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HAMILTONIAN NILPOTENT CENTERS OF LINEAR PLUS CUBIC HOMOGENEOUS POLYNOMIAL VECTOR FIELDS

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Abstract. We provide normal forms and the global phase portraits in the Poincaré disk for all Hamiltonian nilpotent centers of linear plus cubic homogeneous planar polynomial vector fields.

1. Introduction and statement of the main results

Determining limit cycles and distinguishing when a singular point is either a focus or a center are two of the main problems in the qualitative theory of real planar polynomial differential systems. Poincaré in [19], defines a *center* for a vector field on the real plane as a singular point having a neighborhood filled with periodic orbits with the exception of the singular point.

If an analytic system has a center, then after an affine change of variables and a rescaling of the time variable, it can be written in one of the following three forms:

$$\dot{x} = -y + P(x, y), \quad \dot{y} = x + Q(x, y),$$

called a *linear type center*;

$$\dot{x} = y + P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

called a *nilpotent center*;

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

called a *degenerate center*, where $P(x, y)$ and $Q(x, y)$ are real analytic functions without constant and linear terms, defined in a neighborhood of the origin. An algorithm for the characterization of linear type centers is provided by Poincaré [20] and Lyapunov [15], see also Chazy [5] and Moussu [17]. There is also an algorithm for the characterization of the nilpotent and some class of degenerate centers due to Chavarriga *et al.* [4], Giacomini *et al.* [12], and Cima and Llibre [7].

The classification of the centers of polynomial differential systems goes back to Dulac [9], Kapteyn [13, 14], Bautin [1], Zoladek [24], who studied quadratic systems. Also see Schlomiuk [21] for an update on the quadratic centers. For the centers of polynomial differential systems of degree larger than 2 there are many partial results. For instance the linear type centers for cubic polynomial differential systems of the form linear with homogeneous nonlinearities of degree 3 were characterized by Malkin [16], and by Vulpe and Sibiirski [23]. For polynomial differential systems of the form linear with homogeneous nonlinearities of degree greater than 3 the linear type

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centers are not classified, but there are partial results for degree 4 and 5, see for instance Chavarriga and Giné [2, 3]. On the other hand there is still much to do in order to obtain a complete classification of the centers for all polynomial differential systems of degree 3. Some interesting results on some subclasses of cubic systems are those of Rousseau and Schlomiuk [22], and the ones of Zoladek [25, 26].

In this work we classify the global phase portraits of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a nilpotent center at the origin. To do this we will use the Poincaré compactification of polynomial vector fields, see Section 3. We say that two vector fields on the Poincaré disk are *topologically equivalent* if there exists a homeomorphism from one onto the other which sends orbits to orbits preserving or reversing the direction of the flow. Our main result is the following one.

Theorem 1. *If a Hamiltonian planar polynomial vector field with linear plus cubic homogeneous terms has a nilpotent singular point at the origin, then, after a linear change of variables and a rescaling of its independent variable, it can be written as one of the following six classes:*

$$
\begin{array}{lll}
(I) & \dot{x} = ax + by, & \dot{y} = \left(c - \dfrac{a^2}{b+c}\right) x - ay + x^3, \\
(II) & \dot{x} = ax + by - x^3, & \dot{y} = \left(c - \dfrac{a^2}{b+c}\right) x - ay + 3x^2y, \\
(III) & \dot{x} = ax + by - 3x^2y + y^3, & \dot{y} = \left(c - \dfrac{a^2}{b+c}\right) x - ay + 3xy^2, \\
(IV) & \dot{x} = ax + by - 3x^2y - y^3, & \dot{y} = \left(c - \dfrac{a^2}{b+c}\right) x - ay + 3xy^2, \\
(V) & \dot{x} = ax + by - 3\mu x^2y + y^3, & \dot{y} = \left(c - \dfrac{a^2}{b+c}\right) x - ay + x^3 + 3\mu xy^2, \\
(VI) & \dot{x} = ax + by - 3\mu x^2y - y^3, & \dot{y} = \left(c - \dfrac{a^2}{b+c}\right) x - ay + x^3 + 3\mu xy^2,
\end{array}
$$

where $a, b, c, \mu \in \mathbb{R}$ such that either $c = 0$ or $a = b = 0$, with $b + c \neq 0$. Moreover, if the origin is a center, then its global phase portraits is topologically equivalent to one of the 12 phase portraits of Figure 1.

We note that in the above six systems $(I) - (VI)$ a rescaling of both the dependent and the independent variables allows to assume $a = \pm 1$ whenever $a \neq 0$. However we do not use this simplification since it does not help us in our computations.

The global phase portraits of the linear type centers of the Hamiltonian planar vector fields with linear plus cubic homogeneous terms are classified in [8], see also [11]. In addition, it is shown in [7] that the degenerate centers of such vector fields are topologically equivalent to 1.11 of Figure 1. Therefore, this work completes the classification of all centers of these vector fields.

The normal forms provided in Theorem 1 (and also in [8]) will allow to study how many limit cycles can bifurcate from the periodic orbits of the Hamiltonian centers with only linear and cubic terms when they are

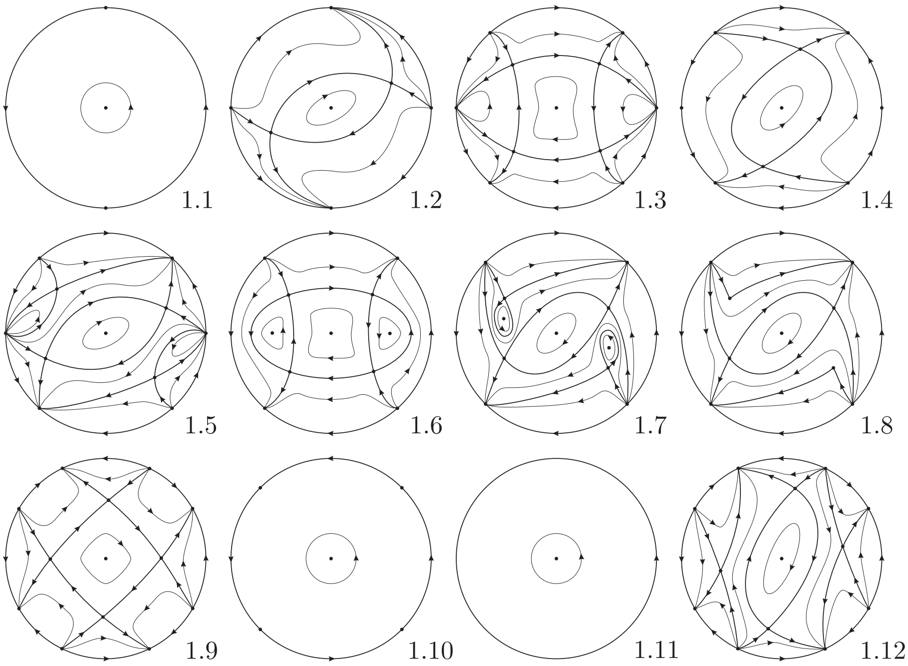


FIGURE 1. Global phase portraits of the vector fields in Theorem 1 which have a nilpotent center at the origin. The separatrices are in bold.

perturbed inside the class of all cubic polynomial differential systems. This last study was made for the quadratic polynomial differential systems, see [6] and the references quoted there.

2. Classification

Doing a linear change of variables and a rescaling of the independent variable, cubic homogeneous systems can be classified into the following ten classes, see [7]:

$$(i) \quad \begin{array}{l} \dot{x} = x(p_1x^2 + p_2xy + p_3y^2), \\ \dot{y} = y(p_1x^2 + p_2xy + p_3y^2), \end{array}$$

$$(ii) \quad \begin{array}{l} \dot{x} = p_1x^3 + p_2x^2y + p_3xy^2, \\ \dot{y} = \alpha x^3 + p_1x^2y + p_2xy^2 + p_3y^3, \end{array}$$

$$(iii) \quad \begin{array}{l} \dot{x} = (p_1 - 1)x^3 + p_2x^2y + p_3xy^2, \\ \dot{y} = (p_1 + 3)x^2y + p_2xy^2 + p_3y^3, \end{array}$$

$$(iv) \quad \begin{array}{l} \dot{x} = p_1x^3 + (p_2 - 3\alpha)x^2y + p_3xy^2, \\ \dot{y} = p_1x^2y + (p_2 + 3\alpha)xy^2 + p_3y^3, \end{array}$$

$$(v) \quad \begin{array}{l} \dot{x} = p_1x^3 + (p_2 - \alpha)x^2y + p_3xy^2 - \alpha y^3, \\ \dot{y} = \alpha x^3 + p_1x^2y + (p_2 + \alpha)xy^2 + p_3y^3, \end{array}$$

$$(vi)\quad \begin{array}{l}\dot{x} = p_1x^3 + (p_2 - 3\alpha)x^2y + p_3xy^2 + y^3,\\ \dot{y} = p_1x^2y + (p_2 + 3\alpha)xy^2 + p_3y^3,\end{array}$$

$$(vii)\quad \begin{array}{l}\dot{x} = p_1x^3 + (p_2 - 3\alpha)x^2y + p_3xy^2 - \alpha y^3,\\ \dot{y} = p_1x^2y + (p_2 + 3\alpha)xy^2 + p_3y^3,\end{array}$$

$$(viii)\quad \begin{array}{l}\dot{x} = p_1x^3 + (p_2 - 3\mu)x^2y + p_3xy^2 + y^3,\\ \dot{y} = x^3 + p_1x^2y + (p_2 + 3\mu)xy^2 + p_3y^3,\end{array} \qquad \mu \in \mathbb{R}.$$

$$(ix)\quad \begin{array}{ll}\dot{x} = p_1x^3 + (p_2 - 3\alpha\mu)x^2y + p_3xy^2 - \alpha y^3, & \mu > -1/3,\\ \dot{y} = \alpha x^3 + p_1x^2y + (p_2 + 3\alpha\mu)xy^2 + p_3y^3, & \mu \neq 1/3,\end{array}$$

$$(x)\quad \begin{array}{l}\dot{x} = p_1x^3 + (p_2 - 3\mu)x^2y + p_3xy^2 - y^3,\\ \dot{y} = x^3 + p_1x^2y + (p_2 + 3\mu)xy^2 + p_3y^3,\end{array} \qquad \mu < -1/3,$$

where $\alpha = \pm 1$. So for studying the cubic planar polynomial vector fields having only linear and cubic terms, it is sufficient to add to the above ten systems a linear part. The following two propositions define the precise forms of the vector fields that we will study.

Proposition 2. *Let X be a cubic planar polynomial vector field having only linear and cubic terms, such that its cubic homogeneous part is given by one of the above ten forms $(i)-(x)$. Then X is Hamiltonian with a Hamiltonian polynomial of degree four if and only if $p_1 = p_2 = p_3 = 0$.*

Proposition 2 is proved in [8]. We note that when the parameters p_1, p_2 and p_3 are all zero, system (i) is not cubic. For this reason, we will restrict our attention to systems $(ii) - (x)$. In addition, in all of these nine Hamiltonian vector fields that we are going to study we can assume $\alpha = 1$ because the systems with $\alpha = -1$ can be obtained from those with $\alpha = 1$ simply by the linear transformation $x \mapsto -x$.

We also note that when $p_1 = p_2 = p_3 = 0$, the only difference between systems (ix) and (x) is the restriction of the parameter μ to the intervals $(-\infty, -1/3)$ and $(-1/3, \infty) \setminus \{1/3\}$. Moreover, when $\mu = 1/3$ system (x) becomes system (v), and when $\mu = -1/3$ it becomes system (iv) via a rotation by $\pi/4$. Therefore we will remove the restriction on μ in system (x) so that it includes systems $(iv), (v)$ and (ix).

We say that a singular point is *non–elementary* if both eigenvalues of the linear part of the vector field at that point are zero, and *elementary* otherwise. A non–elementary singular point is called *degenerate* if its linear part is identically zero, otherwise it is called *nilpotent*.

Proposition 3. *The linear part of each of the ten classes of Hamiltonian cubic planar polynomial vector fields having only linear and cubic homogeneous terms which have a nilpotent singular point at the origin can be chosen to be either*

$$\dot{x} = ax + by, \quad \dot{y} = -(a^2/b)x - ay, \tag{2}$$

or

$$\dot{x} = 0, \quad \dot{y} = cx, \tag{3}$$

where $a, b, c \in \mathbb{R}$ such that $b, c \neq 0$.

Proof. We will give the proof only for system (x) because the remaining cases can be proved in the same way.

Assume that system (x) plus a linear part is Hamiltonian, then by Proposition 2, it can be written as

$$\dot{x} = ax + by - 3\mu x^2 y - y^3,$$
$$\dot{y} = cx + dy + x^3 + 3\mu x y^2,$$

for some real constants a, b, c, d. Since X is Hamiltonian, we have $d = -a$. The eigenvalues of the linear part of this system at the origin are

$$\lambda_{1,2} = \pm\sqrt{a^2 + bc}.$$

In order for the origin to be nilpotent, these eigenvalues must be equal to zero. So, if $b \neq 0$ we get $c = -(a^2/b)$. If $b = 0$ we have $a = 0$ with $c \neq 0$ because the linear part of the system at the origin cannot be zero. □

In order to write these vector fields in a more compact way, we will write their linear part as

$$\dot{x} = ax + by, \quad \dot{y} = \left(c - \frac{a^2}{b+c}\right) x - ay,$$

with $b + c \neq 0$ and the condition that either $c = 0$ or $a = b = 0$.

In short, Hamiltonian planar polynomial vector fields having only linear and cubic terms which have a nilpotent singular point at the origin can be classified into the six vector fields $(I) - (VI)$ given in Theorem 1. Having determined the forms of the vector fields which we are going to study, using the following proposition we will find necessary and sufficient conditions so that their origins are centers.

Proposition 4. *If P and Q are homogeneous polynomials of degree m, then system* (1) *has a nilpotent center or a focus at the origin if and only if m is odd and the coefficient of x^m in Q is negative.*

For more details about Proposition 4 and its proof see [4]. We note that this proposition gives necessary and sufficient conditions in order that systems $(I) - (VI)$ of Theorem 1 have a nilpotent center at the origin. To be able to determine these conditions we need to apply a linear change of variables provided in the following proposition.

Proposition 5. *Systems $(I) - (VI)$ can be written in the form* (1) *after applying the change of variables*

$$X = x, \quad Y = ax + by,$$

or

$$X = y, \quad Y = cx,$$

when $b \neq 0$ or $b = 0$, respectively.

Remark 6. Due to the fact that the right hand sides of each of the vector fields $(I) - (VI)$ are odd functions, their phase portraits are symmetric with respect to the origin.

3. Poincaré compactification

In this section we summarize the Poincaré compactification that we shall use for describing the global phase portraits of our Hamiltonian systems. For more details on the Poincaré compactification see Chapter 5 of [10].

Let $\mathbb{S}^2$ be the set of points $(s_1, s_2, s_3) \in \mathbb{R}^3 : s_1^2 + s_2^2 + s_3^2 = 1$. We will call this set the Poincaré sphere. Given a polynomial vector field

$$X = (\dot{x}, \dot{y}) = (P(x, y), Q(x, y))$$

in $\mathbb{R}^2$, it can be extended analytically to the Poincaré sphere by projecting each point $x \in \mathbb{R}^2 = (x_1, x_2, 1) \in \mathbb{R}^3$ onto the Poincaré sphere using a straight line through x and the origin of $\mathbb{R}^3$. In this way we obtain two copies of X: one on the northern hemisphere $\{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_3 > 0\}$ and another on the southern hemisphere $\{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_3 < 0\}$. The equator $\mathbb{S}^1 = \{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_3 = 0\}$ corresponds to the infinity of $\mathbb{R}^2$. The local charts needed for doing the calculations on the Poincaré sphere are

$$U_i = \{\mathbf{s} \in \mathbb{S}^2 : s_i > 0\}, \quad V_i = \{\mathbf{s} \in \mathbb{S}^2 : s_i < 0\},$$

where $\mathbf{s} = (s_1, s_2, s_3)$, with the corresponding local maps

$$\varphi_i(s) : U_i \to \mathbb{R}^2, \quad \psi_i(s) : V_i \to \mathbb{R}^2,$$

such that $\varphi_i(s) = -\psi_i(s) = \left(\dfrac{s_m}{s_i}, \dfrac{s_n}{s_i}\right)$ for $m < n$ and $m, n \neq i$, for $i = 1, 2, 3$. The expression for the corresponding vector field on $\mathbb{S}^2$ in the local chart U_1 is given by

$$\dot{u} = v^d \left[-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right)\right], \quad \dot{v} = -v^{d+1} P\left(\frac{1}{v}, \frac{u}{v}\right); \tag{4}$$

the expression for U_2 is

$$\dot{u} = v^d \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right)\right], \quad \dot{v} = -v^{d+1} Q\left(\frac{u}{v}, \frac{1}{v}\right); \tag{5}$$

and the expression for U_3 is just

$$\dot{u} = P(u, v), \quad \dot{v} = Q(u, v), \tag{6}$$

where d is the degree of the vector field X. The expressions for the charts V_i are those for the charts U_i multiplied by $(-1)^{d-1}$, for $i = 1, 2, 3$. Hence, to study the vector field X, it is enough to study its Poincaré compactification restricted to the northern hemisphere plus $\mathbb{S}^1$, which we will call the Poincaré disk. To draw the phase portraits, we will consider the projection of the Poincaré disk onto $\mathbb{R}^2$ by $\pi(s_1, s_2, s_3) = (s_1, s_3)$.

Finite singular points of X are the singular points of its compactification which are in $\mathbb{S}^2 \setminus \mathbb{S}^1$, and they can be studied using U_3. Infinite singular points of X, on the other hand, are the singular points of the corresponding vector field on the Poincaré disk lying on $\mathbb{S}^1$. Since $s \in \mathbb{S}^1$ is an infinite singular point whenever $-s \in \mathbb{S}^1$ is, and the local behavior of one is that of the other multiplied by $(-1)^{d-1}$, to study the infinite singular points it suffices to look at $U_1|_{v=0}$ and the origin of U_2.

The next theorem by Neumann [18] gives a characterization of two topologically equivalent vector fields in the Poincaré sphere.

Theorem 7 (Neumann's Theorem)**.** *Two continuous flows in $\mathbb{S}^2$ with isolated singular points are topologically equivalent if and only if their separatrix configurations are equivalent.*

This theorem implies that once the separatrices of a vector field in the Poincaré sphere are determined, the global phase portrait of that vector field is obtained up to topological equivalence.

4. Global phase portraits of system (I)

We first assume that $b = 0$. Then system (I) becomes

$$\dot{x} = 0, \quad \dot{y} = cx + x^3.$$

We see that the origin cannot be a center since the y-axis is invariant.

Next we assume that $b \neq 0$. In this case system (I) becomes

$$\dot{x} = ax + by, \quad \dot{y} = -(a^2/b)x - ay + x^3. \tag{7}$$

and it has the Hamiltonian

$$H_1 = -\frac{x^4}{4} + \frac{a^2x^2}{2b} + \frac{by^2}{2} + axy.$$

Now we will apply Proposition 4 to find the necessary and sufficient conditions for the origin to be a center. After doing the linear change of variables suggested in Proposition 5 system (7) becomes

$$\dot{x} = y, \quad \dot{y} = bx^3.$$

Then by Proposition 4, we see that system (7) has a nilpotent center at the origin if and only if $b < 0$.

Now we will find the global phase portraits of system (7) under the restriction $b < 0$. We first investigate the infinite singular points of this system. Using (4), we see that in the local chart U_1 system (7) becomes

$$\begin{aligned}\dot{u} &= -v^2\left(bu^2 + 2au + a^2/b\right) + 1,\\ \dot{v} &= -v^3\left(bu + a\right).\end{aligned}$$

When $v = 0$, there are no singular points on U_1.

Next we will check whether the origin of the local chart U_2 is a singular point. In U_2 we use (5) to get

$$\begin{aligned}\dot{u} &= v^2\left((a^2/b)u^2 + 2au + b\right) - u^4,\\ \dot{v} &= v^3\left((a^2/b)u + a\right) - u^3v,\end{aligned} \tag{8}$$

and we see that the origin is a singular point and that its linear part is zero. We need to do blow-ups to describe the local behavior at this point. We perform the directional blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$ and have

$$\begin{aligned}\dot{u} =& u^2w^2\left((a^2/b)u^2 + 2au + b\right) - u^4,\\ \dot{w} =& -uw^3(au + b).\end{aligned}$$

We eliminate the common factor u between $\dot{u}$ and $\dot{w}$, and get the vector field

$$\begin{aligned}\dot{u} &= uw^2\left((a^2/b)u^2 + 2au + b\right) - u^3,\\ \dot{w} &= -w^3(au + b).\end{aligned} \tag{9}$$

When $u = 0$, because $b < 0$ the only singular point of system (9) is the origin, whose linear part is again zero. Hence we do another blow-up $(u, w) \to (u, z)$ with $z = w/u$, eliminate the common factor u^2, and get the vector field

$$\begin{aligned} \dot{u} &= uz^2\left((a^2/b)u^2 + 2au + b\right) - u, \\ \dot{z} &= -z^3\left((a^2/b)u^2 + 3au + 2b\right) + z. \end{aligned} \tag{10}$$

When $u = 0$, the unique singular point of system (10) is the origin since $b < 0$. The eigenvalues of the linear part of system (10) at the origin are ± 1, hence it is a saddle. Going back through the changes of variables until system (8) as shown in Figure 2, we see that locally the origin of U_2 consists of two hyperbolic sectors.

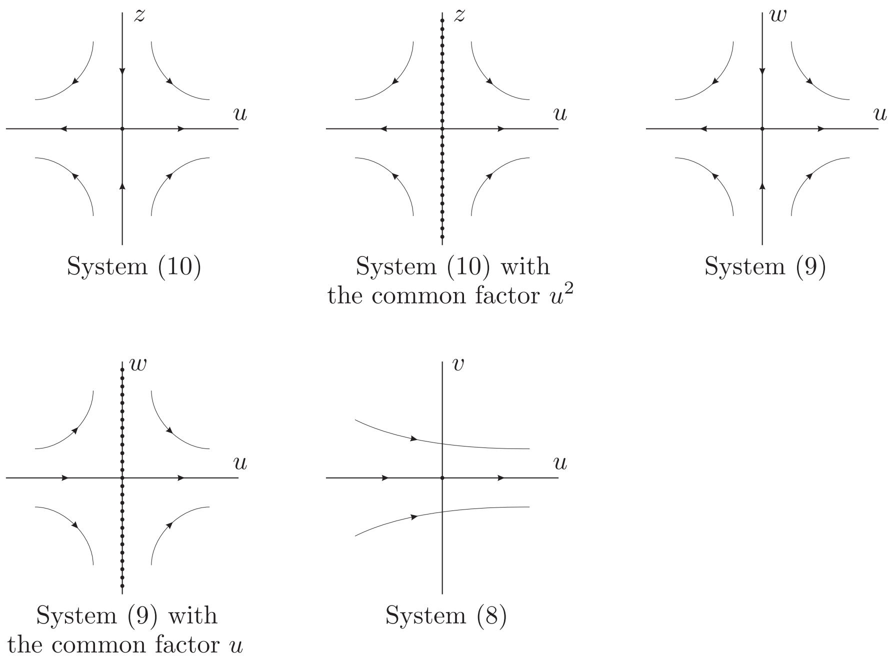


FIGURE 2. Blow-up of the origin of U_2 of system (I) when $b < 0$.

We now look at the finite singular points of system (7) and see that only the origin is singular, which is a center. Therefore the global phase portrait of system (7) is topologically equivalent to the phase portrait 1.1 of Figure 1.

5. Global phase portraits of system (II)

Again we first consider the case $b = 0$. Then system (II) is

$$\dot{x} = x^3, \quad \dot{y} = cx + 3x^2y,$$

The y-axis is an invariant straight line, hence the origin cannot be a center in this case.

Now we consider the case $b \neq 0$. So system (II) writes

$$\dot{x} = ax + by - x^3, \quad \dot{y} = -(a^2/b)x - ay + 3x^2y, \tag{11}$$

and it has the Hamiltonian

$$H_2 = -x^3y + \frac{a^2x^2}{2b} + \frac{by^2}{2} + axy.$$

Note that we can assume $b > 0$ because the linear change $y \mapsto -y$ gives exactly the same system with the opposite sign of the parameter b.

To determine the necessary and sufficient conditions for the origin of system (11) to be a center, using Proposition 5 we rewrite it as

$$\dot{x} = y - x^3, \quad \dot{y} = -4ax^3 + 3x^2y.$$

Then, by Proposition 4 the origin of system (11) is a center if and only if $a > 0$. Therefore we assume $a > 0$ and begin the study of its phase portrait with finding the infinite singular points.

In the local chart U_1 system (11) is

$$\begin{aligned} \dot{u} &= -v^2\left(bu^2 + 2au + a^2/b\right) + 4u, \\ \dot{v} &= -v^3\left(bu + a\right) + v. \end{aligned}$$

When $v = 0$, only the origin of U_1 is singular. The eigenvalues at this point are 4 and 1, meaning that it is a repelling node.

Next, we should check the origin of U_2, in which system (11) becomes

$$\begin{aligned} \dot{u} &= v^2\left((a^2/b)u^2 + 2au + b\right) - 4u^3, \\ \dot{v} &= v^3\left((a^2/b)u + a\right) - 3u^2v. \end{aligned} \tag{12}$$

We see that the origin is singular and its linear part is zero. We need to do blow-up for analyzing the local behavior at this point. Doing the blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$ and eliminating the common factor u we get the system

$$\begin{aligned} \dot{u} &= uw^2\left((a^2/b)u^2 + 2au + b\right) - 4u^2, \\ \dot{w} &= -w^3\left(au + b\right) - uw. \end{aligned} \tag{13}$$

When $u = 0$, the only singular point of system (13) is the origin, whose linear part is again zero. So, we do another blow-up $(u, w) \mapsto (u, z)$ with $z = w/u$, eliminate the common factor u, and obtain

$$\begin{aligned} \dot{u} &= u^2z^2\left((a^2/b)u^2 + 2au + b\right) - 4u, \\ \dot{z} &= -uz^3\left((a^2/b)u^2 + 3au + 2b\right) + 5z. \end{aligned} \tag{14}$$

When $u = 0$, the only singular point of system (14) is the origin, which is a saddle. We trace the changes of variables back to system (12) as shown in Figure 3, and we find out that the origin of U_2 is an attracting node.

Having determined the infinite singular points of system (11), we now compute its finite singular points, which are $\pm\left(\sqrt{4a/3}, \sqrt{4a^3/27b^2}\right)$ plus the origin. The eigenvalues of the linear part of system (11) at these two points are $\pm 4\sqrt{2}a/\sqrt{3}$, which means that they are saddles since $a > 0$.

Now we will determine the global phase portrait according to this local information. The two saddles must be on the boundary of the period annulus of the center at the origin due to the symmetry of the system. Also there are no singular points other than the origin on the axes, on which the

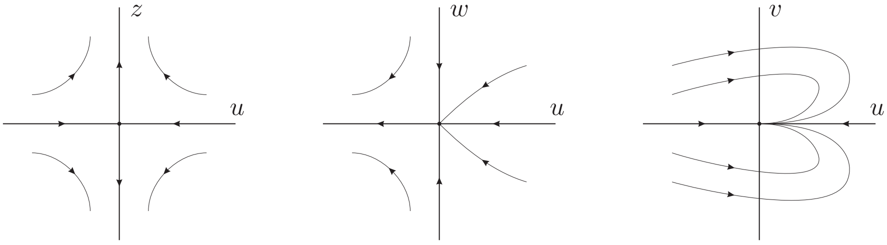


FIGURE 3. Blow-up of the origin of U_2 of system (II) when $a, b > 0$.

Hamiltonian H_2 is quadratic. By the same argument used for system (I), this means that the separatrices passing through saddles cannot cross the axes anymore. Hence we obtain the global phase portrait 1.2 of Figure 1.

6. Global phase portraits of system (III)

We first study the case $b = 0$. Then system (III) becomes

$$\dot{x} = -3x^2y + y^3, \quad \dot{y} = cx + 3xy^2, \tag{15}$$

where $c \neq 0$, and it has the Hamiltonian

$$H_3^1(x, y) = \frac{y^4}{4} - \frac{3x^2y^2}{2} - \frac{cx^2}{2}.$$

Using Proposition 5 we rewrite system (15) as

$$\dot{x} = y + 3x^2y/c, \quad \dot{y} = cx^3 - 3xy^2/c.$$

Then by Proposition 4 the origin is a center if and only if $c < 0$. Hence we assume that $c < 0$.

In U_1 system (15) becomes

$$\dot{u} = cv^2 - u^2(u^2 - 6), \quad \dot{v} = -uv(u^2 - 3). \tag{16}$$

When $v = 0$, there are three singular points on U_1: $(0, 0), (\pm\sqrt{6}, 0)$. The linear part of system (16) is

$$\begin{pmatrix} -4u(u^2-3) & 0 \\ 0 & -u(u^2-3) \end{pmatrix}.$$

Hence the singular points $(\sqrt{6}, 0)$ and $(-\sqrt{6}, 0)$ are attracting and repelling nodes, respectively.

At the origin, however, the linear part is zero. Therefore to describe its local behavior we do the blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$. After eliminating the common factor u between $\dot{u}$ and $\dot{w}$, we obtain the system

$$\dot{u} = cuw^2 - u(u^2 - 6), \quad \dot{w} = -w(cw^2 + 3). \tag{17}$$

When $u = 0$, system (17) has the singular points $(0, 0)$, $\left(0, \pm\sqrt{-3/c}\right)$, all of which are real since $c < 0$. The linear part of system (17) at the points $(0, w)$ is

$$\begin{pmatrix} cw^2+6 & 0 \\ 0 & -3cw^2-3 \end{pmatrix}.$$

So, in addition to the saddle at the origin, the points $\left(0, \pm\sqrt{-3/c}\right)$ are repelling nodes. This time we see that the origin of U_1 has two elliptic sectors and two parabolic sectors, see Figure 4.

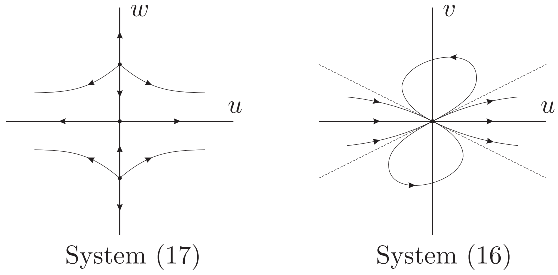


FIGURE 4. Blow-up of the origin of U_1 of system (15) when $c < 0$.

We now look at the origin of U_2, in which system (15) writes

$$\dot{u} = -cv^2 - 6u^2 + 1, \quad \dot{v} = -uv(cv^2 + 3).$$

We see that the origin of U_2 is not a singular point. Hence all the infinite singular points are in U_1 and V_1.

Additionally to the origin, system (15) has the four finite singular points $\left(\pm\sqrt{-c/9}, \pm\sqrt{-c/3}\right)$ on U_3, which are real since $c < 0$. The eigenvalues of the linear part of system (15) are $\pm 2c/3$ at each of them, so they are saddles.

We note that the Hamiltonian H_3^1 has the same value at all of these saddles since it is an even function. Then we claim that all four of them must be on the boundary of the period annulus of the center at the origin. We define *energy levels* of a vector field as the curves on which its Hamiltonian is constant. If there were only two saddles on the mentioned boundary, then a straight line passing through the origin and sufficiently close to the saddles which are not on this boundary would have at least six intersection points with the separatrices of these saddles, which are on the same energy level, see Figure 5. But this is impossible because H_3^1 is only quartic. Hence the claim is proved.

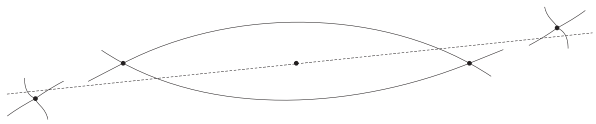


FIGURE 5. The straight line through the origin intersects the separatrices six times.

Moreover, the separatrices of these saddles cross the x-axis exactly twice since H_3^1 is quadratic in x when $y = 0$. Consequently we conclude that the global phase portrait of system (15) is topologically equivalent to 1.3 in Figure 1.

Now we study system (III) when $b \neq 0$. In this case system (III) writes

$$\dot{x} = ax + by - 3x^2y + y^3, \quad \dot{y} = -(a^2/b)x - ay + 3xy^2, \tag{18}$$

and it has the Hamiltonian

$$H_3^2(x,y) = \frac{y^4}{4} - \frac{3x^2y^2}{2} + \frac{a^2x^2}{2b} + \frac{by^2}{2} + axy.$$

We use Proposition 5 to rewrite system (18) as

$$\begin{aligned}\dot{x} &= y - \frac{a(a^2-3b^2)x^3}{b^3} + \frac{3(a^2-b^2)x^2y}{b^3} - \frac{3axy^2}{b^3} + \frac{y^3}{b^3},\\ \dot{y} &= -\frac{a^2(a^2-6b^2)x^3}{b^3} + \frac{3a(a^2-3b^2)x^2y}{b^3} - \frac{3(a^2-b^2)xy^2}{b^3} + \frac{ay^3}{b^3}.\end{aligned}$$

Then by Proposition 4 the origin is a center if and only if

$$a^2/b - 6b > 0 \quad \text{and} \quad a \neq 0. \tag{19}$$

Under these restrictions we first investigate the infinite singular points of (18). In U_1 system (18) becomes

$$\begin{aligned}\dot{u} &= -v^2\left(bu^2 + 2au + (a^2/b)\right) - u^2(u^2-6),\\ \dot{v} &= -v^3\left(bu + a\right) - uv(u^2-3).\end{aligned} \tag{20}$$

When $v = 0$ the singular points are $(0,0), (\pm\sqrt{6}, 0)$. The linear part of system (20) is

$$\begin{pmatrix} -4u(u^2-3) & 0 \\ 0 & -u(u^2-3) \end{pmatrix}.$$

Hence, just like the case $b = 0$, the singular points $(\sqrt{6}, 0)$ and $(-\sqrt{6}, 0)$ are attracting and repelling nodes, respectively.

At the origin, however, the linear part is zero. We do the blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$, eliminate the common factor u between $\dot{u}$ and $\dot{w}$, and obtain the system

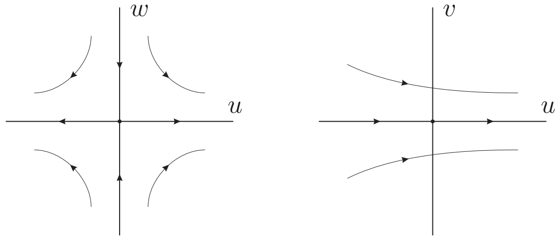
$$\begin{aligned}\dot{u} &= -uw^2\left(bu^2 + 2au + (a^2/b)\right) - u(u^2-6),\\ \dot{w} &= w^3\left(au + (a^2/b)\right) - 3w.\end{aligned} \tag{21}$$

When $u = 0$, system (21) has the singular points $(0,0), \left(0, \pm\sqrt{3b/a^2}\right)$. The linear part of system (21) at the points $(0, w)$ is

$$\begin{pmatrix} -(a^2/b)w^2 + 6 & 0 \\ aw^3 & 3(a^2/b)w^2 - 3 \end{pmatrix}.$$

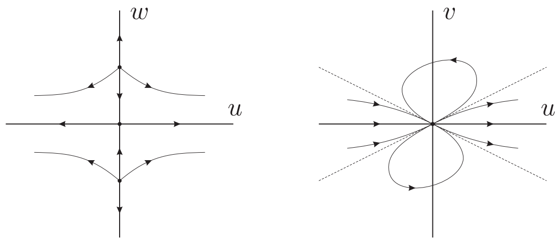
When $b < 0$, we see that $\left(0, \pm\sqrt{3b/a^2}\right)$ are not real, hence the only singular point is the origin, which is a saddle. It is shown in Figure 6 that the origin of U_1 consists of two hyperbolic sectors.

When $b > 0$, all three singular points are real. In addition to the saddle at the origin, the points $\left(0, \pm\sqrt{3b/a^2}\right)$ are repelling nodes. This time we see that the origin of U_1 has two elliptic sectors and two parabolic sectors, see Figure 7.



System (21) System (20)

FIGURE 6. Blow-up of the origin of U_1 of system (III) when $b < 0$.



System (21) System (20)

FIGURE 7. Blow-up of the origin of U_1 of system (III) when $b > 0$.

We now look at the origin of U_2, in which system (18) writes

$$\dot{u} = v^2\left((a^2/b)u^2 + 2au + b\right) - 6u^2 + 1,$$
$$\dot{v} = v^3\left((a^2/b)u + a\right) - 3uv.$$

Hence the origin of U_2 is not a singular point.

The finite singular points of system (18) other than its origin are

$$p_{1,2} = \pm\left(\frac{(3b - A)\sqrt{B - A}}{6\sqrt{6}a}, \frac{\sqrt{B - A}}{\sqrt{6}}\right),$$
$$p_{3,4} = \pm\left(\frac{(3b + A)\sqrt{B + A}}{6\sqrt{6}a}, \frac{\sqrt{B + A}}{\sqrt{6}}\right),$$

where $A = \sqrt{12a^2 + 9b^2}$ and $B = 2a^2/b - 3b$.

When $b < 0$, we see that the expression

$$(2a^2/b - 3b)^2 - (12a^2 + 9b^2) = 4a^2b(a^2/b - 6b) \tag{22}$$

is negative due to (19). Hence $B - A < 0$ and $B + A > 0$. Therefore when $b < 0$ system (18) has only the two finite singular points p_3 and p_4 in addition to the origin. The eigenvalues of the linear part of system (18) at these points are

$$\pm\frac{\sqrt{4a^4 + 15a^2b^2 + 9b^4 + b(5a^2 - 3b^2)\sqrt{12a^2 + 9b^2}}}{3b}$$

We observe that

$$(4a^4 + 15a^2b^2 + 9b^4)^2 - \left(b(5a^2 - 3b^2)\sqrt{12a^2 + 9b^2}\right)^2 = 4a^2(4a^2 + 3b^2)(a^2 - 6b^2)^2 > 0.$$

Therefore p_3 and p_4 are saddles. By the symmetry of (18) they must be on the boundary of the period annulus of the center at the origin. Since the Hamiltonian H_3^2 is quadratic on the x-axis, their separatrices can cross the x-axis only twice. Consequently we obtain the phase portrait 1.4 of Figure 1.

When $b > 0$, on the other hand, (22) is positive and

$$2a^2/b - 3b > a^2/b - 6b > 0.$$

Therefore all of the four finite singular points exist. We already saw that p_3 and p_4 are saddles. Similar computations show that p_1 and p_2 are also saddles. We claim that only two of these saddles can be on the boundary of the period annulus of the center at the origin. This is due to the fact that the Hamiltonian H_3^2 of system (18) is even and that we have

$$H_3^2(p_1) = \frac{2a^4 + 30a^2b^2 - 9b^4 - b(4a^2 + 3b^2)\sqrt{12a^2 + 9b^2}}{72b^2},$$
$$H_3^2(p_3) = \frac{2a^4 + 30a^2b^2 - 9b^4 + b(4a^2 + 3b^2)\sqrt{12a^2 + 9b^2}}{72b^2},$$

which are equal if and only if $b = 0$, which is not the case.

By the same argument used for the case $b < 0$, we see that the saddles which are on the boundary of the period annulus of the center at the origin cannot be connected to any of the other saddles. Therefore the saddles on this boundary have to be connected with the infinite singular points. Due to the fact that the separatrices through these saddles cannot cross the x-axis anymore and that the flow around the origin is clockwise, we get the global phase portrait 1.5 shown in Figure 1.

7. Global phase portraits of system (IV)

As before we first consider the case $b = 0$. Then system (IV) becomes

$$\dot{x} = -3x^2y - y^3, \quad \dot{y} = cx + 3xy^2, \tag{23}$$

where $c \neq 0$, and it has the Hamiltonian

$$H_4^1(x, y) = -\frac{y^4}{4} - \frac{3x^2y^2}{2} - \frac{cx^2}{2}.$$

Using Proposition 5 we rewrite system (23) as

$$\dot{x} = y + 3x^2y/c, \quad \dot{y} = -cx^3 - 3xy^2/c.$$

Then by Proposition 4 the origin is a center if and only if $c > 0$. Hence we will investigate the case $c > 0$.

In U_1 system (23) becomes

$$\dot{u} = cv^2 + u^2(u^2 + 6), \quad \dot{v} = uv(u^2 + 3). \tag{24}$$

When $v = 0$, the only singular point is the origin and its linear part is zero. Therefore to study its local behavior we do the blow-up $(u, v) \mapsto (u, w)$ with

$w = v/u$. Eliminating the common factor u between $\dot u$ and $\dot w$, we obtain the system

$$\dot u = cuw^2 + u(u^2+6), \quad \dot w = -w(cw^2+3). \tag{25}$$

When $u = 0$, system (25) has the unique singular point $(0,0)$ since $c > 0$ and it is a saddle. Hence we see that the origin of U_1 consists of two hyperbolic sectors, see Figure 6.

In U_2 system (23) is expressed as

$$\dot u = -cv^2 - 6u^2 - 1, \quad \dot v = -uv(cv^2+3).$$

The origin of U_2 is not a singular point. Hence the infinite singular points are only the origins of U_1 and V_1.

The finite singular points of (23) are $(0,0)$ and $\left(\pm\sqrt{-c/9}, \pm\sqrt{-c/3}\right)$ on U_3. But because $c > 0$, only the origin is real. As a result we conclude that the global phase portrait of system (23) is topologically equivalent to 1.1 in Figure 1.

Now we study system (IV) when $b \neq 0$. In this case system (IV) writes

$$\dot x = ax + by - 3x^2y - y^3, \quad \dot y = -(a^2/b)x - ay + 3xy^2, \tag{26}$$

and it has the Hamiltonian

$$H_3^2(x,y) = -\frac{y^4}{4} - \frac{3x^2y^2}{2} + \frac{a^2x^2}{2b} + \frac{by^2}{2} + axy.$$

We apply Proposition 5 to system (23) and get

$$\begin{aligned} \dot x &= y + \frac{a(a^2+3b^2)x^3}{b^3} - \frac{3(a^2+b^2)x^2y}{b^3} + \frac{3axy^2}{b^3} - \frac{y^3}{b^3}, \\ \dot y &= \frac{a^2(a^2+6b^2)x^3}{b^3} - \frac{3a(a^2+3b^2)x^2y}{b^3} + \frac{3(a^2+b^2)xy^2}{b^3} - \frac{ay^3}{b^3}. \end{aligned}$$

Then by Proposition 4 the origin is a center if and only if $b < 0$ and $a \neq 0$. Therefore we impose these two conditions on (26) and begin with investigating its infinite singular points.

In U_1 we have

$$\begin{aligned} \dot u &= -v^2\left(bu^2 + 2au + (a^2/b)\right) + u^2(u^2+6), \\ \dot v &= -v^3(bu+a) + uv(u^2+3). \end{aligned} \tag{27}$$

When $v = 0$ the only singular points of (27) is the origin and its linear part is zero. We do the blow-up $(u,v) \mapsto (u,w)$ with $w = v/u$, eliminate the common factor u and obtain the system

$$\begin{aligned} \dot u &= -uw^2\left(bu^2 + 2au + (a^2/b)\right) + u(u^2+6), \\ \dot w &= w^3\left(au + (a^2/b)\right) - 3w. \end{aligned} \tag{28}$$

When $u = 0$, the only singular point of system (28) is the origin, which is a saddle. It is shown in Figure 6 that the origin of U_1 consists of two hyperbolic sectors.

The origin of U_2, in which system (26) writes

$$\begin{aligned} \dot u &= v^2\left((a^2/b)u^2 + 2au + b\right) - 6u^2 - 1, \\ \dot v &= v^3\left((a^2/b)u + a\right) - 3uv, \end{aligned}$$

is clearly not a singular point.

Next we consider the finite singular points of system (26) which are

$$p_{1,2} = \pm\left(\frac{(3b+A)\sqrt{B-A}}{6\sqrt{6}a}, \frac{\sqrt{B-A}}{\sqrt{6}}\right),$$
$$p_{3,4} = \pm\left(\frac{(3b-A)\sqrt{B+A}}{6\sqrt{6}a}, \frac{\sqrt{B+A}}{\sqrt{6}}\right),$$

where $A = \sqrt{-12a^2+9b^2}$ and $B = 2a^2/b+3b$. Due to our assumption that $b<0$, we observe that

$$\begin{aligned} B - A < B + A =& 2a^2/b + 3b + \sqrt{-12a^2+9b^2} \\ <& 2a^2/b + 3b + \sqrt{9b^2} \\ =& 2a^2/b < 0 \end{aligned}$$

Therefore the origin is the only finite singular point of system (26) and we easily see that its global phase portrait is topologically equivalent to 1.1 of Figure 1.

8. Global phase portraits of system (V)

When $b=0$, system (V) becomes

$$\dot{x} = -3x^2y + y^3, \quad \dot{y} = cx + x^3 + 3\mu xy^2, \tag{29}$$

where $c \neq 0$, and it has the Hamiltonian

$$H_5^1(x,y) = \frac{y^4 - x^4}{4} - \frac{3\mu x^2 y^2}{2} - \frac{cx^2}{2}.$$

Using Proposition 5 we rewrite system (29) as

$$\dot{x} = y + 3\mu x^2 y/c + y^3/c, \quad \dot{y} = cx^3 - 3\mu x y^2/c.$$

Then by Proposition 4 the origin is a center if and only if $c<0$. Therefore we assume $c<0$.

As in the previous systems, we first investigate the infinite singular points of system (29). In the local chart U_1 we have

$$\begin{aligned} \dot{u} =& cv^2 - u^4 + 6\mu u^2 + 1, \\ \dot{v} =& -vu(u^2 - 3\mu). \end{aligned} \tag{30}$$

When $v=0$, the real singular points are $\big(\pm\sqrt{3\mu+\sqrt{9\mu^2+1}}, 0\big)$, and the linear part of (30) is

$$\begin{pmatrix} -4u(u^2-3\mu) & 0 \\ 0 & -u(u^2-3\mu) \end{pmatrix},$$

Hence the points $\big(\sqrt{3\mu+\sqrt{9\mu^2+1}}, 0\big)$ and $\big(-\sqrt{3\mu+\sqrt{9\mu^2+1}}, 0\big)$ are respectively attracting and repelling nodes of system (30).

In U_2 system (29) becomes

$$\begin{aligned} \dot{u} =& v^2\big((a^2/b)u^2 + 2au + b\big) - u^4 - 6\mu u^2 + 1, \\ \dot{v} =& v^3\big((a^2/b)u + a\big) - vu(u^2 + 3\mu), \end{aligned}$$

and we see that the origin is not a singular point. Hence system (V) has four infinite singular points, all of which are nodes on U_1 and V_1.

The finite singular points of system (29) are the origin, $\pm\left(\sqrt{-c},0\right)$ and

$$p_i = \pm\left(\pm\sqrt{-c/(1+9\mu^2)},\sqrt{-3c\mu/(1+9\mu^2)}\right).$$

with $i=1,2,3,4$. We observe that when $\mu<0$ the points p_i do not exist, and when $\mu=0$ they are equal to $\pm\left(\sqrt{-c},0\right)$. As a result, system (29) has three finite singular points when $\mu\le 0$, and seven when $\mu>0$. We note that the linear part of system (29) is

$$M_5^1 = \begin{pmatrix} -6\mu xy & 3y^2-3\mu x^2 \\ c+3x^2+3\mu y^2 & 6\mu xy \end{pmatrix}.$$

We first consider the case $\mu\le 0$. We know that the origin is a center. The eigenvalues of M_5^1 at $\pm\left(\sqrt{-c},0\right)$ are $\pm c\sqrt{-6\mu}$. This means that these points are hyperbolic saddles when $\mu<0$, but are degenerate when $\mu=0$. But M_5^1 is not identically zero at these points since

$$c+3x^2+3\mu y^2 = -2c>0.$$

So they are nilpotent. Due to the fact that nilpotent singular points of Hamiltonian vector fields are either saddles, centers or cusps, and that the number of singular points is fixed for $\mu\le 0$, we conclude that the points $\pm\left(\sqrt{-c},0\right)$ are saddles also when $\mu=0$.

Then the global phase portrait in this case is topologically equivalent to 1.2 of Figure 1. This is because the two saddles must be on the boundary of the period annulus of the center at the origin. Their remaining separatrices cannot cross the straight lines passing through the origin and the infinite singular points, namely $y=\pm\sqrt{3\mu+\sqrt{9\mu^2+1}x}$, because the Hamiltonian $H_5^1=-cx^2/2$ is quadratic on these lines. Then taking into account the flow at infinity, we obtain the global phase portrait.

Secondly we look at the case $\mu>0$. In this case the points $\pm\left(\sqrt{-c},0\right)$ are centers. In addition, each p_i is a saddle because the eigenvalues of M_5^1 at these points are $\pm 2c\sqrt{3\mu}/\sqrt{1+9\mu^2}$. Since the Hamiltonian of system (29) is even, $H_5^1(p_i)$ is constant for all i. Then, by the same same argument that we used in system (III) (see Figure 5), we deduce that every p_i must be on the boundary of the period annulus of the center at the origin.

Since the infinite singular points are nodes, the centers can only be created by connecting two adjacent saddles. The flow around the origin is clockwise because $c<0$. In addition, the remaining separatrices of any of the saddles must lie on different sides of the straight line passing through that saddle and the origin, otherwise there would exist another straight line through the origin intersecting both separatrices in six intersection points in the same energy level. Then the flow at infinity ensures that the remaining two centers are formed by connecting adjacent saddles which lie on the same side of the plane with respect to the y-axis. Recalling from the case $\mu\le 0$ that the remaining separatrices of these saddles cannot cross the straight lines passing through the origin and the infinite singular points, we get the global phase portrait 1.6 in Figure 1.

Having established the case $b = 0$, we now investigate the case $b \neq 0$ in which system (V) writes

$$\dot{x} = ax + by - 3\mu x^2 y + y^3, \tag{31a}$$
$$\dot{y} = -(a^2/b)x - ay + x^3 + 3\mu x y^2, \tag{31b}$$

and it has the Hamiltonian

$$H_5^2(x,y) = \frac{y^4 - x^4}{4} - \frac{3\mu x^2 y^2}{2} + \frac{a^2 x^2}{2b} + \frac{by^2}{2} + axy.$$

We remark that without loss of generality we can assume $b > 0$. If we do the linear transformation $(x, y) \mapsto (-y, -x)$, system (V) becomes

$$-\dot{y} = -ay - bx + 3\mu y^2 x - x^3,$$
$$-\dot{x} = (a^2/b)y + ax - y^3 - 3\mu y x^2,$$

which can be rewritten as

$$\begin{aligned} \dot{x} &= -ax - (a^2/b)y + 3\mu x^2 y + y^3, \\ \dot{y} &= bx + ay + x^3 - 3\mu x y^2. \end{aligned} \tag{32}$$

After defining $\bar{a} = -a$, $\bar{\mu} - \mu$, and $\bar{b} = a^2/b$, we see that system (32) is basically system (V) with $b \mapsto -b$. So, we assume $b > 0$.

Using Proposition 5 we rewrite system (31) as

$$\begin{aligned} \dot{x} =& y - \frac{a(a^2 - 3b^2\mu)x^3}{b^3} + \frac{3(a^2 - b^2\mu)x^2 y}{b^3} - \frac{3axy^2}{b^3} + \frac{y^3}{b^3}, \\ \dot{y} =& - \frac{(a^4 - b^4 - 6a^2b^2\mu)x^3}{b^3} + \frac{3a(a^2 - 3b^2\mu)x^2 y}{b^3} - \frac{3(a^2 - b^2\mu)xy^2}{b^3} + \frac{ay^3}{b^3}. \end{aligned}$$

Therefore, by Proposition 4, we impose on system (31) the two conditions

$$a^4 - b^4 - 6a^2b^2\mu > 0 \quad \text{and} \quad b > 0. \tag{33}$$

The infinite singular points of system (31) are the same as those of (29), an attracting node $\big(\sqrt{3\mu + \sqrt{9\mu^2 + 1}}, 0\big)$ and a repelling node $\big(-\sqrt{3\mu + \sqrt{9\mu^2 + 1}}, 0\big)$ on U_1, and the corresponding points on V_1. The origin of U_2 is not a singular point.

The explicit expressions for the finite singular points in terms of the parameters a, b, β are big, and therefore it is hard to study them numerically. For this reason we follow a different approach. We first find the maximum number of finite singular points allowed by the system. We equate (31a) to 0, solve for x and get

$$x_{1,2} = \frac{a \pm \sqrt{a^2 + 12b\mu y^2 + 12\mu y^4}}{6\mu y}. \tag{34}$$

Note that when $y = 0$, (31b) is zero if and only if $x = 0$ or $a = 0$. But if $a = 0$, then (31b) is zero only when $x = 0$. So we conclude that the only finite singular point on the x-axis is the origin, which we know is a center. Hence we can assume that $y \neq 0$. Then it remains to study the case $\mu = 0$ separately, which we will consider later on, but we first assume that $\mu \neq 0$.

If we substitute (34) into (31b) we obtain

$$\dot{y}_{1,2} = -\frac{1}{54b\mu^3y^3}\Big(a\big(a^2b + 9\mu(b^2 - a^2\mu)y^2 + 9b\mu(1-3\mu^2)y^4\big) \\ \pm \sqrt{a^2 + 12b\mu y^2 + 12\mu y^4}\big(a^2b + 3\mu(b^2 - 3a^2\mu)y^2 + 3b\mu(1+9\mu^2)y^4\big)\Big),$$

where $\dot{y}_1$ and $\dot{y}_2$ denote $\dot{y}$ with x_1 and x_2 substituted, respectively. Then the maximum number of roots of the product $\dot{y}_1\dot{y}_2$ will give an upper bound for the number of finite singular points. So we multiply $\dot{y}_1$ and $\dot{y}_2$ and obtain

$$\frac{-1}{27b^2\mu^3}\big(b^2(1+9\mu^2)y^6 + 3b(1+9\mu^2)(b^2 - 2a^2\mu + 3b^2\mu^2)y^4 \\ + 3(b^4 + 3a^4\mu^2 + 6b^4\mu^2 - 18a^2b^2\mu^3)y^2 - b(a^4 - b^4 - 6a^2b^2\mu)\big). \tag{35}$$

We see that (35) cannot be identically zero because its constant term is different from zero due to (33), so it has at most six real roots. This means that for $\mu \neq 0$, all the finite singular points of system (31) are isolated and that there are at most six of them without taking into account the origin.

If $\mu = 0$, on the other hand, system (31) writes

$$\dot{x} = ax + by + y^3, \quad \dot{y} = -(a^2/b)x - ay + x^3.$$

We can easily compute its finite singular points and see that, additionally to the origin, it has only two. Therefore we conclude that system (31) has at most six finite singular points.

Next we will count the indices of the known singular points, both finite and infinite, and then deduce some conditions on the remaining finite singular points of the system. We will use two important theorems in the process. One is the well known *Poincaré Formula* for the index of a singular point of a planar vector field, and the other is the famous *Poincaré-Hopf Theorem* for vector fields in the 2-dimensional sphere. For details about these theorems, see Chapters 1 and 6 of [10].

Theorem 8 (Poincaré Formula)**.** *Let q be an isolated singular point having the finite sectorial decomposition property. Let e, h and p denote the number of elliptic, hyperbolic and parabolic sectors of q, respectively. Then the index of q is $(e-h)/2+1$.*

Theorem 9 (Poincaré-Hopf Theorem)**.** *For every tangent vector field on $\mathbb{S}^2$ with a finite number of singular points, the sum of the indices of the singular points is 2.*

Corollary 10. *The index of a saddle, a center and a cusp are -1, 1 and 0, respectively.*

Remark 11. Nilpotent singular points of Hamiltonian planar polynomial vector fields are either saddles, centers, or cusps (for more details see chapters 2 and 3 of [10], specifically sections 2.6 and 3.5)

Observe that when determining the finite singular points by considering their total index, those with index zero are hard to detect. To overcome this difficulty we present the following lemma, which is inspired from Lemma 15 of [8]. A hyperbolic saddle with a loop and a center inside the loop as in Figure 8 will be called a *center–loop*.

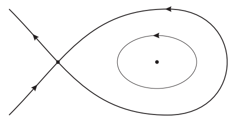


FIGURE 8. A center–loop.

Lemma 12. *Let X_ε be a real Hamiltonian planar polynomial vector field having only linear and cubic terms. Then X_ε can be written as*

$$\dot{x} = a_{10}x + a_{01}y + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3,$$
$$\dot{y} = b_{10}x - a_{10}y + b_{30}x^3 - 3a_{30}x^2y - a_{21}xy^2 - \frac{1}{3}a_{12}y^3 + \varepsilon x.$$

Suppose that p is an isolated singular point of X_ε different from the origin. If $a_{10}^2 + a_{01}b_{10} = 0$ but $a_{01} \neq 0$, then the following statements hold:

(a) *If p is non–elementary, then it is nilpotent.*
(b) *If p is a non–elementary singular point of X_0, then it is an elementary singular point of X_ε with $\varepsilon \neq 0$.*
(c) *If p is a cusp of X_0, then for ε small enough such that $\varepsilon a_{01} < 0$, the origin of X_ε is a linear type center and the local phase portrait of X_ε at p is a center–loop.*

Proof. Let X_ε be the vector field defined in the lemma. It is easy to check that X_ε is Hamiltonian with the Hamiltonian polynomial

$$H_\varepsilon = \frac{a_{03}y^4 - b_{30}x^4}{4} + a_{30}x^3y + \frac{a_{21}x^2y^2}{2} + \frac{a_{12}xy^3}{3} + \frac{a_{01}y^2 - b_{10}x^2}{2} + a_{10}xy.$$

Without loss of generality we can assume that $p = (0, y_0)$, otherwise doing a rotation of the coordinates we can get its x-coordinate to be zero.

Assume that $a_{10}^2 + a_{01}b_{10} = 0$ and that $a_{01} \neq 0$.

We first prove statement (a). At the singular point $(0, y_0)$ we have

$$y_0(a_{01} + a_{03}y_0^2) = -y_0(a_{10} + \frac{1}{3}a_{12}y_0^2) = 0, \tag{36}$$

whereas M_ε, the linear part of X_ε, at $(0, y_0)$ is

$$\begin{pmatrix} a_{10} + a_{12}y_0^2 & a_{01} + 3a_{03}y_0^2 \\ b_{10} - a_{21}y_0^2 + \varepsilon & -a_{10} - a_{12}y_0^2 \end{pmatrix}.$$

Since $y_0 \neq 0$, we see that $(0, y_0)$ is degenerate only if

$$a_{01} + a_{03}y_0^2 = a_{01} + 3a_{03}y_0^2 = 0, \tag{37}$$

which requires $a_{03} = 0$. However, since $a_{01} \neq 0$, equation (37) cannot be satisfied. Therefore we conclude that a non–elementary singular point of X_ε must be nilpotent. So statement (a) is proved.

Now we prove (b). Assume that $(0, y_0)$ is a non–elementary singular point of X_0. We will prove that the eigenvalues of the linear part of X_ε, with $\varepsilon \neq 0$, are different from zero.

The characteristic polynomial at a singular point of X_ε is of the form $\lambda^2+\det(M_\varepsilon)$. So the eigenvalues λ of the linear part of X_ε at a singular point are $\pm\sqrt{-\det(M_\varepsilon)}$. Since we have already assumed that the eigenvalues of M_0 at $(0, y_0)$ are zero, the only nonzero terms in the determinant of M_ε at the same point are those having a factor of ε. Hence the eigenvalues of M_ε at $(0, y_0)$ are

$$\lambda = \pm\sqrt{\varepsilon(3a_{03}y_0^2 + a_{01})}, \tag{38}$$

where $\varepsilon \neq 0$. Then the eigenvalues are zero only if we have (37), but we have already shown that it is not possible. Therefore $(0, y_0)$ is an elementary singular point of X_ε.

Finally we prove part (c). Assume that $(0, y_0)$ is a cusp of X_0 and that $\varepsilon a_{01} < 0$. Then the eigenvalues of M_ε at the origin of X_ε are $\pm\sqrt{\varepsilon a_{01}}$, purely imaginary by assumption. Since X_ε is Hamiltonian, we conclude that the origin is a linear type center.

Due to the fact that $y_0 \neq 0$, we have $a_{01} = -a_{03}y_0^2$, see (36). Then, by (38), the eigenvalues of M_ε at $(0, y_0)$ are

$$\pm\sqrt{\varepsilon(3a_{03}y_0^2 + a_{01})} = \pm\sqrt{\varepsilon(-2a_{01})},$$

both of which are real by assumption. Hence $(0, y_0)$ is a saddle. In addition, since $(0, y_0)$, which was a cusp with index zero, is now a saddle having index -1, new singular points must emerge in a neighborhood W_ε of $(0, y_0)$ to keep the total index of the vector field fixed. Because of the symmetry of the system, there can be at most three new singular points in W_ε so that the total number of finite singular points does not exceed 9. Since X_ε is Hamiltonian, these singular points can only be saddles, centers or cusps. Therefore in W_ε there are additionally to the saddle at $(0, y_0)$ either (i) one center, (ii) one center and one cusp, (iii) one center and two cusps, or (iv) two centers and one saddle. Our claim is that (i) is the only realizable case and that $(0, y_0)$ is the saddle of a center–loop.

Because of the continuity of system X_ε with respect to ε, the new separatrices of $(0, y_0)$ must be arbitrarily close to $(0, y_0)$ for small ε, therefore they cannot go to any other singular point outside W_ε. Note that in all the possibilities $(i) - (iv)$, there exists a center with $(0, y_0)$ on the boundary of its period annulus. Then we see that $(0, y_0)$ cannot be on the boundary of the period annulus of the center at the origin. Otherwise we could find a straight line l through the origin intersecting the boundary of the period annulus of the new center twice, which would, in fact, have at least three intersection points with the separatrices of $(0, y_0)$, the other being on the boundary of the period annulus of the center at the origin, see Figure 9. Then, due to the symmetry of the system with respect to the origin, there would be six points on l all of which are on the same energy level. Clearly this is not possible since the Hamiltonian H_ε is a quartic polynomial.

If $(0, y_0)$ is not on the boundary of the period annulus of the center at the origin, then there must be other saddles on that boundary. This means that system X_0 has at least five finite singular points. This immediately eliminates the possibilities (iii) and (iv), otherwise the number of finite

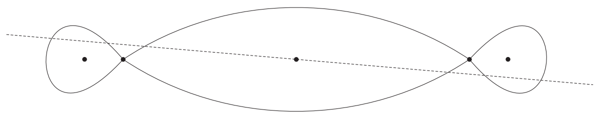


FIGURE 9. The straight line through the origin intersects the separatrices six times.

singular points exceeds the maximum of 9. Furthermore, by the same arguments used for $(0, y_0)$, the cusp in case (ii) would also lead to the existence of more singular points. Therefore we dismiss case (ii) also, proving our claim. □

With the help of this lemma, we will be able to detect possible cusps of the vector fields by studying the center–loops of the Hamiltonian linear type centers obtained by the perturbation in statement (c) of Lemma 12.

We continue determining the finite singular points of system (31). By Theorem 8, the four nodes at infinity and the centers at the origins of U_3 and V_3 have total index 6 on the Poincaré sphere. Then by Theorem 9, the remaining finite singular points on the Poincaré disk have to have total index -2.

By statement (a) of Lemma 12, these singular points are either elementary or nilpotent. Hence they are either centers, saddles or cusps. We claim that at most two finite singular points of system (31) which are different from the origin are nilpotent. To prove this claim we compute the Gröbner basis of the system of three equations obtained by equating (31a), (31b) and the determinant of the linear part of system (31) to zero. Recall that $y \neq 0$. We obtain sixteen equations, two of which are a quadratic polynomial of y only, and another polynomial linear in x. This proves our claim.

In short, by Corollary 10, finite singular points of system (31) other than the origin can be either (i) two saddles, (ii) two saddles and two cusps, or (iii) four saddles and two centers.

Consider case (i). The two saddles must be on the boundary of the period annulus of the center at the origin. Their remaining separatrices cannot cross the straight lines passing through the origin and the infinite singular points, namely $y = \pm\sqrt{3\mu + \sqrt{9\mu^2+1}x}$, because the Hamiltonian H_5^2 is quadratic on these lines. Then, due to the flow at infinity, we get a global phase portrait which is topologically equivalent to 1.2 of Figure 1. We remark that this phase portrait is achieved for the values $a = 2$, $b = 1$ and $\mu = 1/4$.

In case (iii), we compute the Gröbner basis for the system of three equations $\dot{x} = \dot{y} = 0$ and $H_5^2 = c$ for some $c \in \mathbb{R}$. Of the 52 equations that we obtain, one is a quadratic polynomial of only y and another is linear in x. Therefore we conclude that the Hamiltonian H_5^2 cannot have the same value at the finite singular points which are not symmetric with respect to the origin. As a result, only two of the saddles can be on the boundary of the center at the origin. Then the remaining separatrices of these saddles either go to infinity or come back to the same saddles in one of the ways shown in Figure 10.

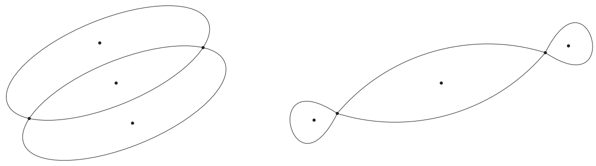


FIGURE 10. Two saddles forming the two centers.

The first figure cannot be realized due to the fact that (31a) does not change on the positive y-axis since $b > 0$. We see that the second figure cannot be achieved either, see Figure 9. Hence we conclude that the remaining saparatrices of these saddles go to infinite singular points, of course without crossing the straight lines passing through the origin and the infinite singular points.

Because of the symmetry of system (31) and the flow near infinity, the remaining finite singular points must be symmetric with respect to the origin, and also there must be a saddle on the boundary of the period annulus of each of the centers, creating a center–loop. We observe that a center–loop may exist only if a straight line l_1 passing through the origin and the saddle of the center–loop intersects the separatrices of this saddle at exactly on point, namely the saddle itself. Otherwise we can find another straight line l_2 passing through the origin and sufficiently close to the saddle of the center–loop such that it intersects the saddle's separatrices at least three times, see Figure 11. Then, because of the symmetry of the vector field with respect to the origin, there would be six points on l_2 which are on the same energy level. But we know that this is not possible since system (31) is cubic.

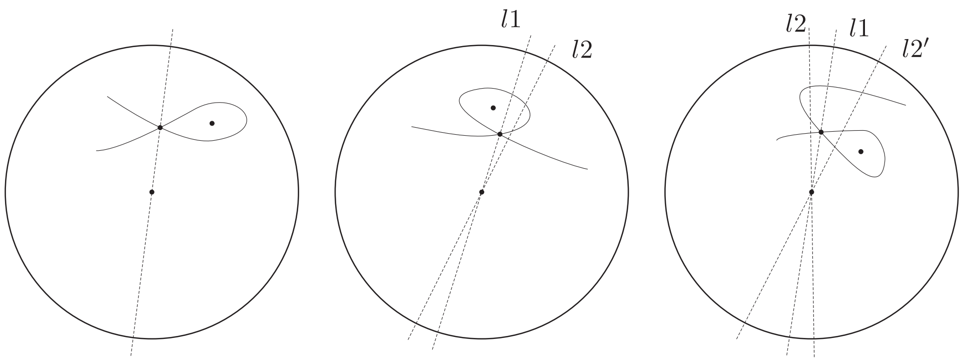


FIGURE 11. Center–loop configuration.

In short, there is only one way, up to topological equivalence, for the center–loops of system (31) to be formed, and we obtain the global phase portrait 1.7 of Figure 1. For $a = 5$, $b = 1$ and $\mu = 1/4$ such a phase portrait is a realized.

Finally we consider case (ii). By statement (c) of Lemma 12, system (31) becomes a Hamiltonian linear type center which has a center–loop. We already know by our previous work that the global phase portrait of this perturbed system is topologically equivalent to 1.7 in Figure 1 also. Therefore the only possible global phase portrait in this case is the portrait 1.8 shown in Figure 1. The particular global phase portrait examples that we provided for the cases (i) and (iii) ensure that the phase portrait 1.8 is actually realized when $b = 1$ and $\mu = 1/4$ for some $a \in (2, 5)$.

9. Global phase portraits of system (VI)

When $b = 0$, system (VI) becomes

$$\dot{x} = -3\mu x^2 y - y^3, \quad \dot{y} = cx + x^3 + 3\mu x y^2, \tag{39}$$

and it has the Hamiltonian

$$H_6^1(x, y) = -\frac{y^4 + x^4}{4} - \frac{3\mu x^2 y^2}{2} - \frac{cx^2}{2}.$$

Using Proposition 4 we see that system (39) has a center at the origin if and only if $c > 0$.

In the local chart U_1 system (39) writes

$$\begin{aligned} \dot{u} &= cv^2 + u^4 + 6u^2\mu + 1, \\ \dot{v} &= uv(u^2 + 3\mu). \end{aligned} \tag{40}$$

When $v = 0$, the singular points of system (40)are $\left(\pm\sqrt{-3\mu \pm \sqrt{9\mu^2 - 1}}, 0\right)$. Therefore, in U_1 there are four singular points if $\mu < -1/3$, two if $\mu = -1/3$, and none if $\mu > -1/3$.

In U_2 system (39) becomes

$$\begin{aligned} \dot{u} &= -cu^2v^2 - u^4 - 6u^2\mu - 1, \\ \dot{v} &= -uv(u^2 + cv^2 + 3\mu), \end{aligned}$$

and we see that the origin is not a singular point. Hence all the infinite singular points are on the local charts U_1 and V_1. The existence of these singular points depend on the parameter μ, so we will investigate the phase portraits in the corresponding subcases.

The finite singular points of system (39) are the origin, $\pm\left(\sqrt{-c}, 0\right)$ and

$$p_i = \pm\left(\pm\sqrt{c/9\mu^2 - 1)}, \sqrt{-3c\mu/(9\mu^2 - 1)}\right). \tag{41}$$

with $i = 1, 2, 3, 4$, for $\mu \neq \pm 1/3$. The points $\pm\left(\sqrt{-c}, 0\right)$ are not real since $c > 0$. When $\mu > -1/3$ none of the p_i are real, whereas they are all real and are saddles when $\mu < -1/3$.

In addition, when $\mu = -1/3$ system (39) becomes

$$\dot{x} = y(x^2 - y^2), \quad \dot{y} = cx + x(x^2 - y^2),$$

hence the only singular point is the origin since $c > 0$. Similarly when $\mu = 1/3$ only the origin is a singular point.

In short, system (39) does not have any finite singular points other than the origin if $\mu \geq -1/3$, however, it has the points p_i for $i = 1, 2, 3, 4$ otherwise.

We first assume that $\mu < -1/3$. Then all of the four singular points on U_1 are real such that $\big(\sqrt{-3\mu + \sqrt{9\mu^2 - 1}}, 0\big)$ and $\big(-\sqrt{-3\mu - \sqrt{9\mu^2 - 1}}, 0\big)$ are repelling nodes, whereas the other two points $\big(\sqrt{-3\mu - \sqrt{9\mu^2 - 1}}, 0\big)$ and $\big(-\sqrt{-3\mu + \sqrt{9\mu^2 - 1}}, 0\big)$ are attracting ones.

We have $H_6^1(p_i) = H_6^1(p_{(i+1)})$ for $i = 1, 2, 3$ because the Hamiltonian is an even function. Then all of them must be on the boundary of the period annulus of the center at the origin, see Figure 5. The remaining separatrices of the saddles cannot cross the straight lines passing through the origin and the infinite singular points because $H_6^1 = -cx^2/2$ on these lines. Therefore all these separatrices must go to the infinite singular points as it is shown in the global phase portrait 1.9 of Figure 1.

Next we assume that $\mu = -1/3$. The linear part of system (40) at both singular points $(\pm 1, 0)$ is zero. So we need blow-ups to understand the local behavior at these points. We will do the computations for the point $(1, 0)$, and the other point $(-1, 0)$ can be studied in the same way.

First we move $(1, 0)$ to the origin by the shift $u \mapsto u + 1$, and get the system

$$\dot{u} = cv^2 + u^2(u+2)^2, \quad \dot{v} = uv(u+1)(u+2).$$

Now if we do the blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$ and eliminate the common factor u, we get the system

$$\dot{u} = u(u+2)^2 + cuw^2, \quad \dot{w} = -w(u + cw^2 + 2). \tag{42}$$

We see that since $c > 0$, the only singular point of system (42) when $u = 0$ is the origin, which is a saddle. Therefore the singular point $(1, 0)$ of system (40) has two hyperbolic sectors, see Figure 6.

Performing the same procedure for the point $(-1, 0)$ reveals that the local behavior around this point is the same as that of $(1, 0)$. Since there are no finite singular points in this case, we get the global phase portrait 1.10 of Figure 1.

Finally, when $\mu > -1/3$ system (39) has neither infinite nor finite singular points. Since $c > 0$, we get the global phase portrait 1.11 of Figure 1.

We have finished studying system (VI) in the case $b = 0$, and now we assume that $b \neq 0$. Then system (VI) writes

$$\dot{x} = ax + by - 3\mu x^2 y - y^3, \tag{43a}$$

$$\dot{y} = -(a^2/b)x - ay + x^3 + 3\mu x y^2, \tag{43b}$$

and it has the Hamiltonian

$$H_6(x, y) = -\frac{y^4 + x^4}{4} - \frac{3\mu x^2 y^2}{2} + \frac{a^2 x^2}{2b} + \frac{by^2}{2} + axy.$$

By Propositions 5 and 6 the origin is a center if and only if

$$\frac{a^4 + b^4 + 6a^2 b^2 \mu}{b} < 0. \tag{44}$$

The infinite singular points of system (43) are the same as those of system (39).

We will study the finite singular points in the same way as we did in the case $b \neq 0$ of system (V). We first find an upper bound for the number of finite singular points. We equate (43a) to zero and solve for x and obtain

$$x_{1,2} = \frac{a \pm \sqrt{a^2 + 12b\mu y^2 - 12\mu y^4}}{6\mu y},$$

if $\mu \neq 0$ (we will handle the case $\mu = 0$ separately). Note that the only singular point of system (43) on the x-axis is the origin, so we can assume $y \neq 0$. If we substitute these into (43b) and multiply the two respective functions, we obtain a polynomial in y of degree six which cannot be identically zero. This means that system (43) has at most six other finite singular points on the Poincaré disk when $\mu \neq 0$.

When $\mu = 0$ the center condition (44) requires that $b < 0$. If we compute the finite singular points of system (43) in this case we see that only the origin is singular. Therefore we conclude that, in any case, system (43) has at most six finite singular points.

We will now determine the global phase portraits of system (43) by considering the different values of μ that lead to different phase portraits at infinity.

When $\mu < -1/3$, the infinite singular points on U_1 are the repelling nodes $\left(\sqrt{-3\mu + \sqrt{9\mu^2 - 1}}, 0\right)$ and $\left(-\sqrt{-3\mu - \sqrt{9\mu^2 - 1}}, 0\right)$, and the attracting nodes $\left(\sqrt{-3\mu - \sqrt{9\mu^2 - 1}}, 0\right)$ and $\left(-\sqrt{-3\mu + \sqrt{9\mu^2 - 1}}, 0\right)$. Then the infinite singular points and the centers at the origins of U_3 and V_3 have total index 10 in the Poincaré sphere. Hence the remaining finite singular points in the Poincaré disk must have total index -4. Using Gröbner basis and statement (a) of Lemma 12 we see that there can be at most two finite nilpotent singular points. Then the possibilities are: (i) four saddles, and (ii) four saddles and two cusps.

By statement (c) of Lemma 12, case (ii) would require the existence of a Hamiltonian linear type center with eight inifinite singular points and center–loops in the finite region. We know from [8] that such a vector field does not exist. Therefore we discard case (ii).

In case (i) using Gröbner basis we see that H_6^2 cannot have a fixed value at no more than two finite singular points. Consequently, there are exactly two saddles on the boundary of the period annulus of the center at the origin. Their remaining separatrices cannot cross the straight lines passing through the origin and the infinite singular points as in the case $b = 0$. Therefore we conclude that the global phase portrait of (43) is topologically to equivalent to 1.12 of Figure 1.

We now look at the case $\mu = -1/3$. We remark that the center condition (44) becomes equivalent to $b < 0$ and $a^2 \neq b^2$. We saw that the infinite singular points $(\pm 1, 0)$ on U_1 have two hyperbolic sectors. Therefore the total index of the infinite singular points and the centers at the origins of U_3 and V_3 is 2 on the Poincaré sphere. Hence, in the Poincaré disk, the total index of the remaining possible six finite singular points must be 0. Then, other than the origin, there are either (*i*) no more singular points, (*ii*) two cusps, (*iii*) two saddles and two centers, (*iv*) two saddles, two centers and two cusps.

We claim that only case (*i*), whose global phase portrait is topologically equivalent to 1.10 of Figure 1, is realizable. We can immediately eliminate case (*iv*) by Lemma 12 because a Hamiltonian linear type center with four infinite singular points and center–loops does not exist as it is shown in [8]. Case (*ii*) is also easily eliminated by Lemma 12, see Figure 9. So it only remains to show that case (*iii*) cannot be realized.

Since the infinite singular points have hyperbolic sectors, the finite region of a phase portrait in case (*iii*) must be one of the two possibilities shown in Figure 10. The first figure is not possible due to the fact that (43a) does not change sign on the positive y-axis since $b < 0$. The second figure is clearly not possible either, see Figure 9. Hence the claim is proved and the case $\mu = -1/3$ is finished.

Finally, when $\mu > -1/3$, system (43) has no infinite singular points. Then the remaining possible six finite singular points on the Poincaré disk must have total index 0. Hence we have the following possibilities: (*i*) no singular points, (*ii*) 2 cusps, (*iii*) two saddles and two centers, (*iv*) two saddles, two centers, and two cusps. Observe that these are exactly the same possible cases that we had when $\mu = -1/3$.

We claim that (43) does not have any additional finite singular points in this case and that its global phase portrait is 1.11 of Figure 1. We prove our claim in the next paragraph. We note that the center condition (44) is equivalent to $b < 0$ when $\mu > -1/3$.

We can eliminate cases (*ii*) and (*iii*) by the same reasoning that we used for $\mu = -1/3$. Considering case (*iv*) we see that a Hamiltonian linear type center which has no infinite singular points but has center–loops exists, whose global phase portrait is topologically equivalent to the first figure in Figure 12. This suggests that a global phase portrait in case (*iv*) may exist only if it is topologically equivalent to the second figure in Figure 12. But (43a) is strictly negative on the positive y-axis since $b < 0$. Hnece (43) cannot have such a phase portrait. This proves our claim, finishing the proof of Theorem 1.

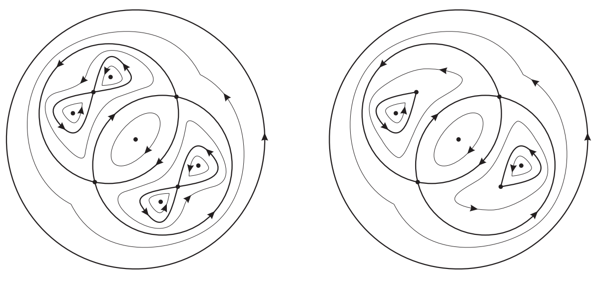


FIGURE 12. A possible cusp for system (VI) when $b \neq 0$.

Acknowledgments

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