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ON THE LIMIT CYCLES OF THE FLOQUET DIFFERENTIAL EQUATIONS

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ABSTRACT. We provide sufficient conditions for the existence of limit cycles for the Floquet differential equations $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \varepsilon(B(t)\mathbf{x}(t)+b(t))$, where $\mathbf{x}(t)$ and $b(t)$ are column vectors of length n, A and $B(t)$ are $n\times n$ matrices, the components of $b(t)$ and $B(t)$ are T–periodic functions, the differential equation $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$ has a plane filled with T–periodic orbits, and ε is a small parameter. The proof of this result is based on averaging theory.

1. Introduction

The linear first order differential equation

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + b(t), \tag{1}$$

where $\mathbf{x}(t)$ and $b(t)$ are column vectors of length n, $A(t)$ is an $n \times n$ matrix, and $A(t)$ and $b(t)$ are periodic with period T or simply T–*periodic* (i.e. $A(t+T) = A(t)$ and $b(t+T) = b(t)$ for all $t \in \mathbb{R}$), is called a *Floquet differential equation*. For more details on the Floquet differential equations see [3], and for some applications of it to the study of limit cycles see [11]. These differential equations have been studied intensively and have many applications. As far as we know there are no general results on the existence or non–existence of limit cycles for the Floquet differential equations. The objective of this paper is to provide sufficient conditions for the existence of limit cycles for a subclass of Floquet differential equations.

A *limit cycle* of the differential equation (1) is a periodic orbit isolated in the set of all periodic orbits of the differential equation (1). To obtain analytically limit cycles of a differential equation is in general a very difficult problem, many times impossible. If the averaging theory can be applied to the differential equation (1), then it reduces this difficult problem to find the zeros of a nonlinear function. It is known that in

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general the averaging theory for finding limit cycles does not provide all the limit cycles of the differential equation.

The averaging theory (see for instance [10]) gives a quantitative relation between the solutions of some nonautonomous differential system and the solutions of its autonomous averaged differential system. In particular, it allows to study the periodic orbits of a nonautonomous differential system in function of the periodic orbits of the averaged one, see for more details [1, 2, 7, 6, 10, 12]. For more information about the averaging theory see section 2.

In this paper we want to study the existence of limit cycles for the subclass

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \varepsilon(B(t)\mathbf{x}(t) + b(t)) \tag{2}$$

of Floquet differential equations (1), where

- (H1) $\mathbf{x}(t)$ and $b(t)$ are column vectors of length n,
- (H2) A and $B(t)$ are $n \times n$ matrices,
- (H3) $b(t)$ and $B(t)$ are $2\pi/\omega$–periodic with ω a positive real number,
- (H4) A has a pair of purely imaginary eigenvalues $\pm\omega i$ and any other eigenvalue of A is not a rational multiple of ωi, and
- (H5) ε is a small parameter.

Note that the assumption (H4) implies that the unperturbed linear differential equation

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) \tag{3}$$

has a plane filled of periodic orbits with period $2\pi/\omega$. What we want to know is: *Are there periodic orbits of this plane which become limit cycles of the perturbed differential equation* (2) *for* $\varepsilon \neq 0$ *sufficiently small?*

Working with the new time $T = \omega t$, that we shall continue denoting by t, we can assume without loss of generality that

- (H3') $b(t)$ and $B(t)$ are 2π–periodic, and
- (H4') A has a pair of purely imaginary eigenvalues $\pm i$ and any other eigenvalue of A is not a rational multiple of i.

Again it is not restrictive to assume

- (H6) the matrix A is in its real Jordan normal form and that its first block of the Jordan normal form is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Otherwise we do a linear change of variables for writing the matrix A in that real Jordan normal form, and the form of the differential equation (2) remains.

We denote the components of the vector $b(t) = (b_k(t))$ and of the matrix $B(t) = (b_{ij}(t))$, for $k, i, j = 1, \dots, n$. Since the functions $b_k(t)$ and $b_{ij}(t)$ are 2π–periodic we expand them in Fourier series as follows

$$
\begin{aligned}
b_k(t) = &\ b_k^0 + \sum_{n=1}^{\infty} \left(b_k^{2n-1} \sin(2nt) + b_k^{2n} \cos(2nt) \right), \\
b_{ij}(t) = &\ b_{ij}^0 + \sum_{n=1}^{\infty} \left(b_{ij}^{2n-1} \sin(2nt) + b_{ij}^{2n} \cos(2nt) \right).
\end{aligned} \tag{4}
$$

For more details in Fourier series see for instance [4].

We shall use the following expressions

$$
\begin{aligned}
X = &\ 2\,[(b_1^1 - b_2^2)(2b_{12}^0 - 2b_{21}^0 - b_{11}^3 + b_{22}^3 + b_{12}^4 + b_{21}^4) \\
&+(b_2^1 + b_1^2)(2b_{11}^0 + 2b_{22}^0 - b_{12}^3 - b_{21}^3 - b_{11}^4 + b_{22}^4)]\,, \\
Y = &\ 2\,[(b_2^1 + b_1^2)(2b_{12}^0 - 2b_{21}^0 + b_{11}^3 - b_{22}^3 - b_{12}^4 - b_{21}^4) \\
&-(b_1^1 - b_2^2)(2b_{11}^0 + 2b_{22}^0 + b_{12}^3 + b_{21}^3 + b_{11}^4 - b_{22}^4)]\,, \\
\Delta = &\ (-b_{11}^3 + b_{22}^3 + b_{12}^4 + b_{21}^4)^2 + (b_{12}^3 + b_{21}^3 + b_{11}^4 - b_{22}^4)^2 \\
&-4(b_{12}^0 - b_{21}^0)^2 - 4(b_{11}^0 - b_{22}^0).
\end{aligned} \tag{5}
$$

Our main result is the following one.

Theorem 1. *Consider the Floquet differential equation* (2) *under the assumptions* $(H1)$, $(H2)$, $(H3')$, $(H4')$, $(H5)$ *and* $(H6)$. *If* $(X^2 + Y^2)\Delta \neq 0$, *then for* $\varepsilon \neq 0$ *sufficiently small there exists a limit cycle* $\mathbf{x}(t, \varepsilon) = (x_1(t, \varepsilon), x_2(t, \varepsilon), x_3(t, \varepsilon), \dots, x_n(t, \varepsilon))$ *of equation* (2) *such that when* $\varepsilon \to 0$ *it tends to the periodic solution*

$$
\left(\frac{X}{\Delta} \cos t - \frac{Y}{\Delta} \sin t, \frac{Y}{\Delta} \cos t + \frac{X}{\Delta} \sin t, 0, \dots, 0 \right) \tag{6}
$$

of the differential equation (3).

Theorem 1 is proved in section 3. As we shall see its proof only uses a result of averaging theory and linear algebra.

We remark that in the expressions of X, Y and Δ only appears coefficients of Fourier corresponding to the trigonometric functions 1, $\sin t$, $\cos t$, $\sin(2t)$ and $\cos(2t)$ of the expansions in Fourier series of the six functions $b_k(t)$ and $b_{ij}(t)$ with $k, i, j = 1, 2$. All the other functions and coefficients which appear in the differential equation (2) do not play any role for the existence of the limit cycle found in Theorem 1.

2. Basic results on averaging theory

In this section we present the basic result from the averaging theory that we shall need for proving the main results of this paper.

We consider the problem of the bifurcation of T–periodic solutions from differential equations of the form

$$\dot{\mathbf{x}} = F_0(t,\mathbf{x}) + \varepsilon F_1(t,\mathbf{x}) + \varepsilon^2 F_2(t,\mathbf{x},\varepsilon), \tag{7}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. Here the functions $F_0, F_1 : \mathbb{R}\times\Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R}\times\Omega\times(-\varepsilon_0,\varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T–periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed differential equation

$$\dot{\mathbf{x}} = F_0(t,\mathbf{x}), \tag{8}$$

has a submanifold of dimension k of periodic solutions. A solution of this problem is given using the averaging theory.

Let $\mathbf{x}(t,\mathbf{z},\varepsilon)$ be the solution of the differential equation (8) such that $\mathbf{x}(0,\mathbf{z},\varepsilon) = \mathbf{z}$. We write the linearization of the unperturbed differential equation along a periodic solution $\mathbf{x}(t,\mathbf{z},0)$ as

$$\dot{\mathbf{y}} = D_{\mathbf{x}}F_0(t,\mathbf{x}(t,\mathbf{z},0))\mathbf{y}, \tag{9}$$

where $\mathbf{y}$ is an $n\times n$ matrix. In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear periodic differential equation (9), and by $\xi : \mathbb{R}^k\times\mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; i.e. $\xi(x_1,\dots,x_n) = (x_1,\dots,x_k)$.

We assume that there exists a k–dimensional submanifold $\mathcal{Z}$ of Ω filled with T–periodic solutions of (8), which is contained in the subspace $\mathbb{R}^k$ of the first k coordinates of $\mathbb{R}^n$. Then an answer to the problem of bifurcation of T–periodic solutions from the periodic solutions contained in $\mathcal{Z}$ of the differential equation (7) is given in the following result.

Theorem 2. *Let W be an open and bounded subset of $\mathbb{R}^k$, and let $\beta : \mathrm{Cl}(W) \to \mathbb{R}^{n-k}$ be a $\mathcal{C}^2$ function. We assume that*

(i) *$\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha,\beta(\alpha)),\ \alpha\in\mathrm{Cl}(W)\}\subset\Omega$ and that for each $\mathbf{z}_\alpha\in\mathcal{Z}$ the solution $\mathbf{x}(t,\mathbf{z}_\alpha,0)$ of (8) is T–periodic;*

(ii) *for each $\mathbf{z}_\alpha\in\mathcal{Z}$ there is a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (9) such that the matrix $M^{-1}_{\mathbf{z}_\alpha}(0) - M^{-1}_{\mathbf{z}_\alpha}(T)$ has in the upper right corner the $k\times(n-k)$ zero matrix, and in the lower right corner a $(n-k)\times(n-k)$ matrix Δ_α with $\det(\Delta_\alpha)\neq 0$.*

We consider the function $\mathcal{F} : \mathrm{Cl}(W) \to \mathbb{R}^k$

$$\mathcal{F}(\alpha) = \xi \left(\frac{1}{T} \int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha, 0)) dt \right). \tag{10}$$

If there exists $a \in W$ *with* $\mathcal{F}(a) = 0$ *and* $\det((d\mathcal{F}/d\alpha)(a)) \neq 0$, *then there is a* T*–periodic solution* $\mathbf{x}(t, \varepsilon)$ *of differential equation* (7) *such that* $\mathbf{x}(0, \varepsilon) \to \mathbf{z}_a$ *as* $\varepsilon \to 0$.

Theorem 2 goes back to Malkin [8] and Roseau [9], for a shorter and easier proof see [2].

3. Proof of Theorem 1

We will apply the averaging theory described in section 2 for studying the limit cycles of equation (2). More precisely we shall analyze which periodic orbits of equation (3) can be continued to limit cycles of equation (2) with $\varepsilon \neq 0$ sufficiently small.

Now we define the elements of section 2 and of Theorem 2 corresponding to our differential equation (2). Clearly, we have that $\Omega = \mathbb{R}^n$ and $T = 2\pi$.

We write (2) in the form (7) and we get that F_0, F_1 and F_2 are given by

$$\begin{aligned} F_0(t, \mathbf{x}) &= A\mathbf{x}, \\ F_1(t, \mathbf{x}) &= B(t)\mathbf{x} + b(t), \\ F_2(t, \mathbf{x}, \varepsilon) &= 0 \end{aligned}$$

Moreover, we also have that

$$\begin{aligned} k &= 2, \\ \alpha &= (x_1^0, x_2^0), \\ W &= \{\mathbf{x} \in \mathbb{R}^n : 0 < ||\mathbf{x}|| < r\}, \text{ where } r \text{ is arbitrarily large}, \\ \beta(x_1^0, x_2^0) &= (0, \ldots, 0) \in \mathbb{R}^{n-2}, \text{ and} \\ \mathbf{z}_\alpha &= (x_1^0, x_2^0, 0, \ldots, 0), \\ \xi : \mathbb{R}^n \to \mathbb{R}^2 & \text{ is } \xi(x_1, \ldots, x_n) = (x_1, x_2). \end{aligned}$$

The 2π–periodic solutions $\mathbf{x}(t, \mathbf{z}_\alpha, 0)$ of equation (3) with initial conditions $\mathbf{z}_\alpha$ at $t = 0$ are

$$\mathbf{x}(t, \mathbf{z}_\alpha, 0) = (x_1^0 \cos t - x_2^0 \sin t, x_2^0 \cos t + x_1^0 \sin t, 0, \ldots, 0). \tag{11}$$

Note that all these solutions are 2π–periodic. Moreover, these periodic solutions of the unperturbed equation fill out the whole plane $(x_1, x_2, 0, \ldots, 0)$ except the origin.

Since the matrix A is in its real Jordan form and satisfies $(H6)$ it can be written in blocks as

$$A = \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ 0 & 0 & \vdots & \vdots \\ 0 & 0 & \dots & A_r \end{pmatrix}.$$

The blocks A_ℓ for $\ell = 2, \dots, r$ are square matrices whose diagonals are over the diagonal of the matrix A, and these blocks are formed by the following four types of matrices

$$(\lambda), \quad \begin{pmatrix} \lambda & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix}, \tag{12}$$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \begin{pmatrix} a & -b & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ b & a & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & a & -b & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & b & a & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & a & -b \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & b & a \end{pmatrix}, \tag{13}$$

and outside these blocks the matrix A is filled with zeros. Of course these blocks are associated to eigenvalues λ and $a \pm bi$ with $b \neq 0$ of the matrix A. For more information on the real Jordan form of a real matrix A see [5].

The differential equation (9) corresponding to our differential equation (2) is

$$\dot{\mathbf{y}} = A\mathbf{y}, \tag{14}$$

with $\mathbf{y}$ an $n \times n$ matrix. Then, it is well known that the fundamental matrix of (14) is

$$M_{\mathbf{z}}(t) = e^{tA} = \begin{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} & 0 & \dots & 0 \\ 0 & e^{tA_2} & \dots & 0 \\ 0 & 0 & \vdots & \vdots \\ 0 & 0 & \dots & e^{tA_r} \end{pmatrix},$$

satisfying that $M_{\mathbf{z}}(0)$ is the identity matrix of $\mathbb{R}^n$. Of course, we have that

$$M_{\mathbf{z}}^{-1}(t) = e^{-tA} = \begin{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} & 0 & \dots & 0 \\ 0 & e^{-tA_2} & \dots & 0 \\ 0 & 0 & \vdots & \vdots \\ 0 & 0 & \dots & e^{-tA_r} \end{pmatrix}.$$

Therefore

$$M_{\mathbf{z}}^{-1}(0) - M_{\mathbf{z}}^{-1}(2\pi) = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 0 & \dots & 0 \\ 0 & I_2 - e^{-2\pi A_2} & \dots & 0 \\ 0 & 0 & \vdots & \vdots \\ 0 & 0 & \dots & I_r - e^{-2\pi A_r} \end{pmatrix},$$

where I_ℓ is the $n_\ell \times n_\ell$ identity matrix if A_ℓ is an $n_\ell \times n_\ell$ matrix for $\ell = 2, \dots, r$. Note that in the upper right corner $k \times (n-k) = 2 \times (n-2)$ of the previous matrix we have the zero matrix.

Now using the notations of the statement of Theorem 2 we have that

$$\det \Delta_\alpha = \prod_{l=2}^{r} \det(I_\ell - e^{-2\pi A_\ell}).$$

We start by proving that we must have that the determinant of the matrix Δ_α is different from zero.

If the matrix A_ℓ is of one of the forms of the matrices in (12), then we would get that $\det(I_\ell - e^{-2\pi A_\ell}) = 0$ if and only if $\lambda = 0$, but this is not possible because then $\lambda = 0$ would be a rational multiple of i and this contradicts condition $(H4')$.

Now assume that the matrix A_ℓ is of the form of one of the matrices in (13). Then we get that $e^{-2\pi A_\ell}$ is either

$$e^{-2\pi a} \begin{pmatrix} \cos(2\pi b) & \sin(2\pi b) \\ -\sin(2\pi b) & \cos(2\pi b) \end{pmatrix}, \tag{15}$$

or

$$
e^{-2\pi a}\begin{pmatrix}
\cos(2\pi b) & \sin(2\pi b) & \dots & * & * \\
-\sin(2\pi b) & \cos(2\pi b) & \dots & * & * \\
0 & 0 & \dots & * & * \\
0 & 0 & \dots & * & * \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & * & * \\
0 & 0 & \dots & * & * \\
0 & 0 & \dots & \cos(2\pi b) & \sin(2\pi b) \\
0 & 0 & \dots & -\sin(2\pi b) & \cos(2\pi b)
\end{pmatrix}. \tag{16}
$$

Note that in (16) in general the *'s are real numbers, but they do not play any role in the computation of the determinant of the matrix $I_\ell - e^{-2\pi A_\ell}$, so we do not write them explicitly.

If $e^{-2\pi A_\ell}$ is given by (15), then we get that the determinant is given by

$$
\begin{aligned}
\det(I_\ell - e^{-2\pi A_\ell}) &= 1 - 2e^{-2a\pi}\cos(2b\pi) + e^{-4a\pi} \\
&\geq 1 - 2e^{-2a\pi} + e^{-4a\pi} \\
&= (1 - e^{-2a\pi})^2.
\end{aligned}
$$

Thus, we get that $\det(I_\ell - e^{-2\pi A_\ell}) = 0$ if and only if $\cos(2b\pi) = 1$ and $a = 0$, but then we would get that the eigenvalue $a + b\,i$ would be a rational multiple of i and this contradicts $(H4')$.

If $e^{-2\pi A_\ell}$ is given by (16), then there exist a positive integer s such that

$$
\begin{aligned}
\det(I_\ell - e^{-2\pi A_\ell}) &= (1 - 2e^{-2a\pi}\cos(2b\pi) + e^{-4a\pi})^s \\
&\geq (1 - 2e^{-2a\pi} + e^{-4a\pi})^s \\
&= (1 - e^{-2a\pi})^{2s}.
\end{aligned}
$$

Therefore, by the same arguments than in the previous case $\det(I_\ell - e^{-2\pi A_\ell})$ cannot be zero.

In short, we have that $\det \Delta_\alpha \neq 0$. Consequently all the assumptions of Theorem 2 hold, consequently we can apply it to our differential equation (2). Then, we must compute the function $\mathcal{F}(\alpha) = (f_1(x_1^0, x_2^0), f_2(x_1^0, x_2^0))$ defined in (10).

Let m and n non–negative integers. Taking into account that

$$\int_0^{2\pi} \cos(2mt)\sin(2nt)dt = 0 \text{ for all } m \text{ and } n,$$

$$\int_0^{2\pi} \cos(2mt)\cos(2nt)dt = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n, \end{cases}$$

$$\int_0^{2\pi} \sin(2mt)\sin(2nt)dt = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n, \end{cases}$$

a tedious but easy computation shows that

$$\begin{aligned} f_1(x_1^0, x_2^0) = & \frac{1}{4}\big(2(b_2^1 + b_1^2) + (2b_{11}^0 + 2b_{22}^0 + b_{12}^3 + b_{21}^3 + b_{11}^4 - b_{22}^4)x_1^0 \\ & +(2b_{12}^0 - 2b_{21}^0 - b_{11}^3 + b_{22}^3 + b_{12}^4 + b_{21}^4)x_2^0\big), \\ f_2(x_1^0, x_2^0) = & \frac{1}{4}\big(2(b_2^2 - b_1^1) - (2b_{12}^0 - 2b_{21}^0 + b_{11}^3 - b_{22}^3 - b_{12}^4 - b_{21}^4)x_1^0 \\ & +(2b_{11}^0 + 2b_{22}^0 - b_{12}^3 - b_{21}^3 - b_{11}^4 + b_{22}^4)x_2^0\big). \end{aligned}$$

Then, the system $f_1(x_1^0, x_2^0) = f_2(x_1^0, x_2^0) = 0$ is a linear system in the variables x_1^0 and x_2^0 whose determinant is Δ as defined in (5), and by assumption we have that $\Delta \neq 0$ (see this assumption in the statement of Theorem 1). Therefore, it is immediate to check that the unique solution of this linear system is

$$(x_1^0, x_2^0) = \left(\frac{X}{\Delta}, \frac{Y}{\Delta}\right),$$

where X and Y are defined in (5). Note that since $(X, Y) \neq (0, 0)$ (because by assumptions $X^2 + Y^2 \neq 0$), the previous solution provides the initial conditions of the periodic solution (11) or equivalently (6) of the unperturbed equation (3) which can be continued to a limit cycle of the perturbed equation (2) with $\varepsilon \neq 0$, because the Jacobian

$$\det\left(\left.\frac{\partial(f_1, f_2)}{\partial(x_1^0, x_2^0)}\right|_{(x_1^0,x_2^0)=\left(\frac{X}{\Delta},\frac{Y}{\Delta}\right)}\right) = -\frac{\Delta}{16} \neq 0.$$

Consequently, the rest of the proof of Theorem 1 follows immediately from the statement of Theorem 2.

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