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PERIODIC SOLUTIONS OF *EL NIÑO* MODEL THROUGH THE VALLIS DIFFERENTIAL SYSTEM

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Abstract. By rescaling the variables, the parameters and the periodic function of the Vallis differential system we provide sufficient conditions for the existence of periodic solutions and we also characterize their kind of stability. The results are obtained using averaging theory.

1. Introduction

The Vallis system, introduced by Vallis [10] in 1988, is a periodic non–autonomous 3–dimensional system that models the atmosphere dynamics in the tropics over the Pacific Ocean, related to the yearly oscillations of precipitation, temperature and wind force. Denoting by x the wind force, by y the difference of near–surface water temperatures of the east and west parts of the Pacific Ocean, and by z the average near–surface water temperature, the Vallis system is

$$\begin{aligned} \frac{dx}{dt} &= -ax + by + au(t), \\ \frac{dy}{dt} &= -y + xz, \\ \frac{dz}{dt} &= -z - xy + 1, \end{aligned} \tag{1}$$

where the parameters a and b are positive and $u(t)$ is some C^1 $T-$periodic function that describes the wind force under season motions of air masses.

Although this model neglects some effects like Earth's rotation, pressure field and wave phenomena, it provides a correct description of the observed processes and recovers many observed El Niño properties. The properties of El Niño phenomena are studied analytically in [10] and [9]. More specifically, in [10] it is shown that taking $u \equiv 0$, it

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is possible to observe the presence of chaos by considering $a = 3$ and $b = 102$. Additionally, in [9] it is proved that exists a chaotic attractor for system (1) after a Hopf bifurcation. This chaotic characteristic can be easily understanding if we observe that there exist a strong similarity between system (1) and Lorenz system, which becomes more clear under the replacement of z by $z + 1$ in (1).

In [4] the authors examine the localization problem of compact invariant sets of nonlinear time–varying systems and apply the results to the Vallis system (1). In [3] it was generalized the localization method for invariant compact sets of the autonomous dynamical system studied in [4] to the case of a nonautonomous system and solved the localization problem for system (1).

In this paper we provide sufficient conditions in order that system (1) has periodic orbits, and additionally we characterize the stability of these periodic orbits. As far as we know, the study of existence of periodic orbits in the Vallis system has not been considered in the literature, with the exception of the Hopf bifurcation studied in [9].

We observe that, the method used here for studying the periodic orbits can be applied to any periodic non–autonomous differential system. Indeed, in [5] the authors applied this method in order to prove the existence of periodic solutions in a periodic FitzHugh–Nagumo system.

This paper is organized as follows. In this section we state the main results. Next, in section 2 we prove the results on the periodic solutions in the Vallis system using averaging theory. Finally, in section 3 we give a brief summary of the results that we use from averaging theory.

From now on, unless we say the contrary, we will call

$$I = \int_0^T u(s)ds.$$

Now we state our main results.

Theorem 1. *For $\varepsilon > 0$ sufficiently small, $I \neq 0$ and assuming that $a \neq b$ and $(a, b, u(t)) = (\varepsilon A, \varepsilon B, \varepsilon U(t))$ the Vallis system* (1) *has a T–periodic solution $(x(t, \varepsilon), y(t, \varepsilon), z(t, \varepsilon))$ such that*

$$(x(0, \varepsilon), y(0, \varepsilon), z(0, \varepsilon)) = (\gamma_1 + O(\varepsilon), \gamma_1 + O(\varepsilon), 1 + O(\varepsilon)),$$

where $\gamma_1 = aI/(T(b - a)\varepsilon)$. Moreover this periodic orbit is stable if $a > b$ and unstable if $a < b$.

Theorem 2. *For $\varepsilon > 0$ sufficiently small, $I \neq 0$ and assuming that $(a, b) = (\varepsilon^2 A, \varepsilon B)$ the Vallis system* (1) *has a T–periodic solution*

$(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ *such that*

$$(x(0,\varepsilon), y(0,\varepsilon), z(0,\varepsilon)) = (\gamma_2 + O(\varepsilon), \gamma_2 + O(\varepsilon), 1 + O(\varepsilon)),$$

where $\gamma_2 = -aI/(Tb\varepsilon)$. *Moreover this periodic orbit is always unstable.*

Theorem 3. *For* $\varepsilon > 0$ *sufficiently small,* $I \neq 0$ *and assuming* $(a, u(t)) = (\varepsilon A, \varepsilon U(t))$ *the Vallis system* (1) *has a* T*–periodic solution* $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ *such that*

$$(x(0,\varepsilon), y_1(0,\varepsilon), z(0,\varepsilon)) = (\gamma_3 + O(\varepsilon), \gamma_3 + O(\varepsilon), 1 + O(\varepsilon)),$$

where $\gamma_3 = I/(T\varepsilon) \neq 0$. *Moreover this periodic orbit is always stable.*

Theorem 4. *For* $\varepsilon > 0$ *sufficiently small,* $I \neq 0$ *and assuming* $(a, u(t)) = (\varepsilon A, \varepsilon U(t))$ *the Vallis system* (1) *has a* T*–periodic solution* $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ *such that*

$$(x(0,\varepsilon), y_1(0,\varepsilon), z(0,\varepsilon)) = (\gamma_3 + O(\varepsilon), O(\varepsilon), 1 + O(\varepsilon)),$$

where $\gamma_3 = I/(T\varepsilon) \neq 0$. *Moreover this periodic orbit is always stable.*

In what follows we consider the function

$$J(t) = \int_0^t u(s)ds.$$

and note that $J(T) = I$. So we have the following result.

Theorem 5. *Consider* $\varepsilon > 0$ *sufficiently small,* $I = 0$, $J(t) \neq 0$ *if* $0 < t < T$ *and assume that* $(a, u(t)) = (\varepsilon A, \varepsilon U(t))$. *Then the Vallis system* (1) *has a* T*–periodic solution* $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ *such that*

$$(x(0,\varepsilon), y_1(0,\varepsilon), z(0,\varepsilon)) = (\gamma_4 + O(\varepsilon), O(\varepsilon), 1 + O(\varepsilon)),$$

where $\gamma_4 = -\dfrac{a}{T\varepsilon^2}\displaystyle\int_0^T J(s)ds \neq 0$. *Moreover this periodic orbit is always stable.*

We observe that the periodic solution presented in each one of the theorems tends to infinity when ε tends to zero. Consequently, these periodic solutions are different from that one found via a Hopf bifurcation in [9].

Moreover, we shall prove that the tools used for proving Theorems 1, 2, 3, 4 and 5 do not provide more periodic solutions outside the ones provided in the mentioned theorems, see Proposition 6

2. Proof of the results

In order to study the periodic solutions of the differential system (1), we start doing a rescaling of the variables (x, y, z), of the function $u(t)$, and of the parameters a and b, as follows

$$
\begin{array}{ccc} x = \varepsilon^{m_1} X, & y = \varepsilon^{m_2} Y, & z = \varepsilon^{m_3} Z, \\ u(t) = \varepsilon^{n_1} U(t), & a = \varepsilon^{n_2} A, & b = \varepsilon^{n_3} B, \end{array} \tag{2}
$$

where ε always is positive and sufficiently small and m_i and n_j are non–negative integers, for all $i, j = 1, 2, 3$. Then, in the new variables (X, Y, Z) system (1) writes

$$
\begin{aligned} \frac{dX}{dt} &= -\varepsilon^{n_2} AX + \varepsilon^{-m_1+m_2+n_3} BY + \varepsilon^{-m_1+n_1+n_2} AU(t), \\ \frac{dY}{dt} &= -Y + \varepsilon^{m_1-m_2+m_3} XZ, \\ \frac{dZ}{dt} &= -Z - \varepsilon^{m_1+m_2-m_3} XY + \varepsilon^{-m_3}. \end{aligned} \tag{3}
$$

Consequently, in order to have non–negative powers of ε, we must impose the conditions

$$
m_3 = 0 \quad \text{and} \quad 0 \leq m_2 \leq m_1 \leq L, \tag{4}
$$

where $L = \min\{m_2 + n_3, n_1 + n_2\}$. So system (3) becomes

$$
\begin{aligned} \frac{dX}{dt} &= -\varepsilon^{n_2} AX + \varepsilon^{-m_1+m_2+n_3} BY + \varepsilon^{-m_1+n_1+n_2} AU(t), \\ \frac{dY}{dt} &= -Y + \varepsilon^{m_1-m_2} XZ, \\ \frac{dZ}{dt} &= 1 - Z - \varepsilon^{m_1+m_2} XY. \end{aligned} \tag{5}
$$

Our aim is to find periodic solutions of system (5) for some special values of m_i, n_j, $i, j = 1, 2, 3$, and after going back by the rescaling to guarantee the existence of periodic solutions in system (1). In what follows we consider the case where n_2 and n_3 are positives and $m_2 = m_1 < n_1 + n_2$. These conditions lead to the proofs of Theorems 1, 2 and 3. For this reason we present this proofs together in order to avoid repetitive arguments.

Proof of Theorems 1, 2 *and* 3*:* We start considering system (5) with n_2 and n_3 positives and $m_2 = m_1 < n_1 + n_2$. So we have

$$
\begin{aligned}
\frac{dX}{dt} &= -\varepsilon^{n_2} AX + \varepsilon^{n_3} BY + \varepsilon^{-m_1+n_1+n_2} AU(t), \\
\frac{dY}{dt} &= -Y + XZ, \\
\frac{dZ}{dt} &= 1 - Z - \varepsilon^{2m_1} XY.
\end{aligned}
\tag{6}
$$

Now we apply the averaging method to the differential system (6). Denoting again the variables (X, Y, Z) by (x, y, z), and using the notation of section 3 we have $\mathbf{x}= (x, y, z)^T$ and

$$
F_0(t, \mathbf{x}) = \begin{pmatrix} 0 \\ -y + xz \\ 1 - z \end{pmatrix}. \tag{7}
$$

We start considering the system

$$
\dot{\mathbf{x}} = F_0(t, \mathbf{x}). \tag{8}
$$

Its solution $\mathbf{x}(t, \mathbf{z}, 0) = (x(t), y(t), z(t))$ such that $\mathbf{x}(0, \mathbf{z}, 0) = \mathbf{z} = (x_0, y_0, z_0)$ is

$$
\begin{aligned}
x(t) &= x_0, \\
y(t) &= (1 - e^{-t}(1 + t))x_0 + e^{-t} y_0 + e^{-t} t x_0 z_0, \\
z(t) &= 1 - e^{-t} + e^{-t} z_0.
\end{aligned}
$$

In order that $\mathbf{x}(t, \mathbf{z}, 0)$ is a periodic solution we must choose $y_0 = x_0$ and $z_0 = 1$. This implies that for every point of the straight line $x = y$, $z = 1$ passes a periodic orbit that lies in the phase space $(x, y, z, t) \in \mathbb{R}^3 \times \mathbb{S}^1$. Here and what follows $\mathbb{S}^1$ is the interval $[0, T]$ identifying T with 0.

Observe that, using the notation of section 3, we have $n = 3$, $k = 1$, $\alpha = x_0$ and $\beta(x_0) = (x_0, 1)$, and consequently $\mathcal{M}$ is an one–dimensional manifold given by $\mathcal{M} = \{(x_0, x_0, 1) \in \mathbb{R}^3 : x_0 \in \mathbb{R}\}$. The fundamental matrix $M_{\mathbf{z}_{x_0}}(t)$ of (8) satisfying that $M_{\mathbf{z}_{x_0}}(0)$ is the identity of $\mathbb{R}^3$ is

$$
\begin{pmatrix} 1 & 0 & 0 \\ 1 - \cosh t + \sinh t & e^{-t} & e^{-t} t x_0 \\ 0 & 0 & e^{-t} \end{pmatrix},
$$

and its inverse matrix $M^{-1}_{\mathbf{z}_{x_0}}(t)$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1-e^t & e^t & -e^t t x_0 \\ 0 & 0 & e^t \end{pmatrix}.$$

Moreover, once the matrix $M^{-1}_{\mathbf{z}_{x_0}}(0) - M^{-1}_{\mathbf{z}_{x_0}}(T)$ has a zero matrix 1×2 in the upper right corner and a lower right corner 2×2 matrix Δ_{x_0}

$$\begin{pmatrix} 1-e^T & e^T T x_0 \\ 0 & 1-e^T \end{pmatrix},$$

with $\det(\Delta_{x_0}) = (1-e^T)^2 \neq 0$ since $T \neq 0$, we can apply averaging theory described in section 3. Indeed, considering the function F given by the vector field of system (6) subtracted by F_0, then the components of the function $M^{-1}_{\mathbf{z}_{x_0}}(t)F(t, \mathbf{x}(t, \mathbf{z}, 0))$ are

$$\begin{aligned} g_1(x_0,t) &= -\varepsilon^{n_2} A x_0 + \varepsilon^{n_3} B x_0 + \varepsilon^{-m_1+n_1+n_2} A U(t), \\ g_2(x_0,t) &= \varepsilon^{2m_1} e^t t x_0^3 + (1-e^t) g_1(x_0,t), \\ g_3(x_0,t) &= -\varepsilon^{2m_1} e^t x_0^2. \end{aligned}$$

To apply averaging theory of first order we need to consider only terms up to order ε. Analysing the expressions of g_1, g_2 and g_3 we note that these terms depend on the values of m_1 and n_j, for each $j = 1,2,3$. In fact, we just need to study the integral of g_1, once $k = 1$. Moreover, studying the function g_1, we observe that the only possibility to obtain an isolated zero by integrating g_1 is assuming that $n_1 + n_2 - m_1 = 1$. Otherwise, if we denote $\mathcal{F} = (f_1, f_2, f_3)$, the only x_0 such that $f_1(x_0, y_0, z_0) = 0$ is $x_0 = 0$, and this value is not interesting because the point $(0,0,1)$ is the equilibrium point of system (8). The same occurs if n_2 and n_3 are greater than 1 simultaneously. This analysis reduces the existence of possible periodic solutions to the following cases:

(p_1) $n_2 = 1$ and $n_3 = 1$;
(p_2) $n_2 > 1$ and $n_3 = 1$;
(p_3) $n_2 = 1$ and $n_3 > 1$.

In the case (p_1), we have $M^{-1}_{\mathbf{z}_{x_0}}(t)F_1(t, x(t,z,0)) = -Ax_0 + Bx_0 + AU(t)$ and then

$$f_1(x_0) = (-A+B)Tx_0 + AI\varepsilon^{-n_1}.$$

Consequently, if $A \neq B$, then $f_1(x_0) = 0$ implies

$$x_0 = \frac{AI\varepsilon^{-n_1}}{T(B-A)}.$$

Therefore, if we take $n_2 = n_3 = 1$, then going back through the rescaling (2) and considering $n_1 = 1$, x_0 becomes

$$x_0 = \frac{(a\varepsilon^{-1})(I\varepsilon^{-1})}{T\varepsilon^{-1}(b-a)} = \frac{aI}{T(b-a)\varepsilon}.$$

Moreover, we note that $f_1'(x_0) = -A+B = (-a+b)/\varepsilon \neq 0$, so the periodic orbit corresponding to x_0 is stable if $a > b$ and unstable otherwise. So this concludes the proof of Theorem 1.

Analogously the function f_1 in the cases (p_2) and (p_3) are respectively

$$f_1(x_0) = TBx_0 + AI\varepsilon^{-n_1} \qquad \text{and} \qquad f_1(x_0) = -TAx_0 + AI\varepsilon^{-n_1}.$$

In the first case, the condition $f_1(x_0) = 0$ implies

$$x_0 = \frac{-AI\varepsilon^{-n_1}}{TB}.$$

Now observe that we have $n_2 > 1$ and $n_3 = 1$. So, going back by the rescaling, we obtain

$$x_0 = \frac{(-a\varepsilon^{-n_2})(I\varepsilon^{-n_1})}{Tb\varepsilon^{-1}} = \frac{aI}{Tb\varepsilon^{n_1+n_2-1}}.$$

and consequently, choosing $n_1 = 0$ and $n_2 = 2$, we get $x_0 = aI/(Tb\varepsilon)$. Note also that $f_1'(x_0) = TB = Tb/\varepsilon$, then the periodic orbit corresponding to x_0 is always unstable. Thus Theorem 2 is proved.

Finally, in the case (p_3), $f_1(x_0) = 0$ implies $x_0 = I\varepsilon^{-n_1}/T$. So, taking $n_1 = 1$, we have $x_0 = I/(T\varepsilon)$. Additionally, in this case $f_1'(x_0) = -Ta/\varepsilon$, since $n_2 = 1$ and $A = \varepsilon^{-n_1}a$. Consequently the periodic solution that comes from x_0 is always stable. This proves Theorem 3. □

Proof of Theorem 4: As the last proof, we start considering a more general case on the powers of ε in (5) taking $n_2 > 0$ and $m_2 < m_1 < L$. In this case, system (24) writes

$$F_0(t, \mathbf{x}) = \begin{pmatrix} 0 \\ y \\ 1-z \end{pmatrix}, \tag{9}$$

whose solution $\mathbf{x}(t, \mathbf{z}, 0)$ satisfying $\mathbf{x}(0, \mathbf{z}, 0) = \mathbf{z}$ is given by

$$(x(t), y(t), z(t)) = (x_0, e^{-t}y_0, 1 - e^{-t} + e^{-t}z_0).$$

This solution is periodic if $y_0 = 0$ and $z_0 = 1$. In this case for every point of the straight line $y = 0$, $z = 1$ passes a periodic orbit that lies in the phase space $(x, y, z, t) \in \mathbb{R}^3 \times \mathbb{S}^1$. We observe that, using the notation of section 3, in this case we have $n = 3$, $k = 1$, $\alpha = x_0$ and $\beta(\alpha) = (0, 1)$. Consequently $\mathcal{M}$ is an one–dimensional manifold given by $\mathcal{M} = \{(x_0, 0, 1) \in \mathbb{R}^3 : x_0 \in \mathbb{R}\}$.

The fundamental matrix $M_{\mathbf{z}_{x_0}}(t)$ of (24) with F_0 given by (9) satisfying $M_{\mathbf{z}_{x_0}}(0) = Id_3$ and its inverse $M^{-1}_{\mathbf{z}_{x_0}}(t)$ are given by

$$M_{\mathbf{z}_{x_0}}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \quad \text{and} \quad M^{-1}_{\mathbf{z}_{x_0}}(t) \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix}.$$

Moreover, once the matrix $M^{-1}_{\mathbf{z}_{x_0}}(0) - M^{-1}_{\mathbf{z}_{x_0}}T$ has a zero matrix 1×2 in the upper right corner and a lower right corner 2×2 matrix Δ_{x_0}

$$\begin{pmatrix} 1 - e^T & 0 \\ 0 & 1 - e^T \end{pmatrix},$$

with $\det(\Delta_{x_0}) = (1 - E^T)^0 \neq 0$, we can apply averaging theory described in section 3. Again, using the notation introduced in the last proof, since $k = 1$ we will look only to the integral of the first coordinate of $\mathcal{F} = (f_1, f_2, f_3)$. In this case we have

$$g_1(x_0, y_0, z_0, t) = -\varepsilon^{n_2} A x_0 + \varepsilon^{-m_1+n_1+n_2} A U(t).$$

Comparing this function g_1 with the same function obtained in the proof of Theorem 1, it is easy to see that this case correspond to the case (p_3) of the last proof. Then, in order to have periodic solutions, we need choose $n_2 = 1$ and $n_1 + n_2 - m_1 = 1$. So, following the steps of the case (p_3) in the last proof, by choosing $n_1 = 1$ and coming back by the rescaling (2) to system (1), Theorem 3 is proved. □

Proof of Theorem 5*:* We start considering system (5) with $n_3 = 2$, $n_2 > 0$, $m_1 = n_1 + n_2$ and $m_2 < m_1 < m_2 + n_3$. With these conditions system (5) writes

$$\begin{aligned} \frac{dX}{dt} &= -\varepsilon^{n_2} A X + \varepsilon^{m_2 - n_1 - n_2 + n_3} B Y + A U(t), \\ \frac{dY}{dt} &= -Y + \varepsilon^{-m_2 + n_1 + n_2} X Z, \\ \frac{dZ}{dt} &= 1 - Z - \varepsilon^{m_2 + n_1 + n_2} X Y. \end{aligned} \tag{10}$$

In order to use the averaging theory described in section 3 we denote again the variables (X, Y, Z) by (x, y, z). So we have $\mathbf{x} = (x, y, z)^T$ and

$$F_0(t, \mathbf{x}) = \begin{pmatrix} AU(t) \\ -y \\ 1 - z \end{pmatrix}. \tag{11}$$

Now we note that the solution $\mathbf{x}(t, \mathbf{z}, 0) = (x(t), y(t), z(t))$ such that $\mathbf{x}(0, \mathbf{z}, 0) = \mathbf{z} = (x_0, y_0, z_0)$ of the system

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}). \tag{12}$$

is

$$x(t) = x_0 + \int_0^t AU(s)ds, \quad y(t) = e^{-t}y_0, \quad z(t) = 1 - e^{-t} + e^{-t}z_0.$$

Observe that, since $I = 0$ and $J(t) \neq 0$ for $0 < t < T$, in order that $\mathbf{x}(t, \mathbf{z}, 0)$ is a periodic solution we need to fix $y_0 = 0$ and $z_0 = 1$. This implies that for every point in a neighbourhood of x_0 in the straight line $y = 0$, $z = 1$ passes a periodic orbit that lies in the phase space $(x, y, z, t) \in \mathbb{R}^3 \times \mathbb{S}^1$.

Following the notation of section 3, we obtain $n = 3$, $k = 1$, $\alpha = x_0$ and $\beta(x_0) = (0, 1)$. Hence $\mathcal{M}$ is an one–dimensional manifold $\mathcal{M} = \{(x_0, 0, 1) \in \mathbb{R}^3 : x_0 \in \mathbb{R}\}$ and the fhe fundamental matrix $M_{\mathbf{z}_{x_0}}(t)$ of (12) satisfying that $M_{\mathbf{z}_{x_0}}(0)$ is the identity of $\mathbb{R}^3$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}.$$

It is easy to see that the matrix $M^{-1}_{\mathbf{z}_{x_0}}(0) - M^{-1}_{\mathbf{z}_{x_0}}(T)$ has a 1×2 zero matrix in the upper right corner and a 2×2 lower right corner matrix

$$\Delta_{x_0} = \begin{pmatrix} 1 - e^T & 0 \\ 0 & 1 - e^T \end{pmatrix},$$

with $\det(\Delta_{x_0}) = (1 - e^T)^2 \neq 0$. Then the hypotheses of Theorem 7 are satisfied. Moreover, the components of the function $M^{-1}_{\mathbf{z}_{x_0}}(t)F(t, x(t, \mathbf{z}, \mathbf{0}))$ are

$$g_1(x_0, t) = -\varepsilon^{n_2} A\left(x_0 + \int_0^t AU(s)ds\right) + AU(t),$$

$$g_2(x_0, t) = \varepsilon^{-m_2+n_1+n_2}\left(x_0 + \int_0^t AU(s)ds\right)e^t,$$

$$g_3(x_0, t) = 0.$$

Taking n_1 and n_2 equal to one and observing that $k = 1$ and $n = 3$, we are interested only in the first component of the function $F_1 = (F_{11}, F_{12}, F_{13})$ described in section 3. Indeed, applying the averaging theory and considering $\mathcal{F}(x_0) = (f_1(x_0), f_2(x_0), f_3(x_0))$, we will study the first component of the integral

$$\mathcal{F}(x_0) = \int_0^T M_{\mathbf{z}_{x_0}}^{-1}(t, \mathbf{z}_{x_0}) F_{11}(t, \mathbf{x}(t, \mathbf{z}_{x_0}))dt.$$

Function F_{11} writes

$$F_{11} = -A\left(x_0 + \int_0^t AU(s)ds\right)$$

and then, coming back by the rescaling function f_1 is

$$\begin{aligned} f_1(x_0) &= \int_0^T -A\left(x_0 + \int_0^t AU(s)ds\right)dt \\ &= -\frac{a}{\varepsilon}Tx_0 - \frac{a^2}{\varepsilon^3}\int_0^T J(s)ds. \end{aligned}$$

Therefore, doing $f_1(x_0) = 0$ we obtain

$$x_0 = -\frac{a}{T\varepsilon^2}\int_0^T J(s)ds \neq 0.$$

Moreover, $f_1'(x_0) = -(a/\varepsilon)T < 0$, since a and ε are positives. Consequently the $T-$periodic orbit detected by averaging theory is always stable. This ends the proof. □

The following shows that there is no other configuration of power of ε defining the rescaling (2) for which we can detect other periodic solutions of system (1) using the averaging theory.

Proposition 6. *By using averaging theory the Vallis system* (1) *has no periodic solutions different from the ones presented in Theorems* 1, 2, 3, 4 *and* 5.

In order to prove Proposition 6 we will study all possible powers of ε in system (5). Indeed, we consider the set $P = \{n_2, -m_1 + m_2 + n_3, -m_1 + n_1 + n_2, m_1 - m_2\}$ of the relevant powers of ε in this system (see (4)) and observe that each integer of P must be non–negative. Therefore, we will study each one of the 16 possible combinations of values of the elements of P taking into account conditions (4). We start considering $n_2 > 0$. Then we have the following eight cases:

Case 1: $n_2 > 0$, $m_1 = m_2$, $n_3 = 0$ and $m_1 = n_1 + n_2$,
Case 2: $n_2 > 0$, $m_1 = m_2$, $n_3 = 0$ and $m_1 < n_1 + n_2$,

Case 3: $n_2 > 0$, $m_1 = m_2$, $n_3 > 0$ and $m_1 = n_1 + n_2$,
Case 4: $n_2 > 0$, $m_1 = m_2$, $n_3 > 0$ and $m_1 < n_1 + n_2$,
Case 5: $n_2 > 0$, $m_1 > m_2$, $n_3 > 0$ and $m_1 = n_1 + n_2$,
Case 6: $n_2 > 0$, $m_1 > m_2$, $n_3 > 0$ and $m_1 < n_1 + n_2$,
Case 7: $n_2 > 0$, $m_1 > m_2$, $m_1 < m_2 + n_3$ and $m_1 = n_1 + n_2$,
Case 8: $n_2 > 0$, $m_1 > m_2$, $m_1 < m_2 + n_3$ and $m_1 < n_1 + n_2$.

The remainder cases from 9 to 16 are the same than the cases from 1 to 8, respectively, taking $n_2 = 0$ instead of $n_2 > 0$.

We observe that the case 4 was studied in Theorems 1, 2 and theorem 3. Additionally, Theorem 4 concerns to case 8 and Theorem 5 concerns to case 7. Thus we will eliminate these cases of the proof of Proposition 6. In the other cases we will prove that some hypotheses of averaging method presented in section 3 do not hold.

Proof of Proposition 6*:* First we prove the proposition using system (5) in case 2. Indeed, considering $\mathbf{x} = (X, Y, Z)$, in case 2 system (24) is

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}) = (BY, -Y + XZ, 1 - Z)^T. \tag{13}$$

The last differential equation is uncoupled and its solution $Z(t)$ is $Z(t) = 1 - e^{-t} + e^{-t}Z_0$. It is easy to see that $Z_0 = 1$ is the only value of Z_0 for which $Z(t)$ is periodic. Now, substituting the solution $Z(t)$ in the second differential equation of (13) and solving the system of differential equations $\dot{X} = BY$, $\dot{Y} = -Y + X$ we get

$$X(t) = \frac{1}{2C}\left(C_1 e^{\frac{1}{2}(-C-1)t} + C_2 e^{\frac{1}{2}(C-1)t}\right),$$
$$Y(t) = \frac{1}{2C}\left(C_3 e^{\frac{1}{2}(-C-1)t} + C_4 e^{\frac{1}{2}(C-1)t}\right),$$

where $C = \sqrt{1 + 4B} > 1$, $C_1 = (C-1)X_0 - 2BY_0$, $C_2 = (C+1)X_0 + 2BY_0$, $C_3 = -2X_0 + (C+1)Y_0$ and $C_4 = 2X_0 + (C-1)Y_0$.

Without lost of generality, we will study the conditions that turn the solution $X(t)$ into a periodic function. In order to do this, we need to choose C_1 and C_2 equal to zero once $C > 1$. Fixing $C_1 = 0$ we obtain

$$X_0 = \frac{-2BY_0}{C-1}.$$

Substituting this value into C_2 we obtain $(1 + 4B + C)Y_0$ which is positive unless $Y_0 = 0$. On the other hand, the value $Y_0 = 0$ implies $X_0 = 0$, and since $Z_0 = 1$ we have the equilibrium point $(0, 0, 1)$ of system (13). This implies that system (13) has no periodic solutions, and then the averaging method described in section 3 cannot be applied in this case. Moreover, there is no lost of generality when we study only the solution $X(t)$ because if one of the solutions $X(t)$, $Y(t)$ or $Z(t)$ for (24) is not periodic, system (13) cannot have a periodic solution. We will use this fact to conclude the proof of Proposition 6 in some other cases below.

In what follows we prove Proposition 6 for system (5) in case 10. Indeed, observe that system (24) now writes

$$\dot{\mathbf{x}} = (-AX + BY, -Y + XZ, 1 - Z)^T. \tag{14}$$

As before, we take the solution $Z(t) = 1 - e^{-t} + e^{-t}Z_0$ of $\dot{Z} = 1 - Z$ and replace this solution with $Z_0 = 1$ in the others differential equations of (14). Therefore the solutions $X(t)$ is

$$X(t) = \frac{1}{2D}\left(D_1 e^{\frac{1}{2}(-A-1-D)t} + C_2 e^{\frac{1}{2}(-A-1+D)t}\right)$$

where $D = \sqrt{(A-1)^2 + 4B} > 0$, $D_1 = (A - 1 + D)X_0 - 2BY_0$ and $D_2 = (-A + 1 + D)X_0 + 2BY_0$. We note that this expression is very similar to the expression of the solution $X(t)$ of system (5) in case 2 just taking A as zero. Moreover, it is possible to show that the same arguments used in case 2 are also true in this case, and consequently the averaging method does not apply to system (5) in case 10.

In case 12 we have

$$\dot{\mathbf{x}} = (-AX, -Y + XZ, 1 - Z)^T, \tag{15}$$

so the solutions $X(t)$ and $Z(t)$ for this systems are, respectively, $X(t) = e^{-At}X_0$ and $Z(t) = 1 - e^{-t} + e^{-t}Z_0$. Then, choosing $X_0 = 0$ and $Z_0 = 1$ in order that $X(t)$ and $Z(t)$ be periodic, the solution $Y(t)$ becomes $Y(t) = e^{-t}Y_0$ and consequently we have to fix $Y_0 = 0$. However, the point $(X_0, Y_0, Z_0) = (0, 0, 1)$ is the equilibrium point of system (15), and consequently system (15) has no periodic solutions. Thus, in case 12 again we cannot apply the averaging theory.

In case 14 system (24) is

$$\dot{\mathbf{x}} = (-AX + BY, -Y, 1 - Z)^T, \tag{16}$$

whose solutions $Y(t)$ and $Z(t)$ starting at Y_0 and Z_0 are, respectively, $Y(t) = e^{-t}Y_0$ and $Z(t) = 1 - e^{-t} + e^{-t}Z_0$. These solutions are periodic if $Y_0 = 0$ and $Z_0 = 1$, and with these values the solution $X(t)$ writes $X(t) = e^{-At}X_0$. So, since $A \neq 0$, we need to take $X_0 = 0$ to have $X(t)$ periodic. The conclusion of Proposition 6 in this case follows as in case 12.

For proving Proposition 6 in case 16 we observe that the solutions $X(t)$, $Y(t)$ and $Z(t)$ of system (24) given by

$$\dot{\mathbf{x}} = (-AX, -Y, 1-Z)^T, \tag{17}$$

are $X(t) = e^{-At}X_0$, $Y(t) = e^{-t}Y_0$ and $Z(t) = 1-e^{-t}+e^{-t}Z_0$, whose values X_0, Y_0 and Z_0 for which these solutions are periodic are $X_0, Y_0, Z_0) = (0,0,1)$. So, as before we cannot apply the averaging theory.

Now we prove that the averaging method does not work in system (5) in case 5. In fact, in this case the solutions $X(t)$, $Y(t)$ and $Z(t)$ of system (24) starting at (X_0, Y_0, Z_0) is

$$(X(t), Y(t), Z(t)) \quad = \quad \left(X_0 + \int_0^t AU(s)ds, e^{-t}Y_0, 1-e^{-t}+e^{-t}Z_0 \right),$$

where now we suppose that $\int_0^T u(s)ds = 0$ in order that $x(t)$ be a $T-$periodic solution. Looking to the expressions of $Y(t)$ and $Z(t)$, it is easy to see that $Y_0 = 0$ and $Z_0 = 1$ are the only values of Y_0 and Z_0 for which $Y(t)$ and $Z(t)$ are periodic. We observe that using the notation of section 3, we have $k = 1$, $n = 3$ and the fundamental matrix $M_{\mathbf{z}_{X_0}}(t)$ is

$$\begin{pmatrix} 1 & B(1-\cosh(t)+\sinh(t)) & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix},$$

and its inverse matrix is

$$M^{-1}_{\mathbf{z}_{X_0}}(t) = \begin{pmatrix} 1 & B(1-e^t) & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix}.$$

So the matrix $M^{-1}_{\mathbf{z}_{X_0}}(0) - M^{-1}_{\mathbf{z}_{X_0}}T$ does not have a 1×2 zero matrix in the upper right corner, since

$$M^{-1}_{\mathbf{z}_{X_0}}(0) - M^{-1}_{\mathbf{z}_{X_0}}T = \begin{pmatrix} 0 & B(-1+e^T) & 0 \\ 0 & 1-e^T & 0 \\ 0 & 0 & 1-e^T \end{pmatrix}.$$

Then, since B is positive, we cannot apply averaging method in case 5.

Case 6 is similar to case 5. In fact, the solution $x(t)$, $y(t)$ and $z(t)$ of system (24) starting at (x_0, y_0, z_0) eliminating the non periodic terms is

$$(X(t), Y(t), Z(t)) = (X_0, 0, 1),$$

and following the steps of section 3 we obtain the same matrix $M^{-1}_{\mathbf{z}_{X_0}}(t)$ of case 5. Hence, we cannot apply the averaging method in this case.

Next we prove Proposition 6 in case 3. The solutions $X(t)$, $Y(t)$ and $Z(t)$ of system (24) are

$$\begin{aligned}
X(t) &= X_0 + \int_0^t AU(s)ds,\\
Y(t) &= X_0 - e^{-t}X_0 + e^{-t}Y_0 + \int_0^t AU(s)ds - e^{-t}\int_0^t Ae^sU(s)ds,\\
Z(t) &= 1 - e^{-t} + e^{-t}Z_0.
\end{aligned}$$

where we suppose that $I = 0$ in order that $X(t)$ be a $T-$periodic solution. Observe that if $I \neq 0$, $X(t)$ is not periodic and then we cannot apply averaging method. Indeed, the expression of $Z(t)$ implies that $Z_0 = 1$ is the only value of Z_0 for which $Z(t)$ is periodic. Moreover, we take $X_0 = Y_0 + W$. Thus, the solutions $X(t)$, $Y(t)$ and $Z(t)$ become

$$\begin{aligned}
X(t) &= Y_0 + W + A\int_0^t U(s)ds,\\
Y(t) &= Y_0 + W + A\int_0^t U(s)ds - e^{-t}\left(W - A\int_0^t e^sU(s)ds\right),\\
Z(t) &= 1,
\end{aligned}$$

where $\int_0^t U(s)ds$ is periodic because $I = 0$. Add, we suppose that $e^{-t}(W - A\int_0^t e^sU(s)ds)$ is periodic, for some $W \in \mathbb{R}$. As before, if there is no W such that $e^{-t}(W - A\int_0^t e^sU(s)ds)$ is periodic, so $Y(t)$ is also no–periodic and averaging theory does not apply. Consequently, the solution $Y(t)$ is $T-$periodic.

We note that considering $U(t) = \cos\, t$ and $W = A/2$ the solutions $X(t)$, $Y(t)$ and $Z(t)$ are periodic, because with this considerations we have

$$\begin{aligned}
X(t) &= Y_0 + (A/2) + A\sin\, t,\\
Y(t) &= (1/2)(A + 2Y_0 - A\cos\, t + A\sin\, t),\\
Z(t) &= 1,
\end{aligned}$$

which is periodic. However, we will considerer the general case instead of this particular case $U(t) = \cos\, t$ and $W = A/2$. Hence, by following the lines of section 3, we have $k = 1$, $n = 3$ and the fundamental matrix

$M_{\mathbf{z}_{Y_0}}(t)$ is

$$\begin{pmatrix} e^t & 1-e^t & e^t E(A,W,Y_0,t) \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{pmatrix},$$

where $E(A,W,Y_0,t) = \int_0^t e^{-2s}(Y_0 + W + A\int_0^s U(w)dw)ds$. Its inverse matrix $M^{-1}_{\mathbf{z}_{Y_0}}(t)$ is

$$\begin{pmatrix} e^{-t} & 1-\cosh t + \sinh t & -e^t E(A,W,Y_0,t) \\ 0 & 1 & 0 \\ 0 & 0 & e^{t} \end{pmatrix}.$$

Then the matrix $M^{-1}_{\mathbf{z}_{Y_0}}(0) - M^{-1}_{\mathbf{z}_{Y_0}}T$ does not have a 2×2 lower right matrix

$$\Delta_{Y_0} = \begin{pmatrix} 0 & 0 \\ 0 & 1-e^T \end{pmatrix},$$

whose determinant is zero. Then we cannot apply averaging method in case 3.

We study system (5) in case 1. Now system (24) is

$$(\dot{X},\dot{Y},\dot{Z}) = (BY + AU(t), -Y + XZ, 1-Z)^T. \tag{18}$$

The last differential equation is uncoupled and its solution $Z(t)$ is $Z(t) = 1 - e^{-t} + e^{-t}Z_0$. As before $Z_0 = 1$ turn $Z(t)$ into a periodic solution. Now, substituting the solution $Z(t)$ in the second differential equation of (18) with $Z_0 = 1$ and solving the system of differential equations $\dot{X} = BY + AU(t)$, $\dot{Y} = -Y + X$ we get solutions very similar to the ones obtained in case 2. In fact, calling $X_2(t)$ and $Y_2(t)$ the solutions of case 2, for case 1 the solutions $X(t)$ is

$$X(t) = X_2(t) + (1/2C)g_2(A,B,t)e^{\frac{1}{2}(-C-1)t},$$

where g_2 is

$$\frac{A}{2C}(-1+C+Ce^{Ct})\int_0^t e^{-\frac{1}{2}(-1+C)s}\left(1+C+(-1+C)e^{Cs}\right)U(s)ds +$$

$$\frac{2AB}{C}(1-e^{Ct})\int_0^t e^{-\frac{1}{2}(-1+C)s}\left(-1+e^{Cs}\right)U(s)ds.$$

We observe that g_2 does not depend neither X_0 nor Y_0. For this reason, we cannot eliminate the non–periodic terms of $X_2(t)$ through the expression $(1/2C)g_2(A,B,t)e^{\frac{1}{2}(-C-1)t}$, whatever the function $g_2(A,B,t)$ chosen. So, as we see in case 2 we must choose $(X_0,Y_0) = (0,0)$ in order that $X_2(t)$ be periodic. Since $Z_0 = 1$, system (18) has no submanifold of periodic solutions as demands the averaging theory described in section 3.

Case 9 is similar to case 1 in the sense that there is no choice of X_0, Y_0 and Z_0 in such way that the solution of the system

$$\dot{\mathbf{x}} = (-AX + BY + AU(t), -Y + XZ, 1 - Z)^T. \tag{19}$$

corresponding to system (5) in case 9 has a submanifold of periodic solutions. As before, $Z(t) = 1 - e^{-t} + e^{-t}Z_0$ is the solution of the last differential equation of system (19) and the value Z_0 for which this solution is periodic is $Z_0 = 1$. Substituting this solution into system (19) and solving it, we obtain a solution similar to the solution $X_{10}(t)$ corresponding to system (14) in case 10. We have

$$X(t) = X_{10}(t) + g_{10}(A, B, t),$$

where g_{10} is

$$\frac{A}{2D}(-1 + A\left(1 - e^{Dt}\right) + D + (1 + D)e^{Dt}) \int_0^t e^{\frac{1}{2}(1+A-D)s} (1 + D+$$
$$(-1 + D)e^{Ds} + A\left(-1 + e^{Ds}\right)\right) U(s)ds + \frac{2AB}{D}(1 - e^{Dt})$$
$$\int_0^t e^{\frac{1}{2}(1+A-D)s} \left(-1 + e^{Ds}\right) U(s)ds.$$

Note that g_{10} does not depend neither X_0 nor Y_0. The conclusion of this case follows from the fact that X_{10} and Y_{10} are non periodic unless $(X_0, Y_0) = (0, 0)$, and using the same arguments of the proof of case 1.

Now we consider system (5) in case 11

$$\dot{\mathbf{x}} = (-AX + AU(t), -Y + XZ, 1 - Z)^T. \tag{20}$$

Considering $A \neq 1$, as before we have $Z(t) = 1 - e^{-t} + e^{-t}Z_0$ and choose $Z_0 = 1$ because $Z(t)$ must be periodic. The solution $X(t)$ is

$$X(t) = e^{-At}X_0 + A\int_0^t e^{As}U(s)ds.$$

This means that we must take $X_0 = 0$ to have $X(t)$ periodic. Substituting $X_0 = 0$ and $Z(t) = 1$ em $Y(t)$, it becomes

$$Y(t) = Y_0e^{-t} + \frac{1}{A-1}e^{-(A+1)t}h_{11}(A, t),$$

where now h_{11} has no Y_0 terms and writes

$$A(e^{At} - e^t)\int_0^t e^{As}U(s)ds + Ae^{At}\int_0^t \left(e^s - e^{As}\right) U(s)ds.$$

How $A \neq 1$ and h_{11} does not depends on Y_0 we cannot eliminate the non periodic term $Y_0 e^{-t}$ of $Y(t)$ unless we take $Y_0 = 0$. Consequently, as cases 1 and 9 the averaging theory does not work in case 11.

Moreover, if $A = 1$, we have the same solutions $X(t)$ and $Z(t)$. So, considering again $X_0 = 0$ and $Z_0 = 1$ the solution $Y(t)$ becomes $Y(t) = e^{-t}Y_0 + h(t)$, where h does not depend on Y_0. Consequently, considering $A = 1$ again we cannot eliminate the non periodic term $Y_0 e^{-t}$ of $Y(t)$ unless $Y_0 = 0$ and consequently averaging cannot be applied.

Cases 13 and 15 are similar. System (24) for system (5) in cases 13 and 15 are, respectively,

$$\dot{\mathbf{x}} = (-AX + BY + AU(t), -Y, 1 - Z)^T, \tag{21}$$

and

$$\dot{\mathbf{x}} = (-AX + AU(t), -Y, 1 - Z)^T. \tag{22}$$

In booth cases solutions $Y(t)$ and $Z(t)$ are $Y(t) = e^{-t}Y_0$ and $Z(t) = 1 - e^{-t} + e^{-t}Z_0$. So, in order to be $Y(t)$ and $Z(t)$ periodic we take $Y_0 = 0$ and $Z_0 = 1$ and these values turn the solution $X(t)$ into the form

$$X(t) = e^{-At}X_0 + A\int_0^t e^s U(s)ds.$$

Again, once $g(t) = \displaystyle\int_0^t e^s U(s)ds$ does not depend on X_0 it is not possible to eliminate the non–periodic term $e^{-At}X_0$ from $X(t)$ unless we take $X_0 = 0$ and then booth systems (21) and (22) do not have a submanifold $\mathcal{M}$ filled with periodic solutions. Hence averaging theory cannot be applied in cases 13 and 15. □

3. The Averaging Theory for Periodic Orbits

Now we present the basic results on the averaging theory of first order that we need to proving our results.

Consider the problem of bifurcation of T–periodic solutions from differential systems of the form

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \tag{23}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. Here the functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$, T–periodic in the first variable and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}), \tag{24}$$

has a submanifold of periodic solutions. A solution of this problem is given using averaging theory.

Indeed, let $\mathbf{x}(t, \mathbf{z}, \varepsilon)$ be the solution of system (24) such that $\mathbf{x}(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. We write the linearization of the unperturbed system along a periodic solution $\mathbf{x}(t, \mathbf{z}, 0)$ as

$$\dot{\mathbf{y}} = D_{\mathbf{x}} F_0(t, \mathbf{x}(t, \mathbf{z}, 0))\mathbf{y}. \tag{25}$$

Then we have the following theorem.

Theorem 7. *Assume there exists a k–dimensional submanifold $\mathcal{M}$ filled with T–periodic solutions of* (24)*. Let V be an open and bounded subset of $\mathbb{R}^k$ and let $\beta : Cl(V) \to \mathbb{R}^{n-k}$ be a $\mathcal{C}^2$ function. We assume that*

(i) $\mathcal{M} = \{\mathbf{z}_\alpha = (\alpha, \beta(\alpha)); \alpha \in Cl(V)\}$ *and that for each $\mathbf{z}_\alpha \in \mathcal{M}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha)$ of* (24) *is T–periodic;*

(ii) *for each $\mathbf{z}_\alpha \in \mathcal{M}$ there is a fundamental matrix $M_{\mathbf{z}_\alpha}$ of* (25) *such that the matrix $M^{-1}_{\mathbf{z}_\alpha}(0) - M^{-1}_{\mathbf{z}_\alpha}(T)$ has in the upper right corner the $k \times (n-k)$ zero matrix, and in the lower right corner a $(n-k) \times (n-k)$ matrix Δ_α with $\det \Delta_\alpha \neq 0$.*

Define $\mathcal{F} : Cl(V) \to \mathbb{R}^k$ as

$$\mathcal{F}(\alpha) = \int_0^T M^{-1}_{\mathbf{z}_\alpha}(t, \mathbf{z}_\alpha) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha))dt.$$

Then the following statements hold

(a) *If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and $\det((\partial \mathcal{F}/\partial \alpha)(a)) \neq 0$, then there exist a T–periodic solution $\mathbf{x}(t, \varepsilon)$ of system* (23) *such that $\mathbf{x}(t, \varepsilon) \to \mathbf{z}_\alpha$ when $\varepsilon \to 0$.*

(b) *The type of stability of the periodic solution $\mathbf{x}(t, \varepsilon)$ is given by the eigenvalues of the Jacobian matrix $((\partial \mathcal{F}/\partial \alpha)(a))$.*

In fact, the result of Theorem (7) is a classical result due to Malkin [6] and Roseau [7]. For a shorter proof of Theorem 7, item (a), see [1].

For additional information on averaging theory see the book [8].

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