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Integrability and algebraic entropy of k-periodic non-autonomous Lyness recurrences*

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Abstract

This work deals with non-autonomous Lyness type recurrences of the form

$$x_{n+2} = \frac{a_n + x_{n+1}}{x_n},$$

where $\{a_n\}_n$ is a k-periodic sequence of positive numbers with minimal period k. We treat such non-autonomous recurrences via the autonomous dynamical system generated by the birational mapping $F_{a_k} \circ F_{a_{k-1}} \circ \cdots \circ F_{a_1}$ where F_a is defined by $F_a(x, y) = (y, \frac{a+y}{x})$. For the cases $k \in \{1, 2, 3, 6\}$ the corresponding mappings have a rational first integral. By calculating the dynamical degree we show that for $k = 4$ and for $k = 5$ generically the dynamical system in no longer rationally integrable. We also prove that the only values of k for which the corresponding dynamical system is rationally integrable for all the values of the involved parameters, are $k \in \{1, 2, 3, 6\}$.

1 Introduction and main results

Consider the non-autonomous Lyness difference equations of the form

$$x_{n+2} = \frac{a_n + x_{n+1}}{x_n}, \tag{1}$$

where $\{a_n\}_n$ is a k-periodic sequence of positive numbers. Such recurrences have been studied in [6], [8], [12], [14], [15] and recently in [7].

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For each k, the *composition mappings* are

$$F_{a_k,\ldots,a_2,a_1} := F_{a_k} \circ \cdots \circ F_{a_2} \circ F_{a_1} \tag{2}$$

where each F_{a_i} is defined by

$$F_{a_i}(x,y) = \left(y, \frac{a_i + y}{x} \right)$$

and $a_1, a_2, \ldots, a_k$ are the k elements of the cycle.

For the sake of shortness, we also will use the notation $F_{[k]} := F_{a_k,\ldots,a_2,a_1}$.

For instance, when $k = 2$, by setting

$$a_n = \begin{cases} a & \text{for} \quad n = 2\ell + 1, \\ b & \text{for} \quad n = 2\ell, \end{cases} \tag{3}$$

we get

$$F_{b,a}(x,y) = F_b \circ F_a(x,y) = \left(\frac{a+y}{x}, \frac{a+bx+y}{xy} \right),$$

and when $k = 3$,

$$a_n = \begin{cases} a & \text{for} \quad n = 3\ell + 1, \\ b & \text{for} \quad n = 3\ell + 2, \\ c & \text{for} \quad n = 3\ell, \end{cases} \tag{4}$$

and

$$F_{c,b,a}(x,y) = F_c \circ F_b \circ F_a(x,y) = \left(\frac{a+bx+y}{xy}, \frac{a+bx+y+cxy}{y\,(a+y)} \right).$$

Clearly the study of the dynamics of the recurrences given by (1) can be deduced from the dynamics generated by the composition mappings (2). It is known that for the cases $k \in \{1,2,3,6\}$ and for all values of the parameters, the mappings $F_a, F_{b,a}, F_{c,b,a}$ and $F_{f,e,d,c,b,a}$ have a rational first integral :

$$\begin{aligned} V_a(x,y) &= \frac{a+(a+1)x+(a+1)y+x^2+y^2+x^2y+xy^2}{xy}, \\ V_{b,a}(x,y) &= \frac{ab+(a+b^2)x+(b+a^2)y+bx^2+ay^2+ax^2y+bxy^2}{xy}, \\ V_{c,b,a}(x,y) &= \frac{ac+(a+bc)x+(c+ab)y+bx^2+by^2+cx^2y+axy^2}{xy}, \\ V_{f,e,d,c,b,a}(x,y) &= \frac{af+(a+bf)x+(f+ae)y+bx^2+ey^2+cx^2y+dxy^2}{xy}. \end{aligned}$$

In this paper we prove that for $k \notin \{1,2,3,6\}$ the corresponding mapping $F_{[k]}$ does not have a rational first integral for all the values of the involved parameters. This result has also been stated in [7] from a numerical point of view. Here we give an analytical proof.

As we mentioned above, the mappings $F_{a_k,\ldots,a_2,a_1} : \mathbb{C}^2 \to \mathbb{C}^2$ are birational mappings. A birational mapping is a mapping F with rational components such that there exists an algebraic curve V and another rational mapping G such that $F \circ G = G \circ F = id$ in $\mathbb{C}^2 \setminus V$. We are going to use the embedding $(x_1, x_2) \longmapsto [x_0 : x_1 : x_2] \in P\mathbb{C}^2$ to extend the mappings to the projective space $P\mathbb{C}^2$ in the usual way, by getting a polynomial homogeneous mapping which has an associated degree, called the degree of the mapping. Let d_n be the degree of $F^n = F \circ \cdots \circ F$. The *dynamical degree* of F is defined as

$$\delta(F) = \lim_{n\to\infty} (d_n)^{\frac{1}{n}}.$$

And its logarithm is called the *algebraic entropy* of F.

It is known that the existence of a foliation of the space by algebraic invariant curves implies that the dynamical degree is one, see [4] for instance, also [11]. In order to prove our results about the non-integrability of the mapping, we use the method of calculating the dynamical degree.

We want to emphasize that the method that we implement allows us to know the sequence d_n for the mappings under consideration. When the growth of this sequence is exponential the calculation of the degrees of the iterates quickly becomes unfeasible.

The results that we get are the following:

Theorem 1. *For $k \notin \{1, 2, 3, 6\}$ there are some values of the parameters for which the mapping $F_{[k]}$ does not have a rational first integral.*

To prove Theorem 1 we begin by studying the cases $k = 4$ and $k = 5$. We calculate the dynamical degree of $F_{[4]}$ and $F_{[5]}$ for general values of the parameters. As usual we say that a set of k parameters $a_1, a_2, \ldots, a_k$ is generic if $\{(a_1, a_2, \ldots, a_k) \in \mathbb{C}^k\}$ is an open and dens subset of $\mathbb{C}^k$ with the usual topology. The results that we get are contained in Propositions 2 and 3.

Proposition 2. *For a generic set of the values of the parameters $a, b, c, d \in \mathbb{C}$, the dynamical degree of the mapping $F_{d,c,b,a} = F_d \circ F_c \circ F_b \circ F_a$ is the largest root of the polynomial $z^3 - 2z^2 - 3z - 1$, which approximately is* 3.079595625.

We emphasize that the above result is valid for generic values of the parameters and that for other values of the parameters the entropy can be changed. For example, if we take $a = \alpha, b = \alpha\beta, c = \beta, d = 1/\alpha$ we get a mapping that is topologically conjugated to F_{1/ac^2}, and so it is rationally integrable with zero entropy. Also the case $a = \alpha^2, b = \alpha, c = 1/\alpha, d = 1/\alpha^2$ has zero entropy: the corresponding mapping is 5-periodic (see [7]).

If $k = 5$, for general values of the parameters the entropy is approximately $\ln(4.079595625) \approx 1.405997872$, as shown in the next proposition.

Proposition 3. *For a generic set of the values of the parameters* $a, b, c, d, e \in \mathbb{C}$, *the dynamical degree of the mapping* $F_{e,d,c,b,a} = F_e \circ F_d \circ F_c \circ F_b \circ F_a$ *is the largest root of the polynomial* $z^3 - 5\,z^2 + 4\,z - 1$, *which approximately is* 4.079595625.

Now let a, b, c, d such that the mapping $F_{d,c,b,a}$ has dynamical degree equal to the largest root of the polynomial $z^3 - 2\,z^2 - 3\,z - 1$, and consider $F_{d,c,b,a,d,c,b,a} = F_{d,c,b,a} \circ F_{d,c,b,a}$. Then the dynamical degree of such a mapping is the square of the dynamical degree of $F_{d,c,b,a}$ (see Lemma 6 in section 5), and so it is also greater than 1. This remark proves the theorem for the case $k = 8$. To prove the theorem for the case $k = 9$ we recall that the Lyness mapping with $a = 1$ is five periodic and we take into account $F_{1,1,1,1,1,d,c,b,a} = F_{1,1,1,1,1} \circ F_{d,c,b,a}$. In fact this mapping actually is $F_{d,c,b,a}$ itself, but it proves that there are values of the parameters when $k = 9$ such that the corresponding mapping has not zero entropy.

In order to cover all the cases we also need some families with positive entropy when $k = 7$ and $k = 11$. We do not the general cases because the computations are too large.

Proposition 4. *For all* $a \neq 0$, $a^p \neq 1$ *for all* $p \in \mathbb{N}$, *the dynamical degree of the one-parametric family of mappings* $F_{a,1,a,1,a,1,a}$ *is the largest root of the polynomial* $z^3 - 3z^2 + z - 1$, *which approximately is* 2.769292354.

Proposition 5. *For all* $a \notin \{-1, 0, 1\}$ *the dynamical degree of the one-parametric family of mappings* $F_{a,1,a,a,1,a,1,a,a,1,a}$ *is the largest root of the polynomial* $z^3 - 2z^2 - 3z - 1$ *which approximately is* 3.079595625.

The paper is organized as follows. In Section 2 we introduce the tools which we use in the following sections. Section 3 is devoted to prove Propositions 2 and 3 while in Section 4 we prove Propositions 4 and 5. Finally, in Section 5 we prove the main Theorem.

2 Preliminary Results

Given a birational mapping $f[x_0 : x_1 : x_2] = [f_1[x_0 : x_1 : x_2], f_2[x_0 : x_1 : x_2], f_3[x_0 : x_1 : x_2]]$ from $P\mathbb{C}^2$ to $P\mathbb{C}^2$, we consider the indeterminacy set $\mathcal{I}(f)$ of points where f is not well-defined as a continuous mapping. This set is given by:

$$\mathcal{I}(f) = \{[x_0 : x_1 : x_2] \in P\mathbb{C}^2 : f_1[x_0 : x_1 : x_2] = 0, f_2[x_0 : x_1 : x_2] = 0, f_3[x_0 : x_1 : x_2] = 0]\}.$$

On the other hand, if we consider one irreducible component V of the determinant of the Jacobian of f, it is known (see [10]) that $f(V)$ reduces to a point, which belongs to the indeterminacy set of the inverse of f. The set of these curves which are sent to a single points is called the *excepcional locus* of f and it is denoted by $\mathcal{E}(f)$.

Given a point $p \in \mathbb{C}^2$, we are going to consider (X, π), the blowing-up of $\mathbb{C}^2$ at the point p. If $p = (0,0) \in \mathbb{C}^2$ (if not we do a translation) then

$$X \;=\; \{((x,y),[u:v]) \in \mathbb{C}^2 \times P\mathbb{C}^1 \;:\; xv = yu\},$$

and

$$\pi : X \longrightarrow \mathbb{C}^2$$

is the projection on the first component:

$$\pi\left((x,y),[u:v]\right) \;=\; (x,y).$$

We notice that

$$\pi^{-1}p = \pi^{-1}(0,0) = \{((0,0),[u:v])\} := E_p \simeq P\mathbb{C}^1$$

and if $q = (x,y) \neq (0,0)$, then

$$\pi^{-1}q = \pi^{-1}(x,y) = ((x,y),[x:y]) \in X.$$

Given the point $((0,0),[u:v]) \in E_p$ (resp. $((x,y),[x:y])$) we are going to represent it by $[u:v]_{E_p}$ (resp. by $(x,y) \in \mathbb{C}^2$ or by $[1:x:y] \in P\mathbb{C}^2$ if it is convenient).

As usual, given a curve C on $\mathbb{C}^2$, the *strict transform* of C is the adherence of $\pi^{-1}(C \setminus \{p\})$, in the Zariski topology, and we denote it by $\hat{C}$.

Indeterminacy sets and excepcional locus can also be defined if we consider meromorphic functions defined on complex manifolds. If X is a complex manifold we are going to consider the Picard group of X, denoted by $\mathcal{P}ic(X)$. Then $\mathcal{P}ic(P\mathbb{C}^2)$ is generated by the class of L, where L is a generic line in $P\mathbb{C}^2$. In this work we deal with complex manifolds X obtained after performing a finite sequence of blowing-ups. If the base points of the blow-ups are $\{p_1, p_2, \ldots, p_k\} \subset P\mathbb{C}^2$ and $E_i := \pi^{-1}\{p_i\}$ then it is known that $\mathcal{P}ic(X)$ is generated by $\{\hat{L}, E_1, E_2, \ldots, E_k\}$, see [2, 3]. Furthermore $\pi : X \longrightarrow P\mathbb{C}^2$ induces a morphism of groups $\pi^* : \mathcal{P}ic(P\mathbb{C}^2) \longrightarrow \mathcal{P}ic(X)$, with the property that for any complex curve $C \subset P\mathbb{C}^2$,

$$\pi^*(C) = \hat{C} + \sum m_i\, E_i, \tag{5}$$

where m_i is the algebraic multiplicity of C at p_i (see [1]).

On the other hand, if f is a birational mapping defined on $P\mathbb{C}^2$, then there is a natural extension of f on X, which we denote by $\tilde{f}$. And $\tilde{f}$ induces a morphism of groups, $\tilde{f}^* : \mathcal{P}ic(X) \to \mathcal{P}ic(X)$ just by taking the class of the preimage. The interesting thing here is that

$$\tilde{f}^*(\hat{L}) \;=\; d\,\hat{L} \,+\, \sum_{i=1}^{k} c_i\, E_i \quad , \quad c_i \in \mathbb{Z}$$

where d is the degree of f. By iterating f, we get the corresponding formula by changing f by f^n and d by d_n. In order to deduce the behaviour of the sequence d_n it is convenient to deal with mappings $\tilde{f}$ such that

$$(\tilde{f^n})^* = (\tilde{f}^*)^n. \tag{6}$$

Mappings $\tilde{f}$ satisfying condition (6) are called *Algebraically Stable mappings* (AS for short), (see [11]).

It is known (see Theorem 0.1 of [11]) that one can always arrange for a birational mapping to be AS considering an extension of f. If it is the case and we call $\mathcal{X}(x) = x^k + \sum_{i=0}^{k-1} c_i\, x^i$ the characteristic polynomial of $A := (\tilde{f}^*)$, then since $\mathcal{X}(A) = 0$ and d_i is the $(1,1)$ term of A^i we get that

$$d_k = -(c_0 + c_1 d_1 + c_2 d_2 + \cdots + c_{k-1} d_{k-1}),$$

i. e., the sequence d_n satisfies a homogeneous linear recurrence with constant coefficients.

In order to get AS mappings we will use the following useful result showed by Fornaess and Sibony in [13]:

$$\text{If for every excepcional curve } C \text{ and all } n \geq 0\,,\ \tilde{f}^n(C) \notin \mathcal{I}(\tilde{f})\,, \text{ then } \tilde{f} \text{ is AS.} \tag{7}$$

3 Proof of Propositions 2 and 3

Proof of Proposition 2

We consider the family of mappings $F_{d,c,b,a}(x,y) := F_d \circ F_c \circ F_b \circ F_a(x,y)$ which has the following expression:

$$F_{d,c,b,a}(x,y) = \left(\frac{cxy + bx + a + y}{y\,(a+y)}, \frac{x\left(dya + dy^2 + cxy + bx + a + y\right)}{(a+y)\,(bx + a + y)} \right).$$

By extending it to $P\mathbb{C}^2$ we get the mapping $f[x_0 : x_1 : x_2]$ with components

$$\begin{aligned}
f_1[x_0 : x_1 : x_2] &= x_0 x_2 (a x_0 + x_2)(a x_0 + b x_1 + x_2),\\
f_2[x_0 : x_1 : x_2] &= x_0 (a x_0 + b x_1 + x_2)(a x_0^2 + b x_0 x_1 + x_0 x_2 + c x_1 x_2),\\
f_3[x_0 : x_1 : x_2] &= x_1 x_2 (a x_0^2 + b x_0 x_1 + (1 + ad) x_0 x_2 + c x_1 x_2 + d x_2^2).
\end{aligned}$$

In order to find the excepcional locus of f we calculate the determinant of the jacobian of f, which we call j_f and it is given by

$$\begin{aligned}
j_f &= 4 x_0 x_2 (a x_0 + x_2)(a x_0 + b x_1 + x_2)^2 (a x_0^2 + b x_0 x_1 + x_0 x_2 + c x_1 x_2)\\
&(a x_0^2 + b x_0 x_1 + (1 + ad) x_0 x_2 + c x_1 x_2 + d x_2^2).
\end{aligned}$$

For $i = 0,1,2,3,4,5$ let S_i be defined by $g_i[x_0, x_1, x_2] = 0$ with:

$$\begin{aligned}
S_0 &:= \{x_0 = 0\},\\
S_1 &:= \{x_2 = 0\},\\
S_2 &:= \{ax_0 + x_2 = 0\},\\
S_3 &:= \{ax_0 + bx_1 + x_2 = 0\},\\
S_4 &:= \{ax_0^2 + bx_0x_1 + x_0x_2 + cx_1x_2 = 0\},\\
S_5 &:= \{ax_0^2 + bx_0x_1 + (1+ad)x_0x_2 + cx_1x_2 + dx_2^2 = 0\}.
\end{aligned}$$

We see that for generic values of the parameters the curves $g_i[x_0, x_1, x_2] = 0$ are irreducible and distinct.

Applying the mapping we see that each S_i collapses to A_i, where

$$A_0 := [0:0:1]\,, A_1 := [0:1:0]\,, A_2 := [0:b:-a]\,, A_4 := [c:0:-d]\,, A_5 := [1:-d:0]$$

and S_3 collapses to A_0 too.

On the other hand the indeterminacy set of f is given by the five points:

$$O_0 := [0:0:1]\,, O_1 := [0:1:0]\,, O_2 := [0:-d:c]\,, O_3 := [b:-a:0]\,, O_4 := [1:0:-a].$$

We observe that $A_0 = O_0\,,\ A_1 = O_1$ and that for generic values of the parameters $A_i \neq O_j$ for $i = 2,4,5\,,\ j = 2,3,4$.

Let X be the space we get after blowing up the two points O_0, O_1. Now we are going to extend the mapping f to X on a continous way. To this end we identify $E_0 := \pi^{-1}(O_0)$ with $P\mathbb{C}^1$ in the following way: given $[u:v] \in P\mathbb{C}^1$, we associate the point

$$[u:v]_{E_0} := \lim_{t\to 0} \pi^{-1}[tu:tv:1] \in E_0. \tag{8}$$

From now on we are going to identify a set $S \subset P\mathbb{C}^2$ with the set $\pi^{-1}(S) \subset X$.

To determine the mapping on $S_0 = \{x_0 = 0\}$ let $x = [0:x_1:x_2] = \lim_{t\to 0}[t:x_1:x_2] \in S_0$. We assign:

$$\tilde{f}(x) = \lim_{t\to 0} f[t:x_1:x_2] \;=\; \lim_{t\to 0}[tx_2^2(bx_1+x_2) : tcx_1x_2(bx_1+x_2) : x_1x_2^2(cx_1+dx_2)].$$

Assuming $x_1x_2(cx_1 + dx_2) \neq 0$ we get

$$\tilde{f}(x) = \lim_{t\to 0}\left[t\,\frac{bx_1+x_2}{x_1(cx_1+dx_2)} : t\,\frac{c(bx_1+x_2)}{x_2(cx_1+dx_2)} : 1\right]$$

and we identify this point with:

$$\left[\frac{bx_1+x_2}{x_1(cx_1+dx_2)} : \frac{c(bx_1+x_2)}{x_2(cx_1+dx_2)}\right]_{E_0} \equiv [x_2 : cx_1]_{E_0}.$$

If $x_1x_2(cx_1+dx_2)=0$ we have the points $O_0=[0:0:1], O_1=[0:1:0]$ and $O_2=[0:-d:c]$. Then we have that

$$\tilde{f}: S_0 \setminus \mathcal{I}(f) \longrightarrow E_0$$

is defined trough

$$\tilde{f}\,[0:x_1:x_2]=[x_2:cx_1]_{E_0}. \tag{9}$$

To determine the mapping $\tilde{f}$ on E_0 we consider a point $[u:v]_{E_0}$ in the fibre E_0 as shown in (8). We need to evaluate $f[tu:tv:1]$. Its three components are given by

$$\begin{aligned}
&t\,u\,(dtu+1)\,(dtu+ctv+1),\\
&tu\left(at^2u^2+bt^2vu+tu+ctv\right)(atu+btv+1)\,,\\
&tv\left(ctv+bt^2vu+at^2u^2+d+tuda+tu\right).
\end{aligned}$$

Hence, $\lim_{t\to 0} f[tu:tv:1]=[u:0:dv]$. Calling $T_1=\{x_1=0\}$ we have that

$$\tilde{f}: E_0 \longrightarrow T_1$$

is given by

$$\tilde{f}[u:v]_{E_0}=[u:0:dv]. \tag{10}$$

On the other hand we notice that the action of f on T_1 is given by

$$f[x_0:0:x_2]=[x_0x_2(ax_0+x_2)^2 : x_0^2(ax_0+x_2)^2:0].$$

If $x_0(ax_0+x_2)=0$ we get the points $[0:0:1]=O_0$ and $[1:0:-a]=O_4$. And if $x_0(ax_0+x_2)\neq 0$, then $f[x_0:0:x_2]=[x_2:x_0:0]$. Hence,

$$\tilde{f}: T_1 \setminus \mathcal{I}(f) \longrightarrow S_1$$

is given by

$$\tilde{f}[x_0:0:x_2]=[x_2:x_0:0]. \tag{11}$$

The same type of arguments and computations allow us to extend f to S_1 and $E_1:=\pi^{-1}(O_1)$. We get that:

$$\tilde{f}: S_1 \setminus \mathcal{I}(f) \longrightarrow E_1$$

is defined by:

$$\tilde{f}[x_0:x_1:0]=[ax_0:x_1]_{E_1}, \tag{12}$$

and

$$\tilde{f}: E_1 \longrightarrow S_0$$

is given by

$$\tilde{f}[u:v]_{E_1}=[0:bu:v]. \tag{13}$$

Hence, from (9), (10), (11), (12) and (13) we see that we get a cycle between these complex 1-dimensional manifolds:

$$S_0 \longrightarrow E_0 \longrightarrow T_1 \longrightarrow S_1 \longrightarrow E_1 \longrightarrow S_0. \tag{14}$$

On the other hand, since f maps S_3 to A_0, we have to extend f to S_3. The result that we get is that S_3 still collapses:

$$\tilde{f} : S_3 \setminus \mathcal{I}(f) \longrightarrow [b : -c]_{E_0}. \tag{15}$$

Hence, S_3 is still excepcional for $\tilde{f}$.

From the above calculations we have that the indeterminacy and the excepcional sets of $\tilde{f}$ are:

$$\mathcal{I}(\tilde{f}) = \{O_2, O_3, O_4\} \quad \text{and} \quad \mathcal{E}(\tilde{f}) = \{S_2, S_3, S_4, S_5\}, \tag{16}$$

where we again identify sets and points on $P\mathbb{C}^2$ with the equivalent points in X.

We claim that, for generic values of the parameters, $\tilde{f}^n(S_i)$ is never an indeterminacy point of $\tilde{f}$ for $i = 2, 3, 4, 5$. To prove the claim we have to follow the orbits of S_i for $i = 2, 3, 4, 5$. From (14) we observe that each one of the manifolds S_0, E_0, T_1, S_1 and E_1 are invariant by $\tilde{f}^5$. The calculations are the following:

- **The orbit of $\mathbf{S}_2$.**

  Since $f(S_2) = A_2 = [0 : b : -a]$ we have that generically:

$$\begin{aligned}
\tilde{f}^{5k}(A_2) &= \tilde{f}^{5k}[0 : b : -a] = [0 : a^{k-1}b^{k+1}c^kd^k : -1] \notin \mathcal{I}(\tilde{f}),\\
\tilde{f}^{5k+1}(A_2) &= \tilde{f}^{5k}[-a : bc]_{E_0}] = [-1 : a^{k-1}b^{k+1}c^{k+1}d^k]_{E_0} \notin \mathcal{I}(\tilde{f}),\\
\tilde{f}^{5k+2}(A_2) &= \tilde{f}^{5k}[-a : 0 : bcd] = [-1 : 0 : a^{k-1}b^{k+1}c^{k+1}d^{k+1}] \notin \mathcal{I}(\tilde{f}),\\
\tilde{f}^{5k+3}(A_2) &= \tilde{f}^{5k}[bcd : -a : 0] = [a^{k-1}b^{k+1}c^{k+1}d^{k+1} : -1 : 0] \notin \mathcal{I}(\tilde{f}),\\
\tilde{f}^{5k+4}(A_2) &= \tilde{f}^{5k}[bdc : -1]_{E_1} = [a^kb^{k+1}c^{k+1}d^{k+1} : -1]_{E_1} \notin \mathcal{I}(\tilde{f}).
\end{aligned}$$

- **The orbit of $\mathbf{S}_4$.** Since $f(S_4) = A_4 = [-b : 0 : a]$ we have that:

$$\begin{aligned}
\tilde{f}^{5k}(A_4) &= \tilde{f}^{5k}[c : 0 : -d] = [1 : 0 : -a^kb^kc^{k-1}d^{k+1}] \notin \mathcal{I}(\tilde{f}),\\
\tilde{f}^{5k+1}(A_4) &= \tilde{f}^{5k}[d : -c : 0] = [: a^kb^kc^{k-1}d^{k+1} : -1 : 0] \notin \mathcal{I}(\tilde{f}),\\
\tilde{f}^{5k+2}(A_4) &= \tilde{f}^{5k}[-ad : c]_{E_1} = [-a^{k+1}b^kc^{k-1}d^{k+1} : -1]_{E_1} \notin \mathcal{I}(\tilde{f}),\\
\tilde{f}^{5k+3}(A_4) &= \tilde{f}^{5k}[0 : -abd : c] = [0 : -a^{k+1}b^{k+1}c^{k-1}d^{k+1} : 1] \notin \mathcal{I}(\tilde{f}),\\
\tilde{f}^{5k+4}(A_4) &= \tilde{f}^{5k}[c : -abcd]_{E_0} = [c : -(abcd)^{k+1}]_{E_0} \notin \mathcal{I}(\tilde{f}).
\end{aligned}$$

Very similar computations show that generically $\tilde{f}^n(S_i) \notin \mathcal{I}(\tilde{f})$ for $i = 3, 5$.

Hence, using (7) we see that $\tilde{f} : X \longrightarrow X$ is AS.

In order to compute the matrix of $\tilde{f}^* : Pic(X) \longrightarrow Pic(X)$ we take into account the action of

$$\pi^* : Pic(P\mathbb{C}^2) = <\hat{L}> \longrightarrow Pic(X) = <\hat{L}, E_0, E_1> .$$

Since $O_0, O_1 \in S_0$ and $\deg(S_0) = 1$, from (5), we have that

$$\pi^*(S_0) = \hat{S}_0 + E_0 + E_1. \tag{17}$$

Similarly

$$\pi^*(S_1) = \hat{S}_1 + E_1, \tag{18}$$

and

$$\pi^*(S_3) = \hat{S}_3. \tag{19}$$

But in $Pic(P\mathbb{C}^2)$, all the curves of degree one are equivalent to L, i. e., $\pi^*(S_i) = \pi^*(L) = \hat{L}$ for $i = 0, 1, 3$. It implies that

$$\hat{S}_0 = \hat{L} - E_0 - E_1 \ , \ \hat{S}_1 = \hat{L} - E_1 \ \text{ and } \ \hat{S}_3 = \hat{L}.$$

Now we are ready to calculate the matrix of $\tilde{f}^*$. From (9), (15), (17) and (19) we deduce that:

$$\tilde{f}^*(E_0) = \hat{S}_0 + \hat{S}_3 = 2\hat{L} - E_0 - E_1,$$

where $\tilde{f}^*$ acts by taking preimages. Similarly,

$$\tilde{f}^*(E_1) = \hat{S}_1 = \hat{L} - E_1.$$

It only remains to determine $\tilde{f}^*(\hat{L})$ which we recall that $L = \{px_0 + qx_1 + rx_2 = 0\}$ where the parameters p, q, r are generic. Considering the algebraic curve of degree 4:

$$f^{-1}(L) = \{[x_0 : x_1 : x_2] \in P\mathbb{C}^2 : pf_1[x_0 : x_1 : x_2] + qf_2[x_0 : x_1 : x_2] + rf_3[x_0 : x_1 : x_2] = 0\}$$

we have that $\pi^*(f^{-1}(L))$ equals to its strict transform plus $m_1E_0 + m_2E_1$ where m_1, m_2 are the multiplicities of $f^{-1}(L)$ at O_0, O_1 respectively. It can easily be seen that $m_1 = 1$ and $m_2 = 2$. Hence,

$$\tilde{f}^*(\hat{L}) = 4\hat{L} - E_0 - 2E_1,$$

and the matrix of $\tilde{f}^*$ is given by:

$$(\tilde{f}^*) = \begin{pmatrix} 4 & 2 & 1 \\ -1 & -1 & 0 \\ -2 & -1 & -1 \end{pmatrix}.$$

The characteristic polynomial of $(\tilde{f}^*)$ is $\lambda^3 - 2\lambda^2 - 3\lambda - 1$, which has a unique real root

$$\lambda_1 = \frac{1}{6}\sqrt[3]{388 + 12\sqrt{69}} + \frac{26}{3}\frac{1}{\sqrt[3]{388 + 12\sqrt{69}}} + \frac{2}{3} \approx 3.079595625$$

and two complex conjugated roots with modulus less than λ_1. So, Proposition 2 is proved. ∎

We observe that the sequence of the degrees d_n exactly satisfies the recurrence

$$d_{n+3} = 2d_{n+2} + 3d_{n+1} + d_n$$

and since $d_1 = 4, d_2 = 12$ and $d_3 = 37$, the sequence of the degrees is

$$4\,,\,12\,,\,37\,,\,114\,,\,351\,,\,1081\,,\,4059\,,\,11712\,,....$$

Next result deals on the case $k = 5$. The case k multiple of 5 is distinguished from others. This is because in this case, generically, F is well defined on the axes $x = 0$ and $y = 0$, these axes are invariant and $(0,0)$ is a fixed point. Furthermore, when we extend the mapping to the projective space also the line at infinity is invariant.

Proof of Proposition 3

In order to prove Proposition 3 we follow the same procedure as described above. Consider the composition mapping $F_{e,d,c,b,a} = F_e \circ F_d \circ F_c \circ F_b \circ F_a$ which has the following two components:

$$\frac{ax + bx^2 + (ad+1)xy + cx^2y + dxy^2}{(a+y)(a+bx+y)},$$
$$\frac{ea^2y + a\,(eb+1)\,xy + 2\,eay^2 + bx^2y + (eb+1+da)\,xy^2 + ey^3 + cx^2y^2 + dxy^3}{(a+bx+y+cxy)(a+bx+y)}.$$

Let f be the extension of the above mapping to $P\mathbb{C}^2$. Then the homogeneous components of $f[x_0, x_1, x_2] = [f_1 : f_2 : f_3]$ are the following:

$$\begin{aligned}
f_1 &= x_0(ax_0 + x_2)(ax_0 + bx_1 + x_2)(ax_0^2 + bx_0x_1 + x_0x_2 + cx_1x_2),\\
f_2 &= x_1(ax_0^2 + bx_0x_1 + (1+ad)x_0x_2 + cx_1x_2 + dx_2^2)(ax_0^2 + bx_0x_1 + x_0x_2 + cx_1x_2),\\
f_3 &= x_2(ax_0 + x_2)(a^2ex_0^3 + a(1+be)x_0^2x_1 + 2aex_0^2x_2 + bx_0x_1^2 + (1+be+ad)x_0x_1x_2\\
&\quad + ex_0x_2^2 + cx_1^2x_2 + dx_1x_2^2).
\end{aligned}$$

Since the jacobian of this mapping is:

$$\begin{aligned} j_f &= 5(ax_0+x_2)^2(ax_0+bx_1+x_2)(ax_0^2+bx_0x_1+x_0x_2+cx_1x_2)^2(ax_0^2+bx_0x_1+ \\ &\quad (1+ad)x_0x_2+cx_1x_2+dx_2^2)(a^2ex_0^3+a(1+be)x_0^2x_1+2aex_0^2x_2+bx_0x_1^2 \\ &\quad +(1+be+ad)x_0x_1x_2+ex_0x_2^2+cx_1^2x_2+dx_1x_2^2), \end{aligned}$$

we have the following five exceptional curves S_i where we define S_i as $g_i[x_0,x_1,x_2]=0$ for $i\in\{1,2,3,4,5\}$:

$$\begin{aligned} S_1 &:= \{ax_0^2+bx_0x_1+x_0x_2+cx_1x_2=0\}, \\ S_2 &:= \{ax_0+x_2=0\}, \\ S_3 &:= \{ax_0+bx_1+x_2=0\}, \\ S_4 &:= \{ax_0^2+bx_0x_1+(1+ad)x_0x_2+cx_1x_2+dx_2^2=0\}, \\ S_5 &:= \{a^2ex_0^3+a(1+be)x_0^2x_1+2aex_0^2x_2+bx_0x_1^2+(1+be+ad)x_0x_1x_2+ex_0x_2^2 \\ &\quad +cx_1^2x_2+dx_1x_2^2=0\}. \end{aligned}$$

Each S_i collapses to A_i where $A_i\in\mathcal{I}(f^{-1})$ and they are as follows:

$$A_1:=[0:0:1]\,,A_2:=[0:1:0]\,,A_3:=[0:-c:b]\,,A_4:=[-d:0:e]\,,A_5:=[1:-e:0].$$

We can see that for generic values of the parameters all these $g_i^{\prime}$s are distinct and irreducible. The indeterminacy points of f are:

$$O_1:=[0:0:1]\,,O_2:=[0:1:0]\,,O_3:=[0:-d:c]\,,O_4:=[-b:a:0]\,,O_5:=[1:0:-a].$$

We observe that $O_1=A_1, O_2=A_2$ and that generically $O_i\neq A_j$ for all $i,j=3,4,5$. To regularize our f so that $\tilde{f}$ is AS we need to follow the orbit of each A_i so that we can see if it reaches an indeterminacy point of f. In our case we see that $S_1\to A_1=O_1$ and $S_2\to A_2=O_2$, while $S_3\to A_3\in\{x_0=0\}$, $S_4\to A_4\in\{x_1=0\}$ and $S_5\to A_5\in\{x_2=0\}$. Also we observe that the straight lines $\{x_0=0\}$, $\{x_1=0\}$ and $\{x_2=0\}$ are invariant under f. More precisely, $\forall k\in\mathbb{N}$:

$$\begin{aligned} f^k[0:x1:x2] &= [0:c^kx_1:x_2], \\ f^k[x_0:0:x2] &= [x_0:0:e^kx_2], \\ f^k[x_0:x1:0] &= [a^kx_0:x_1:0]. \end{aligned}$$

Therefore, $\forall k \in \mathbb{N}$:

$$\begin{aligned} f^k(A_3) &= [0 : c^{k+1} : -b], \\ f^k(A_4) &= [-d : 0 : e^{k+1}], \\ f^k(A_5) &= [a^k : -e : 0]. \end{aligned}$$

We observe that for generic values of the parameters the orbits of A_3, A_4 and A_5 will never reach any indeterminacy point of f. Hence we have only two points O_1 and O_2 which we need to blow up. Let X be the space we get after blowing up O_1 and O_2 and let E_1 and E_2 be the exceptional fibres correspondingly. Let $\tilde{f}$ be the corresponding extension of f to X. To determine $\tilde{f}$ on S_1, S_2, E_1 and E_2 after some computations we see that:

$$\tilde{f} : S_1 \setminus \mathcal{I}(f) \to [c : -d]_{E_1} \quad , \quad \tilde{f} : S_2 \setminus \mathcal{I}(f) \to [b : -a]_{E_2}$$

and

$$\tilde{f}[u : v]_{E_1} = [u : dv]_{E_1} \quad , \quad \tilde{f}[u : v]_{E_2} = [bu : v]_{E_2}.$$

This means that S_1 and S_2 are still exceptional for $\tilde{f}$ and that E_1, E_2 are invariant for $\tilde{f}$. But it is easy to see that $\tilde{f}^k(S_1) = [c : -d^k]_{E_1}$ and $\tilde{f}^k(S_2) = [b^k : -a]_{E_2}$ hence for generic values of parameters the points $[c : d^k]_{E_1}$ and $[b^k : -a]_{E_2}$, for all k, can never reach any indeterminacy point of $\tilde{f}$. It is now clear that $\tilde{f}$ is AS.

In order to compute the matrix $\tilde{f}^* : Pic(X) \longrightarrow Pic(X)$ we first take into account

$$\pi^* : Pic(P\mathbb{C}^2) = < \hat{L} > \longrightarrow Pic(X) = < \hat{L}, E_1, E_2 > .$$

Since $\deg(S_1) = 2$ and $O_1, O_2 \in S_1$ with multiplicity 1, from (5), we have that

$$\pi^*(S_1) = \hat{S}_1 + E_1 + E_2. \tag{20}$$

Similarly

$$\pi^*(S_2) = \hat{S}_2 + E_2, \tag{21}$$

and

$$\pi^*(f^{-1}(L)) = \widehat{f^{-1}}(L) + E_1 + 2E_2, \tag{22}$$

because O_2 has multiplicity 2 in the components of $f[x_0 : x_1 : x_2]$. Now in $Pic(P\mathbb{C}^2)$,

$$\hat{S}_1 = \hat{2L} - E_1 - E_2 \quad \text{and} \quad \hat{S}_2 = \hat{L} - E_2.$$

Therefore by using the above equations we have:

$$\begin{aligned}\tilde{f}^*(\hat{L}) &= 5\hat{L} - E_1 - 2E_2,\\ \tilde{f}^*(E_1) &= \hat{S}_1 + E_1 = 2\hat{L} - E_2,\\ \tilde{f}^*(E_2) &= \hat{S}_2 + E_2 = \hat{L}.\end{aligned}$$

The matrix of $\tilde{f}^*$ is:

$$(\tilde{f}^*) = \begin{pmatrix} 5 & 2 & 1 \\ -1 & 0 & 0 \\ -2 & -1 & 0 \end{pmatrix}.$$

The characteristic polynomial of $(\tilde{f}^*)$ is $\lambda^3 - 5\lambda^2 + 4\lambda - 1$, which has the unique real root

$$\lambda_1 \cong 4.079595625$$

and two complex conjugated roots with modulus less than λ_1. Hence proposition 3 is proved.

∎

We can see that the sequence of the degrees d_n satisfies the recurrence

$$d_{n+3} = 5d_{n+2} - 4d_{n+1} + d_n.$$

Since $d_1 = 5, d_2 = 21$ and $d_3 = 86$, the sequence of the degrees is

$$5\,,\,21\,,\,86\,,\,351\,,\,1432\,,\,5842\,,\,23833\,,\ldots.$$

4 Proof of Propositions 4 and 5

In this section we give two 1−parametric families which are some subfamilies of the general $F_{[7]}$ and $F_{[11]}$ families. The general cases are avoided as they involve tedious and much larger calculations. We will prove that in general $F_{[7]}$ and $F_{[11]}$ are not integrable by giving a proof for the $\delta > 1$ for these two 1−parametric subfamilies.
For the case $k = 7$ we calculate the entropy of $F_{a,1,a,1,a,1,a}$. In general these mappings have two excepcional curves which have degrees 4 and 5, but they have genus zero. As it is well known, the curves of genus zero have a rational parametrization. The existence of such a parametrizations has been useful to deduce the behavior of the induced mapping in the Picard group.

Proof of Proposition 4

For the case $k = 7$ we consider the one parametric family

$$F_{a,1,a,1,a,1,a} = F_a \circ F_1 \circ F_a \circ F_1 \circ F_a \circ F_1 \circ F_a.$$

The extension f of the mapping $F_{a,1,a,1,a,1,a}$ in $P\mathbb{C}^2$ has the form:

$$f[x_0, x_1, x_2] = [x_1\, x_2\, g_1\, g_2\, g_3 : x_2\, g_3\, g_4\, g_5 : x_0\, g_1\, g_6\, g_7],$$

where

$$\begin{aligned}
g_1 &= ax_0^2 + x_0x_1 + x_0x_2 + ax_1x_2, \\
g_2 &= ax_0^2 + x_0x_1 + (1+a)x_0x_2 + ax_1x_2 + x_2^2, \\
g_3 &= a^3x_0^3 + a(a+1)x_0^2x_1 + 2a^2x_0^2x_2 + x_0x_1^2 + ax_0x_2^2 + ax_1^2x_2 + x_1x_2^2 + (1+2a)x_0x_1x_2, \\
g_4 &= a^2x_0^4 + 2ax_0^3x_1 + a(a^2+2)x_0^3x_2 + x_0^2x_1^2 + (2a^2+1)x_0^2x_2^2 + (2+a+2a^2)x_0^2x_1x_2 + \\
&\quad (1+a)x_0x_1^2x_2 + (3a+1)x_0x_1x_2^2 + ax_0x_2^3 + ax_1^2x_2^2 + x_1x_2^3, \\
g_5 &= ax_0 + x_2, \\
g_6 &= ax_0 + x_1 + x_2, \\
g_7 &= a^3x_0^5 + a^2(2+a)x_0^4x_1 + a^2(3+a^2)x_0^4x_2 + a(a+2)x_0^3x_1^2 + 3a(1+a^2)x_0^3x_2^2 + \\
&\quad a(4+3a+3a^2)x_0^3x_1x_2 + (1+3a+2a^2+2a^3)x_0^2x_1^2x_2 + (2+3a+7a^2)x_2^2x_1x_2^2 + \\
&\quad ax_0^2x_1^3 + (3a^2+1)x_0^2x_2^3 + 2a^2x_0x_1^3x_2 + (1+2a+3a^2+a^3)x_0x_1^2x_2^2 + (1+5a)x_0x_1x_2^3 + \\
&\quad ax_0x_2^4 + a^3x_1^3x_2^2 + a(a+1)x_1^2x_2^3 + x_1x_2^4.
\end{aligned}$$

The jacobian of f is $j_f = 9\, x_2\, {g_1}^2\, g_2\, {g_3}^2\, g_4\, g_5\, g_6\, g_7$. So, we have the following eight exceptional curves S_i for $i = 1, 2, \ldots, 8$:

$$S_1 := \{x_2 = 0\},\ S_2 := \{g_5 = 0\},\ S_3 := \{g_6 = 0\},\ S_4 := \{g_2 = 0\},$$
$$S_5 := \{g_4 = 0\},\ S_6 := \{g_7 = 0\},\ S_7 := \{g_1 = 0\}, S_8 := \{g_3 = 0\}.$$

Each S_i collapses to A_i where $A_i \in \mathcal{I}(f^{-1})$ for $i = 1, 2, \ldots, 7$ and S_8 collapses to A_1 too, where:

$$A_1 := [0:0:1]\,, A_2 := [-1:0:1]\,, A_3 := [a:-1:0]\,, A_4 := [0:a:-1]\,,$$

$$A_5 := [1:0:-a]\,, A_6 := [1:-a:0]\,, A_7 := [0:1:0].$$

The indeterminacy points of f are

$$O_1 := [0:0:1]\,, O_2 := [1:0:-a]\,, O_3 := [-a:0:1]\,, O_4 := [0:1:0]\,,$$

$$O_5 := [0:1:-a]\ , O_6 := [-1:1:0]\ , O_7 := [1:-a:0].$$

In order to regularize our f we see that in this family we have the following situation:

$$\begin{aligned}
&S_1 \to A_1 = O_1,\\
&S_2 \to A_2 \in S_4,\\
&S_3 \to A_3 \in S_1,\\
&S_4 \to A_4 \to e_1 \in S_1,\\
&S_5 \to A_5 = O_2,\\
&S_6 \to A_6 = O_7,\\
&S_7 \to A_7 = O_4,\\
&S_8 \to A_1 = O_1,
\end{aligned}$$

where $e_1 = [-a^2 : 1 : 0]$. Therefore we need to blow up A_1, A_5, A_6 and A_7. Let X be the space we get after blowing up the points A_1, A_5, A_6 and A_7 and let E_1, E_2, E_3 and E_4 be the corresponding exceptional fibres. Let $\tilde{f}$ be the extension of f on X.

We begin by determining $\tilde{f}$ on the curves S_1, S_7 and S_8. After some calculations we see that $\tilde{f}$ sends $S_1 = \{x_2 = 0\}$ to the fibre E_1 in the following way:

$$[x_0 : x_1 : 0] \in S_1 \to [x_1 : a^2 x_0]_{E_1},$$

and that S_7 and S_8 are still excepcional for $\tilde{f}$, because

$$S_7 \to [1 : -a]_{E_4} \quad \text{and} \quad S_8 \to [a : -1]_{E_1}.$$

On the other hand, on E_1 and E_4, $\tilde{f}$ acts as:

$$\tilde{f}[u:v]_{E_1} = [v:u]_{E_4} \quad \text{and} \quad \tilde{f}[u:v]_{E_4} = [v:0:a\,u] \in \{x_1 = 0\} := T_1 \in \mathcal{E}(f^{-1}).$$

The mapping $\tilde{f}$ also sends E_2 and E_3 on some excepcional curves of f^{-1}, which we call T_2 and T_3 respectively.

To determine $\tilde{f}$ on S_5 and S_6 we observe that the algebraic curves which define S_5 and S_6 have genus $g = 0$. In fact, $S_5 = \{[x_0, x_1, x_2] : g_4 = 0\}$ can be parametrized trough $\varphi(t)$ where

$$\begin{aligned}
\varphi(t) \;=\; & [(t+a^2)(t+t^2+a+ta) : -1+3\,a+t+t^2-4\,a^3+a^5+ta^4+ta-2\,t^2a\\
& -2\,ta^3-t^3a-4\,ta^2-2\,t^2a^2 : -(a^2+2\,a+2\,ta-1+t+t^2)(t+t^2+a+ta)],
\end{aligned}$$

and $S_6 = \{[x_0, x_1, x_2] : g_7 = 0\}$ can be parametrized through $\phi(t)$ where the three components of $\phi(t)$ are:

$$(a-1)\,((a-1)\,t+a^4-a^3-a^2+a-1)((a-1)\,t-a-a^2+a^3)(\left(2\,a-a^2-1\right)t^2+\left(a^2-1\right)t+a^5-2\,a^4+2\,a^2-2\,a),$$

$$(1-a)\left(p_3(a)t^3+p_2(a)t^2+p_1(a)t+p_0(a)\right)\left((a^2-a)t+a^3-a^2-2a+1\right),$$

and

$$(-1+(a-1)\,t)\left((2\,a-a^2-1)t^2+(-a^3-a+3\,a^2-1)\right)\left(t+a^5-3\,a^4+3\,a^2+2\,a^3+1-5\,a\right)$$

where

$$\begin{aligned}
p_3(a): &= -a^3+3\,a^2-3\,a+1,\\
p_2(a): &= a^3-3\,a+2,\\
p_1(a): &= a^7-a^6-5\,a^5+9\,a^4-a^3-9\,a^2+6\,a\\
p_0(a): &= a^9-3\,a^8+a^7+4\,a^6-5\,a^5+4\,a^4-a^3-4\,a^2+5\,a-1.
\end{aligned}$$

From these parametrizations it is very easy to see that f maps S_5 to $A_5=[1:0:-a]$ and S_6 to $A_6=[1:-a:0]$.

On the other hand to determine $\tilde{f}$ on S_5, we consider the following perturbation:

$$\lim_{s\to 0}[\varphi_1(t)+sh_1(s),\varphi_2(t)+sh_2(s),\varphi_3(t)+sh_3(s)],$$

where h_1,h_2,h_3 are analytic functions on s near $s=0$. Applying F we get:

$$\tilde{f}(\varphi(t))=[1:(1+t)s+o(2):-a+(a^2-1)(t+a+1)s+o(2)]\;\equiv[(1+t):(a^2-1)(t+a+1)]_{E_2},$$

where $o(2)$ means terms of ordre two in s.

In a similar way, the expression of $\tilde{f}$ on S_6 is

$$\tilde{f}(\phi(t))=[\left(a^2-1\right)(-1+(a-1)t):(a-1)t+a^4-a^3-a^2+a-1]_{E_3}.$$

From the above calculations we have the following indeterminacy points of $\tilde{f}$:

$$\mathcal{I}(\tilde{f})=\{O_3,O_5,O_6\}.$$

And the excepcional locus of $\tilde{f}$ is

$$\mathcal{E}(\tilde{f})=\{S_2,S_3,S_4,S_7,S_8\}.$$

Hence we observe the following:

$$\begin{aligned}
S_2\to A_2\to A_4\to e_1\in S_1,\\
S_3\to A_3\in S_1,\\
S_4\to A_4\to e_1\in S_1.
\end{aligned}$$

Also S_7 collapses to the single point $[1:-a]_{E_4}$ and S_8 collapses to $[-a:1]_{E_1}$. We observe that

$$S_1\to E_1\to E_4\to\{x_1=0\}\to\{x_0=0\}\to S_1=\{x_2=0\}\,, \tag{23}$$

where $\tilde{f}[0:x_1:x_2]=[ax_1:x_2:0]$ and $\tilde{f}[x_0:0:x_2]=[0:ax_2:x_0]$. Hence we get that for all

$k \in \mathbb{N}:$

$$
\begin{aligned}
\tilde{f}^{5k}[0 : x_1 : x_2] &= [0 : a^{4k}x_1 : x_2],\\
\tilde{f}^{5k}[x_0 : 0 : x_2] &= [x_0 : 0 : a^{4k}x_2],\\
\tilde{f}^{5k}[x_0 : x_1 : 0] &= [a^{4k}x_0 : x_1 : 0],\\
\tilde{f}^{5k}[u : v]_{E_1} &= [u : a^{4k}v]_{E_1},\\
\tilde{f}^{5k}[u : v]_{E_4} &= [a^{4k}u : v]_{E_4}.
\end{aligned}
$$

Therefore by using the cycle (23) and by the help of the above equations we have that for all $k \in \mathbb{N}:$

$$
\begin{aligned}
\tilde{f}^{5k}(A_3) &= [-a^{4k+1} : 1 : 0],\\
\tilde{f}^{5k}(e_1) &= [-a^{4k+2} : 1 : 0],\\
\tilde{f}^{5k}[-a : 1]_{E_1} &= [1 : -a^{4k-1}]_{E_1},\\
\tilde{f}^{5k}[1 : -a]_{E_4} &= [-a^{4k-1} : 1]_{E_4}.
\end{aligned}
$$

Above equations show that if $a \neq 0$ and $a^p \neq 1$ for all $p \in \mathbb{N}$, then the excepcional curves S_2, S_3, S_4, S_7 and S_8 can never reach any indeterminacy point of $\tilde{f}$. It is now clear that $\tilde{f}$ is AS. Thus we have the following situation:

$$
\begin{aligned}
&S_1 \to E_1 \to E_4,\\
&S_5 \to E_2 \to T_1,\\
&S_6 \to E_3 \to T_2,\\
&S_7 \to E_4 \to T_3,\\
&S_8 \to E_1,
\end{aligned}
$$

where $\{T_1, T_2, T_3\} \subset \mathcal{E}(f^{-1})$.

To compute the matrix of $\tilde{f}^* : Pic(X) \longrightarrow Pic(X)$ we first compute

$$\pi^* : Pic(P\mathbb{C}^2) = <\hat{L}> \longrightarrow Pic(X) = <\hat{L}, E_1, E_2, E_3, E_4> .$$

We can see the degrees of the exceptional curves and the multiplicities of the base points passing through them in the following table:

From the values of table 1 we have the following:

$$
\begin{aligned}
&\tilde{f}^*(\hat{L}) = 9\hat{L} - 2E_1 - 5E_2 - 4E_3 - 4E_4,\\
&\tilde{f}^*(E_1) = \hat{S}_1 + \hat{S}_8 = 4\hat{L} - E_1 - 2E_2 - 2E_3 - 2E_4,\\
&\tilde{f}^*(E_2) = \hat{S}_5 = 4\hat{L} - E_1 - 2E_2 - 2E_3 - 2E_4,\\
&\tilde{f}^*(E_3) = \hat{S}_6 = 5\hat{L} - E_1 - 3E_2 - 2E_3 - 2E_4,\\
&\tilde{f}^*(E_4) = \hat{S}_7 + E_1 = 2\hat{L} - E_2 - E_3 - E_4.
\end{aligned}
$$

Table 1: Multiplicities at Indeterminacy points of Exceptional curves and $f^{-1}(L)$.

Curves	degree	O_1	O_2	O_4	O_7
S_1	1	0	0	1	1
S_5	4	1	2	2	2
S_6	5	1	3	2	2
S_7	2	1	1	1	1
S_8	3	1	2	1	1
$f^{-1}(L)$	9	2	5	4	4

Hence the matrix $\tilde{f}^*$ can be seen as:

$$(\tilde{f}^*) = \begin{pmatrix} 9 & 4 & 4 & 5 & 2 \\ -2 & -1 & -1 & -1 & 0 \\ -5 & -2 & -2 & -3 & -1 \\ -4 & -2 & -2 & -2 & -1 \\ -4 & -2 & -2 & -2 & -1 \end{pmatrix}.$$

The characteristic polynomial of $(\tilde{f}^*)$ is $\lambda^2\left(\lambda^3 - 3\lambda^2 + \lambda - 1\right)$ which has the unique real non-zero root

$$\lambda_1 \cong 2.769292354$$

and the other roots have modulus less than λ_1. Hence proposition 4 is proved. ∎

We can see that the sequence of the degrees d_n exactly satisfies the recurrence

$$d_{n+3} = 3d_{n+2} - d_{n+1} + d_n.$$

Since $d_1 = 9, d_2 = 25$ and $d_3 = 67$, the sequence of the degrees is

$$9\,,\ 25\,,\ 67\,,\ 185\,,\ 513\,,\ 1093\,,\ 2951\,,\ldots$$

Proof of Proposition 5. We now illustrate an example for the case $k = 11$. Consider the 1-parametric family of mappings $F_{a,1,a,a,1,a,1,a,a,1,a}$:

$$F_{a,1,a,a,1,a,1,a,a,1,a}(x,y) = \left(y, \frac{(a+y)^3}{x(ay+1)^2}\right).$$

Let f be the extension of the above mapping to $P\mathbb{C}^2$. Then the homogeneous components of $f[x_0, x_1, x_2] = [f_1 : f_2 : f_3]$ are the following:

$$\begin{aligned} f_1 &= x_0(x_0^2 x_1 + 2a x_0 x_1 x_2 + a^2 x_1 x_2^2), \\ f_2 &= x_1 x_2 (x_0 + a x_2)^2, \\ f_3 &= x_0 (a x_0 + x_2)^3. \end{aligned}$$

Since the jacobian of this mapping is:

$$j_f = 4x_1x_0(ax_0+x_2)^3(x_0+ax_2)^4,$$

we have the following four exceptional curves S_i where we define S_i as $g_i[x_0,x_1,x_2]=0$ for $i \in \{1,2,3\}$:

$$\begin{aligned}S_1 &= \{x_1=0\},\\ S_2 &= \{x_0=0\},\\ S_3 &= \{ax_0+x_2=0\},\\ S_4 &= \{x_0+ax_2=0\}.\end{aligned}$$

Each S_i for $i \in \{1,2,3\}$ collapses to A_i where $A_i \in \mathcal{I}(f^{-1})$ and they are as follows:

$$A_1 := [0:0:1]\,, A_2 := [0:1:0]\,, A_3 := [1:-a:0]\,,$$

and S_4 goes to A_1 too. We observe that all these g_i's are distinct and irreducible for the required values of a. The indeterminacy points of f are

$$O_1 := [0:0:1]\,, O_2 := [0:1:0]\,, O_3 := [1:0:-a]\,.$$

In order to regularize our f we see that in this family we have the following situation:

$$\begin{aligned}&S_1 \to A_1 = O_1,\\ &S_2 \to A_2 = O_2,\\ &S_3 \to A_3 \in \{x_2=0\} \in \mathcal{E}(f^{-1}),\\ &S_4 \to A_1 = O_1.\end{aligned}$$

We see that $\{x_2=0\} \to [x_1:0:x_0] \in S_1$. Therefore we need to blow up A_1, A_2 and then also follow the orbits of S_3, S_4 to see if they reach any indeterminacy point of f. Let X be the space we get after blowing up the points A_1, A_2 and let E_1, E_2 be the corresponding exceptional fibres. Let $\tilde{f}$ be the extension of f on X. To determine the $\tilde{f}$ on S_1, S_2, E_1, E_2 after some calculations, we see that under the action of $\tilde{f}$:

$$\begin{aligned}&S_1 \to [x_0:x_2]_{E_1},\\ &S_2 \to [a^2x_1:x_2]_{E_2},\\ &\tilde{f}[u:v]_{E_1} = [0:a^2v:u] \in S_2,\\ &\tilde{f}[u:v]_{E_2} = [u:v:0] \in \{x_2=0\}.\end{aligned}$$

From the above calculations we have the following indeterminacy point of $\tilde{f}$:

$$\mathcal{I}(\tilde{f}) = \{\pi^{-1}(O_3)\}.$$

And the excepcional locus of $\tilde{f}$ is

$$\mathcal{E}(\tilde{f}) = \{\ \pi^{-1}(S_3), \pi^{-1}(S_4)\}.$$

We observe that $S_3 \to A_3 \to [a:0:-1] \to [a:-1]_{E_1}$. Now we are going to perform the calculations on $S_4 = \{x_0 + a\,x_2 = 0\}$. By calling $t = x_0 + a\,x_2$, the points of S_4 can be described as $\lim_{t\to 0}[t - a\,x_2 : x_1 : x_2]$. And

$$F[t-a\,x_2:x_1:x_2] = [-a\,x_1\,t^2 + o(t^3) : x_1\,t^2 + o(t^3) : (a^2-1)^6\,a\,x_2^3 + o(t)] \ \equiv\ [a:-1]_{E_1} \qquad (24)$$

Hence $\tilde{f}$ mappings $S_4 \to [a : -1]_{E_1}$, and we observe that it does so with multiplicity 2 if $a \notin \{-1, 0, 1\}$.

Thus we now need to follow the orbit of the point $[a : -1]_{E_1}$ to see if S_3 and S_4 can reach any indeterminacy point of $\tilde{f}$. By doing the same calculations as we did in the case $k = 4$ we observe the following:

$$\tilde{f}^{5k}[a : -1]_{E_1} = [-1 : a^{4k-1}]_{E_1} \quad \forall k \in \mathbb{N}.$$

From the previous calculation it is clear that for of $a \notin \{-1, 0, 1\}$ the curves $\pi^{-1}(S_3)$ and $\pi^{-1}(S_4)$ can never reach any indeterminacy point of $\tilde{f}$. It is now clear that $\tilde{f}$ is AS. Thus we have the following situation:

$$S_1 \to E_1,$$
$$S_2 \to E_2,$$
$$S_4 \to E_1.$$

To compute the matrix of $\tilde{f}^* : Pic(X) \longrightarrow Pic(X)$ we first compute

$$\pi^* : Pic(P\mathbb{C}^2) = < \hat{L} > \longrightarrow Pic(X) = < \hat{L}, E_1, E_2 > .$$

Now $A_1 \in S_1$, $A_1, A_2 \in S_2$ and $A_2 \in S_4$. As we said before S_4 reaches the point $[a : -1]_{E_1}$ with multiplicity 2. Whereas the point $A_2 \in f^{-1}(L)$ has multiplicity 3. Thus we have the following set of equations:

$$\tilde{f}^*(\hat{L}) = 4\hat{L} - E_1 - 3E_2,$$
$$\tilde{f}^*(E_1) = \hat{S}_1 + 2\hat{S}_4 = 3\hat{L} - E_1 - 2E_2,$$
$$\tilde{f}^*(E_2) = \hat{S}_2 = \hat{L} - E_1 - E_2.$$

The matrix of $\tilde{f}^*$ thus can be seen as:

$$(\tilde{f}^*) = \begin{pmatrix} 4 & 3 & 1 \\ -1 & -1 & -1 \\ -3 & -2 & -1 \end{pmatrix}.$$

The characteristic polynomial of $(\tilde{f}^*)$ is $\lambda^3 - 2\lambda^2 - 3\lambda - 1$ which has the unique real root

$$\lambda_1 \cong 3.079595625$$

and other roots with modulus less than λ_1. Hence proposition 5 is proved. ∎

In this case the sequence of the degrees d_n satisfies the recurrence

$$d_{n+3} = 2d_{n+2} + 3d_{n+1} + d_n$$

and since $d_1 = 4, d_2 = 10$ and $d_3 = 33$, the sequence of the degrees is

$$4\,,\,10\,,\,33\,,\,100\,,\,309\,,\,951\,,\,2929\,,\ldots$$

5 Proof of main Theorem

Lemma 6. *Let $\delta(F)$ be the dynamical degree of the birational mapping F. Then $\delta(F^2) = \delta(F)^2$.*

Proof

The dynamical degree of F is given as

$$\delta(F) = \lim_{n\to\infty} (d_n)^{\frac{1}{n}}.$$

Let K_n be the degree of mapping $(F^2)^n$. Then the dynamical degree of the mapping F^2 is

$$\delta(F^2) = \lim_{n\to\infty} (K_n)^{\frac{1}{n}}.$$

Observe that the sequence $K_n^{1/n} = d_{2n}^{1/n} = \left(d_{2n}^{1/2n}\right)^2$. Applying limits on both sides as n tends to infinity we get the result. ∎

Proof of Theorem 1

Consider $F^{[4]} := F_{d,c,b,a}$, $F^{[5]} := F_{e,d,c,b,a}$ with generic values of $a, b, c, d, e \in \mathbb{C}$ and $F^{[7]} := F_{a,1,a,1,a,1,a}$, $F^{[11]} := F_{a,1,a,a,1,a,1,a,a,1,a}$ with generic values of $a \in \mathbb{C}$.

From Propositions 2-3-4-5 we know that the dynamical degree of $F^{[4]}, F^{[5]}, F^{[7]}, F^{[11]}$ is greater than one. Hence Theorem 1 is proved for $k \in \{4, 7, 8, 11\}$. In order to find examples of mappings for others values of k, we recall that the Lyness mapping with $a = 1$ is five-periodic and we consider

$$\begin{aligned}
F^{[5m]} &= \left(F^{[5]}\right)^m \ , \ m \geq 1 \\
F^{[5m+1]} &= F^{m-2}_{1,1,1,1,1} \circ F^{[11]} \ , \ m \geq 2, \\
F^{[5m+2]} &= F^{m-1}_{1,1,1,1,1} \circ F^{[7]} \ , \ m \geq 1, \\
F^{[5m+3]} &= F^{m-1}_{1,1,1,1,1} \circ F^{[4]} \circ F^{[4]} \ , \ m \geq 1, \\
F^{[5m+4]} &= F^{m}_{1,1,1,1,1} \circ F^{[4]} \ , \ m \geq 0.
\end{aligned}$$

Using lemma 6 we conclude that the above mappings have dynamical degree greater than one. ∎

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