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COFACTORS AND EQUILIBRIA FOR POLYNOMIAL VECTOR FIELDS

ANTONI FERRAGUT[1] AND JAUME LLIBRE[2]

Abstract. We provide a relationship between the existence of equilibrium points of differential systems and the cofactors of invariant algebraic curves and exponential factors of the system.

1. Introduction and statement of the main results

As usual $\mathbb{C}[x, y]$ denotes the ring of all polynomials in the variables x and y with coefficients in the complex numbers $\mathbb{C}$.

A *complex planar polynomial differential system of degree d* is a differential system of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where $P, Q \in \mathbb{C}[x, y]$ are coprime and d is the maximum of the degrees of the polynomials P and Q. The vector field associated to system (1) is

$$\mathcal{X}(x, y) = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}.$$

Of course we say that $\mathcal{X}$ is a *polynomial vector field of degree d*. In what follows we shall talk indistinctly of the polynomial differential system (1) or of its vector field $\mathcal{X}$.

Let U be an open and dense subset of $\mathbb{C}^2$. A *first integral* of X in U is a locally non–constant analytic function $H : U \to \mathbb{C}$, possibly multi–valued, which is constant on all the solutions of X contained in U, i.e. $\mathcal{X}H = 0$ in the points of U. In this case we also say that X is *integrable* on U.

As usual the points $p = (x, y)$ such that $P(p) = Q(p) = 0$ are called the *equilibria* of the system (1).

Let $f \in \mathbb{C}[x, y]$. The algebraic curve $f = 0$ is *invariant* if there exists a polynomial $K \in \mathbb{C}[x, y]$, called the *cofactor* of f, such that

$$\mathcal{X}f = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = K\, f.$$

Clearly the degree of the cofactor K is at most $d - 1$, see for instance [5].

Clearly if p is an equilibrium point of system (1) then either $K(p) = 0$ or $f(p) = 0$.

Let $f, g \in \mathbb{C}[x, y]$ be coprime polynomials. The function $F = e^{g/f}$ is an *exponential factor* of system (1) if there exists a polynomial $L \in \mathbb{C}[x, y]$ of degree at most $d - 1$, called the *cofactor* of F, such that $\mathcal{X}F = L\, F$. In this case if f is not a constant, then $f = 0$ is an invariant algebraic curve, see [1]. The notion of exponential factor is due to Christopher

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[1]. The exponential factors $e^{g/f}$ appears either when the invariant algebraic curve $f = 0$ has multiplicity greater than one if f is not a constant, or when the invariant straight line of the infinity has multiplicity greater than one if f is a non–zero contant. For more details on exponential factors see [3].

It is clear from the definition of exponential factor that if p is an equilibrium point of system (1) such that $f(p) \neq 0$, then $L(p) = 0$.

Let f and g be two analytic functions. As usual the *Jacobian* of f and g is

$$J(f,g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x},$$

and ∇f denotes the *gradient* of f, i.e. $(\partial f/\partial x, \partial f/\partial y) = (f_x, f_y)$.

If $f = 0$ and $g = 0$ are two invariant algebraic curves of system (1) with f and g coprime polynomials, and $p \in \{f = 0\} \cap \{g = 0\}$, then p is an equilibrium point of system (1).

In this paper we deal with the points which are intersection either of two cofactors, or of one cofactor and one invariant algebraic curve. These cofactors are from invariant algebraic curves or from exponential factors. Our main result is the following theorem.

Theorem 1. *Consider the polynomial differential system* (1) *and let p be a point of $\mathbb{C}^2$. Assume that we have two functions F and G such that*

(*i*) *either $F = 0$ and $G = 0$ are two invariant algebraic curves with $F(p) \neq 0$ and $G(p) \neq 0$;*

(*ii*) *or $F = e^{g_1/f_1}$ and $G = e^{g_2/f_2}$ are two exponential factors with $f_1(p) \neq 0$ and $f_2(p) \neq 0$;*

(*iii*) *or $F = 0$ is an invariant algebraic curve with $F(p) \neq 0$ and $G = e^{g/f}$ is an exponential factor with $f(p) \neq 0$.*

Then the following statements hold.

(*a*) *Let K_F and K_G be the cofactors of F and G, respectively. If $p \in \{K_F = 0\} \cap \{K_G = 0\}$ is such that $J(F,G)(p) \neq 0$, then p is an equilibrium point of system* (1).

(*b*) *In case* (*iii*) *if $f = F$ then the Jacobian condition $J(F,G)(p) \neq 0$ reduces to $J(F,g)(p) \neq 0$.*

(*c*) *Let $F = 0$ be an invariant algebraic curve and let K_G be the cofactor of either an invariant algebraic curve $G = 0$, or an exponential factor $G = e^{g/f}$ with $f(p) \neq 0$. If $p \in \{F = 0\} \cap \{K_G = 0\}$ is such that $J(F,G)(p) \neq 0$, then p is an equilibrium point of system* (1).

(*d*) *The condition $J(F,G)(p) \neq 0$ in statement* (*a*) *is necessary in the three cases* (*i*), (*ii*) *and* (*iii*). *Also the condition $J(F,G)(p) \neq 0$ in statement* (*c*) *is necessary.*

Theorem 1 is proved in Section 2, where we provide examples of polynomial differential systems satisfying all the different assumptions and statements.

Theorem 1 can be extended to polynomial differential systems in $\mathbb{C}^n$ having n functions each one of them being either an invariant algebraic hypersurface or an exponential factor, see [6] for more details on invariant algebraic hypersurfaces and exponential factors of polynomial differential systems in $\mathbb{C}^n$.

Let $f = 0$ and $g = 0$ be two algebraic curves without singular points such that if they intersect at some point, then the intersection is transversal. Suppose that $(f_x, f_y) = 1$ and

$(g_x, g_y) = 1$. According to Lemma 7 in [2], if $f = 0$ and $g = 0$ are invariant algebraic curves of a polynomial differential system (1), then this system can be written as

$$\dot{x} = Afg - Ef_yg - Ffg_y, \quad \dot{y} = Bfg + Ef_xg + Ffg_x, \tag{2}$$

for some polynomials A, B, E, F. Let $\mathcal{Y}$ be vector field associated to the differential system $\dot{x} = A$, $\dot{y} = B$.

Proposition 2. *The following statements hold for the polynomial differential system* (2)*.*

(*a*) *The cofactors of the invariant algebraic curve $f = 0$ and $g = 0$ of system* (2) *are*

$$K_f = \mathcal{Y}(f)g - FJ(f, g) \quad \text{and} \quad K_g = \mathcal{Y}(g)f - FJ(f, g),$$

respectively, where $J(f, g)$ is the Jacobian of f and g.

(*b*) *The rational function f/g is a first integral of system* (2) *if and only if it is a first integral of the vector field $\mathcal{Y}$.*

Proposition 2 is proved in Section 3.

2. Proof of Theorem 1

Proof of statement (*a*) *of Theorem* 1. From the definition of invariant algebraic curve and exponential factor, we have

$$PF_x + QF_y = K_F F,$$
$$PG_x + QG_y = K_G G.$$

Evaluating these equalities at the point p, we have

$$\begin{pmatrix} F_x(p) & F_y(p) \\ G_x(p) & G_y(p) \end{pmatrix} \begin{pmatrix} P(p) \\ Q(p) \end{pmatrix} = A \begin{pmatrix} P(p) \\ Q(p) \end{pmatrix} = 0.$$

Since $J(F, G)(p) \neq 0$, the unique solution of the linear system $A\mathbf{z} = 0$ with $\mathbf{z} \in \mathbb{C}^2$ is the zero solution. Hence $P(p) = Q(p) = 0$, and consequently p is an equilibrium point. □

Proof of statement (*b*) *of Theorem* 1. From the equalities

$$\begin{aligned} J(F, e^{g/F}) &= \frac{e^{g/F}}{F^2}\left[F_x(g_yF - gF_y) - F_y(g_xF - gF_x)\right] \\ &= \frac{e^{g/F}}{F}\left[F_xg_y - F_yg_x\right] = \frac{e^{g/F}J(F, g)}{F}, \end{aligned}$$

statement (*b*) follows immediately. □

Proof of statement (*c*) *of Theorem* 1. The proof is completely similar to the proof of statement (a). □

Before proving statement (d) we shall provide examples of polynomial differential systems satisfying all the different assumptions and statements (a), (b) and (c) of Theorem 1.

The following example satisfies all the assumptions of statement (*a*) under the assumption (*i*) of Theorem 1.

Example 1. *Consider the cubic polynomial differential system*

$$\dot{x} = x(1-2x+x^2+xy+y^2), \qquad \dot{y} = y(-x+x^2+xy+y^2).$$

The straight lines $F = x = 0$ and $G = y = 0$ are invariant for this system. Their respective cofactors are $K_F = 1-2x+x^2+xy+y^2$ and $K_G = -x+x^2+xy+y^2$. We note that for the point $p = (1,-1)$ we have $F(p), G(p) \neq 0$, $p \in \{K_f = 0\} \cap \{K_g = 0\}$ and $J(F,G)(p) = 1$. Of course p is an equilibrium point.

We shall use next lemma later on.

Lemma 3. *Consider the polynomial differential system of degree 8*

$$\begin{aligned} \dot{x} &= 3-6x^4+3x^8-4x^4y^3-4x^5y^3-y^4+x^4y^4-4y^7+2y^8, \\ \dot{y} &= -2+9x^4+4x^7-3x^8+4y^4+4x^3y^4-9x^4y^4-4x^3y^5-2y^8. \end{aligned} \tag{3}$$

This system has the invariant algebraic curve $F = x^4+y^4-1 = 0$ and the three exponential factors $G_1 = e^{\frac{g}{F}}$, $G_2 = e^x$ and $G_3 = e^y$ where $g = -1-x+xy^4+y^5$. Their cofactors are

$$\begin{aligned} K_F &= 4(3x^7-7y^3x^4-2y^4x^3-3x^3-2y^7+2y^3), \\ K_{G_1} &= -4y^7-12x^3y^5-24x^4y^4+20x^3y^4+9y^4-12x^5y^3+16x^4y^3 \\ &\quad -8y^3+9x^4+12x^3+3, \\ K_{G_2} &= 3x^8-4y^3x^5+y^4x^4-4y^3x^4-6x^4+2y^8-4y^7-y^4+3, \\ K_{G_3} &= -3x^8+4x^7-9y^4x^4+9x^4-4y^5x^3+4y^4x^3-2y^8+4y^4-2. \end{aligned}$$

Proof. It follows easily from direct computations. □

Now we provide an example satisfying all the assumptions of statement (a) under the assumption (ii) of Theorem 1.

Example 2. *Consider the planar polynomial differential system* (3). *The point $p = (-0.927255..., 0.630709...)$ belongs to $\{K_{G_2} = 0\} \cap \{K_{G_3} = 0\}$ and $J(F,G)(p) = 0.743381....$ Note that the denominators f_k of $G_k = e^{g_k/f_k}$ can be taken equal to 1. Moreover p is an equilibrium point of system* (3).

The next example satisfies statement (a) under the assumption (iii) and statement (b) of Theorem 1.

Example 3. *Consider the cubic polynomial differential system*

$$\begin{aligned} \dot{x} &= -3-6x-2x^2+6y+8xy-4y^2, \\ \dot{y} &= 6+8x+3x^2-12y-6xy+5y^2. \end{aligned}$$

This system has the invariant straight line $F = x-y+1 = 0$ and the exponential factor $G = e^{(x+3y-2y^2)/F}$. Their respective cofactors are $K_F = -9-5x+9y$ and $K_G = 15+12x-12y-12xy+2y^2$. The intersection of the cofactors consists of the two points

$$p_{\pm} = \left(\frac{9}{98}\left(-4 \pm \sqrt{114}\right), \frac{1}{98}\left(78 \pm 5\sqrt{114}\right) \right).$$

It is easy to check that $F(p_\pm), G(p_\pm) \neq 0$, $J(F,G)(p_\pm) \neq 0$ and that $p_\pm$ are two of the equilibria of this differential system.

The following example satisfies statement (c) of Theorem 1 when $G = 0$ is an invariant algebraic curve.

Example 4. *Consider the polynomial differential system*

$$\begin{aligned}\dot{x} &= 2 + x + 3y - 2x^2 - 2x^3 - 2x^2y - xy^2 + y^3,\\ \dot{y} &= (x-y)(1+y^2).\end{aligned} \tag{4}$$

The straight lines $F = 1 + x + y = 0$ *and* $G = 1 + y^2 = 0$ *are invariant, their respective cofactors are* $K_F = 2(1 - x^2)$ *and* $K_G = 2(x - y)y$. *The point* $p = (-1/2, -1/2)$ *belongs to* $\{F = 0\} \cap \{K_G = 0\}$ *and* $J(F, G)(p) = -1 \neq 0$. *Moreover* p *is an equilibrium point of system* (4).

Finally the next example satisfies statement (c) of Theorem 1 when G is an exponential factor.

Example 5. *Consider the polynomial differential system* $\dot{x} = x$, $\dot{y} = y^2(x + y - 1)$. *This system has the invariant algebraic curve* $F = x = 0$ *and the exponential factor* $G = \exp 1/y$ *with cofactor* $K_G = 1 - x - y$. *The point* $p = (0, 1)$ *belongs to* $\{F = 0\} \cap \{K_G = 0\}$ *and* $J(F, G)(p) = -e \neq 0$. *Moreover it is easy to check that* p *is an equilibrium point of the system.*

Next we prove statement (d) of Theorem 1.

Proof of statement (d) *of Theorem* 1. We shall provide examples of all the situations in which the condition on the Jacobian does not hold and the corresponding statements of Theorem 1 fails.

First we claim that under the assumption (i) we need that $J(F, G)(p) \neq 0$ in order that the conclusion of statement (a) holds. Now we prove the claim. Consider the polynomial differential system (4). We note that for $p = (1, 0)$ we have $F(p), G(p) \neq 0$ and $p \in \{K_F = 0\} \cap \{K_G = 0\}$. Moreover, $\nabla G(p) = 0$ and consequently $J(F, G)(p) = 0$. Finally, it is easy to check that p is not an equilibrium point of the differential system. Hence the claim is proved.

Now we claim that under the assumption (ii) we need that $J(F, G)(p) \neq 0$ in order that the conclusion of statement (a) holds. We shall prove the claim. Consider the polynomial differential system (3) and the point $p = (-1.061785..., 0.795990...)$ of $\{K_{G_1} = 0\} \cap \{K_{G_2} = 0\}$. If we write $G_k = e^{g_k/f_k}$ then $f_1(p) = F(p) = 0.672453...$ and $f_2(p) = 1$. Moreover $J(F, G)(p) = 0$ and it is easy to check that p is not an equilibrium point of the differential system. So the claim is proved.

We claim that under the assumption (iii) we need that $J(F, G)(p) \neq 0$ in order that the conclusion of statement (a) holds. We shall prove the claim. Consider the polynomial differential system $\dot{x} = x(x - 2y + y^2)$, $\dot{y} = (-1 + x)y^2$ and the point $p = (1, 2)$. This system has the invariant algebraic curve $F = x(y - x) = 0$ with cofactor $K_F = 2x - 3y + y^2$ and the exponential factor $G = \exp 1/y$ with cofactor $K_G = 1 - x$. Observe that $K_F(p) = K_G(p) = 0$. Moreover $F(p) \neq 0$, $J(F, G)(p) = 0$ and it is easy to check that p is not an equilibrium point of the differential system. So the claim is proved.

We claim that under the assumptions of statement (c) when $G = 0$ is an invariant algebraic curve we need that $J(F, G)(p) \neq 0$ in order that the conclusion of statement (c) holds. We shall prove the claim. Consider the polynomial differential system (4) and the point $p = (-1, 0)$ belongs to $\{F = 0\} \cap \{K_G = 0\}$ and $J(F, G)(p) = 0$. Then it is easy to check that p is not an equilibrium point of system (4).

Finally we claim that under the assumptions of statement (c) when G is an exponential factor we need that $J(F, G)(p) \neq 0$ in order that the conclusion of statement (a) holds.

We shall prove the claim. Consider the polynomial differential system (3) and the point $p = (1, 0)$ belongs to $\{F = 0\} \cap \{K_G = 0\}$ and $J(F, G)(p) = 0$. Then it follows easily that p is not an equilibrium point of system (4). □

3. Proof of Proposition 2

In this section we prove the two statements of Proposition 2.

Proof of statement (a) of Proposition 2. From the definition of invariant algebraic curve and from system (2) we have

$$\begin{aligned} K_f f = & \; P f_x + Q f_y \\ = & \; A f g f_x - F f f_x g_y + B f g f_y + F f f_y g_x \\ = & \; (A f_x + B f_y) f g - F f (f_x g_y - f_y g_x), \\ = & \; (\mathcal{Y} f) f g - F f J(f, g). \end{aligned}$$

Therefore $K_f = (\mathcal{Y} f) g - F J(f, g)$. A similar expression can be obtained for the invariant algebraic curve $g = 0$. □

For proving statement (b) of Proposition 2 we need the following auxiliary result due to Darboux, see for more details statement (i) of Theorem 8.7 of [4].

Lemma 4. *Suppose that the polynomial differential system* (1) *has two invariant algebraic curves* $f = 0$ *and* $g = 0$ *with cofactors* K_f *and* K_g *such that* $K_f = K_g$. *Then* f/g *is a first integral of system* (1).

Proof of statement (b) of Proposition 2. Assume that f/g is a first integral of the vector field $\mathcal{Y}$. Then

$$\mathcal{Y}\left(\frac{f}{g}\right) = \frac{(\mathcal{Y} f) g - (\mathcal{Y} g) f}{g^2} = \frac{K_f + F J(f, g) - K_g - F J(f, g)}{g^2} = \frac{K_f - K_g}{g^2} = 0. \qquad (5)$$

So $K_f = K_g$. Therefore, since $f = 0$ and $g = 0$ are invariant algebraic curves of system (2) by Lemma 4 we get that f/g is a first integral of system (2).

Suppose now that f/g is a first integral of system (2). If we denote by $\mathcal{X}$ the vector field associated to system (2) and taking into account that $f = 0$ and $g = 0$ are invariant algebraic curves of $\mathcal{X}$, we have

$$\mathcal{X}\left(\frac{f}{g}\right) = \frac{(\mathcal{X} f) g - (\mathcal{X} g) f}{g^2} = \frac{K_f f g - K_g f g}{g^2} = \frac{(K_f - K_g) f}{g} = 0.$$

Therefore $K_f = K_g$. Hence, from the computations (5) it follows that $\mathcal{Y}(f/g) = 0$, consequently f/g is a first integral of $\mathcal{Y}$. □

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[1] Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, (ETSEIB), Av. Diagonal, 647, 08028, Barcelona, Catalonia, Spain

E-mail address: `Antoni.Ferragut@upc.edu`

[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, Edifici C, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: `jllibre@mat.uab.cat`