon)}.$$

On the other hand, for the Chapliguin–Caratheodory sleigh in absence of active forces from the (25) we have

$$J_C \ddot{x} = \varepsilon\mu, \quad m\ddot{y} = \sin x\mu, \quad m\ddot{z} = -\cos x\mu, \quad \varepsilon\dot{x} + \sin x\dot{y} - \cos x\dot{z} = 0.$$

Hence, after integration we obtain the system

$$\dot{x} = qC_0 \cos\theta, \quad \dot{y} = C_0(\sin\theta \cos x - q\varepsilon \cos\theta \sin x), \quad \dot{z} = C_0(\sin\theta \sin x + q\varepsilon \cos\theta \cos x),$$

where $\theta = q\varepsilon x + C$ and $q^2 = \dfrac{m}{J_C + m\varepsilon^2}$, which are a particular case of equations (110) with $\nu_2 = C_0 \sin\theta$, $\nu_3 = C_0 q \cos\theta$. Clearly in this case $2||\mathbf{v}||^2 = (J_C + m\varepsilon^2)\nu_3^2(x, \varepsilon\,) + m\nu_2^2(x, \varepsilon\,) = mC_0^2 = 2(-U + h)$, and the equation $\partial_x \nu_2 = \varepsilon\nu_3$ holds. Thus the proposition follows. □

9.4. **Gantmacher's system.** We shall illustrate this case in the following system which we call *Gantmacher's system* (see for more details [18]).

Two material points m_1 and m_2 with equal masses are linked by a metal rod with fixed length l and small mass. The system can move only in the vertical plane and so the speed of the midpoint of the rod is directed along the rod. It is necessary to determine the trajectories of the material points m_1 and m_2.

Let $(q_1,\, r_1)$ and $(q_2,\, r_2)$ be the coordinates of the points m_1 and m_2. Introducing the following change of coordinates: $x_1 = \dfrac{q_2 - q_1}{2}$, $x_2 = \dfrac{r_1 - r_2}{2}$, $x_3 = \dfrac{r_2 + r_1}{2}$, $x_4 = \dfrac{q_1 + q_2}{2}$, we obtain the mechanical system with configuration space $\mathrm{Q} = \mathbb{R}^4$, Lagrangian function $L = \dfrac{1}{2}\displaystyle\sum_{j=1}^{4} \dot{x}_j^2 - gx_3$, and constraints are $x_1\dot{x}_1 + x_2\dot{x}_2 = 0$, $x_1\dot{x}_3 - x_2\dot{x}_4 = 0$. The equations of motion (25) obtained from the d'Alembert-Lagrange principle are

$$\ddot{x}_1 = \mu_1 x_1, \quad \ddot{x}_2 = \mu_1 x_2, \quad \ddot{x}_3 = -g + \mu_2 x_1, \quad \ddot{x}_4 = -\mu_2 x_2, \tag{115}$$

where μ_1, μ_2 are the Lagrangian multipliers which we determine as follows

$$\mu_1 = -\frac{\dot{x}_1^2 + \dot{x}_2^2}{x_1^2 + x_2^2}, \quad \mu_2 = \frac{\dot{x}_2\dot{x}_4 - \dot{x}_1\dot{x}_3 + gu_1}{x_1^2 + x_2^2}. \tag{116}$$

After the integration of (115) we obtain (for more details see [18])

$$\dot{x}_1 = -\dot{\varphi}x_2, \quad \dot{x}_2 = \dot{\varphi}x_1, \quad \dot{x}_3 = \frac{f}{r}x_2, \quad \dot{x}_4 = \frac{f}{r}x_1, \tag{117}$$

where $(\varphi,\, r)$ are the polar coordinates: $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$ and f is a solution of the equation $\dot{f} = -\dfrac{2g}{r}x_2$.

To construct the differential systems (39) and (28) we determine the 1-forms Ω_j for $j = 1, 2, 3, 4$ as follow (see [43])

$$\Omega_1 = x_1 dx_1 + x_2 dx_2, \quad \Omega_2 = x_1 dx_3 - x_2 dx_4,$$
$$\Omega_3 = x_1 dx_2 - x_2 dx_1, \quad \Omega_4 = x_2 dx_3 + x_1 dx_4.$$

Here Ω_1 and Ω_2 are given by the constraints, and Ω_3 and Ω_4 are chosen in order that the determinant $|\Upsilon|$ becomes nonzero, and if it can be chosen constant will be the ideal situation. Hence we obtain that $|\Upsilon| = -(x_1^2+x_2^2)^2 = -\frac{l^2}{4} \neq 0$. By considering that in this case $N = 4$ and $M = 2$ then from (35) we obtain

$$\mu_{4,2} = x_2\partial_{x_3}\nu_3 - x_1\partial_{x_4}\nu_3 + x_2\partial_{x_1}\nu_4 + x_1\partial_{x_2}\nu_4 = 0. \tag{118}$$

Differential equations (39) take the form

$$\dot{x}_1 = -\nu_3 x_2, \quad \dot{x}_2 = \nu_3 x_1, \quad \dot{x}_3 = \nu_4 x_2, \quad \dot{x}_4 = \nu_4 x_1. \tag{119}$$

It is easy to show that the functions ν_3, ν_4 :

$$\nu_3 = g_3(x_1^2+x_2^2), \qquad \nu_4 = \sqrt{\frac{2(-gx_3+h)}{(x_1^2+x_2^2)} - g_3^2(x_1^2+x_2^2)}, \tag{120}$$

where g and h are constants, and g_3 is an arbitrary function in the variable $x_1^2+x_2^2$, are solutions of (118) as a consequence from the relation (37) we have

$$2||\mathbf{v}||^2 = (x_1^2+x_2^2)(\nu_3^2+\nu_4^2) = 2(-g\,x_3+h) = 2(-U+h).$$

The solutions of (119) with ν_3 and ν_4 given in (120) are

$$\begin{aligned} &x_1 = r\cos\alpha, \quad x_2 = r\sin\alpha, \quad \alpha = \alpha_0 + g_3(r)t, \\ &x_3 = u_3^0 + \frac{g}{2g_3(r)}t - \frac{g}{4g_3^2(r)}\sin 2\alpha - -\frac{\sqrt{2g}C}{g_3(r)}\cos\alpha, \\ &x_4 = -h + \frac{r^2 g_3^2(r)}{2g} + \left(\frac{\sqrt{g}}{\sqrt{2}g_3(r)}\sin\alpha + C\right)^2, \end{aligned} \tag{121}$$

where C, r, α_0, u_3^0, h, are arbitrary constants, g_3 is an arbitrary on r function.

To compare these solutions with the solutions obtained from (117) we observe that they coincide. We note that we have obtained the trajectories of the masses m_1 and m_2 solving the differential equations of first order (119) with the functions (120).

Finally we observe that for the Gantmacher system the system (31) takes the form

$$\ddot{x}_1 = \Lambda_1 x_1, \quad \ddot{x}_2 = \Lambda_1 x_2, \quad \ddot{x}_3 = -g + \Lambda_2 x_1, \quad \ddot{x}_4 = \Lambda_2 x_2, \tag{122}$$

and admits as solutions the ones given in (121) (see Remark 18).

Remark 36. *From these examples we give a partial answer to the questions stated in Remark 18. Differential equations generated by the vector field* (28) *under the conditions* (33) *can be applied to study the behavior of the nonholonomic systems with linear constraints with respect to the velocity (at least for certain class of such system). Is it possible to apply this mathematical model to describe the behavior of the nonholonomic systems with linear constraints with respect to velocity in general? For the moment we have no answer to this question.*

10. Proof of Theorem 19. Applications

Proof of Theorem 19. In this case we obtain that the vector field (28) is

$$\mathbf{v} = -\frac{1}{|\Upsilon|}\begin{vmatrix} df_1(\partial_1) & \dots & df_1(\partial_N) & 0 \\ \vdots & \dots & \vdots & \vdots \\ df_{N-1}(\partial_1) & \dots & df_{N-1}(\partial_N) & 0 \\ df_N(\partial_1) & \dots & df_N(\partial_N) & \nu_N \\ \partial_1 & \dots & \partial_N & 0 \end{vmatrix} \tag{123}$$

$$= \frac{\nu_N}{|\Upsilon|}\begin{vmatrix} df_1(\partial_1) & \dots & df_1(\partial_N) \\ \vdots & \dots & \vdots \\ df_{N-1}(\partial_1) & \dots & df_{N-1}(\partial_N) \\ \partial_1 & \dots & \partial_N \end{vmatrix} = \tilde{\nu}\{f_1,\dots,f_{N-1},*\}.$$

Condition (32) in this case takes the form $\Lambda_N = A_{NN}\nu_N = 0$. Since the matrix A is antisymmetric, then $A_{NN} = 0$. On the other hand from $\Lambda_j = A_{Nj}\nu_N$, for $j = 1,\dots,N-1$, we deduce that system (31) takes the form

$$\begin{aligned}\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_j} - \frac{\partial T}{\partial x_j} &= F_j = \frac{\partial}{\partial x_j}\left(\frac{1}{2}||\mathbf{v}||^2\right) + \sum_{k=1}^{N-1}\Lambda_k df_k(\partial_j) \\ &= \frac{\partial}{\partial x_j}\left(\frac{1}{2}||\mathbf{v}||^2\right) + \nu_N\sum_{k=1}^{N-1}A_{Nk}df_k(\partial_j).\end{aligned}$$

From these relations we obtain the proof of statement (a) of the theorem.

The statement (b) follows trivially from the previous result.

The proof of statement (c) follows by considering that under the assumption (44) we have

$$\left\langle\frac{\partial S}{\partial \mathbf{x}},\frac{\partial \Psi}{\partial \mathbf{x}}\right\rangle = \varrho\begin{vmatrix} df_1(\partial_1) & \dots & df_1(\partial_N) \\ \vdots & \dots & \vdots \\ df_{N-1}(\partial_1) & \dots & df_{N-1}(\partial_N) \\ d\Psi(\partial_1) & \dots & d\Psi(\partial_N) \end{vmatrix} = \varrho\{f_1,\dots,f_{N-1},\Psi\},$$

where Ψ and $\varrho = \varrho(x_1,\dots,x_N)$ are an arbitrary functions. Hence the 1-form associated to the vector field $\mathbf{v}$ is $\sigma = \langle\mathbf{v}, d\mathbf{x}\rangle = \left\langle\nu\frac{\partial S}{\partial \mathbf{x}}, d\mathbf{x}\right\rangle = \nu\, dS$ where $\nu = \dfrac{\tilde{\nu}}{\varrho}$ (see (82)). Thus $d\sigma = d\nu\wedge dS$ and consequently from (84) we have

$$\begin{aligned}\imath_{\mathbf{v}}d\sigma &= \sum_{n,j=1}^{N}\upsilon_n\left(\frac{\partial p_n}{\partial x_j} - \frac{\partial p_j}{\partial x_n}\right)dx_j = d\nu(\mathbf{v})dS - dS(\mathbf{v})d\nu \\ &= \mathbf{v}(\nu)dS - \mathbf{v}(S)d\nu = \left\langle\mathbf{v}(\mathbf{x}),\frac{\partial \nu}{\partial \mathbf{x}}\right\rangle dS - \left\langle\mathbf{v}(\mathbf{x}),\frac{\partial S}{\partial \mathbf{x}}\right\rangle d\nu \\ &= \frac{1}{2}\left(\left\langle\frac{\partial \nu^2}{\partial \mathbf{x}},\frac{\partial S}{\partial \mathbf{x}}\right\rangle dS - \left\|\frac{\partial S}{\partial \mathbf{x}}\right\|^2 d\nu^2\right).\end{aligned}$$

After some computations, we deduce that the field of force $\mathbf{F}$ which in view of (86) admits the representation $F_j = \dfrac{\partial}{\partial x_j}\left(\dfrac{1}{2}||\mathbf{v}||^2\right) + \imath_{\mathbf{v}}d\sigma(\partial_j)$. Hence we obtain (45).

If the curve is given by intersection of the hyperplane $f_j = x_j$ for $j = 1,\dots,N-1$, then the condition (44) takes the form

$$\sum_{k=1}^{N}\tilde{G}_{\alpha k}\frac{\partial S}{\partial x_k} = 0, \quad \alpha = 1,\dots,N-1, \tag{124}$$

where $\tilde{G}$ is the inverse matrix of the matrix G.

By solving these equations with respect to $\dfrac{\partial S}{\partial x_k}$ for $k = 1, \dots, N-1$ we obtain

$$\begin{aligned}\frac{\partial S}{\partial x_k} &= \frac{\frac{\partial S}{\partial x_N}}{\Delta}\begin{vmatrix}\tilde{G}_{11} & \dots & \tilde{G}_{1,k-1} & -\tilde{G}_{1N} & \tilde{G}_{1,k+1} & \dots & \tilde{G}_{1,N-1}\\ \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots\\ \tilde{G}_{1,N-1} & \dots & \tilde{G}_{N-1,k-1} & -\tilde{G}_{N-1,N} & \tilde{G}_{N-1,k+1} & \dots & \tilde{G}_{N-1,N-1}\end{vmatrix}\\ &:= L_k\frac{\partial S}{\partial x_N}.\end{aligned}$$

By using these relations and in view of (124), after some computations by considering that $\sum_{n=1}^{N} L_n\tilde{G}_{Nn} = |\tilde{G}|$ we deduce that

$$\left\langle\frac{\partial S}{\partial \mathbf{x}}, \frac{\partial F}{\partial \mathbf{x}}\right\rangle := \sum_{j,k=1}^{N}\tilde{G}_{jk}\frac{\partial S}{\partial x_k}\frac{\partial F}{\partial x_j} = \sum_{j=1}^{N}\tilde{G}_{Nk}\frac{\partial S}{\partial x_k}\frac{\partial F}{\partial x_N} = \frac{|\tilde{G}|}{\Delta}\frac{\partial S}{\partial x_N}\frac{\partial F}{\partial x_N}. \tag{125}$$

Consequently we obtain the following expression for the equations (46)

$$\begin{aligned}-\frac{\partial h}{\partial \mathbf{x}} &= \left\langle\frac{\partial}{\partial \mathbf{x}}\left(\frac{\nu^2}{2}\right), \frac{\partial S}{\partial \mathbf{x}}\right\rangle\frac{\partial S}{\partial \mathbf{x}} - \left\|\frac{\partial S}{\partial \mathbf{x}}\right\|^2\frac{\partial}{\partial \mathbf{x}}\left(\frac{\nu^2}{2}\right)\\ &= \frac{|\tilde{G}|}{\Delta}\frac{\partial S}{\partial x_N}\left(\frac{\partial}{\partial x_N}\left(\frac{\nu^2}{2}\right)\frac{\partial S}{\partial \mathbf{x}} - \left(\frac{\partial S}{\partial x_N}\right)\frac{\partial}{\partial \mathbf{x}}\left(\frac{\nu^2}{2}\right)\right).\end{aligned} \tag{126}$$

In view of (125) we obtain that the potential function V takes the form

$$V = \frac{\nu^2}{2}\left\|\frac{\partial S}{\partial \mathbf{x}}\right\|^2 - h(f_1, \dots, f_{N-1}) = \frac{\nu^2}{2}\frac{|\tilde{G}|}{\Delta}\left(\frac{\partial S}{\partial x_N}\right)^2 - h(x_1, \dots, x_{N-1}).$$

We observe that if $\tilde{G}_{\alpha N} = 0$ for $\alpha = 1, \dots, N-1$, then $|\tilde{G}| = \Delta\,\tilde{G}_{NN}$ and $S_N = x_N = c_N$ is a family of hyperplanes orthogonal to the hyperplanes $f_j = x_j = c_j$ for $j = 1, \dots, N-1$. After integrating (126) we obtain that

$$V = \frac{1}{2}\tilde{G}_{NN}\nu^2 - h = \left(g(x_N) - \sum_{j=1}^{N-1}\int h(x_1, \dots, x_{N-1})\frac{\partial}{\partial x_j}\left(\frac{1}{\tilde{G}_{NN}}\right)dx_j\right)\tilde{G}_{NN},$$

where $g = g(x_N)$ and $h = h(x_1, \dots, x_{N-1})$ are arbitrary functions.

Clearly if $\nu = \nu(S)$. Then $\sigma = d\Phi(S)$ where $\Phi = \int \nu(S)dS$. Therefore $d\sigma = 0$. So $\imath_{\mathbf{v}}d\sigma = 0$. The proof of statement (c) follows.

Now we prove statement (d). We use the homotopy formula $\mathrm{L}_{\mathbf{v}} = \imath_{\mathbf{v}}d + d\imath_{\mathbf{v}}$, see [19]. The condition (43) in view of (85) is equivalent to

$$\imath_{\mathbf{v}}d\sigma = \sum_{j=1}^{N-1}\Lambda_j df_j = \nu_N\sum_{j=1}^{N-1}A_{Nj}df_j = -dh.$$

Thus $\mathrm{L}_{\mathbf{v}}\sigma = \imath_{\mathbf{v}}d\sigma + d\imath_{\mathbf{v}}\sigma = -dh + d\sigma(\mathbf{v}) = -dh + d||\mathbf{v}||^2 = d\left(||\mathbf{v}||^2 - h\right)$, here we use the relation $\sigma(\mathbf{v}) =< \mathbf{v}, \mathbf{v} >= ||\mathbf{v}||^2$. Hence, if $g^t_{\mathbf{v}}$ is the flow of $\mathbf{v}$ and γ is a closed curve on Q, then the integral $I = \int_{g^t_{\mathbf{v}}(\gamma)}\sigma$ is a function on t. In view of the well- known formula (see [26]) $\dot{I} = \int_{g^t_{\mathbf{v}}(\gamma)}\mathrm{L}_{\mathbf{v}}\sigma$, we obtain that $\dot{I} = 0$. In short Theorem 19 is proved. □

In the two following subsections we illustrate the statement (c) of Theorem 19.

10.1. **Inverse Stäckel's problem.** Let

$$f_j = f_j(\mathbf{x}) = \sum_{k=1}^{n} \int \frac{\varphi_{kj}(x_k)}{\sqrt{K_k(x_k)}} dx_k = c_j, \quad j = 1, 2, \ldots, N-1, \tag{127}$$

be a given $N-1$ -parametric family of orbits in the configuration space Q of the mechanical system with N degrees of freedom and kinetic energy

$$T = \frac{1}{2} \sum_{j=1}^{N} \frac{\dot{x}_j^2}{A_j}, \tag{128}$$

where $K_k(x_k) = 2\Psi_k(x_k) + 2\sum_{j=1}^{N} \alpha_j \varphi_{kj}(x_k)$, α_k, for $k = 1, 2, \ldots, N$ are constants, $\Psi_k = \Psi_k(x_k)$ are arbitrary functions and $A_j = A_j(\mathbf{x})$ such that

$$\frac{\{\varphi_1, \ldots, \varphi_{N-1}, x_j\}}{\{\varphi_1, \ldots, \varphi_{N-1}, \varphi_N\}} = A_j, \tag{129}$$

for $j = 1, 2, \ldots, N$. Here $d\varphi_\alpha = \sum_{k=1}^{N} \varphi_{k\alpha}(x_k) dx_k, \quad \varphi_{k\alpha} = \varphi_{k\alpha}(x_k)$, for $k = 1, \ldots, N$, $\alpha = 1, \ldots, N$ are arbitrary functions.

From (128) follows that the metric G is diagonal with $G_{jj} = \dfrac{1}{A_j}$.

The *inverse Stäckel problem* is the problem of determining the potential field of force that under which any curve of the family (127) is a trajectory of the mechanical system. The solution is as follows (see [43]).

Proposition 37. *For a mechanical system with a configuration space* Q *and kinetic energy* (128)*, the potential field of force* $\boldsymbol{F} = \dfrac{\partial V}{\partial \mathbf{x}}$*, for which the family of curves* (127) *are trajectories is*

$$V = -U = \nu^2(S) \left(\frac{\{\varphi_1, \ldots, \varphi_{N-1}, \Psi\}}{\{\varphi_1, \ldots, \varphi_{N-1}, \varphi_N\}} + \alpha_N \right) - h_0, \tag{130}$$

where $S = \int \sum_{j=1}^{N} \sqrt{\Psi_k(x_k) + \sum_{k=1}^{N} \alpha_j \varphi_{kj}(x_k)}\, dx_k = \int \sum_{k=1}^{N} \dfrac{dx_k}{q_k(x_k)}$ *is a function such that the hypersurface* $S = c_N$ *is orthogonal to the given hypersurfaces* $f_j = c_j$.

Proof. After some tedious computations we get the equality

$$\frac{\{f_1, \ldots, f_{N-1}, *\}}{\{f_1, \ldots, f_{N-1}, f_N\}} = \frac{\begin{vmatrix} q_1 d\varphi_1(\partial_1) & \ldots & q_N d\varphi_1(\partial_N) \\ \vdots & & \vdots \\ q_1 d\varphi_{N-1}(\partial_1) & \ldots & q_N d\varphi_{N-1}(\partial_N) \\ \partial_1 & \ldots & \partial_N \end{vmatrix}}{\prod_{j=1}^{N} q_j \{\varphi_1, \ldots, \varphi_N\}}$$

$$= \sum_{j=1}^{N} \left(\frac{A_j}{q_j} \partial_j \right) = \sum_{j=1}^{N} \left(A_j \frac{\partial S}{\partial x_j} \partial_j \right),$$

From (123) we have $\mathbf{v}(\mathbf{x}) = \nu G^{-1} \dfrac{\partial S}{\partial \mathbf{x}}$, hence in view of the first identity of (58) we obtain

$$\left\langle \frac{\partial S}{\partial \mathbf{x}}, \frac{\partial f_j}{\partial \mathbf{x}} \right\rangle = \sum_{k=1}^{N} A_k \varphi_{kj} = \sum_{k=1}^{N} A_k \frac{\partial \varphi_j}{\partial x_k} = \frac{\{\varphi_1, \ldots, \varphi_{N-1}, \varphi_j\}}{\{\varphi_1, \ldots, \varphi_{N-1}, \varphi_N\}} = 0,$$

for $j = 1, \ldots, N-1$, thus we obtain the orthogonality of the surfaces.

On the other hand from the relation

$$\begin{aligned}\|\mathbf{v}\|^2 = & \ \nu^2 \sum_{k=1}^{N} A^k (K_k(x_k))^2 = \nu^2 \sum_{k=1}^{N} A_k \left(2\Psi_k(x_k) + 2\sum_{j=1}^{N} \alpha_j \varphi_{kj}(x_k) \right) \\ = & \ 2\nu^2 \sum_{k=1}^{N} A^k \Psi_k(x_k) + 2\nu^2 \sum_{j=1}^{N} \alpha_j \sum_{k=1}^{N} A_k \varphi_{kj}(x_k) \\ = & \ 2\nu^2 \left(\frac{\{\varphi_1, \ldots, \varphi_{N-1}, \Psi\}}{\{\varphi_1, \ldots, \varphi_{N-1}, \varphi_N\}} + \sum_{j=1}^{N} \alpha_j \frac{\{\varphi_1, \ldots, \varphi_{N-1}, \varphi_j\}}{\{\varphi_1, \ldots, \varphi_{N-1}, \varphi_N\}} \right) \\ = & \ 2\nu^2 \left(\frac{\{\varphi_1, \ldots, \varphi_{N-1}, \Psi\}}{\{\varphi_1, \ldots, \varphi_{N-1}, \varphi_N\}} + \alpha_N \right),\end{aligned}$$

here we used the first identity of (58), where $d\Psi = \displaystyle\sum_{j=1}^{N} \Psi_k(x_k) dx_k$. We observe that if we choose $\nu = \nu(S)$, then from (46) we obtain that the field of force which generates the given family of orbits (127) is potential with potential function given by (130). In particular if $\nu = 1$ and $h_0 = \alpha_N$ then we obtain the classical *Stäckel potential* (see [13]). □

We observe that from (129) (130) follows that the metric G and potential function U can be determined from the given functions (127).

10.2. **Inverse Problem of two fixed centers.** The next example is a particular case of the inverse Stäckel's problem. This problem is called the *inverse problem of two fixed centers* (for more details see [43]).

Let P be a particle of infinitesimal mass which is attracted by two fixed centers C_0 and C_1 with mass m_0 and m_1 respectively. We select the coordinates so that the origin coincides with the center of mass and the x–axis passing through the points C_0 and C_1. Denoting by r_0, r_1 and $2c$ the distances between $C_0(x_0, 0, 0$ and $P(x, y, z)$, $C_1(x_1, 0, 0)$ and $P(x, y, z)$ and $C_0(x_0, 0, 0)$ and $C_1(x_1, 0, 0)$ respectively, we obtain that

$$r_0 = \sqrt{(x-x_0)^2 + y^2 + z^2}, \quad r = \sqrt{(x-x_1)^2 + y^2 + z^2}, \quad 2c = |x_1 - x_0|.$$

Then we have a particle with configuration space $\mathbb{R}^3$ and Lagrangian function

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \left(\frac{m_0}{r_0} + \frac{m_1}{r_1}\right) f,$$

where f is the attraction constant (see [12]).

After the coordinate change

$$x = \frac{m_0 - m_1}{m_1 + m_0} c + c\,\lambda\,\mu, \quad y = c\sqrt{(\lambda^2-1)(1-\mu^2)}\,\cos w, \quad z = c\sqrt{(\lambda^2-1)(1-\mu^2)}\,\sin w$$

we obtain

$$L = \frac{c^2(\lambda^2-\mu^2)}{4(\lambda^2-1)}\dot{\lambda}^2 - \frac{c^2(\lambda^2-\mu^2)}{4(1-\mu^2)}\dot{\mu}^2 + \frac{c^2(\lambda^2-1)(1-\mu^2)}{2}\dot{w}^2 - f\frac{(m_0+m_1)\lambda + (m_1-m_0)\mu}{c(\lambda^2-\mu^2)},$$

and $r_0 = c(\lambda + \mu)$, $\quad r_1 = c(\lambda - \mu)$, where $1 \le \lambda < +\infty$, $\quad -1 \le \mu \le 1$, $\quad 0 \le w \le 2\pi$.

Clearly the matrix $\tilde{G}$ in this case is

$$\tilde{G} = \begin{pmatrix} \dfrac{2(\lambda^2-1)}{c^2(\lambda^2-\mu^2)} & 0 & 0 \\ 0 & \dfrac{2(1-\mu^2)}{c^2(\lambda^2-\mu^2)} & 0 \\ 0 & 0 & \dfrac{1}{c^2(\lambda^2-1)(1-\mu^2)} \end{pmatrix}.$$

The inverse problem of the two fixed centers involves the construction of the potential field of forces for which the given family of curves

$$\begin{aligned} f_1(\lambda,\,\mu,\,w) \quad &= \int \frac{d\lambda}{\sqrt{R_2(\lambda)}} - \int \frac{d\mu}{\sqrt{R_1(\mu)}} = c_1, \\ f_2(\lambda,\,\mu,\,w) \quad & w - \frac{A}{2}\left(\int \frac{d\lambda}{(\lambda^2-1)\sqrt{R_2(\lambda)}} + \int \frac{d\mu}{(1-\mu^2)\sqrt{R_1(\mu)}}\right) = c_2, \end{aligned}$$

are formed by trajectories of the equations of motion, where R_1 and R_2 are functions such that

$$\begin{aligned} R_1(\mu) \quad &= h_0c^2\mu^4 + fc(m_0-m_1)\mu^3 + (a_2-h_0c^2)\mu^2 - fc(m_0-m_1)\mu - \frac{A^2}{2} - a_2 \\ R_2(\lambda) \quad &= h_0c^2\lambda^4 + fc(m_0+m_1)\lambda^3 + (a_2-h_0c^2)\lambda^2 - fc(m_0+m_1)\lambda - \frac{A^2}{2} - a_2, \end{aligned}$$

here C, h_0, f, A and a_2 are real constants.

After some computations we deduce

$$\begin{aligned} \{f_1, f_2, F\} \quad &= -\frac{1}{\sqrt{R_1(\mu)}}\partial_\lambda F - \frac{1}{\sqrt{R_2(\lambda)}}\partial_\mu F - \frac{A(\lambda^2-\mu^2)}{2\sqrt{R_1(\mu)R_2(\lambda)}(\lambda^2-1)(1-\mu^2)}\partial w F = \\ &= -\frac{c^2(\lambda^2-\mu^2)}{2\sqrt{R_1(\mu)R_2(\lambda)}}\Big(\frac{2(\lambda^2-1)}{c^2(\lambda^2-\mu^2)}\big(\frac{\sqrt{R_2(\lambda)}}{(\lambda^2-1)}\partial_\lambda F\big) + \frac{2(1-\mu^2)}{c^2(\lambda^2-\mu^2)}\big(\frac{\sqrt{R_1(\mu)}}{(1-\mu^2)}\partial_\mu F\big) \\ &+ \frac{1}{c^2(\lambda^2-1)(1-\mu^2)}\big(A\,\partial_w F\big)\Big) := \varrho\left\langle \frac{\partial S}{\partial \mathbf{x}}, \frac{\partial F}{\partial \mathbf{x}}\right\rangle \\ &= \varrho(\tilde{G}_{11}\partial_\lambda S\partial_\lambda F + \tilde{G}_{22}\partial_\mu S\partial_\mu F + \tilde{G}_{33}\partial_w S\partial_w F), \end{aligned}$$

where F is an arbitrary function, and

$$\varrho = -\frac{c^2(\lambda^2-\mu^2)}{2\sqrt{R_1(\mu)R_2(\lambda)}}, \quad S(\lambda,\,\mu,\,w) = \int \frac{\sqrt{R_1(\mu)}}{(1-\mu^2)}d\mu + \int \frac{\sqrt{R_2(\lambda)}}{(\lambda^2-1)}d\lambda + A\,w.$$

Hence from (130) we obtain

$$V = \frac{1}{2}\nu^2(S)\left\|\frac{\partial S}{\partial \mathbf{x}}\right\|^2 - h_0 = \frac{\nu^2}{c^2}\Big(\frac{R_1(\mu)}{(1-\mu^2)(\lambda^2-\mu^2)} + \frac{R_2(\lambda)}{(\lambda^2-1)(\lambda^2-\mu^2)} + \frac{A^2}{(\lambda^2-1)(1-\mu^2)}\Big) - h_0.$$

In view of the equalities

$$\begin{aligned} \frac{R_1(\mu)}{1-\mu^2} &= -h_0c^2\mu^2 + (m_1-m_0)cf\mu - a_2 - \frac{A^2}{2(1-\mu^2)}, \\ \frac{R_2(\lambda)}{\lambda^2-1} &= h_0c^2\lambda^2 + (m_0+m_1)cf\lambda + a_2 - \frac{A^2}{2(\lambda^2-1)} \end{aligned}$$

we deduce that

$$U = \nu^2\Big(h_0 + \frac{(m_0+m_1)\lambda + (m_1-m_0)\mu}{c(\lambda^2-\mu^2)} f\Big) - h_0.$$

By taking $\nu = 1$, then $U = f\,\dfrac{(m_0+m_1)\lambda + (m_1-m_0)\mu}{c(\lambda^2-\mu^2)}$, which coincides with the well known potential (see [12, 13].)

10.3. **Joukovski's example.** We shall study a mechanical systems with three degrees of freedom. If we denote by $x_1 = p$, $x_2 = q$, $x_3 = r$, then we consider the mechanical system with kinetic energy

$$T = \frac{1}{2r^2}\left(\dot p^2 - 2p\,\dot p\,\dot r + \dot q^2 - 2q\,\dot q\,\dot r + \left(\frac{p^2+q^2}{r^2} + r^2\right)\dot r^2\right).$$

Consequently the matrix $\tilde G$ is such that

$$\tilde G = \begin{pmatrix} \dfrac{p^2+r^4}{r^2} & \dfrac{pq}{r^2} & \dfrac{p}{r} \\ \\ \dfrac{pq}{r^2} & \dfrac{q^2+r^4}{r^2} & \dfrac{q}{r} \\ \\ \dfrac{p}{r} & \dfrac{q}{r} & 1 \end{pmatrix}. \ .$$

Then we get $|\tilde G| = r^4$, $\quad \Delta = p^2+q^2+r^4$. We determine the field of force derived from the potential-energy function (48) in such a way that the family of curves $p = c_1$, $q = c_2$ can be freely described by a particle with kinetic energy T.

In this case equations (124) are

$$\begin{aligned} \tilde g_{11}\frac{\partial S}{\partial p} + \tilde g_{12}\frac{\partial S}{\partial q} + \tilde g_{13}\frac{\partial S}{\partial r} &= \frac{p^2+r^4}{r^2}\frac{\partial S}{\partial p} + \frac{pq}{r^2}\frac{\partial S}{\partial q} + \frac{p}{r}\frac{\partial S}{\partial r} = 0,\\ \tilde g_{21}\frac{\partial S}{\partial p} + \tilde g_{22}\frac{\partial S}{\partial q} + \tilde g_{23}\frac{\partial S}{\partial r} &= \frac{q^2+r^4}{r^2}\frac{\partial S}{\partial q} + \frac{pq}{r^2}\frac{\partial S}{\partial p} + \frac{q}{r}\frac{\partial S}{\partial r} = 0. \end{aligned}$$

The solutions of these partial differential equations are $S = S\left(\dfrac{p^2+q^2}{r^2} - r^2\right)$,where S is an arbitrary function in the variable $\dfrac{p^2+q^2}{r^2} - r^2$.

Without loss of generality below we consider that $S = \dfrac{p^2+q^2}{r^2} - r^2$. Hence after some computations we obtain that conditions (126) take the form

$$\frac{\partial h}{\partial p} = \frac{2p}{r}\frac{\partial \nu^2}{\partial r} + \frac{(p^2+q^2+r^4)}{r^2}\frac{\partial \nu^2}{\partial p}, \quad \frac{\partial h}{\partial q} = \frac{2q}{r}\frac{\partial \nu^2}{\partial r} + \frac{(p^2+q^2+r^4)}{r^2}\frac{\partial \nu^2}{\partial q}. \tag{131}$$

From the compatibility conditions of these equations we obtain that $h = h(p^2+q^2)$, $\quad \nu = \nu(p^2+q^2, r)$. In the coordinates $\xi = p^2+q^2$, $r = r$ the conditions (131) write

$$\frac{\partial h}{\partial \xi} = \frac{1}{r^2}\left(r\frac{\partial \nu^2}{\partial r} + 2(\xi + r^4)\frac{\partial \nu^2}{\partial \xi}\right). \tag{132}$$

Thus, from (48), the potential function takes the form

$$V = \frac{1}{2}\nu^2(\xi, r)\left(\frac{\xi}{r^2} + r^2\right) - h(\xi), \tag{133}$$

where $\nu = \nu(\xi, r)$ and $h = h(\xi)$ are solutions of (132).

We shall look for the solution $h = h(\xi)$ of (132) when the function ν^2 is given by

$$\nu^2 = \Psi\left(\frac{\xi}{r^2} - r^2\right) + \sum_{j=-\infty}^{+\infty} a_j(\xi) r^j.$$

where the series is a formal Laurent series, and $\Psi = \Phi(\frac{\xi}{r^2} - r^2)$ is an arbitrary function.

By inserting ν^2 in (132) we obtain

$$\sum_{j=-\infty}^{+\infty}\left(ja_j + 2\xi\frac{da_j}{d\xi} + 2\frac{da_{j-4}}{d\xi}\right) r^j = \frac{r^2}{2}\frac{dh}{d\xi}.$$

We choose the coefficients a_j satisfying

$$\begin{aligned} ja_j + 2\xi\frac{da_j}{d\xi} + 2\frac{da_{j-4}}{d\xi} &= 0 \Longleftrightarrow (j-2)a_j + \frac{d}{d\xi}\left(2\xi a_j + 2a_{j-4}\right), \text{ for } \quad j \neq 2, \\ 2a_2 + 2\xi\frac{da_2}{d\xi} + 2\frac{da_{-2}}{d\xi} &= \frac{dh}{2d\xi} \Longleftrightarrow \frac{d}{d\xi}\left(2\xi a_2 + 2a_{-2} - \frac{h}{2}\right) = 0. \end{aligned}$$

Consequently the potential function (133) takes the form

$$V = 4\left(\Psi\left(\frac{\xi}{r^2} - r^2\right) + \sum_{j=-\infty}^{+\infty} a_j(\xi) r^j\right)\left(\frac{\xi}{r^2} + r^2\right) - 4\xi a_2 - 4a_{-2} - h_0.$$

If we change $p = xz, \quad q = yz, \quad r = z$ where x, y, z are the cartesian coordinates, then in these coordinates the kinetic and potential function takes the form respectively

$$\begin{aligned} T = &\ \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right), \\ V = &\ 4\left(\Psi\left(x^2 + y^2 - z^2\right) + \sum_{j=-\infty}^{+\infty} a_j(z^2(x^2+y^2))z^j\right)\left(x^2 + y^2 + z^2\right) - \\ &\ 4z^2(x^2+y^2)a_2(z^2(x^2+y^2)) - 4a_{-2}(z^2(x^2+y^2)) - h_0. \end{aligned}$$

Clearly if $a_j = 0$ for $j \in \mathbb{Z}$ then we obtain the potential $V = \Psi\left(x^2 + y^2 - z^2\right)\left(x^2 + y^2 + z^2\right) - h_0$ obtained by Joukovski in [22]. On the other hand, if $\Psi = 0$, $a_j = 0$ for $j \in \mathbb{Z} \setminus 2$ and $4a_2 = a$ then we obtain the potentials $V = az^4 - h_0$ given in [43].

11. Proof of Theorem 20, 21 and 25

Proof of Theorem 20. Under the assumptions of Corollary 5 taking the N of the corollary as $2N$, introducing the notations $y_j = x_{N+j}$, and choosing $g_{N+j} = x_j$ for $j = 1, \ldots, N$, we obtain that the differential systems (9) takes the form

$$\dot{x}_j = \lambda_{N+j}, \quad \dot{y}_j = \sum_{k=1}^{N} \lambda_{N+k} \frac{\{f_1, \ldots, f_N, x_1, \ldots, x_{k-1}, y_j, x_{k+1}, \ldots, x_N\}}{\{f_1, \ldots, f_N, x_1, \ldots, x_N\}}, \tag{134}$$

for $j = 1, 2, \ldots, N$. These equations are the most general differential equations which admits N independent first integrals and satisfy the condition $\{f_1, \ldots, f_N, x_1, \ldots, x_N\} \neq 0$.

The proof of Theorem 20 is obtained by choosing the arbitrary functions λ_{N+j} as follows $\lambda_{N+j} = \{H, x_j\}^*$, where H is the Hamiltonian function for $j = 1, \ldots, N$. From the identity (61) with $G = y_k, \quad f_{N+j} = x_j \quad$ for $\quad j = 1, \ldots, N$, we obtain that differential system (134) can be rewritten as

$$
\begin{aligned}
\dot x_j = & \ \{H, x_j\}^*, \\
\dot y_j = & \ \sum_{k=1}^{N}(\{H, x_k\}^*)\frac{\{f_1, \ldots, f_N, x_1, \ldots, x_{k-1}, y_j, x_{k+1}, \ldots, x_N\}}{\{f_1, \ldots, f_N, x_1, \ldots, x_N\}} \\
= & \ \{H, y_j\}^* - \sum_{k=1}^{N}\{H, f_k\}^*\frac{\{f_1, \ldots, f_{k-1}, y_j, f_{k+1}, \ldots, f_N, x_1, \ldots, \ldots, x_N\}}{\{f_1, \ldots, f_N, x_1, \ldots, x_N\}} \\
& + \sum_{k=1}^{N} W_j \frac{\{f_1, \ldots, f_N, x_1, \ldots, x_{k-1}, y_j, x_{k+1}, \ldots, x_N\}}{\{f_1, \ldots, f_N, x_1, \ldots, x_N\}}.
\end{aligned}
$$

Clearly if the first integrals are in involution and $W_j = 0$, then we obtain that the Hamiltonian system with Hamiltonian $H = H(f_1, \ldots, f_N)$ is integrable by quadratures.

Now we shall prove the equations (53). Since $\{f_1, \ldots, f_N, x_1, \ldots, \ldots, x_N\} = 0$ and $\{f_1, \ldots, f_N, x_1, \ldots, , x_{N-1}, y_1\} \neq 0$. Taking $W_j = 0$ for $j = 1, \ldots, N-1$ and $\lambda_{N+j} = \dfrac{\partial H}{\partial y_j} = \{H, x_j\}^*$, for $j = 1, \ldots, N-1$, where H is the Hamiltonian function and in view of the identity (61) with $G = x_N$, $f_{N+j} = x_j$ for $j = 1, \ldots, N-1$, $f_{2N} = y_1$, and $G = y_j$, $f_{N+j} = x_j$ for $j = 1, \ldots, N-1$, $f_{2N} = y_1$, we obtain that differential system (134) can be rewritten as

$$
\begin{aligned}
\dot x_j = & \ \{H, x_j\}^*, \quad \text{for} \quad j = 1, \ldots, N-1, \\
\dot x_N = & \ \sum_{k=1}^{N-1}\{H, x_k\}^*\frac{\{f_1, \ldots, f_N, x_1, \ldots, x_{k-1}, x_N, x_{k+1}, \ldots, y_1\}}{\{f_1, \ldots, f_N, x_1, \ldots, , x_{N-1}, y_1\}} \\
& + \lambda_{2N}\frac{\{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, x_N\}}{\{f_1, \ldots, f_N, x_1, \ldots, , x_{N-1}, y_1\}} \\
= & \ \{H, x_N\}^* - \sum_{k=1}^{N}\{H, f_j\}^*\frac{\{f_1, \ldots, f_{k-1}, x_N, f_{k+1}, \ldots, f_N, x_1, \ldots, \ldots, y_1\}}{\{f_1, \ldots, f_N, x_1, \ldots, , x_{N-1}, y_1\}} \\
& + (\lambda_{2N} - \{H, y_1\}^*)\,\frac{\{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, x_N\}}{\{f_1, \ldots, f_N, x_1, \ldots, , x_{N-1}, y_1\}}, \\
\dot y_1 = & \ \lambda_{2N}, \\
\dot y_j = & \ \sum_{k=1}^{N-1}\{H, x_k\}^*\frac{\{f_1, \ldots, f_N, x_1, \ldots, x_{k-1}, y_j, x_{k+1}, \ldots, x_N\}}{\{f_1, \ldots, f_N, x_1, \ldots, , x_{N-1}, y_1\}} \\
& + \lambda_{2N}\frac{\{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_j\}}{\{f_1, \ldots, f_N, x_1, \ldots, , x_{N-1}, y_1\}} \\
= & \ \{H, y_j\} - \sum_{k=1}^{N}\{H, f_j\}^*\frac{\{f_1, \ldots, f_{k-1}, y_j, f_{k+1}, \ldots, f_N, x_1, \ldots, \ldots, x_N\}}{\{f_1, \ldots, f_N, x_1, \ldots, , x_{N-1}, y_1\}} \\
& + (\lambda_{2N} - \{H, y_1\}^*)\,\frac{\{f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_k\}}{\{f_1, \ldots, f_N, x_1, \ldots, , x_{N-1}, y_1\}}.
\end{aligned}
$$

Therefore by choosing $\lambda_{2N} = \{H, y_1\}^* + \lambda\{f_1, \ldots, f_N, x_1, \ldots, , x_{N-1}, y_1\}$, we get the differential system (53).

In view of the identity (59) with $G = f_j$ from (53) we obtain the relations

$$
\dot f_k = \sum_{j=1}^{N}\frac{\partial f_k}{\partial y_j}\{f_1, \ldots, f_N, x_1, \ldots, y_j\} = \frac{\partial f_k}{\partial x_N}\{f_1, \ldots, f_N, x_1, \ldots, x_N\} = 0.
$$

Differential system (53) when $\{H, f_j\} = 0$ for $j = 1, \dots, N$ is the standard Hamiltonian system with the constraints $\{f_1, \dots, f_N, x_1, \dots, x_N\} = 0$. □

11.1. **Neumar-Moser integrable system.** We shall illustrate these theorem in the Neumann-Moser's integrable system.

Now we study the case when we have N independent involutive first integrals of the form

$$f_\nu = (Ax_\nu + By_\nu)^2 + C\sum_{j\neq\nu}^{N} \frac{(x_\nu y_j - x_j y_\nu)^2}{a_\nu - a_j}, \tag{135}$$

for $\nu = 1, \dots, N$, where A, B and C are constants such that $C(A^2+B^2) \neq 0$. Thus we study the constrained Hamiltonian system $\left(\mathbb{R}^{2N},\, \Omega^2, \mathbb{M}, , H\right)$.

The case when $A = 0$, $B = 1$, $C = 1$ and $A = 1$, $B = 0$, $C = 1$ was study in particular in [34]. The case when $AB \neq 0$ was introduced in [41]). In particular if $C = (A + B)^2$ then from (135) we obtain that $f_\nu = A^2 f_\nu^{(1)} + B^2 f_\nu^{(2)} + 2AB f_\nu^{(3)}$ where

$$f_\nu^{(1)} = x_\nu^2 + \sum_{j\neq\nu}^{N} \frac{(x_\nu y_j - x_j y_\nu)^2}{a_\nu - a_j},\ f_\nu^{(2)} = y_\nu^2 + \sum_{j\neq\nu}^{N} \frac{(x_\nu y_j - x_j y_\nu)^2}{a_\nu - a_j},\ f_\nu^{(3)} = x_\nu y_\nu + \sum_{j\neq\nu}^{N} \frac{(x_\nu y_j - x_j y_\nu)^2}{a_\nu - a_j}.$$

It is easy to show that the following relations hold $\{f_k^{(\alpha)}, f_m^{(\alpha)}\}^* = 0,\quad$ for $\quad \alpha = 1, 2, 3, \quad m, k = 1, \dots, N$, i.e. are in involution.

After some computations we obtain that $\{f_1, \dots, f_N, x_1, \dots, x_N\} \neq 0$ if $B \neq 0$. Then taking in (51) $H = H(f_1, \dots, f_N)$, and $W_j = 0$ for $j = 1, \dots, N$ we obtain a completely integrable Hamiltonian system $\dot{x}_j = \{H,\, x_j\}^*, \quad \dot{y}_j = \{H,\, y_j\}^*$.

If $B = 0$ then $\{f_1, \dots, f_N, x_1, \dots, x_N\} = 0$ Then taking in (53) $H = H(f_1, \dots, f_N), \quad W_j = 0$ for $j = 1, \dots, N$ and in view of the relations $\{f_1, \dots, f_N, x_1, \dots, x_{N-1}, y_j\} = \varrho(x)\, x_j$ for $j = 1, \dots, N$, for convenient function $\varrho = \varrho(\mathbf{x})$, we obtain the differential system

$$\dot{\mathbf{x}} = \{H,\, \mathbf{x}\}^*, \quad \dot{\mathbf{y}} = \{H,\, \mathbf{y}\}^* + \tilde{\lambda}\mathbf{x}, \tag{136}$$

where $\tilde{\lambda} = \varrho\lambda$. In particular for $N = 3$ we deduced that

$$\begin{aligned} \{f_1, f_2, f_3, x_1, x_2, x_3\} = {}& 0, \quad \{f_1, f_2, f_3, x_1, x_2, y_1\} = \frac{K}{\Delta} x_3 x_1, \\ \{f_1, f_2, f_3, , x_1, x_2, y_2\} = {}& \frac{K}{\Delta} x_3 x_2, \quad \{f_1, f_2, f_3, x_1, x_2, y_3\} = \frac{K}{\Delta} x_3 x_3, \end{aligned}$$

where $\Delta = (a_1 - a_2)(a_2 - a_3)(a_1 - a_3)$, and K is a convenient function. Thus the differential system (136) with $\varrho = \dfrac{K x_3}{\Delta}$ describes the behavior of the particle with Hamiltonian $H = H(f_1, f_2, f_3)$ and constrained to move on the sphere $x_1^2 + x_2^2 + x_3^2 = 1$.

If we take $H = \dfrac{1}{2}(a_1 f_1 + a_2 f_2 + a_3 f_3) = \dfrac{1}{2}\left(||\mathbf{x}||^2||\mathbf{y}||^2 - \langle \mathbf{x},\, \mathbf{y}\rangle^2 + a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2\right)$ and $\lambda = \Psi(x_1^2 + x_1^2 + x_1^2)$, then from equations (136) we deduce that the equations of motion of a particle on a 3-dimensional sphere, with an anisotropic harmonic potential (Neumann's problem). This system is one of the best understood integrable systems of classical mechanics.

Proof of Theorem 21. The differential systems (9) under the assumptions of Theorem 21 takes the form

$$\begin{aligned} \dot{x}_j = &\ \lambda_{N+j}, \quad \text{for} \quad j = 1, 2, \ldots, N-r, \\ \dot{x}_n = &\ \sum_{k=N+1}^{2N} \lambda_k \frac{\{f_1, \ldots, f_N, x_1, \ldots, x_{k-1}, x_n, x_{k+1}, \ldots, x_N\}}{\{f_1, \ldots, f_N, x_1, \ldots, x_N\}}, \\ &\ \text{for} \quad n = N-r+1, \ldots, N, \\ \dot{y}_m = &\ \sum_{k=N+1}^{2N} \lambda_k \frac{\{f_1, \ldots, f_N, x_1, \ldots, x_{k-1}, y_n, x_{k+1}, \ldots, x_N\}}{\{f_1, \ldots, f_N, x_1, \ldots, x_N\}},, \\ &\ \text{for} \quad m = 1, 2, \ldots, N. \end{aligned} \tag{137}$$

These equations are the most general differential equations which admits $N+r$ first integrals which satisfies the condition $\{f_1, \ldots, f_{N+r}, x_1, \ldots, x_{N-r}\} \neq 0$.

By choosing in (49) the arbitrary functions $W_j = 0$ and $\lambda_{N+j} = \{H, x_j\}^*$ for $j = 1, \ldots, N-r$, where H is the Hamiltonian and by using the identity (61) with $G = x_k, \quad f_{N+r+j} = x_j \quad \text{for} \quad j = 1, \ldots, N-r$, and $G = y_k, \quad f_{N+r+j} = x_j \quad \text{for} \quad j = 1, \ldots, N-r$, we obtain that differential system (137) can be rewritten as

$$\begin{aligned} \dot{x}_j = &\ \{H, x_j\}^* \quad \text{for} \quad j = 1, 2, \ldots, N-r, \\ \dot{x}_k = &\ \sum_{j=1}^{N-r} \{H, x_j\}^* \frac{\{f_1, \ldots, f_{N+r}, x_1, \ldots, x_{j-1}, x_k, x_{j+1}, \ldots, x_{N-r}\}}{\{f_1, \ldots, f_{N+r}, x_1, \ldots, x_{N-r}\}} \\ = &\ \{H, x_k\}^* - \\ &\ \sum_{k=1}^{N+r} \{H, f_j\}^* \frac{\{f_1, \ldots, f_{j-1}, x_k, f_{j+1}, \ldots, f_{N+r}, x_1, \ldots, \ldots, x_{N-r}\}}{\{f_1, \ldots, f_{N+r}, x_1, \ldots, x_{N-r}\}}, \\ &\ \text{for} \quad k = N-r+1, \ldots, N, \\ \dot{y}_j = &\ \sum_{k=1}^{N-r} \{H, x_k\}^* \frac{\{f_1, \ldots, f_{N+r}, x_1, \ldots, x_{k-1}, y_j, x_{k+1}, \ldots, x_N\}}{\{f_1, \ldots, f_{N+r}, x_1, \ldots, x_{N-r}\}} \\ = &\ \{H, y_j\}^* - \\ &\ \sum_{k=1}^{N+r} \{H, f_k\}^* \frac{\{f_1, \ldots, f_{k-1}, y_j, f_{k+1}, \ldots, f_{N+r}, x_1, \ldots, \ldots, x_{N-r}\}}{\{f_1, \ldots, f_{N+r}, x_1, \ldots, x_{N-r}\}}, \end{aligned}$$

for $j = 1, 2, \ldots, N$. Hence we get the differential system (54). □

Proof of Theorem 25. Analogously to the proof of Theorem 3 from formula (78), denoting by $(\partial_1, \ldots, \partial_{2N}) = (\partial_{x_1}, \ldots, \partial_{x_N}, \partial_{y_1}, \ldots, \partial_{y_N})$, and taking the arbitrary functions $\lambda_{N+j} = \{\tilde{H}, x_j\}^*$ for $j = 1, \ldots, N$, where $\tilde{H}$ is the Hamiltonian function, from identity (61) with $f_j = g_j, \quad f_{N+j} = x_j, \quad G = y_j$, for $j = 1, \ldots, N$, we obtain the differential system (55). This is the proof of the Theorem 25 □

11.2. **Gantmacher system.** We shall illustrate Theorem 25 in the nonholonomic system study in subsection 9.4. Thus we shall study the constrained Hamiltonian system $(\mathbb{R}^8, \Omega^2, \mathcal{M}_2, H)$ with $\mathcal{M}_2 = \{g_1 = x_1y_1 + x_2y_2 = 0, \quad g_2 = x_1y_3 - x_2y_4 = 0.$. We choose the arbitrary functions g_j for $j = 3, \ldots, 8$ as follows

$$g_3 = x_1y_2 - x_2y_1, \quad g_4 = x_2y_3 + x_1y_4, \quad g_{j+4} = x_j, \quad \text{for} \quad j = 1, 2, 3, 4.$$

We apply Theorem 25. In view of the relations

$$\{g_1,g_2,g_3,g_4,x_1,\ldots,x_4\}=-(x_1^2+x_2^2)^2,\quad \{y_1,g_2,g_3,g_4,x_1,\ldots,x_4\}=-x_1(x_1^2+x_2^2),$$
$$\{g_1,y_1,g_3,g_4,x_1,\ldots,x_4\}=0,\quad \{g_1,g_2,y_1,g_4,x_1,\ldots,x_4\}=x_2(x_1^2+x_2^2),$$
$$\{g_1,g_2,g_3,y_1,x_1,\ldots,x_4\}=0,\quad \{g_1,g_2,g_3,g_4,y_1,x_2,x_3,x_4\}=(x_1y_1-x_2y_2)(x_1^2+x_2^2),$$
$$\{g_1,g_2,g_3,g_4,x_1,y_1,x_3,,x_4\}=(x_1y_2+x_2y_1)(x_1^2+x_2^2),\quad \{g_1,g_2,g_3,g_4,x_1,x_2,y_1,x_4\}=0,$$
$$\{g_1,g_2,g_3,g_4,x_1,x_2,x_3,y_4\}=0.$$

In a similar form we can obtain the remain determinant. Thus system (55) takes the form

$$\begin{aligned}
\dot{x}_j= &\quad \{\tilde{H},x_j\}^*,\quad \text{for}\quad j=1,2,3,4,\\
\dot{y}_1= &\quad \{\tilde{H},y_1\}^*-\frac{x_1\{H,g_1\}^*}{x_1^2+x_2^2}-(\lambda_3-\{H,g_3\}^*)\,\frac{x_2}{x_1^2+x_2^2},\\
\dot{y}_2= &\quad \{\tilde{H},y_2\}^*-\frac{x_2\{H,g_1\}^*}{x_1^2+x_2^2}+(\lambda_3-\{H,g_3\}^*)\,\frac{x_1}{x_1^2+x_2^2},\\
\dot{y}_3= &\quad \{\tilde{H},y_3\}^*-\frac{x_1\{H,g_2\}^*}{x_1^2+x_2^2}+(\lambda_4-\{H,g_4\}^*)\,\frac{x_2}{x_1^2+x_2^2},\\
\dot{y}_4= &\quad \{\tilde{H},y_4\}^*+\frac{x_2\{H,g_2\}^*}{x_1^2+x_2^2}+(\lambda_4-\{H,g_4\}^*)\,\frac{x_1}{x_1^2+x_2^2}.
\end{aligned}\tag{138}$$

In particular, taking $\lambda_3=\{H,g_3\}^*,\quad \lambda_4=\{H,g_4\}^*$, and $H=\frac{1}{2}\left(y_1^2+y_2^2+y_3^2+y_4^2\right)-gx_3$, thus in view of (116) we obtain

$$\{H,g_1\}^*=y_1^2+y_2^2=-\mu_1(x_1^2+x_2^2),\quad \{H,g_2\}^*=y_1y_3-y_2y_4+gx_1=-\mu_2(x_1^2+x_2^2).$$

Consequently differential equations (56) take the form

$$\begin{aligned}
\dot{x}_1= &\quad y_1,\quad \dot{x}_2=y_2,\quad \dot{x}_3=y_3,\quad \dot{x}_4=y_4,\\
\dot{y}_1= &\quad x_1\mu_1,\quad \dot{y}_2=x_2\mu_1,\quad \dot{y}_3=-g+x_1\mu_2,\quad \dot{y}_4=-x_2\mu_2,
\end{aligned}$$

which coincide with the Hamiltonian form of equations (122).

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