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LOCAL DARBOUX FIRST INTEGRALS OF ANALYTIC DIFFERENTIAL SYSTEMS

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Abstract. In this paper we discuss local and formal Darboux first integrals of analytic differential systems, using the theory of Poincaré-Dulac normal forms. We study the effect of local Darboux integrability on analytic normalization. Moreover we determine local restrictions on classical Darboux integrability of polynomial systems.

1. Introduction

Classically, Darboux integrability was introduced for planar polynomial vector fields, but the notion directly carries over to polynomial vector fields in higher dimension. Thus, we say that a differential equation

$$\dot{x} = P(x) := \begin{pmatrix} P_1(x) \\ \vdots \\ P_n(x) \end{pmatrix} \tag{1}$$

in $\mathbb{C}^n$ with polynomial right-hand side admits a Darboux first integral if there exist pairwise relatively prime polynomials $\phi_1, \ldots, \phi_r$ and nonzero complex numbers $d_1, \ldots, d_r$ such that the identity

$$X_P\left(\phi_1^{d_1} \cdots \phi_r^{d_r}\right) = 0$$

holds. (Here, as usual, $X_P = \sum P_i \frac{\partial}{\partial x_i}$ is the derivation corresponding to P, and the Lie bracket of vector fields P and Q is defined via $X_{[P,Q]} = X_P X_Q - X_Q X_P$.) The existence of a Darboux first integral implies that each ϕ_i is a semi-invariant (also called Darboux polynomial) of P, thus

$$X_P(\phi_i) = L_i \cdot \phi_i, \quad 1 \le i \le r$$

with polynomials L_i (called the cofactors of the ϕ_i); equivalently each zero set of ϕ_i is an invariant set for $\dot{x} = P(x)$. Given semi-invariants ϕ_i with cofactors L_i, existence of a Darboux first integral is equivalent to the identity

$$d_1 \cdot L_1 + \cdots + d_r \cdot L_r = 0.$$

For details and further results we refer to the survey [9] of Darboux integrability for polynomial systems.

In the present paper we will discuss an extension of Darboux integrability to (germs of) complex local analytic vector fields, and to formal vector fields on $\mathbb{C}^n$. Due to the straightening theorem, the existence of local Darboux first integrals is guaranteed near any non-stationary point; therefore we will

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focus on stationary points in the present paper. A fundamental tool in the investigation will be the theory of Poincaré-Dulac normal forms. For systems in normal form with non-nilpotent linear part one obtains precise restrictions on the local analytic or formal functions involved, as well as the possible exponents. Indeed, every Darboux first integral of a system in normal form is also a first integral of the semisimple part of its linearization. Since every formal differential system is formally equivalent to a system in normal form, we obtain Theorem 1 for general formal vector fields. There are, on the one hand, applications to analytic normalizability of analytic systems. In particular A.D. Bruno's "Condition A" [4] is satisfied if and only if there exist $n-1$ independent Darboux first integrals, and we obtain convergence criteria as well as a transparent proof for an upper bound for the number of independent formally meromorphic first integrals (Theorems 2 and 3). On the other hand the local results are applicable to the classical question of Darboux integrability for polynomial vector field. Via local obstructions near stationary points they provide easily applicable criteria for nonexistence of classical Darboux first integrals.

2. Definitions and basics

We first review some properties of formal and convergent power series rings. Details and proofs can be found, for instance, in the monograph by Zariski and Samuel [13], Chapter VII. We denote by $\mathbb{C}[[x_1, \ldots, x_n]]$ the ring of formal power series in n variables. Its elements can be represented in the form

$$\psi = \sum_{k\geq 0} \psi^{(k)}$$

with each $\psi^{(k)}$ a homogeneous polynomial in $x_1, \ldots, x_n$ of degree k. If $\psi \neq 0$ then we call the smallest ℓ with $\psi^{(\ell)} \neq 0$ the *order* of ψ, and denote this number by $o(\psi)$. A formal power series ψ is (multiplicatively) invertible in $\mathbb{C}[[x_1, \ldots, x_n]]$ if and only if $\psi^{(0)} \neq 0$, thus its order equals 0. There is a unique maximal ideal $\mathbf{m} := \{\phi : \phi^{(0)} = 0\}$. Moreover, the ring of formal power series is a unique factorization domain, thus every nonzero non-invertible series is a product of irreducible ones, and this representation is unique up to the ordering of factors and multiplication by invertible series. Mutatis mutandis, these properties carry over to the subring $\mathbb{C}[[x_1, \ldots, x_n]]_c$ of power series that converge in some neighborhood of 0, which may be identified with the ring of (germs of) analytic functions in 0.
Following Shafarevich [10], we call the quotient field $\mathbb{C}((x_1, \ldots, x_n))_c$ of $\mathbb{C}[[x_1, \ldots, x_n]]_c$ the field of (local) meromorphic functions. For the quotient field $\mathbb{C}((x_1, \ldots, x_n))$ of $\mathbb{C}[[x_1, \ldots, x_n]]$ no name seems to be in general use; for the present paper we adopt the name "field of formally meromorphic functions". (In Reference [6] the term "generalized rational functions" was used.)

Finally, we consider on $\mathbb{C}^n$ local analytic or formal differential systems of the form

$$\dot{x} = F(x) = Ax + \sum_{k\geq 2} f^{(k)}(x) \tag{2}$$

with a linear map $A = DF(0)$, and every $f^{(k)}$ a homogeneous vector valued polynomial of degree k. The decomposition

$$A = A_s + A_n, \quad [A_s, A_n] = 0 \tag{3}$$

with semisimple A_s and nilpotent A_n, will be of some relevance. Moreover, we denote the eigenvalues of A (hence of A_s) by

$$\lambda_1, \ldots, \lambda_n \tag{4}$$

counting each eigenvalue according to multiplicity. If A_s is represented by a diagonal matrix then we will tacitly assume that the λ_i in this order are the diagonal entries. To F we associate the Lie derivative X_F, which acts on $\mathbb{C}[[x_1, \ldots, x_n]]$, resp. on $\mathbb{C}[[x_1, \ldots, x_n]]_c$, and the actions extend to the quotient fields.

Definition 1. *Let $\psi \in \mathbb{C}[[x_1, \ldots, x_n]] \setminus \{0\}$ be non-invertible. Then ψ is called a formal semi-invariant (or formal Darboux function) of F (or of* (2)*) if there is a formal power series K (called cofactor) such that*

$$X_F(\psi) = K \cdot \psi.$$

If F, ψ and K are convergent in some neighborhood of 0 *then we will call ψ a local analytic semi-invariant (or local analytic Darboux function).*

Remark 1. (a) Due to the unique factorization property of power series rings, every irreducible factor of a semi-invariant is a semi-invariant. This is proven just as in the polynomial case.
(b) By the Hilbert-Rückert Nullstellensatz, a non-invertible analytic function is a semi-invariant of F if and only if its zero set is invariant for (2). This is proven analogously to the polynomial case.
(c) Every invertible series would automatically satisfy the defining identity for semi-invariants; but in the analytic case the local zero set of an invertible function is empty, hence invertible functions are of no geometric interest in view of (b) above. It is important to notice, however, that multiplying a semi-invariant by an invertible series will yield a semi-invariant.

Now that the notions of semi-invariant have been carried over to power series rings and vector fields, we can proceed to define Darboux integrability. An expression like ψ^d, given a formal power series ψ and a complex number d, makes sense in a suitable differential field extension of $\mathbb{C}((x_1, \ldots, x_n))$, and there is a canonical extension of X_F to such extension fields. If ψ is invertible, $\psi^{(0)} = 1$ with no loss of generality, then $\psi^d \in \mathbb{C}[[x_1, \ldots, x_n]]$ in view of the binomial series expansion

$$(1 + (\psi - 1))^d = \sum_{k \geq 0} \binom{d}{k} (\psi - 1)^k$$

which converges with respect to the $\mathbf{m}$-adic topology on the ring of formal power series. (See Zariski and Samuel [13], Chapter VIII, Section 2. The $\mathbf{m}$-adic topology is induced by the norm $||\psi|| := 2^{-o(\psi)}$ for nonzero ψ. Thus a series $\sum \gamma_k$ of formal power series converges in the $\mathbf{m}$-adic topology if and only if the orders of the γ_k tend to infinity as $k \to \infty$.) In the local analytic case one may proceed without differential field extensions, since one can always reduce the matter to powers in $\mathbb{C}$.

Definition 2. *System* (2) *admits a formal (resp. local) Darboux first integral if there are formal (resp. local analytic) semi-invariants* $\psi_1, \ldots, \psi_r$ *and nonzero complex numbers* $d_1, \ldots, d_r$ *such that*

$$X_F\left(\psi_1^{d_1} \cdots \psi_r^{d_r}\right) = 0.$$

Moreover, one then calls $\psi_1^{d_1} \cdots \psi_r^{d_r}$ *a Darboux first integral of* F*. If all the* d_i *are integers then we speak of a formally meromorphic (resp. meromorphic) first integral.*

Just as for the polynomial case, one may equivalently express the first integral condition via cofactors. If K_i is the cofactor of ψ_i, respectively, then the system admits the Darboux first integral $\psi_1^{d_1} \cdots \psi_r^{d_r}$ if and only if

$$d_1 \cdot K_1 + \cdots + d_r \cdot K_r = 0.$$

3. Darboux first integrals for normal forms

Following the definition given e.g. in [11] we say that system (2) is in *Poincaré-Dulac normal form* if

$$[A_s, F] = 0.$$

Up to a linear transformation this is equivalent to the following coordinate-dependent definition (see e.g. Bruno [4], Bibikov [1]): Assume that A is in Jordan canonical form (thus A_s is diagonal), with eigenvalues $\lambda_1, \ldots, \lambda_n$. Denote by $(e_1, \ldots, e_n)$ the standard basis of $\mathbb{C}^n$ and let $x_1, \ldots, x_n$ be the corresponding coordinates. Call a vector monomial $x_1^{m_1} \cdots x_n^{m_n} e_j$ resonant if $m_1\lambda_1 + \cdots + m_n\lambda_n - \lambda_j = 0$. Then system (2) is in normal form if and only if every $f^{(k)}$ is a linear combination of resonant vector monomials.

Poincaré-Dulac normal forms have been used to establish necessary conditions for the existence of analytic or formal first integrals [7, 8], and for the existence of formally meromorphic first integrals [6]. Applications include results on analytic normalization for analytic systems; see [14, 15] and [8]

We now extend the normal form approach to semi-invariants and Darboux integrability. We first note a few facts. See e.g. the references [4], [1] for the well-known first statement in the following Proposition. The second statement is straightforward.

Proposition 1. *For any formal system* (2) *there exists a "near-identity" formal transformation*

$$\Phi(x) = x + \text{ higher order terms}$$

and a system

$$\dot{x} = F^*(x) = Ax + \text{ higher order terms}$$

in Poincaré-Dulac normal form, such that

$$D\Phi(x)F^*(x) = F(\Phi(x))$$

holds for all x*. A series* ψ *is a semi-invariant of* F*, with cofactor* K*, if and only if* $\psi \circ \Phi$ *is a semi-invariant of* F^**, with cofactor* $K \circ \Phi$*.*

Therefore, regarding formal semi-invariants and Darboux integrability we may assume that the vector field is in normal form. The advantage of normal forms lies in greater transparency when it comes to investigating semi-invariants.

Definition 3. *Let F be in Poincaré-Dulac normal form. A semi-invariant ψ of F, with $X_F(\psi) = K \cdot \psi$, is called* distinguished *if*

$$X_{A_s}(\psi) = K(0) \cdot \psi \text{ and } X_{A_s}(K) = 0.$$

The following was shown in [12], Lemma 2.2.

Lemma 1. *Assume that F is in normal form, and let ϕ be a formal semi-invariant of F. Then there is an invertible series ρ and a distinguished semi-invariant ψ of F such that $\phi = \rho \cdot \psi$. In particular, every semi-invariant of F is also a semi-invariant of A_s.*

Note that Lemma 1 applies in particular to the semi-invariants of A_s. We will require a less restrictive characterization of distinguished semi-invariants for linear systems.

Lemma 2. *Let ψ be a semi-invariant of $\dot{x} = A_s x$, with cofactor K, such that $X_{A_s}(K) = 0$. Then K is constant.*

Proof. For any integer $m > 0$ the restriction of X_{A_s} to the space of homogeneous polynomials of degree m is a semisimple linear map; see [11]. Let $\psi^{(\ell)}$ be the smallest degree nonzero term in ψ and write the representation with homogeneous polynomials as

$$\psi = \sum_{j \geq 0} \psi^{(\ell+j)};$$

likewise, let

$$K = \sum_{i \geq 0} k^{(i)}$$

with each $k^{(i)}$ homogeneous of degree i. For degree $\ell + j$ the semi-invariance condition implies

$$X_{A_s}\left(\psi^{(\ell+j)}\right) = \sum_{i=0}^{j} k^{(i)} \psi^{(\ell+j-i)}.$$

We now prove $X_{A_s}\left(\psi^{(\ell+j)}\right) = k^{(0)}\psi^{(\ell+j)}$ by induction on j. The start $j = 0$ is immediate. Going from $j-1$ to j, and using the induction hypothesis $X_{A_s}\left(\psi^{(\ell+j-i)}\right) = k^{(0)}\psi^{(\ell+j-i)}$ as well as the fact that $X_{A_s}(k^{(i)}) = 0$ for $1 \leq i \leq j$, we see that

$$X_{A_s}\left(\psi^{(\ell+j)}\right) - k^{(0)}\psi^{(\ell+j)} = \sum_{i \geq 1} k^{(i)} \psi^{(\ell+j-i)}$$

lies in the eigenspace of X_{A_s} corresponding to the eigenvalue $k^{(0)}$. On the other hand one has the eigenspace decomposition

$$\psi^{(\ell+j)} = \sum_{\beta} \psi_{\beta}^{(\ell+j)}$$

with the sum extending over the eigenvalues of X_{A_s}. Applying X_{A_s} shows that $X_{A_s}\left(\psi^{(\ell+j)}\right) - k^{(0)}\psi^{(\ell+j)}$ lies in a sum of eigenspaces for eigenvalues different from $k^{(0)}$. Since one has a direct sum decomposition, this implies

$$X_{A_s}\left(\psi^{(\ell+j)}\right) - k^{(0)}\psi^{(\ell+j)} = 0,$$

completing the induction step. □

We next restate a result from [11] on eigenspaces of X_{A_s}.

Lemma 3. (a) *Assume that A_s is represented by a diagonal matrix with respect to the standard basis. Then any homogeneous distinguished semi-invariant ψ of $\dot{x} = A_s x$ is a linear combination of monomials, and the cofactor is equal to*

$$m_1\lambda_1 + \cdots + m_n\lambda_n,$$

if the monomial $x_1^{m_1}\cdots x_n^{m_n}$ appears in ψ with nonzero coefficient.
(b) *For arbitrary semisimple A_s, every cofactor of a distinguished semi-invariant is a linear combination of the eigenvalues with nonnegative integers as coefficients.*

With regard to Darboux first integrals, irreducible analytic and formal semi-invariants are of particular importance. The following (well-known) examples show that their number (up to multiplication by invertible series) may be finite or infinite, depending on the eigenvalues.

Examples. (a) If the eigenvalues $\lambda_1, \ldots, \lambda_n$ are linearly independent over the rationals $\mathbb{Q}$ then the only irreducible distinguished semi-invariants of $A = A_s$ are the coordinate functions $x_1, \ldots, x_n$. This follows directly from the fact that two different monomials $x_1^{k_1}\cdots x_n^{k_n}$ and $x_1^{\ell_1}\cdots x_n^{\ell_n}$ have different cofactors $k_1\lambda_1 + \cdots + k_n\lambda_n$ resp. $\ell_1\lambda_1 + \cdots + \ell_n\lambda_n$. In this case any Poincaré-Dulac normal form is necessarily linear.
(b) If $n = 2m$, $\omega_1, \ldots, \omega_m$ are linearly independent over $\mathbb{Q}$ and $\lambda_{2j-1} = \omega_j$, $\lambda_{2j} = -\omega_j$ then the only irreducible distinguished semi-invariants of $A = A_s$ are the coordinate functions $x_1, \ldots, x_n$. This follows from the fact that every monomial has a unique representation

$$x_1^{k_1}\cdots x_n^{k_n} = x_1^{v_1}\cdots x_n^{v_n}\cdot (x_1x_2)^{u_1}\cdots(x_{n-1}x_n)^{u_m}$$

with $v_{2j-1} = 0$ or $v_{2j} = 0$ for all j, and the cofactor $\sum_{j=1}^{m}(v_{2j-1} - v_{2j})\,\omega_j$ uniquely determines the v_i. Note that in this case there exist nontrivial Poincaré-Dulac normal forms.
(c) In case $n = 2$, $\lambda_2 = 2\cdot\lambda_1 \neq 0$, for every $\beta \in \mathbb{C}$ the semi-invariant $\psi_\beta := x_2 + \beta x_1^2$ is irreducible, and different choices of β yield relatively prime power series. Thus there exist infinitely many irreducible semi-invariants for the cofactor λ_2, and for each choice of pairwise distinct constants $\beta_1, \ldots, \beta_s$ and nonzero constants $e_1, \ldots, e_s$ with $\sum e_j = 0$ one has a Darboux first integral

$$(x_2 + \beta_1 x_1)^{e_1}\cdots(x_2 + \beta_s x_1)^{e_s}.$$

See [11], Theorem 3.2 for further details in dimension 2.

The following is the principal technical result on semi-invariants of normal forms.

Proposition 2. *Assume that F is in normal form, and let $\phi_1^{d_1}\cdots\phi_r^{d_r}$ be a Darboux first integral of* (2). *Then there exist distinguished semi-invariants $\psi_1,\ldots,\psi_r$ such that*

$$\psi_1^{d_1}\cdots\psi_r^{d_r}=\phi_1^{d_1}\cdots\phi_r^{d_r}.$$

In particular the ψ_i are also semi-invariants of A_s and $\psi_1^{d_1}\cdots\psi_r^{d_r}$ is a Darboux first integral of $\dot{x}=A_sx$.

Proof. By Lemma 1, for $i=1,\ldots,r-1$ there exist invertible formal series ρ_i such that $\psi_i(x)=\rho_i^{-1}(x)\phi_i(x)$ are distinguished semi–invariants of system (2). Let $L_i(x)$ be the corresponding cofactor associated with $\psi_i(x)$, i.e. $X_F(\psi_i)(x)=L_i(x)\psi_i(x)$ for $i=1,\ldots,r-1$. Then $X_{A_s}(L_i)=0$ for $1\le i\le r-1$. Define

$$\psi_r=\rho_1^{d_1/d_r}\ldots\rho_{r-1}^{d_{r-1}/d_r}\phi_r,$$

and note that the product of the $\rho_i^{d_i/d_r}$ is an invertible formal power series. Therefore ψ_r is a semi-invariant of F; let L_r be its cofactor. Then the construction of the $\psi_i(x)$ shows that

$$\psi_1^{d_1}\ldots\psi_r^{d_r}=\phi_1^{d_1}\ldots\phi_r^{d_r}$$

and therefore $X_F\left(\psi_1^{d_1}\ldots\psi_r^{d_r}\right)=0$, whence

$$d_1\cdot L_1+\cdots+d_r\cdot L_r=0.$$

Applying X_{A_s} to this relation, one finds $X_{A_s}(L_r)=0$. In order to finish the proof, we still need to show that $X_{A_s}(\psi_r)$ is a constant multiple of ψ_r. By Lemma 1 there exists a series K such that $X_{A_s}(\psi_r)=K\cdot\psi_r$. Therefore one has

$$X_F(K)\cdot\psi_r+KL_r\cdot\psi_r=X_FX_{A_s}(\psi_r)=X_{A_s}X_F(\psi_r)=L_rK\cdot\psi_r,$$

since X_{A_s} and X_F commute by the normal form property, and $X_{A_s}(L_r)=0$. Therefore $X_F(K)=0$ and, by [11], Prop. 1.8, this implies $X_{A_s}(K)=0$. Now Lemma 2 shows that K is constant, which implies $K=L_r(0)$.

As a final step, we notice that $\sum d_iL_i=0$ implies $\sum d_iL_i(0)=0$, whence $\psi_1^{d_1}\cdots\psi_r^{d_r}$ is a first integral of $\dot{x}=A_sx$. □

Our results are applicable to general formal vector fields, as follows:

Theorem 1. *Let system* (2) *be given, with F not necessarily in normal form, and denote by $\lambda_1,\ldots,\lambda_n$ the eigenvalues of $A=DF(0)$.*
(a) *If ψ is a semi-invariant of F, with cofactor K, and θ is the lowest-order nonvanishing homogeneous term of ψ, then θ is a semi-invariant of A_s, and there exist nonnegative integers m_k such that*

$$K(0)=\sum_k m_k\lambda_k,\quad \sum_k m_k>0.$$

(b) *If $\psi_1^{d_1}\cdots\psi_r^{d_r}$ is a Darboux first integral of $\dot{x}=F(x)$ and θ_j is the lowest-order nonvanishing homogeneous term of ψ_j, then $\theta_1^{d_1}\cdots\theta_r^{d_r}$ is a Darboux first integral of $\dot{x}=A_sx$.*

Proof. If F is in Poincaré-Dulac normal form then all assertions follow from Proposition 2 and Lemma 3. Otherwise, by Proposition 1 there exists a near-identity transformation to normal form, and evaluating the lowest-order terms shows the assertions. □

The following consequence of Proposition 2 is of independent interest. In particular, the first statement generalizes Lemma 7 in [6]. The proof is immediate from the prime factorization of numerator and denominator of a formally meromorphic function.

Corollary 1. *Assume that F is in normal form, and let ρ be a formally meromorphic first integral of* (2)*. Then there exist distinguished semi-invariants ψ_1, ψ_2 and a constant β such that*

$$\rho = \frac{\psi_1}{\psi_2}, \quad X_{A_s}(\psi_1) = \beta\psi_1,\ X_{A_s}(\psi_2) = \beta\psi_2;$$

whence ψ_1/ψ_2 is also a formally meromorphic first integral of $\dot{x} = A_s x$. Thus the field of formally meromorphic first integrals of $\dot{x} = F(x)$ is a subfield of the field of formally meromorphic first integrals of $\dot{x} = A_s x$.

4. Applications

4.1. **Integrability and normalizability.** Our first application is with regard to analytic normalization of an analytic system (2). While there always exists a formal transformation to Poincaré-Dulac normal form, the existence of a transformation that converges in some neighborhood of 0 is a highly nontrivial problem. Building on and extending work of Siegel, Pliss and others, Bruno [2, 3] identified two conditions relevant for convergent questions. (See also the monograph [4] for a streamlined presentation.) One of them, called "Condition ω", is a Diophantine condition on the eigenvalues of A. The other (we refer specifically to the version from [4], p. 203) is as follows: *Condition A*: Some formal Poincaré-Dulac normal form of system (2) has the form

$$F^*(x) = (1 + \mu(x))A_s x$$

with a formal series μ such that $\mu(0) = 0$.

Bruno's convergence theorem states: If an analytic system (2) satisfies Conditions ω and A, then it is analytically equivalent to a normal form. The following theorem shows the relation to formal Darboux integrability, and generalizes results from [14, 15] and [8].

Theorem 2. (a) *The vector field* (2) *on $\mathbb{C}^n$, with $A_s \neq 0$, admits $n-1$ independent formal Darboux first integrals if and only if it satisfies Bruno's Condition A. Therefore, Bruno's arithmetic Condition ω on the eigenvalues of A in conjunction with the existence of $n-1$ independent formal Darboux first integrals guarantees convergence.*
(b) *If system* (2) *is analytic and the equation admits $n-1$ independent formally meromorphic first integrals then a convergent transformation to Poincaré-Dulac normal form exists.*

Proof. (i) First preliminary observation: If $\phi_1, \ldots, \phi_r$ are formal series, $d_1, \ldots, d_r$ are complex constants and $\theta := \phi_1^{d_1} \cdots \phi_r^{d_r}$ then obviously

$$D\theta(x) = \theta(x) \cdot (\rho_1(x), \ldots, \rho_n(x))$$

with formally meromorphic entries ρ_j.
(ii) Second preliminary observation: If formal vector fields G and H admit the same $n-1$ independent formal Darboux first integrals $\gamma_1, \ldots, \gamma_{n-1}$ then there is a formally meromorphic function ν such that $H = \nu \cdot G$. To see this, let $\Gamma := (\gamma_1, \ldots, \gamma_{n-1})^{\mathrm{tr}}$ (the exponent "tr" denoting the transpose). By (i) and elementary row operations, the system

$$D\Gamma(x)Q(x) = 0$$

for a formally meromorphic vector field Q, is equivalent to a system

$$R(x)Q(x) = 0$$

with formally meromorphic entries of R. Since the rank of $R(x)$ is equal to $\operatorname{rank} D\Gamma(x) = n-1$, the system has a one-dimensional solution space. Thus any two solutions are linearly dependent over $\mathbb{C}((x_1, \ldots, x_n))$.
(iii) Recall that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A_s; see (4). To prove sufficiency in part (a), use Lemma 3 and note that the linear equation $\sum m_i\lambda_i = 0$ admits $n-1$ independent solutions in $\mathbb{C}^n$. To prove necessity, we may assume that F is in normal form. Let $\gamma_1, \ldots, \gamma_{n-1}$ be independent Darboux first integrals of F. By Proposition 2 these are also first integrals for A_s, and by (i) we see that

$$Ax + \sum_k f^{(k)}(x) = F(x) = \rho(x) \cdot A_s x$$

with some formally meromorphic ρ. From this $\rho = 1 + \cdots$ follows as in [8], Theorem 9.
(iv) Similar to the proof of [15], Theorem 1.3 or [8], Theorem 9, one sees that there exist some nonzero constant α and integers λ_j^* such that $\lambda_j = \alpha \cdot \lambda_j^*$ holds for every eigenvalue of A. This ensures that Condition ω is satisfied, and (b) follows.

□

4.2. **Meromorphic first integrals.** The following result was first stated and proved by Cong, Llibre and Zhang [6], Theorem 1. We will provide a shorter and less technically involved proof here.

Theorem 3. *The number of functionally independent formally meromorphic first integrals of system* (2) *is less than or equal to the rank of the* $\mathbb{Z}$*-module*

$$\Delta := \{(r_1, \ldots, r_n) \in \mathbb{Z}^n : \sum r_i\lambda_i = 0\}.$$

Alternative proof. (i) We may assume that F is in normal form, and therefore by Corollary 1 it is sufficient to show that the field of formally meromorphic first integrals of $\dot{x} = A_s x$ contains at most $\operatorname{rank}\Delta$ independent elements. Assume that $\rho_1, \ldots, \rho_m$ is a maximal independent set of formally meromorphic first integrals of A_s. Then we may assume $\rho_i = \tilde{\psi}_i/\theta_i$ with distinguished semi-invariants $\tilde{\psi}_i$ and θ_i, and $X_{A_s}(\tilde{\psi}_i) = \beta_i\tilde{\psi}_i$ as well as $X_{A_s}(\theta_i) = \beta_i\theta_i$ with some constant β_i, for $1 \leq i \leq m$; see Proposition 2. By taking e.g. the product of the denominators, we may assume that $\rho_i = \psi_i/\theta$ with a common denominator, and

$$X_{A_s}(\theta) = \beta\theta, \quad X_{A_s}(\psi_i) = \beta\psi_i, \quad 1 \le i \le m,$$

for some $\beta \in \mathbb{C}$.
(ii) Let I_0 denote the set of all polynomial first integrals of A_s, and let I_β denote the set of all polynomials ψ such that $X_{A_s}(\psi) = \beta\psi$. Then it is known from [11], Proposition 1.6 and its proof that I_0 is a finitely generated algebra and that I_β is a finitely generated module over I_0.
For reasons of convenience we will from now on assume that A_s is represented by a diagonal matrix. Then every element of I_0 is a linear combination of monomials in I_0, and a monomial $x_1^{k_1} \cdots x_n^{k_n}$ lies in I_0 if and only if $\sum_i k_i\lambda_i = 0$. Likewise, every element of I_β is a linear combination of monomials in I_β, and a monomial lies in I_β if and only if its exponents satisfy $\sum_i k_i\lambda_i = \beta$. Let $\mu_1, \ldots, \mu_s$ be a set of monomial generators for the algebra I_0, and $\nu_1, \ldots, \nu_t$ a set of monomial generators for the module I_β. Moreover, assume that a maximal independent set among the μ_i has cardinality q. Then there exist $\nu_1, \ldots, \nu_r$ such that any maximal set of independent functions among $\mu_1, \ldots, \mu_s, \nu_1, \ldots, \nu_t$ contains $q + r$ elements. Any maximal set of independent functions among the Laurent polynomials

$$\mu_1, \ldots, \mu_s, \frac{\nu_2}{\nu_1}, \ldots, \frac{\nu_t}{\nu_1}$$

then has cardinality $q + r - 1$, and the vector of exponents of any such Laurent polynomial lies in the module Δ. Since algebraic independence of Laurent polynomials is equivalent to linear independence of their exponent vectors, we see that $q + r - 1 \le \operatorname{rank} \Delta$.
(iii) Returning to formal power series, we may represent every series ψ such that $X_{A_s}(\psi) = \beta\psi$ in the form

$$\psi = \sum_{j=1}^{t} \phi_j(\mu_1 \ldots, \mu_s)\nu_j$$

with formal power series ϕ_j in s variables. This holds because every homogeneous part admits a representation of this kind. Therefore

$$\frac{\psi}{\nu_1} = \sum_{j=1}^{t} \phi_j(\mu_1 \ldots, \mu_s)\frac{\nu_j}{\nu_1}$$

With

$$R := \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_m \end{pmatrix}, \quad \rho_j = \frac{\psi_j/\nu_1}{\theta/\nu_1}, \quad S := \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_s \\ \nu_2/\nu_1 \\ \vdots \\ \nu_t/\nu_1 \end{pmatrix},$$

one obtains an identity

$$R = \sum_{j=1}^{t} M_j(\mu_1, \ldots, \mu_s)\frac{\nu_j}{\nu_1} =: \Theta(S),$$

where each M_j has entries in $\mathbb{C}[[y_1, \dots, y_s]]$ with indeterminates $y_1, \dots, y_s$. Note furthermore that Θ is a formal power series in $s+t-1$ variables. We obtain by the chain rule that

$$DR(x) = D\Theta(S(x))DS(x),$$

and by linear algebra $DR(x)$ has rank less than or equal to that of $DS(x)$, which in turn is less than or equal to the rank of Δ. This concludes the proof. □

Remark 2. (a) Actually, one has $q+r-1 = \operatorname{rank}\Delta$, but we did not require this fact in the proof above.
(b) A formally meromorphic first integral of a system in normal form can be written as an infinite series of Laurent polynomials with increasing total degree (see [6]), with each summand a first integral of A_s. But passing to the infinite series would amount to interchanging X_{A_s} with a limiting process. For this reason, we chose the proof variant in part (iii).

4.3. **Classical Darboux integrability.** We now return to polynomial systems and investigate local conditions for Darboux integrability.

Proposition 3. *Assume that the polynomial system* (1) *admits a Darboux first integral*

$$\phi_1^{d_1} \cdots \phi_r^{d_r}$$

with pairwise relatively prime irreducible polynomials ϕ_j and nonzero constants d_j. Let z be a stationary point of P, denote the eigenvalues of $B := DP(z)$ by $\mu_1, \dots, \mu_n$, and let $B = B_s + B_n$ be the semisimple-nilpotent decomposition.
(a) *If P admits no nonconstant analytic first integral in z then $\phi_j(z) = 0$ for some j.*
(b) *Assume that $q \geq 1$, $\phi_1(z) = \cdots = \phi_q(z) = 0$ and $\phi_k(z) \neq 0$ for all $k > q$. Let $\theta_j(x_1 - z_1, \dots, x_n - z_n)$ be the lowest order term of $\phi_j(x_1 - z_1, \dots, x_n - z_n)$, $1 \leq j \leq q$. Then there are nonnegative integers ℓ_{jk} such that*

$$X_{B_s}(\theta_j) = \left(\sum_k \ell_{jk} \mu_k \right) \cdot \theta_j, \quad 1 \leq j \leq q,$$

and $\theta_1^{d_1} \cdots \theta_q^{d_q}$ is a first integral of $\dot{x} = B_s x$.

Proof. (a) If all $\phi_j(z) \neq 0$ then $\phi_1^{d_1} \cdots \phi_r^{d_r}$ is analytic in z.
(b) The function $\rho := \phi_{q+1}^{d_{q+1}} \cdots \phi_r^{d_r}$ is analytic in z with $\rho(z) \neq 0$, hence there is an analytic function σ in some neighborhood of z with $\sigma^{d_q} = \rho$. Now Theorem 1 is applicable to the local Darboux first integral

$$\phi_1^{d_1} \cdots \phi_{q-1}^{d_{q-1}} (\phi_q \sigma)^{d_q} \text{ of } P.$$

□

Since formal Darboux first integrals do not necessarily exist at a stationary point, Proposition 3 may impose local obstructions to the existence of Darboux first integrals built from Darboux polynomials. In particular we note:

Corollary 2. *If P admits $n-1$ independent Darboux first integrals then Bruno's Condition A is satisfied at every stationary point of P.*

Remark 3. Recall that (e.g. by Lemma 1 for cofactor 0 and Lemma 3) there is a nonconstant local analytic first integral in the stationary point z only if there are nonnegative integers $m_1, \ldots, m_n$ such that

$$m_1\mu_1 + \cdots + m_n\mu_n = 0, \quad m_1 + \cdots + m_n > 0.$$

Example. The polynomial equation

$$\begin{array}{rcl} \dot{x}_1 &=& x_1 + \alpha_1 x_1^2 + \alpha_2 x_1 x_2 + \alpha_3 x_2^2 + \cdots \\ \dot{x}_2 &=& 2x_2 + \beta_1 x_1^2 + \beta_2 x_1 x_2 + \beta_3 x_2^2 + \cdots \end{array}$$

(with the dots symbolizing higher order terms) admits no Darboux first integral whenever $\beta_1 \neq 0$.

To verify this, note that a Poincaré-Dulac normal form of the equation can be read off directly, and is equal to

$$\begin{array}{rcl} \dot{x}_1 &=& x_1 \\ \dot{x}_2 &=& 2x_2 + \beta_1 x_1^2. \end{array}$$

By Corollary 2 there exists a formal Darboux first integral only if $\beta_1 = 0$. By Remark 3 there is no local analytic first integral in 0.

Example. Given polynomials α_i and β_i in two variables, the vector field

$$P = \left(\begin{array}{ccccc} -\alpha_1 \cdot (x_2 + x_1^2 - 1) & - & \alpha_2 \cdot x_2 & + & \beta_1 \cdot x_2(x_2 + x_1^2 - 1) \\ & & 2\alpha_2 \cdot x_1 x_2 & + & \beta_2 \cdot x_2(x_2 + x_1^2 - 1) \end{array} \right)$$

admits the polynomial semi-invariants $\phi_1 = x_2$ and $\phi_2 = x_2 + x_1^2 - 1$, and this is the most general polynomial vector field to admit these two semi-invariants; see [5], Theorem 3.8. The vector field admits the two stationary points $(\pm 1,\, 0)$ with eigenvalues $\lambda_1 = -2\alpha_1(1,0)$, $\lambda_2 = 2\alpha_2(1,0)$, resp. $\mu_1 = 2\alpha_1(-1,0)$, $\mu_2 = -2\alpha_2(-1,0)$, as a straightforward computation shows. Moreover the eigenspaces for the eigenvalues λ_2 and μ_2 are tangent to $\phi_1 = 0$. The semi-invariants ϕ_1 and ϕ_2 remain irreducible locally at $(\pm 1,\, 0)$ because they have order one. Now assume that not both λ_i are zero, and that their ratio is not a positive rational number; and assume the same for the μ_i. Then [11], Theorem 3.2 shows that there are just two irreducible local semi-invariants (viz., ϕ_1 and ϕ_2) at both stationary points. Thus, if there exists a Darboux first integral

$$\phi_1^{d_1}\phi_2^{d_2}\phi_3^{d_3}\cdots\phi_r^{d_r}$$

with irreducible, pairwise relatively prime polynomials ϕ_j and complex constants d_j (and possibly $d_1 = 0$ or $d_2 = 0$, while $d_j \neq 0$ for all $j > 2$), then Proposition 3 and Remark 3 show that $\phi_j(\pm 1,\, 0) \neq 0$ for all $j > 2$, $d_1 \neq 0$ or $d_2 \neq 0$ and $d_1\lambda_2 + d_2\lambda_1 = 0$ as well as $d_1\mu_2 + d_2\mu_1 = 0$. In particular this implies the nontrivial condition

$$\det \left(\begin{array}{cc} 2\alpha_2(1,0) & -2\alpha_1(1,0) \\ -2\alpha_2(-1,0) & 2\alpha_1(-1,0) \end{array} \right) = 0.$$

To summarize: Unless one of the stationary points $(\pm 1,\, 0)$ is degenerate (with nilpotent linearization) or dicritical (positive rational eigenvalue ratio), one obtains a necessary condition for the existence of "classical" Darboux first integrals, and this condition imposes restrictions on the exponents

d_1 and d_2. In the dicritical case, as indicated by the previous example, even the consideration of one stationary point may show the non-existence of a Darboux first integral.

To illustrate the feasibility of such an approach we discuss a small specific example involving one complex parameter ν, with $\alpha_1 = -2 - \nu x_1$, $\alpha_2 = x_1$, $\beta_1 = 0$ and $\beta_2 = 1$, thus

$$P = \begin{pmatrix} (2 + \nu x_1) \cdot (x_2 + x_1^2 - 1) - x_1 x_2 \\ 2x_1^2 x_2 + x_2(x_2 + x_1^2 - 1) \end{pmatrix}$$

One finds that the eigenvalues of the linearization at $(1,\, 0)$ are $\lambda_1 = 2(2+\nu)$ and $\lambda_2 = 2$. Therefore the dicritical case occurs if and only if $\nu > -2$ and rational. Likewise, the eigenvalues at $(-1,\, 0)$ are $\mu_1 = 2(-2+\nu)$ and $\mu_2 = 2$, with dicriticality if and only if ν is rational and > 2. Finally, one has

$$\det \begin{pmatrix} \lambda_2 & \lambda_1 \\ \mu_2 & \mu_1 \end{pmatrix} = -16 \neq 0$$

for any choice of ν. Without any further work we see that P cannot admit a (classical) Darboux first integral unless ν is rational and > -2. In the remaining cases, an individual study of the dicritical points will provide additional conditions.

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