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A NEW APPROACH TO THE VAKONOMIC MECHANICS

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Abstract. The aim of this paper is to show that the Lagrange–d'Alembert and its equivalent the Gauss and Appel principle are not the only way to deduce the equations of motion of the nonholonomic systems. Instead of them, here we consider the generalization of the Hamiltonian principle for nonholonomic systems with nonzero transpositional relations.

By applying this variational principle which takes into the account transpositional relations different from the classical ones we deduce the equations of motion for the nonholonomic systems with constraints that in general are nonlinear in the velocity. These equations of motion coincide, except perhaps in a zero Lebesgue measure set, with the classical differential equations deduced with d'Alembert–Lagrange principle.

We provide a new point of view on the transpositional relations for the constrained mechanical systems: the virtual variations can produce zero or non–zero transpositional relations. In particular the independent virtual variations can produce non–zero transpositional relations. For the unconstrained mechanical systems the virtual variations always produce zero transpositional relations.

We conjecture that the existence of the nonlinear constraints in the velocity must be sought outside of the Newtonian model.

All our results are illustrated with precise examples.

1. Introduction

The history of nonholonomic mechanical systems is long and complex and goes back to the 19 century, with important contribution by Hertz [16] (1894) , Ferrers [10] (1871), Vierkandt [51] (1892) and Chaplygin [6] (1897).

The nonholonomic mechanic is a remarkable generalization of the classical Lagrangian and Hamiltonian mechanic. The birth of the theory of dynamics of nonholonomic systems occurred when Lagrangian-Euler formalism was found to be inapplicable for studying the simple mechanical problem of a rigid body rolling without slipping on a plane.

A long period of time has been needed for finding the correct equations of motion of the nonholonomic mechanical systems and the study of the deeper questions associated with the geometry and the analysis of these equations. In particular the integration theory of equations of motion for nonholonomic mechanical systems is not so complete as in the case of holonomic systems. This is due to several reasons. First, the equations of motion of nonholonomic systems have more complex structure than the Lagrange one, which describes the behavior of holonomic systems. Indeed, a holonomic systems can be described by a unique function of its state and time, the Lagrangian function. For the nonholonomic systems this is not possible. Second, the equations of motion of nonholonomic systems in general have no invariant measure, as they have the equations of motion of holonomic systems (see [21, 28, 30, 50]).

One of the most important directions in the development of the nonholonomic mechanics is the research connected with the general mathematical formalism to describe the behavior

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of such systems which differs from the Lagrangian and Hamiltonian formalism. The main problem with the equations of motion of the nonholonomic mechanics has been centered on whether or not these equations can be derived from the Hamiltonian principle in the usual sense, such as for the holonomic systems (see for instance [33]). But there is not doubt that the correct equations of motion for nonholonomic systems are given by the d'Alembert–Lagrange principle.

The general understanding of inapplicability of Lagrange equations and variational Hamiltonian principles to the nonholonomic systems is due to Hertz, who expressed it in his fundamental work *Die Prinzipien der Mechanik in neuem Zusammenhaange dargestellt* [16]. Hertz's ideas were developed by Poincaré in [39]. At the same time various aspects of nonholonomic systems need to be studied such as

(a) The problem of the realization of nonholonomic constraints (see for instance [22, 23]).

(b) The stability of nonholonomic systems (see for instance [35, 43]).

(c) The role of the so called *transpositional relations* (see [19, 34, 35, 42])

$$\delta\frac{d\mathbf{x}}{dt}-\frac{d}{dt}\delta\mathbf{x}=\left(\delta\frac{dx_1}{dt}-\frac{d}{dt}\delta x_1,\ldots,\delta\frac{dx_N}{dt}-\frac{d}{dt}\delta x_N\right),\tag{1}$$

where $\dfrac{d}{dt}$ denotes the differentiation with respect to the time, δ is the virtual variation, and $\mathbf{x}=(x_1,\ldots,x_N)$ is the vector of the generalized coordinates.

Indeed the most general formulation of the Hamiltonian principle is the *Hamilton–Suslov principle*

$$\int_{t_0}^{t_1}\left(\delta\,\tilde{L}-\sum_{j=1}^{N}\frac{\partial\tilde{L}}{\partial\dot{x}_j}\left(\delta\frac{dx_j}{dt}-\frac{d}{dt}\delta x_j\right)\right)dt=0,\tag{2}$$

suitable for constrained and unscontrained Lagrangian systems, where $\tilde{L}$ is the Lagrangian of the mechanical system.. Clearly the equations of motion obtained from the Hamilton–Suslov principle depend on the point of view on the transpositional relations. This fact shows the importance of these relations.

(d) The relation between nonholonomic mechanical systems and *vakonomic mechanical systems.*

There was some confusion in the literature between nonholonomic mechanical systems and variational nonholonomic mechanical systems also called vakonomic mechanical systems. Both kinds of systems have the same mathematical "ingredients": a Lagrangian function and a set of constraints. But the way in which the equations of motion are derived differs. As we observe the equations of motion in nonholonomic mechanic are deduced using d'Alembert–Lagrange's principle. In the case of vakonomic mechanics the equations of motion are obtained through the application of a constrained variational principle. The term vakonomic ("variational axiomatic kind") is due to Kozlov (see [24, 25, 26]), who proposed this mechanics as an alternative set of equations of motion for a constrained Lagrangian systems.

The distinction between the classical differential equations of motion and the equations of motion of variational nonholonomic mechanical systems has a long history going back to the survey article of Korteweg (1899) [20] and discussed in a more modern context in [9, 18, 29, 49]. In these papers the authors have discussed the domain of the vakonomic and nonholonomic mechanics. In the paper *Critics of some mathematical model to describe the behavior of mechanical systems with differential constraints* [18], Kharlamov studied the Kozlov model and in a concrete example showed that the subset of solutions of the studied nonholonomic systems is not included in the set of vakonomic model and proved that the principle of determinacy is not valid in the Kozlov model. In [27] the authors put in evidence the main differences between the d'Alembertian and the vakonomic approaches. From the results obtained in several papers it follows that in general the vakonomic model is not applicable to the nonholonomic constrained Lagrangian systems.

The equations of motion for the constrained mechanical systems deduced by Kozlov (see for instance [2]) from the Hamiltonian principle with the Lagrangian $L : \mathbb{R}\times T\mathrm{Q}\times\mathbb{R}^M \longrightarrow \mathbb{R}$ such that $L = L_0 - \sum_{j=1}^{M}\lambda_j L_j$, where $L_j = 0$ for $j = 1, \dots, M < N$ are the given constraints, and L_0 is the classical Lagrangian. These equations are

$$E_k L = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} - \frac{\partial L}{\partial x_k} = 0 \Longleftrightarrow E_k L_0 = \sum_{j=1}^{M}\left(\lambda_j E_k\, L_j + \frac{d\lambda_j}{dt}\frac{\partial L_j}{\partial \dot{x}_k}\right), \tag{3}$$

for $k = 1, \dots, N$, see for more details [2]. Clearly, equations (3) differ from the classical equations by the presence of the terms $\lambda_j E_k\, L_j$. If the constraints are integrable, i.e. $L_j = \frac{d}{dt} g_j(t, \mathbf{x})$, then the vakonomic mechanics reduces to the holonomic one.

In this paper we give a modification of the vakonomic mechanics. This modification is valid for the holonomic and nonholonomic constrained Lagrangian systems. We apply the generalized constrained Hamiltonian principle with non–zero transpositional relations. By applying this constrained variational principle we deduce the equations of motion for the nonholonomic systems with constraints which in general are nonlinear in the velocity. These equations coincide, except perhaps in a zero Lebesgue measure set, with the classical differential equations deduced from d'Alembert–Lagrange principle.

2. Statement of the main results

In this paper we solve the following *inverse problem of the constrained Lagrangian systems* (see [31])

We consider the constrained Lagrangian systems with configuration space Q and phase space TQ.

Let $L : \mathbb{R}\times T\mathrm{Q}\times\mathbb{R}^M \longrightarrow \mathbb{R}$ be a smooth function such that

$$L(t, \mathbf{x}, \dot{\mathbf{x}}, \Lambda) = L_0(t, \mathbf{x}, \dot{\mathbf{x}}) - \sum_{j=1}^{M}\lambda_j\, L_j(t, \mathbf{x}, \dot{\mathbf{x}}) - \sum_{j=M+1}^{N}\lambda_j^0 L_j(t, \mathbf{x}, \dot{\mathbf{x}}), \tag{4}$$

where $\Lambda = (\lambda_1, \dots, \lambda_M)$ are the additional coordinates (Lagrange multipliers), $L_j : \mathbb{R}\times T\mathrm{Q} \longrightarrow \mathbb{R}$, $\quad (t, \mathbf{x}, \dot{\mathbf{x}}) \longmapsto L_j(t, \mathbf{x}, \dot{\mathbf{x}})$, be smooth functions for $j = 0, \dots, N$, where L_0 is the nonsingular function i.e. $\det\left(\frac{\partial^2 L_0}{\partial \dot{x}_k \partial \dot{x}_j}\right) \neq 0$, and $L_j = 0$, for $j = 1, \dots, M$, are the constraints satisfying

$$\operatorname{rank}\left(\frac{\partial(L_1, \dots, L_M)}{\partial(\dot{x}_1, \dots, \dot{x}_N)}\right) = M \tag{5}$$

in all the points of $\mathbb{R}\times T\mathrm{Q}$, except perhaps in a zero Lebesgue measure set, L_j and λ_j^0 are arbitrary functions and constants respectively, for $j = M+1, \dots, N$.

We must determine the smooth functions L_j, constants λ_j^0 for $j = M+1, \dots, N$ and the matrix A in such a way that the differential equations describing the behavior of the constrained Lagrangian systems and obtained from the the Hamiltonian principle

$$\int_{t_0}^{t_1}\delta\, L = \int_{t_0}^{t_1}\left(\frac{\partial L}{\partial x_j}\delta x_j + \frac{\partial L}{\partial \dot{x}_j}\frac{d}{dt}\delta x_j + \sum_{j=1}^{N}\frac{\partial L}{\partial \dot{x}_j}\left(\delta\frac{dx_j}{dt} - \frac{d}{dt}\delta x_j\right)\right)dt = 0, \tag{6}$$

with transpositional relation given by

$$\delta\frac{d\mathbf{x}}{dt} - \frac{d}{dt}\delta\mathbf{x} = A(t, \mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}})\,\delta\mathbf{x}, \tag{7}$$

where $A = A(t, \mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = (A_{\nu j}(t, \mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}))$ is a $N \times N$ matrix,

We give the solutions of this problem in two steps. First we obtain the differential equations along the solutions satisfying (6). Second we shall contrast the obtained equations and classical differential equations which described the behavior of the constrained mechanical systems. The solution of this inverse problem is presented in section 4.

Note that the function L is singular, due to the absence of $\dot{\lambda}$.

We observe that the arbitrariness of the functions L_j, of the constants λ_j^0 for $j = M+1, \ldots, N$, and of the matrix A will play a fundamental role in the construction of the mathematical model which we propose in this paper.

Our main results are the following

Theorem 1. *We assume that $\delta x_\nu(t)$, $\nu = 1, \ldots, N$, are arbitrary functions defined in the interval $[t_0, t_1]$, smooth in the interior of $[t_0, t_1]$ and vanishing at its endpoints, i.e., $\delta x_\nu(t_0) = \delta x_\nu(t_1) = 0$. If (7) holds then the path $\gamma(t) = (x_1(t), \ldots, x_N(t))$ compatible with the constraints $L_j(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) = 0$, for $j = 1, \ldots, M$ satisfies (6) with L given by the formula (4) if and only if it is a solution of the differential equations*

$$D_\nu L := E_\nu L - \sum_{j=1}^{N} A_{\nu j} \frac{\partial L}{\partial \dot{x}_j} = 0, \quad \frac{\partial L}{\partial \lambda_k} = -L_k = 0, \tag{8}$$

for $\nu = 1, \ldots, N$, and $k = 1, \ldots, M$, where $E_\nu = \dfrac{d}{dt}\dfrac{\partial}{\partial \dot{x}_\nu} - \dfrac{\partial}{\partial x_\nu}$. System (8) is equivalent to the following two differential systems

$$\begin{aligned} D_\nu L_0 = &\sum_{j=1}^{M} \left(\lambda_j D_\nu L_j + \frac{d\lambda_j}{dt}\frac{\partial L_j}{\partial \dot{x}_\nu} \right) + \sum_{j=M+1}^{N} \lambda_j^0 D_\nu L_j, \quad L_k = 0 \Longleftrightarrow \\ E_\nu L_0 = &\sum_{k=1}^{N} A_{jk} \frac{\partial L_0}{\partial \dot{x}_k} + \sum_{j=1}^{M} \left(\lambda_j D_\nu L_j + \frac{d\lambda_j}{dt}\frac{\partial L_j}{\partial \dot{x}_\nu} \right) + \sum_{j=M+1}^{N} \lambda_j^0 D_\nu L_j, \quad L_k = 0. \end{aligned} \tag{9}$$

for $\nu = 1, \ldots, N$ and $k = 1, \ldots, M$.

Theorem 2. *Using the notation of Theorem 1 let*

$$L = L(t, \boldsymbol{x}, \dot{\boldsymbol{x}}, \Lambda) = L_0(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) - \sum_{j=1}^{M} \lambda_j L_j(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) - \sum_{j=M+1}^{N} \lambda_j^0 L_j(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) \tag{10}$$

be the Lagrangian and let $L_j(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) = 0$ be the independent constraints for $j = 1, \ldots, M < N$, and let λ_k^0 be the arbitrary constants for $k = M+1, \ldots, N$, $L_k : \mathbb{R} \times TQ \longrightarrow \mathbb{R}$ for $k = M+1, \ldots, N$ arbitrary functions such that

$$|W_1| = \det W_1 = \det \left(\frac{\partial(L_1, \ldots, L_N)}{\partial(\dot{x}_1, \ldots, \dot{x}_N)} \right) \neq 0,$$

except perhaps in a zero Lebesgue measure set $|W_1| = 0$. We determine the matrix A satisfying

$$W_1 A = \Omega_1 := \begin{pmatrix} E_1 L_1 & \ldots & E_N L_1 \\ \vdots & \ldots & \vdots \\ \vdots & \ldots & \vdots \\ E_1 L_N & \ldots & E_N L_N \end{pmatrix}. \tag{11}$$

Then the differential equations (9) *become*

$$
(12)\qquad \begin{aligned}
D_\nu L_0 = & \sum_{\alpha=1}^{M} \dot{\lambda}_\alpha \frac{\partial L_\alpha}{\partial \dot{x}_\nu} \quad \textit{for} \quad \nu = 1, \ldots, N \\
\Longleftrightarrow & \quad \frac{d}{dt}\frac{\partial L_0}{\partial \dot{\boldsymbol{x}}} - \frac{\partial L_0}{\partial \boldsymbol{x}} = \left(W_1^{-1}\Omega_1\right)^T \frac{\partial L_0}{\partial \dot{\boldsymbol{x}}} + W_1^T \frac{d\lambda}{dt},
\end{aligned}
$$

where $\dfrac{\partial}{\partial \dot{\boldsymbol{x}}} = \left(\dfrac{\partial}{\partial \dot{x_1}}, \ldots, \dfrac{\partial}{\partial \dot{x_N}}\right)^T$, $\dfrac{\partial}{\partial \boldsymbol{x}} = \left(\dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_N}\right)^T$, $\lambda = (\lambda_1, \ldots, \lambda_M, 0, \ldots, 0)^T$, *and the transpositional relation* (7) *becomes*

$$
(13)\qquad \delta \frac{d\boldsymbol{x}}{dt} - \frac{d}{dt}\delta \boldsymbol{x} = \left(W_1^{-1}\Omega_1\right)\delta \boldsymbol{x}.
$$

Theorem 3. *Using the notation of Theorem 1 let*

$$
(14)\qquad L(t, \boldsymbol{x}, \dot{\boldsymbol{x}}, \Lambda) = L_0(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) - \sum_{j=1}^{M} \lambda_j L_j(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) - \sum_{j=M+1}^{N-1} \lambda_j^0 L_j(t, \boldsymbol{x}, \dot{\boldsymbol{x}})
$$

be the Lagrangian and $L_j(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) = 0$ *be the independent constraints for* $j = 1, \ldots, M < N$, *and let* λ_j^0 *be arbitrary constants, for* $j = M+1, \ldots, N-1$ *and* $\lambda_N^0 = 0$, $L_j : \mathbb{R} \times TQ \longrightarrow \mathbb{R}$ *for* $j = M+1, \ldots, N-1$ *arbitrary functions, and* $L_N = L_0$ *such that*

$$
|W_2| = \det W_2 = \det\left(\frac{\partial(L_1, \ldots, L_{N-1}, L_0)}{\partial(\dot{x}_1, \ldots, \dot{x}_N)}\right) \neq 0,
$$

except perhaps in a zero Lebesgue measure set $|W_2| = 0$. *We determine the matrix* A *satisfying*

$$
(15)\qquad W_2 A = \Omega_2 := \begin{pmatrix}
E_1 L_1 & \ldots & E_N L_1 \\
\vdots & \ldots & \vdots \\
E_1 L_{N-1} & \ldots & E_N L_{N-1} \\
0 & \ldots & 0
\end{pmatrix}.
$$

Then the differential equations (9) *become*

$$
(16)\qquad \frac{d}{dt}\frac{\partial L_0}{\partial \dot{\boldsymbol{x}}} - \frac{\partial L_0}{\partial \boldsymbol{x}} = W_2^T \frac{d}{dt}\tilde{\lambda},
$$

where $\lambda := \tilde{\lambda} = \left(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_M, 0, \ldots, 0\right)^T$, *and the transpositional relation* (7) *becomes*

$$
(17)\qquad \delta \frac{d\boldsymbol{x}}{dt} - \frac{d}{dt}\delta \boldsymbol{x} = \left(W_2^{-1}\Omega_2\right)\delta \boldsymbol{x},
$$

The proofs of Theorems 1, 2 and 3 are given in section 5.

Theorem 4. *Under the assumptions of Theorem 2 and assuming that*

$$
\begin{aligned}
x_\alpha = & \quad x_\alpha, \quad x_\beta = y_\beta \quad \boldsymbol{x} = (x_1, \ldots, x_{s_1}) \quad \boldsymbol{y} = (y_1, \ldots, y_{s_2}), \\
L_\alpha = & \quad \dot{x}_\alpha - \Phi_\alpha(\boldsymbol{x}, \boldsymbol{y}, \dot{\boldsymbol{x}}, \dot{\boldsymbol{y}}) = 0, \quad L_\beta = \dot{y}_\beta,
\end{aligned}
$$

for $\alpha = 1, \ldots, s_1 = M$ *and* $\beta = s_1 + 1, \ldots, s_1 + s_2 = N$.

Then $|W_1| = 1$ *and the differential equations* (12) *take the form*

$$
(18)\qquad \begin{aligned}
E_j L_0 = & \sum_{\alpha=1}^{s_1} \left(E_j L_\alpha \frac{\partial L_0}{\partial \dot{x}_\alpha}\right) + \dot{\lambda}_j \quad j = 1, \ldots, s_1, \\
E_k L_0 = & \sum_{\alpha=1}^{s_1} \left(E_k L_\alpha \frac{\partial L_0}{\partial \dot{x}_\alpha} + \dot{\lambda}_\alpha \frac{\partial L_\alpha}{\partial \dot{y}_k}\right) \quad k = 1, \ldots, s_2.
\end{aligned}
$$

or, equivalently (excluding the Lagrange multipliers)

$$E_kL_0=\sum_{\alpha=1}^{s_1}\left(E_kL_\alpha\frac{\partial L_0}{\partial\dot x_\alpha}+\left(E_\alpha L_0-\sum_{\beta=1}^{s_1}\left(E_\alpha L_\beta\frac{\partial L_0}{\partial\dot x_\beta}\right)\right)\frac{\partial L_\alpha}{\partial\dot y_k}\right),\quad k=1,\ldots,s_2. \tag{19}$$

In particular if we choose $L_0=\tilde L(\boldsymbol{x},\boldsymbol{y},\dot{\boldsymbol{x}},\dot{\boldsymbol{y}})-\tilde L(\boldsymbol{x},\boldsymbol{y},\Phi,\dot{\boldsymbol{y}})=\tilde L-L^*$, *where* $\Phi=(\Phi_1,\ldots,\Phi_{s_1})$, *then* (19) *holds if*

$$E_k\tilde L=\sum_{\alpha=1}^{s_1}E_\alpha\tilde L\frac{\partial L_\alpha}{\partial\dot y_k},\quad k=1,\ldots,s_2,$$

and

$$E_k(L^*)=\sum_{\alpha=1}^{s_1}\left(\frac{d}{dt}\left(\frac{\partial\Phi_\alpha}{\partial\dot y_k}\right)-\left(\frac{\partial\Phi_\alpha}{\partial y_k}+\sum_{\nu=1}^{s_1}\frac{\partial\Phi_\alpha}{\partial x_\nu}\frac{\partial\Phi_\nu}{\partial\dot y_k}\right)\right)\Psi_\alpha+\sum_{\nu=1}^{s_1}\frac{\partial L^*}{\partial x_\nu}\frac{\partial\Phi\nu}{\partial\dot y_k}, \tag{20}$$

where $\Psi_\alpha=\left.\dfrac{\partial\tilde L}{\partial\dot x_\alpha}\right|_{\dot x_1=\Phi_1,\ldots,\dot x_{s_1}=\Phi_{s_1}}$. *The transpositional relations* (13) *in this case are*

$$\begin{aligned}\delta\frac{dx_\alpha}{dt}-\frac{d}{dt}\delta x_\alpha&=\sum_{k=1}^{s_2}\left(\sum_{j=1}^{s_1}E_j(L_\alpha)\frac{\partial L_j}{\partial\dot y_k}+E_k(L_\alpha)\right)\delta y_k,\quad\alpha=1,\ldots,s_1,\\ \delta\frac{dy_m}{dt}-\frac{d}{dt}\delta y_m&=0,\quad m=1,\ldots,s_2.\end{aligned} \tag{21}$$

Proposition 5. *Differential equations* (20) *describe the motion of the nonholonomic systems with the constraints* $L_\alpha=\dot x_\alpha-\Phi_\alpha(\boldsymbol{x},\boldsymbol{y},\dot{\boldsymbol{y}})=0$ *for* $\alpha=1,\ldots,s_1$. *In particular if the constraints are given by the formula*

$$\dot x_j=\sum_{k=1}^{s_2}a_{jk}(t,\boldsymbol{x},\boldsymbol{y})\dot y_k+a_j(t,\boldsymbol{x}),\quad j=1,\ldots,s_1, \tag{22}$$

then systems (20) *becomes*

$$E_k(L^*)=\sum_{\alpha=1}^{s_1}\left(\frac{da_{\alpha k}}{dt}-\left(\frac{\partial a_{\alpha m}}{\partial y_k}+\sum_{\nu=1}^{s_1}\frac{\partial a_{\alpha m}}{\partial x_\nu}a_{\nu k}\right)\dot y_m\right)\Psi_\alpha+\sum_{\nu=1}^{s_1}\frac{\partial L^*}{\partial x_\nu}a_{\nu k},$$

which are the classical Voronets differential equations. Consequently equations (20) *are an extension of the Voronets differential equations for the case when the constraints are nonlinear in the velocities.*

Proposition 6. *Differential equations* (20) *describe the motion of the constrained Lagrangian systems with the constraints* $L_\alpha=\dot x_\alpha-\Phi_\alpha(\boldsymbol{y},\dot{\boldsymbol{y}})=0$ *and Lagrangian* $L^*=L^*(\boldsymbol{y},\dot{\boldsymbol{y}})$. *Under these assumptions equations* (20) *take the form*

$$E_k(L^*)=\sum_{\alpha=1}^{s_1}\left(\frac{d}{dt}\left(\frac{\partial\Phi_\alpha}{\partial\dot y_k}\right)-\frac{\partial\Phi_\alpha}{\partial y_k}\right)\Psi_\alpha. \tag{23}$$

In particular if the constraints are given by the formula

$$\dot x_\alpha=\sum_{k=1}^{s_2}a_{\alpha k}(\boldsymbol{y})\dot y_k,\quad\alpha=1,\ldots,s_1, \tag{24}$$

then systems (23) *becomes*

$$E_kL^*=\sum_{j=1}^{s_1}\sum_{r=1}^{s_2}\left(\frac{\partial\alpha_{jk}}{\partial y_r}-\frac{\partial\alpha_{jr}}{\partial y_k}\right)\dot y_r\Psi_j, \tag{25}$$

for $k = 1, \ldots, s_2$, which are the equations which Chaplygin published in the Proceeding of the Society of the Friends of Natural Science in 1897 .

Consequently equations (23) *are an extension of the classical Chaplygin equations for the case when the constraints are nonlinear.*

From (5) and in view of the Implicit Function Theorem, we can locally express the constraints (reordering coordinates if is necessary) as

$$\dot{x}_\alpha = \Phi_\alpha \left(\mathbf{x}, \dot{x}_{M+1}, \ldots, \dot{x}_N\right) \tag{26}$$

for $\alpha = 1, \ldots, M$. We note that Propositions 5 and 6 are also valid for every constrained mechanical systems with constraints locally given by (26), this follows from Theorem 4 changing the notations, see Corollary 22.

The proofs of Theorem 4 and Propositions 5 and 6 is given in section 8.

The next result is the *third point of view on the transpositional relations.*

Corollary 7. *For the constrained mechanical systems the virtual variations can produce zero or non–zero transpositional relations. For the unconstrained mechanical systems the virtual variations always produce zero transpositional relations.*

The proof of this corollary is given in section 9.

We have the following conjecture.

Conjecture 8. *The existence of mechanical systems with nonlinear constraints in the velocity must be sought outside of the Newtonian model.*

This conjecture is supported by several facts see section 9.

The results are illustrated with precise examples.

3. Variational Principles. Transpositional relations

3.1. **Hamiltonian principle.** We introduce the following results, notations and definitions which we will use later on (see [2]).

A *Lagrangian system* is a pair $(\mathrm{Q}, \tilde{L})$ consisting of a smooth manifold Q, and a smooth function $\tilde{L} : \mathbb{R} \times T\mathrm{Q} \longrightarrow \mathbb{R}$, where $T\mathrm{Q}$ is the tangent bundle of Q. The point $\mathbf{x} = (x_1, \ldots, x_N) \in \mathrm{Q}$ denotes the *position* (usually its components are called *generalized coordinates*) of the system and we call each tangent vector $\dot{\mathbf{x}} = (\dot{x}_1, \ldots, \dot{x}_N) \in T_{\mathbf{x}}\mathrm{Q}$ the *velocity* (usually called *generalized velocity*) of the system at the point $\mathbf{x}$. A pair $(\mathbf{x}, \dot{\mathbf{x}})$ is called a *state* of the system. In Lagrangian mechanics it is usual to call Q, the *configuration space*, the tangent bundle $T\mathrm{Q}$ is called the *phase space*, $\tilde{L}$ is the *Lagrange function* or *Lagrangian* and the dimension N of Q is the number of *degrees of freedom*.

Let a_0 and a_1 be two points of Q. The map

$$\begin{aligned} \gamma : [t_0, t_1] \subset \mathbb{R} &\longrightarrow \mathrm{Q}, \\ t &\longmapsto \gamma(t) = (x_1(t), \ldots, x_N(t)), \end{aligned}$$

such that $\gamma(t_0) = a_0$, $\gamma(t_1) = a_1$ is called a *path* from a_0 to a_1. We denote the set of all these path by $\Omega(\mathrm{Q}, a_0, a_1, t_0, t_1) := \Omega$.

We shall derive one of the most simplest and general variational principles the *Hamiltonian principle* (see [40]).

The functional $F : \Omega \longrightarrow \mathbb{R}$ defined by

$$F(\gamma(t)) = \int_{\gamma(t)} \tilde{L} dt = \int_{t_0}^{t_1} \tilde{L}(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)) dt$$

is called the *action.*

We consider the path $\gamma(t) = \mathbf{x}(t) = (x_1(t), \ldots, x_N(t)) \in \Omega$.

Let the *variation* of the path $\gamma(t)$ be defined as a smooth mapping

$$\begin{aligned}\gamma^* : [t_0, t_1] \times (-\tau, \tau) &\longrightarrow \mathrm{Q},\\ (t, \varepsilon) &\longmapsto \gamma^*(t, \varepsilon) = \mathbf{x}^*(t, \varepsilon) = (x_1(t) + \varepsilon\delta x_1(t), \ldots, x_N(t) + \varepsilon\delta x_N(t)),\end{aligned}$$

satisfying

$$\mathbf{x}^*(t_0, \varepsilon) = a_0, \quad \mathbf{x}^*(t_1, \varepsilon) = a_1, \quad \mathbf{x}^*(t, 0) = \mathbf{x}(t).$$

By definition we have

$$\delta\mathbf{x}(t) = \left.\frac{\partial \mathbf{x}^*(t, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}.$$

This function is called the *virtual displacement* or *virtual variation* corresponding to the variation of $\gamma(t)$ and it is a function of time, all its components are functions of t of class $C^2(t_0, t_1)$ and vanish at t_0 and t_1 i.e. $\delta\mathbf{x}(t_0) = \delta\mathbf{x}(t_1) = 0$.

A *varied path* is a path which can be obtained as a variation path.

The *first variation* of the functional F at $\gamma(t)$ is

$$\delta F := \left.\frac{\partial F(\mathbf{x}^*(t, \varepsilon))}{\partial \varepsilon}\right|_{\varepsilon=0},$$

and it is called the *differential* of the functional F (see [2]). The path $\gamma(t) \in \Omega$ is called the *critical point* of F if $\delta F(\gamma(t)) = 0$.

Let $\mathbb{L}$ be the space of all smooth functions $g : \mathbb{R} \times T\mathrm{Q} \longrightarrow \mathbb{R}$. The operator

$$\begin{aligned}E_\nu : \mathbb{L} &\longrightarrow \mathbb{R},\\ g &\longmapsto E_\nu g = \frac{d}{dt}\frac{\partial g}{\partial \dot{x}_\nu} - \frac{\partial g}{\partial x_\nu}, \quad \text{for} \quad \nu = 1, \ldots, N,\end{aligned}$$

is known as the *Lagrangian derivative.*

It is easy to show the following property of the Lagrangian derivative

$$E_\nu \frac{df}{dt} = 0, \tag{27}$$

for arbitrary smooth function $f = f(t, \mathbf{x})$. We observe that in view of (27) we obtain that the Lagrangian derivative is unchanged if we replace the function g by $g + \dfrac{df}{dt}$, for any function $f = f(t, \mathbf{x})$. This reflects the *gauge invariance.* We shall say that the functions $g = g(t, \mathbf{x}, \dot{\mathbf{x}})$ and $\hat{g} = \hat{g}(t, \mathbf{x}, \dot{\mathbf{x}})$ are *equivalently* if $g - \hat{g} = \dfrac{df(t, \mathbf{x})}{dt}$, and we shall write $g \simeq \hat{g}$.

Proposition 9. *The differential of the action can be calculated as follows*

$$\delta F = -\int_{t_0}^{t_1} \sum_{k=1}^{N} \left(E_k \tilde{L} \delta x_k - \frac{\partial \tilde{L}}{\partial \dot{x}_k} \left(\delta \frac{dx_k}{dt} - \frac{d}{dt}\delta x_k \right) \right) dt, \tag{28}$$

where $\mathbf{x} = \mathbf{x}(t)$, $\dot{\mathbf{x}} = \dfrac{d\mathbf{x}}{dt}$, *and* $\tilde{L} = \tilde{L}\left(t, \mathbf{x}, \dfrac{d\mathbf{x}}{dt}\right)$.

Proof. We have that

$$\begin{aligned}
\delta F =& \left.\frac{\partial F\left(\mathbf{x}^*(t,\varepsilon)\right)}{\partial \varepsilon}\right|_{\varepsilon=0}\\
=& \int_{t_0}^{t_1} \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} L\left(t, \mathbf{x}^*(t,\varepsilon), \frac{d}{dt}\left(\mathbf{x}^*(t,\varepsilon)\right)\right) dt = \int_{t_0}^{t_1} \sum_{k=1}^{N} \left(\frac{\partial L}{\partial x_k}\delta x_k + \frac{\partial L}{\partial \dot{x_k}}\delta \dot{x_k}\right) dt\\
=& \int_{t_0}^{t_1} \sum_{k=1}^{N} \left(\frac{\partial L}{\partial x_k}\delta x_k + \frac{\partial L}{\partial \dot{x_k}}\frac{d}{dt}\delta x_k + \frac{\partial L}{\partial \dot{x_k}}\left(\delta\frac{dx_k}{dt} - \frac{d}{dt}\delta x_k\right)\right) dt\\
=& \left.\sum_{k=1}^{N} \frac{\partial L}{\partial \dot{x_k}}\delta x_k\right|_{t=t_0}^{t=t_1} + \int_{t_0}^{t_1} \sum_{k=1}^{N} \left(\left(\frac{\partial L}{\partial x_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x_k}}\right)\delta x_k + \frac{\partial L}{\partial \dot{x_k}}\left(\delta\frac{dx_k}{dt} - \frac{d}{dt}\delta x_k\right)\right) dt.
\end{aligned}$$

Hence, by considering that the virtual variation vanishes at the points $t = t_0$ and $t = t_1$ we obtain the proof of the proposition. ☐

Corollary 10. *The differential of the action for a Lagrangian system* $\left(\mathrm{Q},\, \tilde{L}\right)$ *can be calculated as follows*

$$\delta F = -\int_{t_0}^{t_1} \sum_{k=1}^{N} E_k \tilde{L}\left(t, \mathbf{x}, \frac{d\mathbf{x}}{dt}\right) \delta x_k \, dt.$$

Proof. Indeed, for the Lagrangian system the transpositional relation is equal to zero (see for instance [32] page 29), i.e.

$$\delta\frac{d\mathbf{x}}{dt} - \frac{d}{dt}\delta\mathbf{x} = 0. \tag{29}$$

Thus, from Proposition 9, it follows the proof of the corollary. ☐

The path $\gamma(t) \in \Omega$ is called a *motion* of the Lagrangian systems $\left(\mathrm{Q},\, \tilde{L}\right)$ if $\gamma(t)$ is a critical point of the action F, i.e.

$$\delta F\left(\gamma(t)\right) = 0 \Longleftrightarrow \int_{t_0}^{t_1} \delta \tilde{L}\, dt = 0.$$

This definition is known as the *Hamiltonian variational principle* or *Hamiltonian variational principle of least action* or simple *Hamiltonian principle.*

Now we need the *Lagrange lemma* or *fundamental lemma of calculus of variations* (see for instance [1])

Lemma 11. *Let f be a continuous function of the interval $[t_0,\, t_1]$ satisfying the equation*

$$\int_{t_0}^{t_1} f(t)\zeta(t)dt = 0,$$

for arbitrary continuous function $\zeta(t)$ such that $\zeta(t_0) = \zeta(t_1) = 0$. Then $f(t) \equiv 0$.

Corollary 12. *The Hamiltonian principle for Lagrangian systems is equivalent to the Lagrangian equations*

$$E_\nu \tilde{L} = \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{x_\nu}}\right) - \frac{\partial \tilde{L}}{\partial x_\nu} = 0, \tag{30}$$

for $\nu = 1, \ldots, N$.

Proof. Clearly, if (30) holds, by Corollary 10, $\delta F = 0$. The reciprocal result follows from Lemma 11. ☐

From the formal point of view, the Hamiltonian principle in the form (**??**) is equivalent to the problem of *variational calculus* [13, 40]. However, despite the superficial similarity, they differ essentially. Namely, in mechanics the symbol δ stands for the its virtual variation, i.e., it is not an arbitrary variation but a displacement compatible with the constraints imposed on the systems. Thus only in the case of the holonomic systems, for which the number of degrees of freedom is equal to the number of generalized coordinates, the virtual variations are arbitrary and the Hamiltonian principle (**??**) is completely equivalent to the corresponding problem of the variational calculus. An important difference arises for the systems with nonholonomic constraints, when the variations of the generalized coordinates are connected by the additional relations usually called Chetaev conditions which we give later on.

3.2. **D'Alembert–Lagrange principle.** Let $L_j : \mathbb{R}\times T\mathrm{Q} \longrightarrow \mathbb{R}$ be smooth functions for $j = 1,\ldots,M$. The equations

$$L_j = L_j\left(t,\mathbf{x},\dot{\mathbf{x}}\right) = 0, \quad \text{for} \quad j = 1,\ldots,M < N,$$

with rank $\left(\dfrac{\partial(L_1,\ldots,L_M)}{\partial(\dot{x}_1,\ldots,\dot{x}_N)}\right) = M$ in all the points of $\mathbb{R}\times T\mathrm{Q}$, except perhaps in a zero Lebesgue measure set, define M *independent constraints* for the Lagrangian systems $(\mathrm{Q},\tilde{L})$.

Let $\mathcal{M}^*$ be the submanifold of $\mathbb{R}\times T\mathrm{Q}$ defined by the equations (**??**), i.e.

$$\mathcal{M}^* = \{(t,\mathbf{x},\,\dot{\mathbf{x}}) \in \mathbb{R}\times T\mathrm{Q} : L_j(t,\mathbf{x},\dot{\mathbf{x}}) = 0, \quad \text{for} \quad j = 1,\ldots,M\}.$$

A *constrained Lagrangian system* is a triplet $(\mathrm{Q},\tilde{L},\mathcal{M}^*)$. The number of degree of freedom is $\kappa = dim\mathrm{Q} - M = N - M$.

The constraint is called *integrable* if it can be written in the form $L_j = \dfrac{d}{dt}\left(G_j(t,\mathbf{x})\right) = 0$, for a convenient function G_j. Otherwise the constraint is called *nonintegrable.* According to Hertz [16] the nonintegrable constraints are also called *nonholonomic.*

The Lagrangian systems with nonintegrable constraints are usually called (also following to Hertz) the *nonholonomic mechanical systems,* or *nonholonomic constrained mechanical systems,* and with integrable constraints are called the *holonomic constrained mechanical systems* or *holonomic constrained Lagrangian systems.* The systems free of constraints are called *Lagrangian systems or holonomic systems.*

Sometimes it is also useful to distinguish between constraints that are dependent on or independent of time. Those that are independent of time are called *scleronomic,* and those that depend on time are called *rheonomic.* This therminology can also be applied to the mechanical systems themselves. Thus we say that the constrained Lagrangian systems is scleronomic (reonomic) if the constraints and Lagrangian are time independent (dependent).

The constraints

$$L_k = \sum_{j=1}^{N} a_{kj}\dot{x}_j + a_k = 0, \quad \text{for} \quad k = 1,\ldots,M, \tag{31}$$

where $a_{kj} = a_{kj}(t,\mathbf{x}),\ a_k = a_k(t,\mathbf{x})$, are called *linear constraints with respect to the velocity.* For simplicity we shall call *linear constraints.*

We observe that (31) admits an equivalent representation as a Pfaffian equations (for more details see [38])

$$\omega_k := \sum_{j=1}^{N} a_{kj}dx_j + a_k\,dt = 0.$$

We shall consider only two classes of systems of equations, the equations of constraints linear with respect to the velocity $(\dot{x}_1,\ldots,\dot{x}_N)$, or linear with respect to the differential $(dx_1,\ldots,dx_N,dt)$. In order to study the integrability or nonintegrability problem of the constraints the last representation, a Pfaffian system is the more useful. This is related with the fact that for the given 1-forms we have the Frobenius theorem which provides the necessary and sufficient conditions under which the 1-forms are closed and consequently the given set of constraints is integrable.

The constrains $L_j(t, \mathbf{x}, \dot{\mathbf{x}}) = 0$ are called *perfect constraints* or *ideal* if they satisfy the *Chetaev conditions* (see [7])

$$\sum_{k=1}^{N} \frac{\partial L_\alpha}{\partial \dot{x}_k} \delta x_k = 0, \tag{32}$$

for $\alpha = 1, \ldots, M$.

In what follows, we shall consider only perfect constraints.

If the constraints admit the representation (26) then the Chetaev conditions takes the form

$$\delta x_\alpha = \sum_{k=M+1}^{N} \frac{\partial \Phi_\alpha}{\partial \dot{x}_k} \delta x_k.$$

The virtual variations of the variables x_α for $\alpha = 1, \ldots, M$ are called *dependent variations* and for the variable x_β for $\beta = M+1, \ldots, N$ are called *independent variations.*

We say that the path $\gamma(t) = \mathbf{x}(t)$ is *admissible* with the perfect constraint if $L_j(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)) = 0$.

The admissible path is called the *motion* of the constrained Lagrangian systems $(\mathrm{Q}, \tilde{L}, \mathcal{M}^*)$ if for all $t \in [t_0, t_1]$

$$\sum_{\nu=1}^{N} E_\nu \tilde{L}\left(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)\right) \, \delta x_\nu(t) = 0,$$

for all virtual displacement $\delta \mathbf{x}(t)$ of the path $\gamma(t)$. This definition is known as *d'Alembert–Lagrange principle.*

It is well known the following result (see for instance [2, 5, 14, 35]).

Proposition 13. *The d'Alembert–Lagrange principle for constrained Lagrangian systems is equivalent to the Lagrangian differential equations with multipliers*

$$\begin{aligned} E_j \tilde{L} = & \ \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{x}_j} - \frac{\partial \tilde{L}}{\partial x_j} = \sum_{\alpha=1}^{M} \mu_\alpha \frac{\partial L_\alpha}{\partial \dot{x}_j}, \quad \text{for} \quad j = 1, \ldots, N, \\ L_j(t, \mathbf{x}, \dot{\mathbf{x}}) = & \ 0, \quad \text{for} \quad j = 1, \ldots, M, \end{aligned} \tag{33}$$

where μ_α for $\alpha = 1, \ldots, M$ are the Lagrangian multipliers.

3.3. **The varied path.** The varied path produced in Hamiltonian's principle is not in general an admissible path if the perfect constraints are nonholonomic, i.e. the mechanical systems cannot travel along the varied path without violating the constraints. We prove the following result, which shall play an important role in the all assertions below.

Proposition 14. *If the varied path is an admissible path then, the following relations hold*

$$\sum_{k=1}^{N} \frac{\partial L_\alpha}{\partial \dot{x}_k} \left(\delta \frac{dx_k}{dt} - \frac{d}{dt}\delta x_k \right) = \sum_{k=1}^{N} E_k L_\alpha \, \delta x_k, \tag{34}$$

for $\alpha = 1, \ldots, M$.

Proof. Indeed, the original path $\gamma(t) = \mathbf{x}(t)$ by definition satisfies the Chetaev conditions, and constraints, i.e. $L_j\left(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)\right) = 0$. If we suppose that the variation path $\gamma^*(t) = \mathbf{x}(t) + \varepsilon \delta \mathbf{x}(t)$, also satisfies the constraints i.e.

$$L_j\left(t, \mathbf{x} + \varepsilon \delta \mathbf{x}, \dot{\mathbf{x}} + \varepsilon \delta \dot{\mathbf{x}}\right) = L_j\left(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)\right) + \varepsilon \delta \, L_\alpha\left(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)\right) + \ldots = 0.$$

Thus restricting only to the terms of first order with respect to ε and by the Chetaev conditions we have (for simplicity we omitted the argument)

$$
\begin{aligned}
0 = \quad & \delta\, L_\alpha = \sum_{k=1}^{N} \left(\frac{\partial L_\alpha}{\partial\, x_k}\delta x_k + \frac{\partial L_\alpha}{\partial \dot{x}_k}\delta \dot{x}_k \right), \\
0 = \quad & \sum_{k=1}^{N} \frac{\partial L_\alpha}{\partial \dot{x}_k}\, \delta x_k,
\end{aligned}
\tag{35}
$$

for $\alpha = 1, \ldots, M$. The Chetaev conditions are satisfied at each instant, so

$$
\frac{d}{dt}\left(\sum_{k=1}^{N} \frac{\partial L_\alpha}{\partial \dot{x}_k}\, \delta x_k \right) = \sum_{k=1}^{N} \frac{d}{dt}\left(\frac{\partial L_\alpha}{\partial \dot{x}_k} \right) \delta x_k + \sum_{k=1}^{N} \frac{\partial L_\alpha}{\partial \dot{x}_k} \frac{d}{dt}\delta x_k = 0.
$$

Subtracting these relations from (35) we obtain (34). Consequently if the varied path is an admissible path, then relations (34) must hold. □

From (34) and (7) it follows that the elements of the matrix A satisfy

$$
\sum_{m=1}^{N} \delta x_m \left(E_m L_\alpha - \sum_{k=1}^{N} A_{k\,m} \frac{\partial L_\alpha}{\partial \dot{x}_k} \right) = \sum_{m=1}^{N} \delta x_m D_m L_\alpha = 0, \quad \text{for} \quad \alpha = 1, \ldots, M. \tag{36}
$$

This property will be used below.

Corollary 15. *For the holonomic constrained Lagrangian systems the relations* (34) *hold if and only if*

$$
\sum_{k=1}^{N} \frac{\partial L_\alpha}{\partial \dot{x}_k} \left(\delta \frac{dx_k}{dt} - \frac{d}{dt}\delta x_k \right) = 0, \quad \textit{for} \quad \alpha = 1, \ldots, M. \tag{37}
$$

Proof. Indeed, for holonomic constrained Lagrangian systems the constraints are integrable, consequently in view of (27) we have $E_k L_\alpha = 0$ for $k = 1, \ldots, N$ and $\alpha = 1, \ldots, M$. Thus, from (34), we obtain (37). □

Clearly the equalities (37) are satisfied if (29) holds. We observe that in general for holonomic constrained Lagrangian systems relation (29) cannot hold (see example 2).

3.4. **Transpositional relations.** As we observe in the previous subsection for nonholonomic constrained Lagrangian systems the curves, obtained doing a virtual variation in the motion of the systems, in general are not kinematical possible trajectories when (29) is not fulfilled. This leads to the conclusion that the Hamiltonian principle cannot be applied to nonholonomic systems, as it is usually employed for holonomic systems. The essence of the problem of the applicability of this principle for nonholonomic systems remains unclarified (see [35]). In order to clarify this situation, it is sufficient to note that the question of the applicability of the principle of stationary action to nonholonomic systems is intimately related to the question of transpositional relation.

The key point is that the Hamiltonian principle assumes that the operation of differentiation with respect to the time $\dfrac{d}{dt}$ and the virtual variation δ commute in all the generalized coordinate systems.

For the holonomic constrained Lagrangian systems relations (29) cannot hold (see Corollary 15). For a nonholonomic systems the form of the Hamiltonian principle will depend on the point of view adopted with respect to the transpositional relations.

What are then the correct transpositional relations? Until now, does not exist a common point of view concerning to the commutativity of the operation of differentiation with respect to the time and the virtual variation when there are nonintegrable constraints. Two points of view have been maintained. According to one (supported, for example, by Volterra, Hamel, Hölder, Lurie, Pars,. . .), the operations $\frac{d}{dt}$ and δ commute for all the generalized coordinates, independently if the systems are holonomic or nonholonomic, i.e.

$$\delta\frac{dx_k}{dt}-\frac{d}{dt}\delta x_k=0,\quad \text{for}\quad k=1,\ldots,N.$$

According to the other point of view (supported by Suslov, Voronets, Levi-Civita, Amaldi,. . .) the operations $\frac{d}{dt}$ and δ commute always for holonomic systems, and for nonholonomic systems with the constraints

$$\dot{x}_\alpha=\sum_{j=M+1}^{N}a_{\alpha j}(t,\mathbf{x})\dot{x}_j+a_\alpha(t,\mathbf{x}),\quad \text{for}\quad \alpha=1,\ldots,M.$$

the transpositional relations are equal to zero only for the generalized coordinates $x_{M+1},\ldots,x_N$, (for which their virtual variations are independent). For the remaining coordinates $x_1,\ldots,x_M$, (for which their virtual variations are dependent), the transpositional relations must be derived on the basis of the equations of the nonholonomic constraints, and cannot be identically zero, i.e.

$$\begin{aligned}\delta\frac{dx_k}{dt}-\frac{d}{dt}\delta x_k &= 0,\quad \text{for}\quad k=M+1,\ldots,N\\ \delta\frac{dx_k}{dt}-\frac{d}{dt}\delta x_k &\neq 0,\quad \text{for}\quad k=1,\ldots,M.\end{aligned}$$

The second point of view acquired general acceptance and the first point of view was considered erroneous (for more details see [35]). The meaning of the transpositional relations (1) can be found in [19, 32, 34, 35].

In the results given in the following section play a key role the equalities (34). From these equalities and from the examples it will be possible to observe that the second point of view is correct only for the so called Voronets–Chaplygin systems, and in general for locally nonholonomic systems. There exist many examples for which the independent virtual variations generated non–zero transpositional relations. Thus we propose a third point of view on the transpositional relations: the virtual variations can generate the transpositional relations given by the formula (7) where the elements of the matrix A satisfies the conditions (see formula (36))

$$D_\nu L_\alpha=E_\nu L_\alpha-\sum_{k=1}^{N}A_{k\,\nu}\frac{\partial L_\alpha}{\partial\dot{x}_k}=0,\quad \text{for}\quad \nu=1,\ldots,M,\quad \alpha=1,\ldots,M. \tag{38}$$

we observe that here the $L_\alpha=0$ are constraints which in general are nonlinear in the velocity.

3.5. **Hamiltonian–Suslov principle.** After the introduction of the nonholonomic mechanics by Hertz, it appeared the question of extending to the nonholonomic mechanics the results of the holonomic mechanics. Hertz [16] was the first in studying the problem of applying the Hamiltonian principle to systems with nonintegrable constraints. In [16] Hertz wrote: "Application of Hamilton's principle to any material systems does not exclude that between selected coordinates of the systems rigid constraints exist, but it still requires that these relations could be expressed by integrable constraints. The appearance of nonintegrable constraints is unacceptable. In this case the Hamilton's principle is not valid." Appell [3] in correspondence with Hertz's ideas affirmed that it is not possible to apply the Hamiltonian principle for systems with nonintegrable constraints

Suslov [48] claimed that "Hamilton's principle is not applied to systems with nonintegrable constraints, as derived based on this equation are different from the corresponding equations of Newtonian mechanics".

The applications of the most general differential principle, i.e. the d'Alembert–Lagrange and their equivalent Gauss and Appel principle, is complicated due to the presence of the terms containing the second order derivative. On the other hand the most general variational integral principle of Hamilton is not valid for nonholonomic constrained Lagrangian systems. The generalization of the Hamiltonian principle for nonholonomic mechanical systems was deduced by Voronets and Suslov (see for instance [48, 53]). As we can observe later on from this principle follows the importance of the transpositional relations to determine the correct equations of motion for nonholonomic constrained Lagrangian systems.

Proposition 16. *The d'Alembert–Lagrangian principle for the contrained Lagrangian systems* $\sum_{k=1}^{N}\delta x_k E_k\tilde{L}=0$ *is equivalent to the Hamilton–Suslov principle* (2) *where we assume that* $\delta x_\nu(t)$, $\nu=1,\dots,N$, *are arbitrary smooth functions defined in the interior of the interval* $[t_0,\,t_1]$ *and vanishing at its endpoints, i.e.,* $\delta x_\nu(t_0)=\delta x_\nu(t_1)=0$.

Proof. From the d'Alembert–Lagrangian principle we obtain the identity

$$\begin{aligned}0= &\; -\sum_{k=1}^{N}\delta x_k E_k\tilde{L}=\sum_{k=1}^{N}\delta x_k\frac{\partial\tilde{L}}{\partial x_k}-\sum_{k=1}^{N}\delta x_k\frac{d}{dt}\frac{\partial\tilde{L}}{\partial\dot{x}_k}\\ = &\; \sum_{k=1}^{N}\left(\delta x_k\frac{\partial\tilde{L}}{\partial x_k}+\delta\dot{x}_k\frac{\partial\tilde{L}}{\partial\dot{x}_k}\right)-\sum_{k=1}^{N}\left(\left(\delta\frac{dx_k}{dt}-\frac{d}{dt}\delta x_k\right)\frac{\partial\tilde{L}}{\partial\dot{x}_k}-\frac{d}{dt}\left(\frac{\partial\tilde{L}}{\partial\dot{x}_k}\delta x_k\right)\right)\\ = &\; \delta\tilde{L}-\sum_{k=1}^{N}\left(\left(\delta\frac{dx_k}{dt}-\frac{d}{dt}\delta x_k\right)\frac{\partial\tilde{L}}{\partial\dot{x}_k}-\frac{d}{dt}\left(\frac{\partial\tilde{L}}{\partial\dot{x}_k}\delta x_k\right)\right),\end{aligned}$$

where $\delta\tilde{L}$ is a variation of the Lagrangian $\tilde{L}$. After the integration and assuming that $\delta x_k(t_0)=0$, $\delta x_k(t_1)=0$ we easily obtain (2), which represent the most general formulation of the Hamiltonian principle (*Hamilton–Suslov principle*) suitable for constrained and unconstrained Lagrangian systems. □

Suslov determine the transpositional relations only for the case when the constraints are of Voronets type, i.e. given by the formula (22). Assume that

$$\delta\frac{dy_k}{dt}-\frac{d}{dt}\delta y_k=0,\quad \text{for}\quad k=M+1,\dots,N,$$

Voronets and Suslov deduced that

$$\delta\frac{dx_k}{dt}-\frac{d}{dt}\delta x_k=\sum_{k=1}^{N}B_{kr}\delta y_r-\delta a_k$$

for convenient functions $B_{kr}=B_{kr}\left(t,\mathbf{x},\mathbf{y},\dot{\mathbf{x}},\dot{\mathbf{y}}\right)$, for $r=M+1,\dots,N$ and $k=1,\dots,M$.

Thus we obtain

$$\int_{t_0}^{t_1}\left(\delta\,\tilde{L}-\sum_{k=1}^{N}\frac{\partial\tilde{L}}{\partial\dot{x}_j}\left(\sum_{k=1}^{N}B_{kr}\delta y_r-\delta a_k\right)\right)dt=0,$$

This is the Hamiltonian principle for nonholonomic systems in the Suslov form (see for instance [48]). We observe that the same result was deduced by Voronets in [53].

It is important to observe that Suslov and Voronets require a priori that the independent virtual variations produce the zero transpositional relations. At the sometimes these authors consider only linear constraints with respect to the velocity of the type (22).

3.6. **Modification of the vakonomic mechanics (MVM).** As we observe in the introduction, the main objective of this paper is to construct the variational equations of motion describing the behavior of the constrained Lagrangian systems in which the equalities (34) take place in the most general possible way. We shall show that the d'Alembert–Lagrange principle is not the only way to deduce the equations of motion for the constrained Lagrangian systems. Instead of it we can apply the generalization of the Hamiltonian principle, whereby the motions of such systems are extremals of the *variational Lagrange problem* (see for instance [13]), i.e. the problem of determining the critical points of the action in the class of curves with fixed endpoints and satisfying the constraints. The solution of this problem as we shall see will give the differential equations of second order which coincide with the well–known classical equations of the mechanics except perhaps in a zero Lebesgue measure set.

From the previous section we deduce that in order to generalize the Hamiltonian principle to nonholonomic systems we must take into account the following relations

$$\text{(A)} \qquad \delta L_\alpha = \sum_{j=1}^{N} \left(\frac{\partial L_\alpha}{\partial x_j}\delta x_j + \frac{\partial L_\alpha}{\partial \dot{x}_j}\delta \dot{x}_j \right) = 0 \quad \text{for} \quad \alpha = 1, \ldots, M,$$

$$\text{(B)} \qquad \sum_{j=1}^{N} \frac{\partial L_\alpha}{\partial \dot{x}_j}\delta x_j = 0 \quad \text{for} \quad \alpha = 1, \ldots, M,$$

$$\text{(C)} \qquad \delta \frac{dx_j}{dt} - \frac{d}{dt}\delta x_j = 0 \quad \text{for} \quad j = 1, \ldots, N,$$

where $L_\alpha = 0$ for $\alpha = 1, \ldots, M$ are the constraints.

A lot of authors consider that (C) is always fulfilled (see for instance [32, 38]), together with the conditions (A) and (B). However these conditions are incompatible in the case of the nonintegrable constrains. We observe that these authors deduced that the Hamiltonian principle is not applicable to the nonholonomic systems.

To obtain a generalization of the Hamiltonian principle for the nonholonomic mechanical systems, some of these three conditions must be excluded.

In particular for the Hölder principle conditions (A) is excluded and keep (B) and (C) (see [17]). For the Hamiltonian–Suslov principle condition (A) and (B) hold, and (C) only holds for the independent variations.

In this paper we extend the Hamiltonian principle by supposing that conditions (A) and (B) hold and (C) does not hold . Instead of (C) we consider that (7) holds where elements of matrix A satisfy the relations (38).

4. Solution of the inverse problem of the constrained Lagrangian systems

We shall determine the equations of motion of the constrained Lagrangian systems using the Hamiltonian principle with non zero transpositional relations, whereby the motions of the systems are *extremals* of the *variational Lagrange's problem* (see for instance [13]), i.e. are the critical points of the action functional

$$\int_{t_0}^{t_1} L_0\left(t, \mathbf{x}, \dot{\mathbf{x}}\right)\, dt,$$

in the class of path with fixed endpoints satisfying the independent constraints

$$L_j\left(t, \mathbf{x}, \dot{\mathbf{x}}\right) = 0, \quad \text{for} \quad j = 1, \ldots, M.$$

In the classical solution of the Lagrange problem usually we apply the *Lagrange multipliers method* which consists in the following. We introduce the additional coordinates $\Lambda = (\lambda_1, \ldots, \lambda_M)$, and Lagrangian $\widehat{L} : \mathbb{R} \times T\mathrm{Q} \times \mathbb{R}^M \longrightarrow \mathbb{R}$ given by

$$\widehat{L}\left(t, \mathbf{x}, \dot{\mathbf{x}}, \Lambda\right) = L_0\left(t, \mathbf{x}, \dot{\mathbf{x}}\right) - \sum_{j=1}^{M} \lambda_j\, L_j\left(t, \mathbf{x}, \dot{\mathbf{x}}\right),$$

Under this choice we reduce the Lagrange problem to a variational problem without constraints, i.e. we must determine the extremal of the action functional $\int_{t_0}^{t_1} \widehat{L}\, dt$. We shall study a slight *modification of the Lagrangian multipliers method*. We introduce the additional coordinates $\Lambda = (\lambda_1, \ldots, \lambda_M)$, and the Lagrangian on $\mathbb{R} \times TQ \times \mathbb{R}^M$ given by the formula (4), where we assume that λ_j^0 are arbitrary constants, and L_j are arbitrary functions for $j = M+1, \ldots, N$.

Now we determine the critical points of the action functional $\int_{t_0}^{t_1} L\,(t, \mathbf{x}, \dot{\mathbf{x}}, \Lambda)\, dt$, i.e. we determine the path $\gamma(t)$ such that $\int_{t_0}^{t_1} \delta\,(L\,(t, \mathbf{x}, \dot{\mathbf{x}}, \Lambda))\, dt = 0$ under the additional condition that the transpositional relations are given by the formula (7).

The solution of the inverse problem stated in section 2 is the following. Differential equations obtained from (6) are given by the formula (8) (see Theorem 1). We choose the arbitrary functions L_j in such a away that the matrix W_1 and W_2 given in Theorems 2 and 3 are nonsingular, except perhaps in a zero Lebesgue measure set. The constants λ_j^0 for $j = M+1, \ldots, N$ are arbitrary in Theorem 2, and λ_j^0 for $j = 1, \ldots, N-1$ are arbitrary and $\lambda_N^0 = 0$ in Theorem 3. The matrix A is determined from the equalities (11) and (15) of Theorems 2 and 3 respectively.

Remark 17. *It is interesting to observe that from the solutions of the inverse problem, the constants* λ_j^0 *for* $j = M+1, \ldots, N$ *are arbitrary except in Theorem 3 in which* $\lambda_N^0 = 0$. *Clearly, if* $L_j\,(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) = \dfrac{d}{dt} f_j(t, \boldsymbol{x})$ *for* $j = M+1, \ldots, N$, *then the* $L \simeq \widehat{L}$. *Using the arbitrariness of the constants* λ_j^0 *we can always take that* $\lambda_k^0 = 0$ *if* $L_k\,(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) \neq \dfrac{d}{dt} f_k(t, \boldsymbol{x})$. *Consequently we can always suppose that* $L \simeq \widehat{L}$. *Thus the only difference between the classical and the modified Lagrangian multipliers method consists only on the transpositional relations: for the classical method the virtual variations produce zero transpositional relations (i.e. the matrix* A *is the zero matrix) and for the modified method in general it is determined by the formulae* (7) *and* (36).

A very important subscase is obtained when the constraints are given in the form (Voronets-Chapliguin constraints type) $\dot{x}_\alpha - \Phi_\alpha\,(t, \boldsymbol{x}, \dot{x}_{M+1}, \ldots, \dot{x}_N) = 0$, *for* $\alpha = 1, \ldots, M$. *As we shall show under these assumptions the arbitrary functions are determined as follows:* $L_j = \dot{x}_j$ *for* $j = M+1, \ldots, N$. *Consequently the action of the modified Lagrangian multipliers method and the action of the classical Lagrangian multipliers method are equivalently. In view of* (26) *this equivalence always locally holds for any constrained Lagrangian systems.*

5. Proof of Theorems 1, 2 and 3

Proof of Theorem 1. In view of the equalities

$$\begin{aligned}
\int_{t_0}^{t_1} \delta L\, dt = & \int_{t_0}^{t_1} \sum_{k=1}^{M} \left(\frac{\partial L}{\partial \lambda_k} \delta\lambda_k \right) dt + \int_{t_0}^{t_1} \sum_{j=1}^{N} \left(\frac{\partial L}{\partial x_j} \delta x_j + \frac{\partial L}{\partial \dot{x}_j} \delta \frac{dx_j}{dt} \right) dt \\
= & \int_{t_0}^{t_1} \sum_{k=1}^{M} \left(-L_k \delta\lambda_k \right) dt + \int_{t_0}^{t_1} \sum_{j=1}^{N} \left(\frac{\partial L}{\partial x_j} \delta x_j + \frac{\partial L}{\partial \dot{x}_j} \frac{d}{dt} \delta x_j + \frac{\partial L}{\partial \dot{x}_j} \left(\delta \frac{dx_j}{dt} - \frac{d}{dt} \delta x_j \right) \right) dt \\
= & \int_{t_0}^{t_1} \sum_{k=1}^{M} \left(-L_k \delta\lambda_k \right) dt + \int_{t_0}^{t_1} \sum_{j=1}^{N} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_j} \delta x_j \right) dt \\
& - \int_{t_0}^{t_1} \sum_{j=1}^{N} \left(\left(-\frac{\partial L}{\partial x_j} + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_j} \right) \right) \delta x_j + \frac{\partial L}{\partial \dot{x}_j} \left(\delta \frac{dx_j}{dt} - \frac{d}{dt} \delta x_j \right) \right) dt.
\end{aligned}$$

Consequently

$$\begin{aligned}
\left. \int_{t_0}^{t_1} \delta L\, dt \right|_{L_\nu = 0} = & \int_{t_0}^{t_1} \sum_{j=1}^{N} \left(\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_j} \delta x_j \right) - \left(E_j L - \sum_{k=1}^{N} A_{jk} \frac{\partial L}{\partial \dot{x}_k} \right) \delta x_j \right) dt \\
= & \left. \sum_{j=1}^{N} \frac{\partial T}{\partial \dot{x}_j} \delta x_j \right|_{t=t_0}^{t=t_1} - \int_{t_0}^{t_1} \sum_{j=1}^{N} \left(E_j L - \sum_{k=1}^{N} A_{jk} \frac{\partial L}{\partial \dot{x}_k} \right) \delta x_j dt \\
= & - \int_{t_0}^{t_1} \sum_{j=1}^{N} \left(E_j L - \sum_{k=1}^{N} A_{jk} \frac{\partial L}{\partial \dot{x}_k} \right) \delta x_j dt = 0,
\end{aligned}$$

where $\nu = 1, \ldots, M$. Here we use the equalities $\delta \mathbf{x}(t_0) = \delta \mathbf{x}(t_1) = 0$. Hence if (8) holds then (6) is satisfied. The reciprocal result is proved by choosing

$$\delta x_k(t) = \begin{cases} \zeta(t) & \text{if } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\zeta(t)$ is a positive function in the interval (t_0^*, t_1^*), and it is equal to zero in the intervals $[t_0,\, t_0^*]$ and $[t_1^*,\, t_1]$, and applying Corollary 11.

From the definition (8) we have that

$$D_\nu (fg) = D_\nu f\, g + f\, D_\nu\, g + \frac{\partial f}{\partial \dot{x}_\nu} \frac{dg}{dt} + \frac{df}{dt} \frac{\partial g}{\partial \dot{x}_\nu}, \quad D_\nu a = 0,$$

where a is a constant.

Now we shall write (8) in a more convenient way

$$\begin{aligned}
0 = D_\nu L = & D_\nu \left(L_0 - \sum_{j=1}^{M} \lambda_j L_j - \sum_{j=M+1}^{N} \lambda_j^0 L_j \right) \\
= & D_\nu\, L_0 - \sum_{j=1}^{M} D_\nu \left(\lambda_j L_j \right) - \sum_{j=M+1}^{N} \lambda_j^0 D_\nu\, L_j \\
= & D_\nu\, L_0 - \sum_{j=M+1}^{N} \lambda_j^0 D_\nu\, L_j - \\
& - \sum_{j=1}^{M} \left(D_\nu\, \lambda_j\; L_j + \lambda_j D_\nu\, L_j + \frac{d\lambda_j}{dt} \frac{\partial L_j}{\partial \dot{x}_\nu} + \frac{dL_j}{dt} \frac{\partial \lambda_j}{\partial \dot{x}_\nu} \right).
\end{aligned}$$

From these relations and since the constraints $L_j = 0$ for $j = 1, \ldots, M$, we easily obtain equations (9) or equivalently

$$E_\nu L_0 = \sum_{k=1}^{N} A_{jk} \frac{\partial L_0}{\partial \dot{x}_k} + \sum_{j=1}^{M} \left(\lambda_j D_\nu L_j + \frac{d\lambda_j}{dt} \frac{\partial L_j}{\partial \dot{x}_\nu} \right) + \sum_{j=M+1}^{N} \lambda_j^0 D_\nu \, L_j. \tag{39}$$

Thus the theorem is proved. □

Now we show that the differential equations (39) for convenient functions L_j constants λ_j^0 for $j = M+1, \ldots, N$ and for convenient matrix A describe the motion of the constrained Lagrangian systems.

Proof of Theorem 2. The matrix equation (11) can be rewritten in components as follows

$$\sum_{j=1}^{N} A_{kj} \frac{\partial L_\alpha}{\partial \dot{x}_j} = E_k L_\alpha \Longleftrightarrow D_k L_\alpha = 0, \tag{40}$$

for α, $k = 1, \ldots, N$. Consequently the differential equations (39) become

$$E_\nu L_0 = \sum_{k=1}^{N} \left(A_{\nu k} \frac{\partial L_0}{\partial \dot{x}_k} + \frac{d\lambda_k}{dt} \frac{\partial L_k}{\partial \dot{x}_\nu} \right) \Longleftrightarrow D_\nu L_0 = \sum_{j=1}^{M} \frac{d\lambda_j}{dt} \frac{\partial L_j}{\partial \dot{x}_\nu}, \tag{41}$$

which coincide with the first systems (12).

In view of the condition $|W_1| \neq 0$ we can solve equation (11) with respect to A and obtain $A = W_1^{-1} \Omega_1$. Hence, by considering (40) we obtain the second systems from (12) and the transpositional relation (13). □

Proof of Theorem 3. The matrix equation (15) is equivalent to the systems

$$\begin{aligned} \sum_{j=1}^{N} A_{kj} \frac{\partial L_\alpha}{\partial \dot{x}_j} &= \quad E_k L_\alpha \Longleftrightarrow D_k L_\alpha = 0, \\ \sum_{j=1}^{N} A_{kj} \frac{\partial L_0}{\partial \dot{x}_j} &= \quad 0, \end{aligned}$$

for $k = 1, \ldots, N$, and $\alpha = 1, \ldots, N-1$. Thus, by considering that $\lambda_N^0 = 0$ we deduce that systems (39) takes the form

$$E_\nu L_0 = \sum_{j=1}^{M} \frac{d\tilde{\lambda}_j}{dt} \frac{\partial L_j}{\partial \dot{x}_\nu}.$$

Hence we obtain systems (16). On the other hand from (15) we have that $A = W_2^{-1} \Omega_2$. Hence we deduce that the transpositional relation (7) can be rewritten in the form (17). □

The mechanics basic on the Hamiltonian principle with non–zero transpositional relations given by formula (7), Lagrangian (4) and equations of motion (8) are called here the *modification of the vakonomic mechanics* and we shortly write *MVM*.

From the proofs of Theorems 2 and 3 follows that the relations (36) holds identically in MVM.

Corollary 18. *Differential equations* (12) *are invariant under the change*

$$L_0 \longrightarrow L_0 - \sum_{j=1}^{N} a_j L_j,$$

where the a_j*'s are constants for* $j = 1, \ldots, N$.

Proof. Indeed, from (41) and (40) it follows that

$$D_\nu\left(L_0-\sum_{j=1}^{N}a_jL_j\right)=D_\nu\,L_0-\sum_{j=1}^{N}a_jD_\nu\,L_j=D_\nu\,L_0=\sum_{j=1}^{M}\frac{d\lambda_j}{dt}\frac{\partial L_j}{\partial\dot{x}_\nu}.$$

□

Remark 19. *The following interesting facts follow from Theorems 2 and 3.*

(1) *The equations of motion obtained from Theorem 2 are more general than the equations obtained from Theorem 3. Indeed in* (12) *there are* $N-M$ *arbitrary functions while in* (16) *are* $N-M-1$ *arbitrary functions.*

(2) *If the constraints are linear in the velocity then between the Lagrangian multipliers* μ, $\dfrac{d\lambda}{dt}$ *and* $\dfrac{d\tilde{\lambda}}{dt}$ *there is the following relation*

$$\mu=\frac{d\tilde{\lambda}}{dt}=\left(W_2^{-1}\right)^T\left(W_1^T\frac{d\lambda}{dt}+W_2^{-1}\Omega_1^TW_1^{-T}\frac{\partial L_0}{\partial\dot{\boldsymbol{x}}}\right),$$

where W_1 *and* W_2 *are the matrixes defined in Theorems 2 and 3.*

(3) *If the constraints are linear in the velocity then one of the important question which appear in MVM is related with the arbitrariness functions* L_j *for* $j=M+1,\ldots,N$. *The following question arise: Is it possible to determine these functions in such a way that* $|W_1|$ *or* $|W_2|$ *is non–zero everywhere in* $\mathcal{M}^*$*? If we have a positive answer to this question, then the equations of motion of the MVM give a global behavior of the constrained Lagrangian systems, i.e. the obtained motions completely coincide with the motions obtained from the classical mathematical models. Thus if* $|W_1|\neq 0$ *and* $|W_2|\neq 0$ *everywhere in* $\mathcal{M}^*$ *then we have the equivalence*

$$D_\nu L_0=\sum_{j=1}^{M}\frac{d\lambda_j}{dt}\frac{\partial L_j}{\partial\dot{x}_\nu}\Longleftrightarrow E_\nu L_0=\sum_{j=1}^{M}\frac{d\tilde{\lambda}_j}{dt}\frac{\partial L_j}{\partial\dot{x}_\nu}\Longleftrightarrow E_\nu L_0=\sum_{j=1}^{M}\mu_j\frac{\partial L_j}{\partial\dot{x}_\nu}\tag{42}$$

If the constraints are nonlinear in the velocity and $|W_2|\neq 0$ *everywhere in* $\mathcal{M}^*$ *then we have the equivalence*

$$E_\nu L_0=\sum_{j=1}^{M}\frac{d\tilde{\lambda}_j}{dt}\frac{\partial L_j}{\partial\dot{x}_\nu}\Longleftrightarrow E_\nu L_0=\sum_{j=1}^{M}\mu_j\frac{\partial L_j}{\partial\dot{x}_\nu}\tag{43}$$

The equivalence with respect to the equations $D_\nu L_0=\displaystyle\sum_{j=1}^{M}\frac{d\lambda_j}{dt}\frac{\partial L_j}{\partial\dot{x}_\nu}$ *in general is not valid in this case because the term* $\Omega_1^TW_1^{-T}\dfrac{\partial L_0}{\partial\dot{\boldsymbol{x}}}$ *depend on* $\ddot{\boldsymbol{x}}$.

5.1. **Application of Theorems 2 and 3 to the Appell–Hamel mechanical systems.** As a general rule the constraints studied in classical mechanics are linear with respect to the velocities, i.e. L_j can be written as (31). However Appell and Hamel (see [3, 15]) in 1911, considered an artificial example of nonlinear nonholonomic constrains. A big number of investigations have been devoted to the derivation of the equations of motion of mechanical systems with nonlinear nonholonomic constraints see for instance [8, 15, 35, 36]. The works of these authors do not contain examples of systems with nonlinear nonholonomic constraints differing essentially from the example given by Appell and Hamel.

Corollary 20. *The equivalence* (42) *also holds for the Appell – Hamel system i.e. for the constrained Lagrangian systems*

$$\left(\mathbb{R}^3,\quad \tilde{L}=\frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)-gz,\quad \{\dot{z}-a\sqrt{\dot{x}^2+\dot{y}^2}=0\}\right),$$

where a and g are positive constants.

Proof. The classical equations (33) for the Appell-Hamel system are

$$\ddot{x} = -\frac{a\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\mu, \qquad \ddot{y} = -\frac{a\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\mu, \qquad \ddot{z} = -g+\mu, \tag{44}$$

where μ is the Lagrangian multiplier.

Now we apply Theorem 3. Hence, in order to obtain that $|W_2| \neq 0$ everywhere we choose the functions L_j for $j = 1, 2, 3$ as follows

$$L_1 = \dot{z} - a\sqrt{\dot{x}^2+\dot{y}^2} = 0, \quad L_2 = \arctan\frac{\dot{x}}{\dot{y}}, \quad L_3 = L_0 = \tilde{L}.$$

In this case the matrices W_2, Ω_2 and A are

$$W_2 = \begin{pmatrix} -\frac{a\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} & -\frac{a\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} & 1 \\ \frac{\dot{y}}{\dot{x}^2+\dot{y}^2} & -\frac{\dot{x}}{\dot{x}^2+\dot{y}^2} & 0 \\ \dot{x} & \dot{y} & \dot{z} \end{pmatrix}, \quad |W_2|_{L_1=0} = 1+a^2,$$

$$\Omega_2 = \begin{pmatrix} -\dot{y}q & \dot{x}q & 0 \\ \frac{\ddot{y}\left(\dot{x}^2-\dot{y}^2\right)-2\dot{x}\dot{y}\ddot{x}}{\left(\dot{x}^2+\dot{y}^2\right)^2} & \frac{\ddot{x}\left(\dot{x}^2-\dot{y}^2\right)+2\dot{x}\dot{y}\ddot{y}}{\left(\dot{x}^2+\dot{y}^2\right)^2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the matrix $A|_{L_1=0}$ is

$$\begin{pmatrix} -\frac{\dot{y}\left(a^2\dot{y}\dot{x}\ddot{x}+\left((a^2+1)\dot{y}^2+\dot{x}^2\right)\ddot{y}\right)}{(1+a^2)\left(\dot{x}^2+\dot{y}^2\right)} & \frac{\left(a^2\dot{x}^2+(a^2+1)\left(\dot{y}^2+\dot{x}^2\right)^2\right)\dot{y}\ddot{x}-a^2\dot{x}^3\ddot{y}}{(1+a^2)\left(\dot{x}^2+\dot{y}^2\right)^2} & 0 \\ \frac{\left(a^2\dot{y}^2+(a^2+1)\left(\dot{y}^2+\dot{x}^2\right)\right)\dot{x}\ddot{y}-a^2\dot{y}^3\ddot{x}}{(1+a^2)\left(\dot{x}^2+\dot{y}^2\right)} & -\frac{\dot{x}\left(a^2\dot{x}\dot{y}\ddot{y}+\left((a^2+1)\dot{x}^2+\dot{y}^2\right)\ddot{x}\right)}{(1+a^2)\left(\dot{x}^2+\dot{y}^2\right)^2} & 0 \\ \frac{\dot{y}a\left(\dot{y}\ddot{x}-\dot{x}\ddot{y}\right)}{(1+a^2)\left(\dot{x}^2+\dot{y}^2\right)^{3/2}} & -\frac{\dot{x}a\left(\dot{y}\ddot{x}-\dot{x}\ddot{y}\right)}{(1+a^2)\left(\dot{x}^2+\dot{y}^2\right)^{3/2}} & 0 \end{pmatrix}.$$

By considering that $|W_2|_{L_1=0} = 1+a^2$, we obtain that the equations (16) in this case describe the global behavior of the Appell–Hamel systems and take the form

$$\ddot{x} = -\frac{a\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\dot{\tilde{\lambda}}, \qquad \ddot{y} = -\frac{a\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\dot{\tilde{\lambda}}, \qquad \ddot{z} = -g+\dot{\tilde{\lambda}}. \tag{45}$$

Clearly that this system coincide with classical differential equations (44) with $\dot{\tilde{\lambda}} = \mu$.

After the derivation of the constraint $\dot{z} - a\sqrt{\dot{x}^2+\dot{y}^2} = 0$ along the solutions of (45), we obtain

$$0 = \ddot{z} - a\frac{\ddot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} + a\frac{\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = -g + (1+a^2)\dot{\tilde{\lambda}}.$$

Therefore $\dot{\tilde{\lambda}} = \frac{g}{1+a^2}$. Hence the equations of motion (45) become

$$\ddot{x} = -\frac{ag}{1+a^2}\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}, \qquad \ddot{y} = -\frac{ag}{1+a^2}\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}, \qquad \ddot{z} = -\frac{a^2g}{1+a^2}. \tag{46}$$

In this case the Lagrangian (14) writes

$$L=\frac{1}{2}(\dot x^2+\dot y^2+\dot z^2)-gz-\frac{g\,(t+C)}{1+a^2}(\dot z-a\sqrt{\dot x^2+\dot y^2})-\lambda_2^0\arctan\frac{\dot x}{\dot y},$$

where C and λ_2^0 are an arbitrary constants.

Under the condition $L_1=0$ we obtain that the transpositional relations are

$$\begin{aligned}\delta\frac{dx}{dt}-\frac{d}{dt}\delta x= &\ \frac{\dot y\left((1+a^2)\left(\dot x^2+\dot y^2\right)(\ddot x\delta y-\ddot y\delta x)+a^2\dot x\,(\dot y\ddot x-\dot x\ddot y)\,(\dot x\delta y-\dot y\delta x)\right)}{(1+a^2)\,(\dot x^2+\dot y^2)^2},\\ \delta\frac{dy}{dt}-\frac{d}{dt}\delta y= &\ \frac{\dot x\left((1+a^2)\left(\dot x^2+\dot y^2\right)(\ddot y\delta x-\ddot x\delta y)+a^2\dot y\,(\dot y\ddot x-\dot x\ddot y)\,(\dot x\delta y-\dot y\delta x)\right)}{(1+a^2)\,(\dot x^2+\dot y^2)^2},\\ \delta\frac{dz}{dt}-\frac{d}{dt}\delta z= &\ \frac{a\,(\dot y\ddot x-\dot x\ddot y)\,(\dot x\delta y-\dot x\delta y)}{(1+a^2)\,(\dot x^2+\dot y^2)^{3/2}}.\end{aligned}\tag{47}$$

From this example we obtain that the independent virtual variations δx and δy produce non–zero transpositional relations. This result is not in accordance with with the Suslov point on view on the transpositional relations.

Now we apply Theorem 2. The functions L_0, L_1, L_2 and L_3 are determined as follows

$$L_0=\tilde L,\quad L_1=\dot z-a\,\sqrt{\dot x^2+\dot y^2},\quad L_2=\dot y,\quad L_3=\dot x.$$

Thus the matrix W_1 and Ω_1 are

$$W_1=\begin{pmatrix}-\dfrac{a\dot x}{\sqrt{\dot x^2+\dot y^2}} & -\dfrac{a\dot y}{\sqrt{\dot x^2+\dot y^2}} & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix},\quad \Omega_1=\begin{pmatrix}\dot yq & -\dot xq & 0\\ 0&0&0\\0&0&0\end{pmatrix},$$

where $q=\dfrac{a(\ddot x\dot y-\ddot x\dot y)}{\sqrt{\dot x^2+\dot y^2}^3}$. Therefore $|W_1|=-1$.

Hence, after some computations from (11) we have that

$$A=\begin{pmatrix}0&0&0\\0&0&0\\ \dot yq & -\dot xq & 0\end{pmatrix}.$$

The equations of motion (12) becomes

$$\begin{aligned}\ddot x= &\ -\frac{a^2\dot y}{\dot x^2+\dot y^2}(\dot y\ddot x-\dot x\ddot y)-\frac{a\dot\lambda}{\sqrt{\dot x^2+\dot y^2}}\dot x,\\ \ddot y= &\ -\frac{a^2\dot x}{\dot x^2+\dot y^2}(\dot x\ddot y-\dot y\ddot x)-\frac{a\dot\lambda}{\sqrt{\dot x^2+\dot y^2}}\dot y,\\ \ddot z= &\ -g+\dot\lambda.\end{aligned}\tag{48}$$

By solving these equations with respect to $\ddot x$, $\ddot y$ and $\ddot z$ we obtain the equations

$$\ddot x=-\frac{a\dot x}{\sqrt{\dot x^2+\dot y^2}}\dot\lambda,\qquad \ddot y=-\frac{a\dot y}{\sqrt{\dot x^2+\dot y^2}}\dot\lambda,\qquad \ddot z=-g+\dot\lambda,$$

We observe in this case that $|W_1|=-1$, consequently these equations, obtained from Theorem 2, give a global behavior of the Appell–Hamel systems, i.e. coincide with the classical equations (44) with $\dot\lambda=\dot{\tilde\lambda}=\mu=\dfrac{g}{1+a^2}$.

The transpositional relations (13) can be written as

$$\delta\frac{dx}{dt}-\frac{d}{dt}\delta\,x=0,\quad \delta\frac{dy}{dt}-\frac{d}{dt}\delta\,y=0,\quad \delta\frac{dz}{dt}-\frac{d}{dt}\delta\,z=q\,(\dot y\delta\,x-\dot x\delta\,y)\,.\tag{49}$$

□

From this corollary we observe that the independent virtual variations δx and δy produce non–zero transpositional relations (47) and zero transpositional relations (49).

The Lagrangian (10) in this case takes the form

$$\begin{aligned} L = & \frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) - gz - \frac{g\,(t+C)}{1+a^2}(\dot{z} - a\sqrt{\dot{x}^2+\dot{y}^2}) - \lambda_2^0\dot{y} - \lambda_3^0\dot{x} \\ \simeq & \frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) - gz - \frac{g\,(t+C)}{1+a^2}(\dot{z} - a\sqrt{\dot{x}^2+\dot{y}^2}). \end{aligned}$$

From (34) it follows that

$$\delta\frac{dz}{dt} - \frac{d}{dt}\delta\, z = q\,(\dot{y}\delta\, x - \dot{x}\delta\, y) + \frac{a\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\left(\delta\frac{dx}{dt} - \frac{d}{dt}\delta\, x\right) + \frac{a\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\left(\delta\frac{dy}{dt} - \frac{d}{dt}\delta\, y\right).$$

Therefore this relation holds identically for (47) and (49).

In the next sections we show the importance of the equations of motion (12) and (16) contrasting them with the classical differential equations of nonholonomic mechanics.

6. Modificated vakonomic mechanics versus vakonomic mechanics

Now we show that the equations of the *vakonomic mechanics* (3) can be obtained from equations (9). More precisely, if in (7) we require that all the virtual variations of the coordinates produce the zero transpositional relations, i.e. the matrix A is the zero matrix and we require that $\lambda_j^0 = 0$ for $j = M+1,\ldots,N$, then from (9) by considering that $D_kL = E_kL$, we obtain the vakonomic equations (3), i.e.

$$\begin{aligned} D_\nu L_0 = & \sum_{j=1}^{M}\left(\lambda_j D_\nu L_j + \frac{d\lambda_j}{dt}\frac{\partial L_j}{\partial \dot{x}_\nu}\right) + \sum_{j=M+1}^{N}\lambda_j^0 D_\nu\, L_j \Longrightarrow \\ E_\nu\, L_0 = & \sum_{j=1}^{M}\left(\lambda_j E_\nu\, L_j + \frac{d\lambda_j}{dt}\frac{\partial L_j}{\partial \dot{x}_\nu}\right), \quad \nu = 1,\ldots,N \end{aligned}$$

In the following example in order to contrast Theorems 2 with the vakonomic model we study the *skate or knife edge on an inclined plane.*

Example 1. To set up the problem, consider a plane Ξ with cartesian coordinates x and y, slanted at an angle α. We assume that the y–axis is horizontal, while the x–axis is directed downward from the horizontal and let (x, y) be the coordinates of the point of contact of the skate with the plane. The angle φ represents the orientation of the skate measured from the x–axis. The skate is moving under the influence of the gravity. Here the the acceleration due to gravity is denoted by g. It also has mass m, and the moment inertia of the skate about a vertical axis through its contact point is denoted by J, (see page 108 of [35] for a picture). The equation of nonintegrable constraint is

$$L_1 = \dot{x}\sin\varphi - \dot{y}\cos\varphi = 0. \tag{50}$$

With these notations the Lagrangian function of the skate is

$$\hat{L} = \frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right) + \frac{J}{2}\dot{\varphi}^2 + mg\,x\,\sin\alpha.$$

Thus we have the constrained mechanical systems

$$\left(\mathbb{R}^2\times\mathbb{S}^1, \quad \hat{L} = \frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right) + \frac{J}{2}\dot{\varphi}^2 + mg\,x\,\sin\alpha, \quad \{\dot{x}\sin\varphi - \dot{y}\cos\varphi = 0\}\right).$$

For appropriate choice of mass, length and time units, we reduces the Lagrangian $\hat{L}$ to

$$L_0 = \frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{\varphi}^2\right) + x\,g\sin\alpha,$$

here for simplicity we leave the same notations for the all variables. The question is, *what is the motion of the point of contact?* To answer this question we shall use the vakonomic equations (3) and the equations (12) proposed in Theorem 2.

6.1. **The study of the skate applying Theorem 2.** We determine the motion of the point of contact of the skate using Theorem 2. We choose the arbitrary functions L_2 and L_3 as follows

$$L_2 = \dot{x}\cos\varphi + \dot{y}\sin\varphi, \quad L_3 = \dot{\varphi},$$

in order that the determinant $|W_1| \neq 0$ everywhere in the configuration space.

The Lagrangian (10) becomes

$$\begin{aligned} L(x, y, \varphi, \dot{x}, \dot{y}, \dot{\varphi}, \Lambda) = &\ \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{\varphi}^2\right) + g\sin\alpha x - \lambda(\dot{x}\sin\varphi - \dot{y}\cos\varphi) - \lambda_3^0\dot{\varphi} \\ \simeq &\ \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{\varphi}^2\right) + g\sin\alpha x - \lambda(\dot{x}\sin\varphi - \dot{y}\cos\varphi), \end{aligned}$$

where $\lambda := \lambda_1$.

The matrix W_1 and Ω_1 are

$$W_1 = \begin{pmatrix} \sin\varphi & -\cos\varphi & 0 \\ \cos\varphi & \sin\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad |W_1| = 1,$$

$$\Omega_1 = \begin{pmatrix} \dot{\varphi}\cos\varphi & \dot{\varphi}\sin\varphi & -L_2 \\ -\dot{\varphi}\sin\varphi & \dot{\varphi}\cos\varphi & -L_1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix $A = W_1^{-1}\Omega_1$ becomes

$$A = \left.\begin{pmatrix} 0 & \dot{\varphi} & -\sin\varphi L_2 - \cos\varphi L_1 \\ -\dot{\varphi} & 0 & \cos\varphi L_2 - \sin\varphi L_1 \\ 0 & 0 & 0 \end{pmatrix}\right|_{L_1=0} = \begin{pmatrix} 0 & \dot{\varphi} & -\dot{y} \\ -\dot{\varphi} & 0 & \dot{x} \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence the equation (12) and transpositional relations (13) take the form

$$\ddot{x} + \dot{\varphi}\dot{y} = g\sin\alpha + \dot{\lambda}\sin\varphi, \quad \ddot{y} - \dot{\varphi}\dot{x} = -\dot{\lambda}\cos\varphi, \quad \ddot{\varphi} = 0, \tag{51}$$

and

$$\begin{aligned} \delta\frac{dx}{dt} - \frac{d\delta x}{dt} &= \dot{y}\delta\varphi - \dot{\varphi}\delta y, \\ \delta\frac{dy}{dt} - \frac{d\delta y}{dt} &= \dot{\varphi}\delta x - \dot{x}\delta\varphi, \\ \delta\frac{d\varphi}{dt} - \frac{d\delta\varphi}{dt} &= -L_2\left(\delta x\sin\varphi - \delta y\cos\varphi\right) = 0, \end{aligned} \tag{52}$$

respectively, here we have applied the Lagrange–Chetaev's condition $\sin\varphi\,\delta x - \cos\varphi\,\delta y = 0$.

The initial conditions

$$x_0 = x|_{t=0}, \quad y_0 = y|_{t=0}, \quad \varphi_0 = \varphi|_{t=0}, \quad \dot{x}_0 = \dot{x}|_{t=0}, \quad \dot{y}_0 = \dot{y}|_{t=0}, \quad \dot{\varphi}_0 = \dot{\varphi}|_{t=0},$$

satisfy the constraint, i.e.

$$\sin\varphi_0\dot{x}_0 - \cos\varphi_0\dot{y}_0 = 0. \tag{53}$$

After the derivation of the constraint along the solutions of the equation of motion (51), and using (50) we obtain

$$\begin{aligned} 0 = &\ \sin\varphi\ddot{x} - \cos\varphi\ddot{y} + \dot{\varphi}\left(\cos\varphi\dot{x} + \sin\varphi\dot{y}\right) \\ = &\ \sin\varphi\left(g\sin\alpha + \dot{\lambda}\sin\varphi - \dot{\varphi}\dot{y}\right) - \cos\varphi\left(-\dot{\lambda}\cos\varphi + \dot{\varphi}\dot{x}\right) + \dot{\varphi}\left(\cos\varphi\dot{x} + \sin\varphi\dot{y}\right). \end{aligned}$$

Hence $\dot\lambda = -g\sin\alpha\sin\varphi$. Therefore the differential equations (51) can be written as

$$\ddot x + \dot\varphi\dot y = g\sin\alpha\cos^2\varphi, \quad \ddot x - \dot\varphi\dot x = g\sin\alpha\sin\varphi\cos\varphi, \quad \ddot\varphi = 0. \tag{54}$$

We study the motion of the skate in the following three cases:

(i) $\dot\varphi|_{t=0} = \omega = 0$.
(ii) $\dot\varphi|_{t=0} = \omega \neq 0$.
(iii) $\alpha = 0$.

For the first case ($\omega = 0$), after the change of variables

$$X = \cos\varphi_0\, x - \sin\varphi_0\, y, \quad Y = \cos\varphi_0\, x + \sin\varphi_0\, y,$$

the differential equations (9) and the constraint become

$$\ddot X = 0, \quad \ddot Y = g\sin\alpha\cos\varphi_0, \quad \varphi = \varphi_0, \quad \dot X = 0,$$

respectively. Consequently

$$X = X_0, \quad Y = g\sin\alpha\cos\varphi_0\frac{t^2}{2} + \dot Y_0 t + Y_0, \quad \varphi = \varphi_0,$$

thus the trajectories are straight lines.

For the second case ($\omega \neq 0$), we take $\varphi_0 = \dot y_0 = \dot x_0 = x_0 = y_0 = 0$ in order to simplify the computations. In view of the equality $\dot\varphi = \dot\varphi|_{t=0} = \omega$ and denoting by $'$ the derivation with respect φ we get that (54) become

$$x'' + y' = \frac{g\sin\alpha}{\omega^2}\cos^2\varphi, \quad x'' - x' = \frac{g\sin\alpha}{\omega^2}\sin\varphi\cos\varphi, \quad \varphi' = 1. \tag{55}$$

Which are easy to integrate and we obtain

$$x = -\frac{g\sin\alpha}{4\omega^2}\cos(2\varphi), \quad y = -\frac{g\sin\alpha}{4\omega^2}\sin(2\varphi) + \frac{g}{2\omega^2}\varphi, \quad \varphi = \omega t,$$

which correspond to the equation of the cycloid. Hence the point of contact of the skate follows a cycloid along the plane, but do not slide down the plane.

For the third case ($\alpha = 0$), if $\varphi_0 = 0$, $\omega \neq 0$ we obtain that the solutions of the given differential systems (54) are

$$x = \dot y_0\cos\varphi + \dot x_0\sin\varphi + a, \quad y = \dot y_0\sin\varphi + \dot y_0\cos\varphi + b, \quad \varphi = \varphi_0 + \omega t,$$

where $a = x_0 - \dfrac{\dot y_0}{\omega}$, $b = y_0 + \dfrac{\dot x_0}{\omega}$, which correspond to the equation of the circle with center at (a,b) and radius $\dfrac{\dot x_0^2 + \dot y_0^2}{\omega^2}$.

If $\alpha = 0$ and $\varphi_0 = 0$, $\omega = 0$ then we obtain that the solutions are

$$x = \dot x_0 t + x_0, \quad y = \dot y_0 t + y_0.$$

All these solutions coincide with the solutions obtained from the Lagrangian equations (33) with multipliers (see [2])

$$\ddot x = g\sin\alpha + \mu\sin\varphi, \quad \ddot y - \dot\varphi\dot x = -\mu\cos\varphi, \quad \ddot\varphi = 0,$$

with $\mu = \dot\lambda = -g\sin\alpha\sin\varphi$.

6.2. **The study of the skate applying vakonomic model.** Now we consider instead of Theorem 2 the vakomic model for studying the motion of the skate.

We consider the Lagrangian

$$L(x,y,\varphi,\dot x,\dot y,\dot\varphi,\Lambda) = \frac{1}{2}\left(\dot x^2 + \dot y^2 + \dot\varphi^2\right) + g\,x\,\sin\alpha - \lambda(\dot x\sin\varphi - \dot y\cos\varphi).$$

The equations of motion (3) for the skate are

$$\frac{d}{dt}(\dot x - \lambda\sin\varphi) = 0, \quad \frac{d}{dt}(\dot y + \lambda\cos\varphi) = 0, \quad \ddot\varphi = -\lambda(\dot x\cos\varphi + \dot y\sin\varphi).$$

We shall study only the case when $\alpha = 0$. After integration we obtain the differential systems

$$(56)\quad \begin{aligned} \dot{x} =& \quad \lambda \sin\varphi + a = \cos\varphi\,(a\cos\varphi + b\sin\varphi)\,,\\ \dot{y} =& \quad -\lambda\cos\varphi + b = \sin\varphi\,(a\cos\varphi + b\sin\varphi)\,,\\ \ddot{\varphi} =& \quad (b\cos\varphi - a\sin\varphi)\,(a\cos\varphi + b\sin\varphi) = (b_1^2 + a_2^2)\sin(\varphi+\alpha)\cos(\varphi+\alpha),\\ \lambda =& \quad b\cos\varphi - a\sin\varphi, \end{aligned}$$

where $a = \dot{x}_0 - \lambda_0\sin\varphi_0$, $b = \dot{y}_0 + \lambda_0\cos\varphi_0$ and $\lambda_0 = \lambda|_{t=0}$ is an arbitrary parameter. After the integration of the third equation we obtain that

$$(57)\qquad \int_0^{\varphi} \frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}} = t\sqrt{\frac{h+a^2+b^2}{2}},$$

where h is an arbitrary constant which we choose in such a way that $\kappa^2 = \dfrac{2(a^2+b^2)}{h+a^2+b^2} < 1$.

From (57) we get $\sin\varphi = sn\left(t\sqrt{\dfrac{h+a^2+b^2}{2}}\right)$, $\quad \cos\varphi = cn\left(t\sqrt{\dfrac{h+a^2+b^2}{2}}\right)$, where sn and cn are the Jacobi elliptic functions . Hence, if we take $\dot{x}_0 = 1$, $\dot{y}_0 = \varphi_0 = 0$, then the solutions of the differential equations (56) are

$$(58)\quad \begin{aligned} x =& \quad x_0 + \int_{t_0}^{t}\left(cn\left(t\sqrt{\frac{h+1+\lambda_0^2}{2}}\right) sn\left(t\sqrt{\frac{h+1+\lambda_0^2}{2}}\right) + \lambda_0 sn\left(t\sqrt{\frac{h+1+\lambda_0^2}{2}}\right)\right) dt,\\ y =& \quad y_0 + \int_{t_0}^{t} sn\left(t\sqrt{\frac{h+1+\lambda_0^2}{2}}\right)\lambda_0\, sn\left(t\sqrt{\frac{h+1+\lambda_0^2}{2}}\right) dt,\\ \varphi =& \quad am\left(t\sqrt{\frac{h+1+\lambda_0^2}{2}}\right). \end{aligned}$$

It is interesting to compare this amazing motions with the motions that we obtained above. For the same initial conditions the skate moves sideways along the circles. By considering that the solutions (58) depend on the arbitrary parameter λ_0 we obtain that for the given initial conditions do not exist a unique solution of the differential equations in the vakonomic model. Consequently the principle of determinacy is not valid for vakonomic mechanics with nonintegrable constraints (see the Corollary of page 36 in [2]).

7. Modificated vakonomic mechanics versus Lagrangian and constrained Lagrangian mechanics

7.1. **MVM versus Lagrangian mechanics.** The Lagrangian equations which describe the motion of the Lagrangian systems can be obtained from Theorem 2 by supposing that $M = 0$, i.e. there is no constraints We choose the arbitrary functions L_α for $\alpha = 1,\ldots,N$ as follows

$$L_\alpha = \frac{dx_\alpha}{dt}, \quad \alpha = 1,\ldots,N.$$

Hence the Lagrangian (10) takes the form

$$L = L_0 - \sum_{j=1}^{N}\lambda_j^0\frac{dx_j}{dt} \simeq L_0.$$

In this case we have that $|W_1| = 1$.

By considering the property of the Lagrangian derivative (see (27)) we obtain that Ω_1 is a zero matrix . Hence the matrices A_1 is the zero matrix. As a consequence the equations

(12) become

$$D_\nu L = E_\nu L = E_\nu \left(L_0 - \sum_{j=1}^{N} \lambda_j^0 \dot{x}_j \right) = E\nu L_0 = 0$$

because $L \simeq L_0$. The transpositional relation (13) in this case are $\delta \frac{d\mathbf{x}}{dt} - \frac{d\delta \mathbf{x}}{dt} = 0$, which are the well known relations in the Lagrangian mechanics (see formula (29)).

7.2. **MVM versus constrained Lagrangian systems.** From the equivalences (42) we have that in the case when the constraints are linear in the velocity the equations of motions of the MVM coincide with the Lagrangian equations with multipliers (33) except perhaps in a zero Lebesgue measure set $|W_2| = 0$ or $|W_1| = 0$. When the constraints are nonlinear in the velocity, we have the equivalence (43). Consequently equations of motions of the MVM coincide with the Lagrangian equations with multipliers (33) except perhaps in a zero Lebesgue measure set $|W_2| = 0$.

We illustrate this result in the following example.

Example 2. Let

$$\left(\mathbb{R}^2, \quad L_0 = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - U(x,y), \quad \{2\left(x\dot{x} + y\dot{y}\right) = 0\} \right),$$

be the constrained Lagrangian systems.

In order to apply Theorem 2 we choose the arbitrary function L_1 and L_2 as follow

(a)

$$L_1 = 2\left(x\dot{x} + y\dot{y}\right), \quad L_2 = -y\dot{x} + x\dot{y}.$$

Thus the matrices W_1 and Ω_1 are

$$W_1 = \begin{pmatrix} 2x & 2y \\ -y & x \end{pmatrix}, \quad |W_1| = 2x^2 + 2y^2 = 2, \quad \Omega_1 = \begin{pmatrix} 0 & 0 \\ -2\dot{y} & 2\dot{x} \end{pmatrix}.$$

Consequently equations (12) describe the motion everywhere for the constrained Lagrangian systems.

Equations (12) become

$$\ddot{x} = -\frac{\partial U}{\partial x} + 2\dot{y}\left(y\dot{x} - x\dot{y}\right) + 2x\dot{\lambda}\Big|_{L_1=0} = -\frac{\partial U}{\partial x} + x\left(\dot{\lambda} - 2(\dot{x}^2 + \dot{y}^2)\right),$$

$$\ddot{y} = -\frac{\partial U}{\partial y} - 2\dot{x}\left(y\dot{x} - x\dot{y}\right) + 2y\dot{\lambda}\Big|_{L_1=0} = -\frac{\partial U}{\partial y} + y\left(\dot{\lambda} - 2(\dot{x}^2 + \dot{y}^2)\right),$$

Transpositional relations take the form

$$\delta \frac{dx}{dt} - \frac{d\delta x}{dt} = 2y\left(\dot{y}\delta x - \dot{x}\delta y\right), \quad \delta \frac{dy}{dt} - \frac{d\delta y}{dt} = -2x\left(\dot{y}\delta x - \dot{x}\delta y\right). \tag{59}$$

(b) If we choose $L_2 = \frac{y\dot{x}}{x^2+y^2} - \frac{x\dot{y}}{x^2+y^2} = \frac{d}{dt}\arctan\frac{x}{y}$, then

$$W_1 = \begin{pmatrix} 2x & 2y \\ \frac{y}{x^2+y^2} & -\frac{x}{x^2+y^2} \end{pmatrix}, \quad |W_1| = -2, \quad \Omega_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Equations (12) and transpositional relations become

$$\ddot{x} = -\frac{\partial U}{\partial x} + 2x\dot{\lambda}, \quad \ddot{y} = -\frac{\partial U}{\partial y} + 2y\dot{\lambda},$$

$$\delta \frac{dx}{dt} - \frac{d\delta x}{dt} = 0, \quad \delta \frac{dy}{dt} - \frac{d\delta y}{dt} = 0. \tag{60}$$

respectively.

From this example we obtain that for the holonomic constrained Lagrangian systems the transpositional relations can be non–zero (see (59)), or can be zero (see (60)). We observe that from condition (34) it follows the relation

$$x\left(\delta\frac{dx}{dt}-\frac{d\delta x}{dt}\right)+y\left(\delta\frac{dy}{dt}-\frac{d\delta y}{dt}\right)=0.$$

This equality holds identically if (60) and (59) takes place.

The equations of motions (33) in this case are

$$\ddot{x}=-\frac{\partial U}{\partial x}+2x\,\mu,\quad \ddot{y}=-\frac{\partial U}{\partial y}+2y\,\mu,$$

with $\mu=\dot{\lambda}-2(\dot{x}^2+\dot{y}^2)$.

Example 3. To contrast the MVM with the classical model we apply Theorems 2 to the *Gantmacher's systems* (see for more details [11, 45]).

Two material points m_1 and m_2 with equal masses are linked by a metal rod with fixed length l and small mass. The systems can move only in the vertical plane and so the speed of the midpoint of the rod is directed along the rod. It is necessary to determine the trajectories of the material points m_1 and m_2.

Let $(q_1,\,r_1)$ and $(q_2,\,r_2)$ be the coordinates of the points m_1 and m_2, respectively. Clearly $(q_1-q_2)^2+(r_1-r_2)^2=l^2$. Thus we have a constrained Lagrangian system in the configuration space $\mathbb{R}^4$ with the Lagrangian function $\mathrm{L}=\frac{1}{2}\left(\dot{q}_1^2+\dot{q}_2^2+\dot{r}_1^2+\dot{r}_2^2\right)-g/2(r_1+r_2)$, and with the linear constraints

$$(q_2-q_1)(\dot{q}_2-\dot{q}_1)+(r_2-r_1)(\dot{r}_2-\dot{r}_1)=0,\quad (q_2-q_1)(\dot{r}_2+\dot{r}_1)-(r_2-r_1)(\dot{q}_2+\dot{q}_1)=0.$$

Introducing the following change of coordinates:

$$x_1=\frac{q_2-q_1}{2},\quad x_2=\frac{r_1-r_2}{2},\quad x_3=\frac{r_2+r_1}{2},\quad x_4=\frac{q_1+q_2}{2},$$

we obtain $x_1^2+x_2^2=\frac{1}{4}\left((q_1-q_2)^2+(r_1-r_2)^2\right)=\frac{l^2}{4}$. Hence we have the constrained Lagrangian mechanical systems

$$\left(\mathbb{R}^4,\quad \tilde{L}=\frac{1}{2}\sum_{j=1}^{4}\dot{x}_j^2-gx_3,\quad \{x_1\dot{x}_1+x_2\dot{x}_2=0,\quad x_1\dot{x}_3-x_2\dot{x}_4=0\}\right).$$

The equations of motion (33) obtained from the d'Alembert–Lagrange principle are

$$\ddot{x}_1=\mu_1x_1,\quad \ddot{x}_2=\mu_1x_2,\quad \ddot{x}_3=-g+\mu_2x_1,\quad \ddot{x}_4=-\mu_2x_2, \tag{61}$$

where μ_1, μ_2 are the Lagrangian multipliers such that

$$\mu_1=-\frac{\dot{x}_1^2+\dot{x}_2^2}{x_1^2+x_2^2},\quad \mu_2=\frac{\dot{x}_2\dot{x}_4-\dot{x}_1\dot{x}_3+gx_1}{x_1^2+x_2^2}. \tag{62}$$

For applying Theorem 2 we have the constraints

$$L_1=x_1\dot{x}_1+x_2\dot{x}_2=0,\quad L_2=x_1\dot{x}_3-x_2\dot{x}_4=0,$$

and we choose the arbitrary functions L_3 and L_4 as follows

$$L_3=-x_1\dot{x}_2+x_2\dot{x}_1,\quad L_4=x_2\dot{x}_3+x_1\dot{x}_4.$$

For the given functions we obtain that

$$W_1=\begin{pmatrix} x_1 & x_2 & 0 & 0\\ 0 & 0 & x_1 & -x_2\\ x_2 & -x_1 & 0 & 0\\ 0 & 0 & x_2 & x_1\end{pmatrix},\quad \Omega_1=\begin{pmatrix} 0 & 0 & 0 & 0\\ -\dot{x}_3 & \dot{x}_4 & \dot{x}_1 & -\dot{x}_2\\ -2\dot{x}_2 & 2\dot{x}_1 & 0 & 0\\ -\dot{x}_4 & -\dot{x}_3 & \dot{x}_2 & \dot{x}_1\end{pmatrix}.$$

Therefore $|W_1| = (x_1^2+x_2^2)^2 = \dfrac{l^2}{4} \neq 0$. The matrix A in this case is

$$\begin{pmatrix} \dfrac{2x_2\dot{x}_2}{x_1^2+x_2^2} & -\dfrac{2x_2\dot{x}_1}{x_1^2+x_2^2} & 0 & 0 \\ -\dfrac{2x_1\dot{x}_2}{x_1^2+x_2^2} & \dfrac{2x_1\dot{x}_1}{x_1^2+x_2^2} & 0 & 0 \\ -\dfrac{x_1\dot{x}_3+x_2\dot{x}_4}{x_1^2+x_2^2} & \dfrac{x_1\dot{x}_4-x_2\dot{x}_3}{x_1^2+x_2^2} & \dfrac{x_1\dot{x}_1+x_2\dot{x}_2}{x_1^2+x_2^2} & \dfrac{x_2\dot{x}_1-x_1\dot{x}_2}{x_1^2+x_2^2} \\ \dfrac{x_1\dot{x}_4-x_2\dot{x}_3}{x_1^2+x_2^2} & \dfrac{x_1\dot{x}_3-x_2\dot{x}_4}{x_1^2+x_2^2} & \dfrac{x_2\dot{x}_1-x_1\dot{x}_2}{x_1^2+x_2^2} & \dfrac{x_1\dot{x}_1+x_2\dot{x}_2}{x_1^2+x_2^2} \end{pmatrix}.$$

Consequently differential equations (12) take the form

$$\begin{aligned} \ddot{x}_1 = & \left.\left(\frac{2x_2\dot{x}_1\dot{x}_2-2x_1\dot{x}_2^2-x_1\dot{x}_3^2-x_1\dot{x}_4^2}{x_1^2+x_2^2}+x_1\dot{\lambda}_1\right)\right|_{L_1=L_2=0} \\ = & \; x_1\left(\dot{\lambda}_1-\frac{2\dot{x}_1^2+2\dot{x}_2^2+\dot{x}_3^2+\dot{x}_4^2}{x_1^2+x_2^2}\right), \\ \ddot{x}_2 = & \left.-\left(\frac{-2x_1\dot{x}_1\dot{x}_2+2x_2\dot{x}_2^2+x_2\dot{x}_3^2+x_2\dot{x}_4^2}{x_1^2+x_2^2}+x_2\dot{\lambda}_1\right)\right|_{L_1=L_2=0} \\ = & \; x_2\left(\dot{\lambda}_1-\frac{2\dot{x}_1^2+2\dot{x}_2^2+\dot{x}_3^2+\dot{x}_4^2}{x_1^2+x_2^2}\right), \\ \ddot{x}_3 = & \left.\left(\frac{\dot{x}_3\left(x_1\dot{x}_1+x_2\dot{x}_2\right)-\dot{x}_4\left(x_2\dot{x}_1-x_1\dot{x}_2\right)}{x_1^2+x_2^2}+x_1\dot{\lambda}_2-g\right)\right|_{L_1=L_2=0} \\ = & \; \frac{\dot{x}_4\left(x_2\dot{x}_1-x_1\dot{x}_2\right)}{x_1^2+x_2^2}+x_1\dot{\lambda}_2-g, \\ \ddot{x}_4 = & \left.\left(\frac{\dot{x}_4\left(x_1\dot{x}_1+x_2\dot{x}_2\right)-\dot{x}_3\left(x_2\dot{x}_1-x_1\dot{x}_2\right)}{x_1^2+x_2^2}-x_2\dot{\lambda}_2\right)\right|_{L_1=L_2=0} \\ = & -\frac{\dot{x}_3\left(x_2\dot{x}_1-x_1\dot{x}_2\right)}{x_1^2+x_2^2}-x_2\dot{\lambda}_2. \end{aligned} \tag{63}$$

Derivating the constraints we obtain that the multipliers $\dot{\lambda}_1$ and $\dot{\lambda}_2$ are

$$\dot{\lambda}_1 = \frac{\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2+\dot{x}_4^2}{x_1^2+x_2^2} = \mu_1 + \frac{\dot{x}_3^2+\dot{x}_4^2}{x_1^2+x_2^2}, \quad \dot{\lambda}_2 = \frac{gx_1}{x_1^2+x_2^2} = \mu_2 + \frac{\dot{x}_1\dot{x}_3-\dot{x}_2\dot{x}_4}{x_1^2+x_2^2}.$$

Inserting these values into (63) we deduce

$$\begin{aligned} \ddot{x}_1 = & -\frac{x_1\left(\dot{x}_1^2+\dot{x}_2^2\right)}{x_1^2+x_2^2}, \qquad \ddot{x}_2 = \frac{x_2\left(\dot{x}_1^2+\dot{x}_2^2\right)}{x_1^2+x_2^2}, \\ \ddot{x}_3 = & -g+\frac{x_1\left(\dot{x}_2\dot{x}_4-\dot{x}_1\dot{x}_3+gx_1\right)}{x_1^2+x_2^2}, \quad \ddot{x}_4 = -\frac{x_2\left(\dot{x}_2\dot{x}_4-\dot{x}_1\dot{x}_3+gx_1\right)}{x_1^2+x_2^2}. \end{aligned}$$

These equations coincide with equations (61) everywhere because $|W_1| = \dfrac{l^2}{4}$, where l is the length of the rod.

The transpositional relations in this case are

$$
\begin{aligned}
\delta\frac{dx_1}{dt}-\frac{d\delta x_1}{dt} &= -\frac{2x_2}{x_1^2+x_2^2}\left(\dot{x}_1\delta x_2-\dot{x}_2\delta x_1\right),\\
\delta\frac{dx_2}{dt}-\frac{d\delta x_2}{dt} &= \frac{2x_1}{x_1^2+x_2^2}\left(\dot{x}_1\delta x_2-\dot{x}_2\delta x_1\right),\\
\delta\frac{dx_3}{dt}-\frac{d\delta x_3}{dt} &= \frac{x_1}{x_1^2+x_2^2}\left(\dot{x}_1\delta x_3-\dot{x}_3\delta x_1+\dot{x}_4\delta x_2-\dot{x}_2\delta x_4\right),\\
&\quad+\frac{x_2}{x_1^2+x_2^2}\left(\dot{x}_1\delta x_4-\dot{x}_4\delta x_1+\dot{x}_2\delta x_3-\dot{x}_3\delta x_2\right),\\
\delta\frac{dx_4}{dt}-\frac{d\delta x_4}{dt} &= -\frac{x_2}{x_1^2+x_2^2}\left(\dot{x}_1\delta x_3-\dot{x}_3\delta x_1+\dot{x}_4\delta x_2-\dot{x}_2\delta x_4\right)\\
&\quad+\frac{x_1}{x_1^2+x_2^2}\left(\dot{x}_1\delta x_4-\dot{x}_4\delta x_1+\dot{x}_2\delta x_3-\dot{x}_3\delta x_2\right).
\end{aligned}
\tag{64}
$$

From this example we again get that the virtual variations produce the non–zero transpositional relations.

Remark 21. *From the previous example we observe that the virtual variations produce zero or non–zero transpositional relations, depending on the arbitrary functions which appear in the construction of the proposed mathematical model. Thus, the following question arises: Can be choosen the arbitrary functions L_j for $j = M+1,\ldots,N$ in such a way that for the nonholonomic systems only the independent virtual variations would generate zero transpositional relations?*

The positive answer to this question is obtained locally for any constrained Lagrangian systems and globally for the *Chaplygin-Voronets mechanical systems,* and for the generalization of these systems studied in the next section.

8. MVM and nonholonomic generalized Voronets–Chaplygin systems. Proofs of Theorem 4 and Proposition 5 and 6.

It was pointed out by Chaplygin [6] that in many conservative nonholonomic systems the generalized coordinates

$$
(\mathbf{x},\mathbf{y}) := (x_1,\ldots,x_{s_1},y_1,\ldots,y_{s_2}), \quad s_1+s_2=N,
$$

can be chosen in such a way that the Lagrangian function and the constraints take the simplest form. In particular Voronets in [53] studied the constrained Lagrangian systems with Lagrangian $\tilde{L}=\tilde{L}(\mathbf{x},\mathbf{y},\dot{\mathbf{x}},\dot{\mathbf{y}})$ and constraints (22). This systems is called the *Voronets mechanical systems.*

We shall apply equations (12) to study the generalization of the Voronets systems, which we define now.

The constrained Lagrangian mechanical systems

$$
\Big(\mathrm{Q},\quad \tilde{L}(t,\mathbf{x},\mathbf{y},\dot{\mathbf{x}},\dot{\mathbf{y}}),\quad \{\dot{x}_\alpha-\Phi_\alpha(t,\mathbf{x},\mathbf{y},\dot{\mathbf{y}})=0,\quad \alpha=1,\ldots,s_1\}\Big),
\tag{65}
$$

is called the *generalized Voronets mechanical systems.*

An example of generalized Voronets systems is Appell-Hamel systems analyzed in the previous subsection.

Corollary 22. *Every Nonholonomic constrained Lagrangian mechanical systems locally is a generalized Voronets mechanical systems.*

Proof. Indeed, the independent constraints can be locally represented in the form (26). Thus by introducing the coordinates

$$x_j = x_j, \quad \text{for} \quad j = 1, \ldots, M, \quad x_{M+k} = y_k, \quad \text{for} \quad k = 1, \ldots, N - M,$$

then we have that any constrained Lagrangian mechanical systems is locally a generalized Voronets mechanical systems. □

Proof of Theorem 4. For simplicity we shall study only scleronomic generalized Voronets systems.

To determine equations (12) we suppose that

$$L_\alpha = \dot{x}_\alpha - \Phi_\alpha(\mathbf{x}, \mathbf{y}, \dot{\mathbf{y}}) = 0, \quad \alpha = 1, \ldots, s_1. \tag{66}$$

It is evident from the form of the constraint equations that the virtual variations $\delta\mathbf{y}$, are independent by definition. The remaining variations $\delta\mathbf{x}$, can be expressed in terms of them by the relations (Chetaev's conditions)

$$\delta x_\alpha - \sum_{j=1}^{s_2} \frac{\partial L_\alpha}{\partial \dot{y}_j} \delta y_j = 0, \quad \alpha = 1, \ldots, s_1. \tag{67}$$

We shall apply Theorem 2. To construct the matrix W_1. We first determine $L_{s_1+1}, \ldots, L_{s_1+s_2} = L_N$ as follow:

$$L_{s_1+j} = \dot{y}_j, \quad j = 1, \ldots, s_2.$$

Hence, the Lagrangian (4) becomes

$$L = L_0 - \sum_{j=1}^{s_1} \lambda_j \left(\dot{x}_\alpha - \Phi_\alpha(x, y, \dot{y}) \right) - \sum_{j=s_1+1}^{N} \lambda_j^0 \dot{y}_j \simeq L_0 - \sum_{j=1}^{s_1} \lambda_j \left(\dot{x}_\alpha - \Phi_\alpha(x, y, \dot{y}) \right). \tag{68}$$

The matrices W_1 and W_1^{-1} are

$$\begin{pmatrix} 1 & \ldots & 0 & 0 & a_{11} & \ldots & a_{s_2 1} \\ 0 & \ldots & 0 & 0 & a_{12} & \ldots & a_{s_2 2} \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & \ldots & \vdots & 1 & a_{1 s_1} & \ldots & a_{s_2 s_1} \\ 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \ldots & 0 & 0 & -a_{11} & \ldots & -a_{s_2 1} \\ 0 & \ldots & 0 & 0 & -a_{12} & \ldots & -a_{s_2 2} \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & \ldots & \vdots & 1 & a_{1 s_1} & \ldots & -a_{s_2 s_1} \\ 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}, \tag{69}$$

respectively, where $a_{\alpha j} = \dfrac{\partial L_\alpha}{\partial \dot{y}_j}$, and the matrices Ω_1 and A are

$$A = \Omega_1 := \begin{pmatrix} E_1(L_1) & \ldots & E_{s_1}(L_1) & E_{s_1+1}(L_1) & \ldots & E_N(L_1) \\ \vdots & \ldots & \vdots & \ldots & \ldots & \vdots \\ E_1(L_{s_1}) & \ldots & E_{s_1}(L_{s_1}) & E_{s_1+1}(L_{s_1}) & \ldots & E_N(L_{s_1}) \\ 0 & \ldots & 0 & \ldots & 0 & 0 \\ \vdots & \ldots & \vdots & \ldots & \ldots & \vdots \\ 0 & \ldots & 0 & \ldots & 0 & 0 \end{pmatrix}, \tag{70}$$

respectively. Consequently the differential equations (12) take the form (18).

The transpositional relations (13) in view of (67) take the form (21). As we can observe from (21) the independent virtual variations $\delta\mathbf{y}$ for the systems with the constraints (66) produce the zero transpositional relations. The fact that the transpositional relations are zero follows automatically and it is not necessary to assume it a priori, and it is valid in general for the constraints which are nonlinear in the velocity variables.

We observe that the relations (34) in this case take the form

$$\delta\frac{dx_\alpha}{dt} - \frac{d}{dt}\delta\, x_\alpha + \sum_{m=1}^{s_2}\frac{\partial L_\alpha}{\partial \dot y_m}\left(\delta\frac{dy_m}{dt} - \frac{d}{dt}\delta\, y_m\right) = \sum_{k=1}^{s_1} E_k(L_\alpha)\delta x_k + \sum_{k=1}^{s_2} E_k(L_\alpha)\delta y_k.$$

for $\alpha = 1,\ldots,s_1$. Clearly from (21) these relations hold identically.

From differential equations (18), eliminating the Lagrangian multipliers we obtain equations (19). After some computations we obtain

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot y_k} - \sum_{\alpha=1}^{s_1}\frac{\partial L_\alpha}{\partial \dot y_k}\frac{\partial L_0}{\partial \dot x_\alpha}\right) - \left(\frac{\partial L_0}{\partial y_k} - \sum_{\alpha=1}^{s_1}\frac{\partial L_\alpha}{\partial y_k}\frac{\partial L_0}{\partial \dot x_\alpha}\right) + \\ \sum_{\alpha=1}^{s_1}\left(\frac{\partial L_0}{\partial x_\alpha} - \sum_{\beta=1}^{s_1}\frac{\partial L_\beta}{\partial x_\alpha}\frac{\partial L_0}{\partial \dot x_\beta}\right)\frac{\partial L_\alpha}{\partial \dot y_k} = 0,\end{aligned} \tag{71}$$

for $k = 1,\ldots,s_2$.

By introducing the function $\Theta = L_0|_{L_1=\ldots=L_{s_1}=0}$, equations (71) can be written as

$$\frac{d}{dt}\left(\frac{\partial \Theta}{\partial \dot y_k}\right) - \left(\frac{\partial \Theta}{\partial y_k}\right) + \sum_{\alpha=1}^{s_1}\left(\frac{\partial \Theta}{\partial x_\alpha}\right)\frac{\partial L_\alpha}{\partial \dot y_k} = 0, \tag{72}$$

for $k = 1,\ldots,s_2$. Here we consider that $\frac{d}{dt}\left(\frac{\partial L_\beta}{\partial \dot x_\alpha}\right) = 0$, for $\alpha,\ \beta = 1,\ldots,s_1$.

We shall study the case when equations (72) hold identically, i.e. $\Theta = 0$. We choose

$$L_0 = \tilde L\,(\mathbf{x},\mathbf{y},\dot{\mathbf{x}},\dot{\mathbf{y}}) - \tilde L\,(\mathbf{x},\mathbf{y},\Phi,\dot{\mathbf{y}}) = \tilde L - L^*, \tag{73}$$

being $\tilde L$ the Lagrangian of (65). Now we establish the relations between equations (18) and the classical Voronets differential equations with the Lagrangian function $L^* = \tilde L\Big|_{L_1=\ldots=L_{s_1}=0}$. The functions $\tilde L$ and L^* are determined in such a way that equations (19) take place in view of the equalities

$$E_k\tilde L = \sum_{\alpha=1}^{s_1} E_\alpha \tilde L\frac{\partial L_\alpha}{\partial \dot y_k},$$

and

$$E_k L^* = -\sum_{\alpha=1}^{s_1}\left(-E_k(L_\alpha\,) + \sum_{\nu=1}^{s_1} E_\nu\,(L_\alpha)\frac{\partial L_\nu}{\partial \dot y_k}\right)\frac{\partial \tilde L}{\partial \dot x_\alpha} - \sum_{\nu=1}^{s_1} E_\nu(L^*)\frac{\partial L_\nu}{\partial \dot y_k},$$

for $k = 1,\ldots,s_2$, which in view of equalities $\frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot x_\nu}\right) = 0$ for $\nu = 1,\ldots,s_1$, take the form (20). □

Proof of Proposition 5. Equations (20) describe the motion of the constrained generalized Voronets systems with Lagrangian L^* and constraints (66). The classical Voronets equations for scleronomic systems are easy to obtain from (20) with $\Phi_\alpha = \sum_{k=1}^{s_2} a_{\alpha\, k}(\mathbf{x},\mathbf{y})\dot y_k$. □

Finally by considering Corollary 22 we get that differential equations (20) describe locally the motions of any constrained Lagragian systems.

8.1. **Generalized Chaplygin systems.** The constrained Lagrangian mechanical systems with Lagrangian $\tilde{L} = \tilde{L}(\mathbf{y}, \dot{\mathbf{x}}, \dot{\mathbf{y}})$, and constraints (24) is called the *Chaplygin mechanical systems.*

The constrained Lagrangian systems

$$\left(\mathrm{Q}, \quad \tilde{L}(\mathbf{y}, \dot{\mathbf{x}}, \dot{\mathbf{y}}), \qquad \{\dot{x}_\alpha - \Phi_\alpha(\mathbf{y}, \dot{\mathbf{y}}) = 0, \quad \alpha = 1, \ldots, s_1\}\right)$$

is called the *generalized Chaplygin systems.* Note that now the Lagrangian do not depend on $\mathbf{x}$ and the constraints do not depend on $\mathbf{x}$ and $\dot{\mathbf{x}}$. So, the generalized Chaplygin systems are a particular case of the generalized Voronets system.

Proof of Proposition 6. To determine the differential equations which describe the behavior of the generalized Chaplygin systems we apply Theorem 2, with

$$L_0 = L_0(\mathbf{y}, \dot{\mathbf{x}}, \dot{\mathbf{y}}), \quad L_\alpha = \dot{x}_\alpha - \Phi_\alpha(\mathbf{y}\dot{\mathbf{y}}), \quad L_\beta = \dot{y}_\beta,$$

for $\alpha = 1, \ldots, s_1$ and $\beta = s_1 + 1, \ldots, s_2$ and consequently the matrix W_1 is given by the formula (69) and

$$A = \Omega_1 := \begin{pmatrix} E_1(L_1) & \ldots & E_{s_1}(L_1) & E_{s_1+1}(L_1) & \ldots & E_N(L_1) \\ \vdots & \ldots & \vdots & \ldots & \ldots & \vdots \\ E_1(L_{s_1}) & \ldots & E_{s_1}(L_{s_1}) & E_{s_1+1}(L_{s_1}) & \ldots & E_N(L_{s_1}) \\ 0 & \ldots & 0 & \ldots & 0 & 0 \\ \vdots & \ldots & \vdots & \ldots & \ldots & \vdots \\ 0 & \ldots & 0 & \ldots & 0 & 0 \end{pmatrix}$$

(74)

$$= \begin{pmatrix} 0 & \ldots & 0 & E_{s_1+1}(L_1) & \ldots & E_N(L_1) \\ \vdots & \ldots & \vdots & \ldots & \ldots & \vdots \\ 0 & \ldots & 0 & E_{s_1+1}(L_{s_1}) & \ldots & E_N(L_{s_1}) \\ 0 & \ldots & 0 & \ldots & 0 & 0 \\ \vdots & \ldots & \vdots & \ldots & \ldots & \vdots \\ 0 & \ldots & 0 & \ldots & 0 & 0 \end{pmatrix},$$

Therefore the differential equations (12) take the form

$$\begin{aligned} E_j L_0 = & \; \frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot{x}_\alpha}\right) = \dot{\lambda}_j \quad j = 1, \ldots, s_1, \\ E_k L_0 = & \sum_{\alpha=1}^{s_1}\left(E_k L_\alpha \frac{\partial L_0}{\partial \dot{x}_\alpha} + \dot{\lambda}_\alpha \frac{\partial L_\alpha}{\partial \dot{y}_k}\right) \quad k = 1, \ldots, s_2. \end{aligned} \tag{75}$$

The transpositional relations are

$$\begin{aligned} \delta \frac{dx_\alpha}{dt} - \frac{d}{dt}\delta\, x_\alpha &= \sum_{k=1}^{s_2} E_k(L_\alpha)\delta y_k, \quad \alpha = 1, \ldots, s_1, \\ \delta \frac{dy_m}{dt} - \frac{d}{dt}\delta\, y_m &= 0, \quad m = 1, \ldots, s_2. \end{aligned} \tag{76}$$

By excluding the Lagrangian multipliers from (75) we obtain the equations

$$E_k L_0 = \sum_{\alpha=1}^{s_1}\left(E_k(L_\alpha)\frac{\partial L_0}{\partial \dot{x}_\alpha} + \frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot{x}_\alpha}\right)\frac{\partial L_\alpha}{\partial \dot{y}_k}\right),$$

for $k = 1, \ldots, s_2$.

In this case equations (73) take the form

$$\frac{d}{dt}\left(\frac{\partial\,\Theta}{\partial \dot{y}_k}\right) - \left(\frac{\partial\,\Theta}{\partial y_k}\right) = 0, \tag{77}$$

Analogously to the Voronets case we study the subcase when $\Theta = 0$. We choose $L_0 = \tilde{L}(\mathbf{y}, \dot{\mathbf{x}}, \dot{\mathbf{y}}) - \tilde{L}(\mathbf{y}, \Phi, \dot{\mathbf{y}}) := \tilde{L} - L^*$. We assume that the functions $\tilde{L}$ and L^* are such that

$$E_k L^* = -\sum_{\alpha=1}^{s_1} E_k(L_\alpha)\frac{\partial \tilde{L}}{\partial \dot{x}_\alpha}\Psi_\alpha, \tag{78}$$

where $\Psi_\alpha = \left.\dfrac{\partial \tilde{L}}{\partial \dot{x}_\alpha}\right|_{L_1=\ldots=L_{s_1}=0}$ and

$$E_k(\tilde{L}) = \sum_{\alpha=1}^{s_1} \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{x}_\alpha}\right)\frac{\partial L_\alpha}{\partial \dot{y}_k},$$

for $k = 1, \ldots, s_2$.

By inserting $\dot{x}_j = \displaystyle\sum_{k=1}^{s_2} a_{j\,k}(\mathbf{y})\dot{y}_k, \quad j = 1, \ldots, s_1$, into equations (78) we obtain system (25). Consequently system (78) is an extension of the classical Chaplygin equations when the constraints are nonlinear. □

For the generalized Chaplygin systems the Lagrangian L takes the form

$$L = \tilde{L}(\mathbf{y}, \dot{\mathbf{x}}, \dot{\mathbf{y}}) - \tilde{L}(\mathbf{y}, \Phi, \dot{\mathbf{y}}) - \sum_{j=1}^{s_1}\left(\frac{\partial L^*}{\partial \dot{x}_j} + C_j\right)(\dot{x}_j - \Phi_j(\mathbf{y}, \dot{\mathbf{y}})) - \sum_{j=}^{s_2}\lambda_j^0 \dot{y}_j, \tag{79}$$

for $j = 1, \ldots, s_1$ where the constants C_j for $j = 1, \ldots, s_1$ are arbitrary. Indeed, from (75) follows that

$$\lambda_j = \frac{\partial L_0}{\partial \dot{x}_j} + C_j = \frac{\partial L^*}{\partial \dot{x}_j} + C_j.$$

By inserting in (4) $L_0 = \tilde{L} - L^*$ and λ_j for $j = 1, \ldots, s_1$ we obtain function L of (79).

We note that Vorones and Chaplygin equations with nonlinear constraints in the velocity was also obtained by Rumiansev and Sumbatov (see [44, 47]).

Example 4. We shall illustrate the above results in the following example.

In the Appel's and Hamel's investigations the following mechanical system was analyzed. A weight of mass m hangs on a thread which passes around the pulleys and is wound round the drum of radius a. The drum is fixed to a wheel of radius b which rolls without sliding on a horizontal plane, touching it at the point B with the coordinates $(x_B,\, y_B)$. The legs of the frame that support the pulleys and keep the plane of the wheel vertical slide on the horizontal plane without friction. Let θ be the angle between the plane of the wheel and the Ox axis; φ the angle of the rotation of the wheel in its own plane; and (x, y, z) the coordinates of the mass m. Clearly,

$$\dot{z} = b\dot{\varphi}, \quad b > 0.$$

The coordinates of the point B and the coordinates of the mass are related as follows (see page 223 of [35] for a picture)

$$x = x_B + \rho\cos\theta, \quad y = y_B + \rho\sin\theta.$$

The condition of rolling without sliding leads to the equations of nonholonomic constraints:

$$\dot{x}_B = a\cos\theta\dot{\varphi}, \quad \dot{y}_B = a\sin\theta\dot{\varphi} \quad b > 0.$$

We observe that the constraints $\dot{z} = b\dot{\varphi}$ admits the representation

$$\dot{z} = \frac{b}{a}\sqrt{\dot{x}^2 + \dot{y}^2 - \rho^2\dot{\theta}^2}.$$

Denoting by m_1, A and C the mass and the moments of inertia of the wheel and neglecting the mass of the frame, we obtain the following expression for the Lagrangian function

$$\tilde{L} = \frac{m+m_1}{2}\left(\dot{x}^2+\dot{y}^2\right) + \frac{m}{2}\dot{z}^2 + m_1\rho\dot{\theta}\left(\sin\theta\dot{x} - \cos\theta\dot{y}\right) + \frac{A+m_1\rho^2}{2}\dot{\theta}^2 + \frac{C}{2}\dot{\varphi}^2 - mgz.$$

The equations of the constraints are

$$\dot{x} - a\cos\theta\dot{\varphi} + \rho\sin\theta\dot{\theta} = 0, \quad \dot{y} - a\sin\theta\dot{\varphi} - \rho\cos\theta\dot{\theta} = 0, \quad \dot{z} - b\dot{\varphi} = 0,$$

Now we shall study the motion of this constrained Lagrangian in the coordinates

$$x_1 = x,\, x_2 = y,\, x_3 = \dot{\varphi}, y_1 = \theta,\, y_2 = z.$$

i.e., we shall study the nonholonomic system with Lagrangian

$$\begin{aligned} \tilde{L} = & \ \tilde{L}\left(y_1,\, y_2,\, \dot{x}_1,\, \dot{x}_2,\, \dot{x}_3,\, \dot{y}_1,\, \dot{y}_2\right) \\ = & \ \frac{m+m_1}{2}\left(\dot{x}_1^2+\dot{x}_2^2\right) + \frac{C}{2}\dot{x}_3^2 + \frac{J}{2}\dot{y}_1^2 + \frac{m}{2}\dot{y}_2^2 + m_1\rho\dot{y}_1\left(\sin\, y_1\dot{x}_1 - \cos\, y_1\dot{x}_2\right) - \frac{mg}{b}y_2, \end{aligned}$$

and with the constraints

$$\begin{aligned} l_1 = & \ \dot{x}_1 - \frac{a}{b}\,\dot{y}_2\cos y_1 - \rho\dot{y}_1\sin\, y_1 = 0, \\ l_2 = & \ \dot{x}_2 - \frac{a}{b}\dot{y}_2\sin y_1 + \rho\dot{y}_1\cos\, y_1 = 0, \\ l_3 = & \ \dot{x}_3 - \frac{1}{b}\dot{y}_2 = 0. \end{aligned}$$

Thus we have a classical Chaplygin system. To determine differential equations (78) and the transpositional relations (76) we define the functions:

$$\begin{aligned} L^* = - & \ \tilde{L}|_{l_1=l_2=l_3=0} = \frac{m(a^2+b^2)m + a^2m_1 + C}{2b^2}\dot{y}_2^2 + \frac{m\rho^2+J}{2}\dot{y}_1^2 - \frac{mg}{b}y_2, \\ L_1 = & \ l_1, \quad L_2 = l_2, \quad L_3 = l_3, \quad L_4 = \dot{y}_1, \quad L_5 = \dot{y}_2. \end{aligned}$$

After some computations we obtain that the matrix A (see formulae (74)) in this case becomes

$$A = \begin{pmatrix} 0 & 0 & 0 & -\frac{a}{b}\dot{y}_2\sin\, y_1 & \frac{a}{b}\dot{y}_1\sin\, y_1 \\ 0 & 0 & 0 & \frac{a}{b}\dot{y}_2\cos\, y_1 & -\frac{a}{b}\dot{y}_1\cos\, y_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

thus differential equations (78) take the form

$$\begin{aligned} & \left(m\rho^2+J\right)\ddot{y}_1 + \frac{a\rho m}{b}\dot{y}_1\dot{y}_2 = 0, \\ & \left((m+m_1)a^2+mb^2\right)\ddot{y}_2 - mab\rho\dot{y}_1^2 = -mgb. \end{aligned}$$

Assuming that $(m+2m_1)\rho^2 + J \neq 0$ and by considering the existence of the first integrals

$$\begin{aligned} C_2 = & \ \dot{y}_1\exp\left(-\frac{a\varrho m y_2}{b\left(m\rho^2+J\right)}\right), \\ h = & \ \frac{\left((m+m_1)a^2+mb^2\right)}{2}\dot{y}_2^2 + \frac{b^2\left(m\rho^2+J\right)}{2}\dot{y}_1^2 + mgby_2, \end{aligned}$$

after the integration of these first integrals we obtain

$$\int \frac{\sqrt{(m+m_1)a^2+mb^2}dy_2}{\sqrt{2h-2mgby_2-b^2\,(m\rho^2+J)C_3\exp\left(\dfrac{a\rho\, my_2}{bm\rho^2+J}\right)}}=t+C_1,$$

$$y_1(t)=C_3+C_2\int\exp\left(2\frac{a\rho\, my_2(t)}{bm\rho^2+J}\right)dt.$$

Consequently, if $\rho=0$ then

$$y_1=C_3+C_2t,\qquad \int\frac{\sqrt{(m+m_1)a^2+mb^2}dy_2}{\sqrt{2h-2mgby_2-JC_3}}=t+C_1.$$

Hamel in [15] neglect the mass of the wheel ($m_1=J=C=0$). Under these conditions the previous equations become

$$\begin{aligned}&\rho^2\ddot{y}_1+\frac{a\rho}{b}\dot{y}_1\dot{y}_2=0,\\&(a^2+b^2)\ddot{y}_2-ab\rho\dot{y}_1^2=-gb\end{aligned}$$

Appell and Hamel obtained the example of nonholonomic system with nonlinear constraints by means of the passage to the limit $\rho\to0$. However, as a result of this limiting process, the order of the system of differential equations is reduced, i.e., they become degenerate. In [35] the authors study the motion of the nondegenerate system for $\rho>0$ and $\rho<0$. From these studies it follows that the motion of the nondegenerate system ($\rho\neq0$) and degenerate system ($\rho\to0$) differ essentially. Thus the Appell-Hamel example with nonlinear constraints is incorrect.

The transpositional relations (76) become

$$\begin{aligned}\delta\frac{dx_1}{dt}-\frac{d\delta\, x_1}{dt}&=\frac{a}{b}\sin\, y_1\left(\frac{dy_1}{dt}\delta y_2-\frac{dy_2}{dt}\delta y_1\right),\\ \delta\frac{dx_2}{dt}-\frac{d\delta\, x_2}{dt}&=\frac{a}{b}\cos\, y_1\left((\frac{dy_1}{dt}\delta y_2-\frac{dy_2}{dt}\delta y_1\right),\\ \delta\frac{dx_3}{dt}-\frac{d\delta\, x_3}{dt}&=0,\quad \delta\frac{dy_1}{dt}-\frac{d\delta\, y_1}{dt}=0,\quad \delta\frac{dy_2}{dt}-\frac{d\delta\, y_2}{dt}=0.\end{aligned}$$

Clearly these relations are independent of ϱ, A, C and m_1.

9. Consequences of Theorems 2 and 3 and the proof of Corollary 7.

We observe the following important aspects from Theorems 2 and 3.

(I) Conjecture 8 is supported by the following facts. (a) As a general rule the constraints studied in classical mechanics are linear in the velocities. However Appell and Hamel in 1911, considered an artificial example with a constraint nonlinear in the velocity . As it follows from [35] (see example 2) this constraint does not exist in the Newtonian mechanics.

(b) The idea developed for some authors (see for instance [4]) to construct a theory in Newtonian mechanics, by allowing that the field of force depends on the acceleration, i.e. function of $\ddot{\mathbf{x}}$ as well as of the position $\mathbf{x}$, velocity $\dot{\mathbf{x}}$, and the time t is inconsistent with one of the fundamental postulates of the Newtonian mechanics: when two forces act simultaneously on a particle the effect is the same as that of a single force equal to the resultant of both forces (for more details see [38] pages 11–12). Consequently the forces depending on the acceleration are not admissible in Newtonian dynamics. This does not preclude their appearance in electrodynamics, where this postulate does not hold.

(c) Let T be the kinetic energy of the constrained Lagrangian systems. We consider the generalization of the Newton law: *the acceleration* (*see* [46, 37])

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\boldsymbol{x}}} - \frac{\partial T}{\partial \boldsymbol{x}}$$

is equal to the force $\mathbf{F}$. Then in the differential equations (12) with $L_0 = T$ we obtain that the field of force $\mathbf{F}$ generated by the constraints is

$$\mathbf{F} = \left(W_1^{-1}\Omega_1\right)^T \frac{\partial T}{\partial \dot{\mathbf{x}}} + W_1^T \frac{d}{dt}\lambda := \mathbf{F}_1 + \mathbf{F}_2.$$

The field of force $\mathbf{F}_2 = W_1^T \frac{d}{dt}\lambda = (F_{21}, \ldots, F_{2N})$ is called the *reaction force of the constraints.* What is the meaning of the force

$$\mathbf{F}_1 = \left(W_1^{-1}\Omega_1\right)^T \frac{\partial T}{\partial \dot{\mathbf{x}}}\,? \tag{80}$$

If the constraints are nonlinear in the velocity, then $\mathbf{F}_1$ depends on $\ddot{\mathbf{x}}$. Consequently in Newtonian mechanics does not exist a such field of force. Therefore, the existence of nonlinear constraints in the velocity and the meaning of force $\mathbf{F}_1$ must be sought outside of the Newtonian model.

For example, for the Appel-Hamel constrained Lagrangian systems studied in the previous subsection we have that

$$\mathbf{F}_1 = \left(-\frac{a^2\dot{x}}{\dot{x}^2+\dot{y}^2}(\dot{x}\ddot{y} - \dot{y}\ddot{x}),\, \frac{a^2\dot{y}}{\dot{x}^2+\dot{y}^2}(\dot{x}\ddot{y} - \dot{y}\ddot{x}),\, 0\right).$$

For the generalized Voronets systems and locally for any nonholonomic constrained Lagrangian systems from the equations (18) we obtain that the field of force $\mathbf{F}_1$ has the following components

(81)
$$\begin{aligned} F_{k\,1} = &\sum_{\alpha=1}^{s_1} E_k L_\alpha \frac{\partial L_0}{\partial \dot{x}_\alpha} \\ = &\sum_{j=1}^{N}\sum_{\alpha=1}^{s_1}\left(\frac{\partial^2 L_\alpha}{\partial \dot{x}_k \dot{x}_j}\frac{\partial L_0}{\partial \dot{x}_\alpha}\ddot{x}_j + \frac{\partial^2 L_\alpha}{\partial \dot{x}_k \partial x_j}\frac{\partial L_0}{\partial \dot{x}_\alpha}\dot{x}_j\right) + \sum_{\alpha=1}^{s_1}\frac{\partial^2 L_\alpha}{\partial \dot{x}_k \partial t}\frac{\partial L_0}{\partial \dot{x}_\alpha}, \quad \text{for} \quad k = 1\ldots, N, \quad s_1 = M. \end{aligned}$$

consequently such field of force does not exist in Newtonian mechanics if the constraints are nonlinear in the velocity.

(II) Equations (12) can be rewritten in the form

$$G\ddot{\mathbf{x}} + \mathbf{f}(t, \mathbf{x}, \dot{\mathbf{x}}) = 0, \tag{82}$$

where $G = G(t, \mathbf{x}, \dot{\mathbf{x}})$ is the matrix $(G_{j,k})$ given by

$$G_{jk} = \frac{\partial^2 L_0}{\partial \dot{x}_j \partial \dot{x}_k} - \sum_{n=1}^{N}\frac{\partial A_{nk}}{\partial \ddot{x}_j}\frac{\partial L_0}{\partial \dot{x}_n}, \quad j, k = 1, \ldots, N,$$

and $\mathbf{f}(t, \mathbf{x}, \dot{\mathbf{x}})$ is a convenient vector function. If $\det G \neq 0$ then equation (82) can be solved with respect to $\ddot{\mathbf{x}}$. This implies, in particular that the motion of the mechanical system at time $\bar{t} \in [t_0,\, t_1]$ is uniquely determined, i.e. the *principle of determinacy* (see for instance [2]) holds for the mechanical systems with equation of motion given in (12).

In particular for the Appel-Hamel constrained Lagrangian systems we have (see formula (48)) that

$$\mathbf{x} = (x, y, z)^T, \quad \mathbf{f} = \left(\frac{a\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} \dot{\lambda},\ \frac{a\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \dot{\lambda},\ g - \dot{\lambda} \right)^T$$

$$G = \begin{pmatrix} 1 + \dfrac{a^2\dot{y}^2}{\dot{x}^2+\dot{y}^2} & -\dfrac{a^2\dot{x}\dot{y}}{\dot{x}^2+\dot{y}^2} & 0 \\ -\dfrac{a^2\dot{x}\dot{y}}{\dot{x}^2+\dot{y}^2} & 1 + \dfrac{a^2\dot{x}^2}{\dot{x}^2+\dot{y}^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad |G| = 1 + a^2.$$

So in the Appel–Hamel system the principle of determinacy holds.

(III)

Proof of Corollary 7. From Theorems 2 and 3 (see formulas (13) and (17)) and from all examples which we gave in the previous sections we see that are examples with zero transpositional relations and examples where all they are not zero. By contrasting the MVM with the Lagrangian mechanics we obtain that for the unconstrained Lagrangian systems the transpositional relations are always zero. Thus we have the proof of the corollary. □

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