This is a preprint of: "Hamiltonian linear type centers of linear plus cubic homogeneous polynomial vector fields", Ilker Colak, Jaume Llibre, Clàudia Valls, *J. Differential Equations*, vol. 257, 1623–1661, 2014.
DOI: [10.1016/j.jde.2014.05.024]

HAMILTONIAN NON-DEGENERATE CENTERS OF LINEAR PLUS CUBIC HOMOGENEOUS POLYNOMIAL VECTOR FIELDS

ILKER E. COLAK, JAUME LLIBRE, AND CLAUDIA VALLS

Abstract. We provide normal forms and the global phase portraits in the Poincaré disk for all the Hamiltonian non-degenerate centers of linear plus cubic homogeneous planar polynomial vector fields.

The corresponding author is Jaume Llibre
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona, Catalonia, Spain

Phone +34 935811303
Fax +34 935812790

jllibre@mat.uab.cat

2010 *Mathematics Subject Classification.* Primary: 34C07, 34C08.
Key words and phrases. Hamiltonian non-degenerate center, cubic polynomial, vector fields, phase portrait.

1. Introduction and statement of the main results

In the qualitative theory of real planar polynomial differential systems two of the main problems are the determination of limit cycles and the center-focus problem, i.e. to distinguish when a singular point is either a focus or a center. The notion of *center* goes back to Poincaré in [18]. He defined a center for a vector field on the real plane as a singular point having a neighborhood filled of periodic orbits with the exception of the singular point.

The classification of the centers of polynomial differential systems started with the quadratic ones with the works of Dulac [9], Kapteyn [12, 13], Bautin [2], Zoladek [23]. See Schlomiuk [20] for an update on the quadratic centers. There are many partial results for the centers of polynomial differential systems of degree larger than 2. For instance the centers for cubic polynomial differential systems of the form linear with homogeneous nonlinearities of degree 3 were classified by Malkin [15], and by Vulpe and Sibiirski [22]. For polynomial differential systems of the form linear with homogeneous nonlinearities of degree greater than 3 the centers are not classified, but there are partial results for degree 4 and 5, see for instance Chavarriga and Giné [3, 4]. On the other hand there is a long way to do for obtaining the complete classification of the centers for all polynomial differential systems of degree 3. Some interesting results on some subclasses of cubic systems are those of Rousseau and Schlomiuk [21], and the ones of Zoladek [24, 25].

If an analytic system has a center, then after an affine change of variables and a rescaling of the time variable, it can be written in one of the following three forms:

$$\dot{x} = -y + P(x, y), \quad \dot{y} = x + Q(x, y),$$

called a *non-degenerate center*;

$$\dot{x} = y + P(x, y), \quad \dot{y} = Q(x, y),$$

called a *nilpotent center*;

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

called a *degenerate center*, where $P(x, y)$ and $Q(x, y)$ are real analytic functions without constant and linear terms, defined in a neighborhood of the origin. There is an algorithm for the characterization of non-degenerate centers due to Poincaré [19] and Lyapunov [14], see also Chazy [6] and Moussu [16]. An algorithm for the characterization of the nilpotent and some class of degenerate centers is done in the works of Chavarriga *et al.* [5], Giacomini *et al.* [11], and Cima and Llibre [8].

In this work we classify the global phase portraits of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a non-degenerate center at the origin. To do this we will use the Poincaré compactification of polynomial vector fields, see section 3. We say that two vector fields on the Poincaré disk are *topologically equivalent* if there exists a homeomorphism from one onto the other which sends orbits to orbits preserving or reversing the direction of the flow. Our main result is the following one.

Theorem 1. *Any Hamiltonian non-degenerate center at the origin of a linear plus cubic homogeneous polynomial vector field can be written as one of the following six classes after a linear change of variables and a rescaling of its independent variable:*

$$
\begin{array}{lll}
(I) & \dot{x} = ax + by, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3, \\
(II) & \dot{x} = ax + by - x^3, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + 3x^2y, \\
(III) & \dot{x} = ax + by - 3x^2y + y^3, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + 3xy^2, \\
(IV) & \dot{x} = ax + by - 3x^2y - y^3, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + 3xy^2, \\
(V) & \dot{x} = ax + by - 3\mu x^2y + y^3, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3 + 3\mu xy^2, \\
(VI) & \dot{x} = ax + by - 3\mu x^2y - y^3, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3 + 3\mu xy^2,
\end{array}
$$

where $a, b, \beta, \mu \in \mathbb{R}$ with $b \neq 0$ and $\beta > 0$. Moreover, the global phase portraits of these systems are topologically equivalent to one of the 23 phase portraits of Figure 1.

We note that in the above six systems $(I)-(VI)$ a rescaling of both the dependent and the independent variables allows to assume $a = \pm 1$ whenever $a \neq 0$. However we do not use this simplification since it does not help us in our computations.

The classification done in Theorem 1 will allow to start the study of how many limit cycles can bifurcate from the periodic orbits of the Hamiltonian non-degenerate centers with only linear and cubic terms when they are perturbed inside the class of all cubic polynomial differential systems. This last study was made for the quadratic polynomial differential systems, see the paper [7] and the references quoted there.

2. Classification

Doing a linear change of variables and a rescaling of the independent variable, cubic homogeneous systems can be classified into the following ten

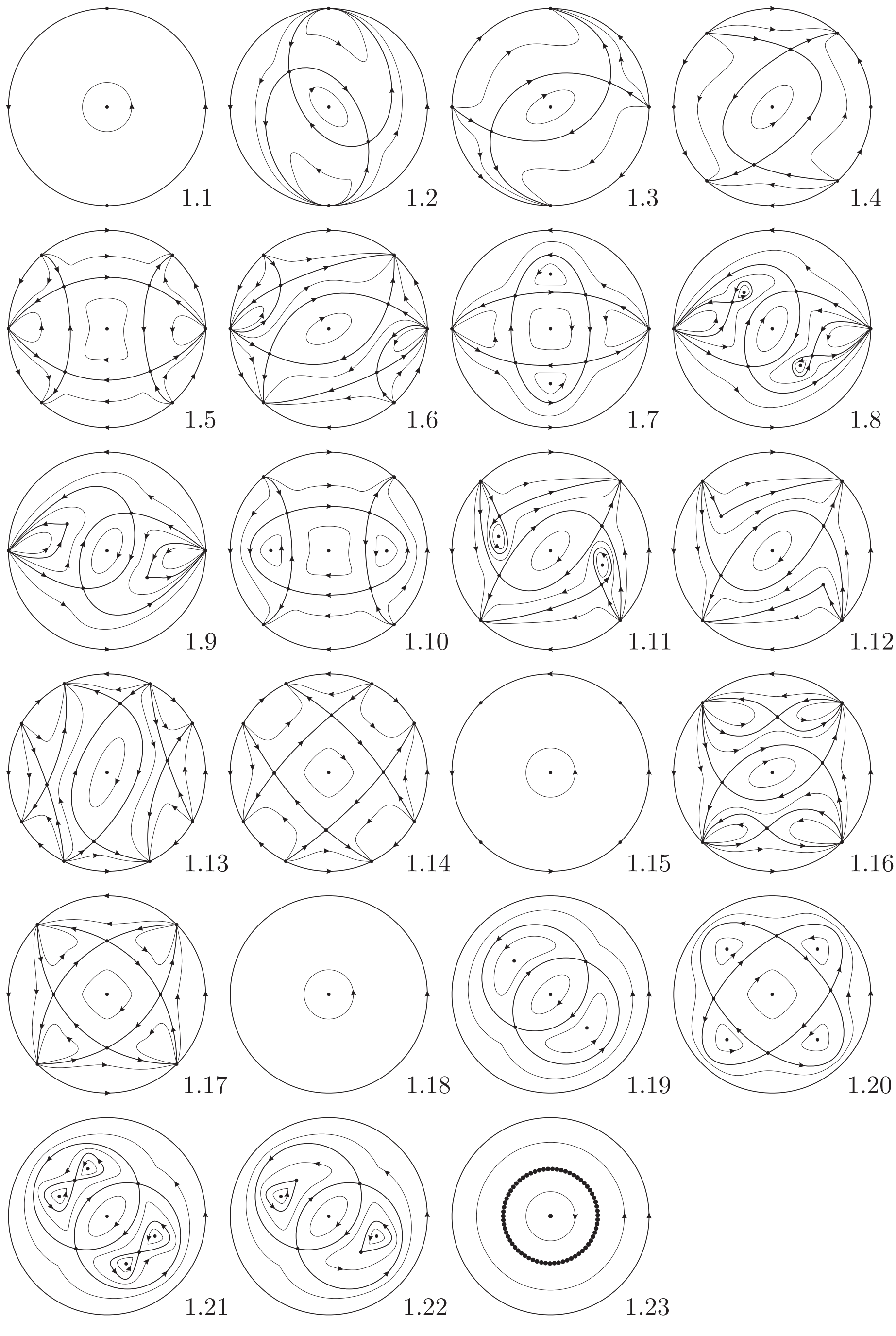

FIGURE 1. Global phase portraits of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a non-degenerate center at the origin. The separatrices are in bold.

classes, see [8]:

$$(i)\qquad \begin{aligned}\dot{x} &= x(p_1x^2+p_2xy+p_3y^2),\\ \dot{y} &= y(p_1x^2+p_2xy+p_3y^2),\end{aligned}$$

$$(ii)\qquad \begin{aligned}\dot{x} &= p_1x^3+p_2x^2y+p_3xy^2,\\ \dot{y} &= \alpha x^3+p_1x^2y+p_2xy^2+p_3y^3,\end{aligned}$$

$$(iii)\qquad \begin{aligned}\dot{x} &= (p_1-1)x^3+p_2x^2y+p_3xy^2,\\ \dot{y} &= (p_1+3)x^2y+p_2xy^2+p_3y^3,\end{aligned}$$

$$(iv)\qquad \begin{aligned}\dot{x} &= p_1x^3+(p_2-3\alpha)x^2y+p_3xy^2,\\ \dot{y} &= p_1x^2y+(p_2+3\alpha)xy^2+p_3y^3,\end{aligned}$$

$$(v)\qquad \begin{aligned}\dot{x} &= p_1x^3+(p_2-\alpha)x^2y+p_3xy^2-\alpha y^3,\\ \dot{y} &= \alpha x^3+p_1x^2y+(p_2+\alpha)xy^2+p_3y^3,\end{aligned}$$

$$(vi)\qquad \begin{aligned}\dot{x} &= p_1x^3+(p_2-3\alpha)x^2y+p_3xy^2+y^3,\\ \dot{y} &= p_1x^2y+(p_2+3\alpha)xy^2+p_3y^3,\end{aligned}$$

$$(vii)\qquad \begin{aligned}\dot{x} &= p_1x^3+(p_2-3\alpha)x^2y+p_3xy^2-\alpha y^3,\\ \dot{y} &= p_1x^2y+(p_2+3\alpha)xy^2+p_3y^3,\end{aligned}$$

$$(viii)\qquad \begin{aligned}\dot{x} &= p_1x^3+(p_2-3\mu)x^2y+p_3xy^2+y^3,\\ \dot{y} &= x^3+p_1x^2y+(p_2+3\mu)xy^2+p_3y^3,\end{aligned}\qquad \mu\in\mathbb{R}.$$

$$(ix)\qquad \begin{aligned}\dot{x} &= p_1x^3+(p_2-3\alpha\mu)x^2y+p_3xy^2-\alpha y^3,\\ \dot{y} &= \alpha x^3+p_1x^2y+(p_2+3\alpha\mu)xy^2+p_3y^3,\end{aligned}\qquad \begin{aligned}&\mu>-1/3,\\ &\mu\neq 1/3,\end{aligned}$$

$$(x)\qquad \begin{aligned}\dot{x} &= p_1x^3+(p_2-3\mu)x^2y+p_3xy^2-y^3,\\ \dot{y} &= x^3+p_1x^2y+(p_2+3\mu)xy^2+p_3y^3,\end{aligned}\qquad \mu<-1/3,$$

where $\alpha=\pm1$. So for studying the cubic planar polynomial vector fields having only linear and cubic terms, it is sufficient to add to the above ten systems a linear part. The following propositions define the precise forms of the vector fields that we will study.

Proposition 2. *Let X be a cubic planar polynomial vector field having only linear and cubic terms, such that its cubic homogeneous part is given by one of the above ten forms $(i)-(x)$. Then X is Hamiltonian with a Hamiltonian polynomial of degree four if and only if $p_1=p_2=p_3=0$.*

Proof. We will give the proof only for system (x) since the other nine cases can be proved in the same way.

Let $X=(P(x,y),Q(x,y))$ be system (x) with some arbitrary linear part, that is

$$\begin{aligned}\dot{x} &= P(x,y)=ax+by+p_1x^3+(p_2-3\mu)x^2y+p_3xy^2-y^3,\\ \dot{y} &= Q(x,y)=cx+dy+x^3+p_1x^2y+(p_2+3\mu)xy^2+p_3y^3,\end{aligned}$$

where $a, b, c, d \in \mathbb{R}$. Let H be its Hamiltonian polynomial of degree 4. We have

$$H_x = -Q, \quad H_y = P,$$

where H_x denotes the partial derivative of H with respect to x. To find H, we first integrate H_y with respect to y and get

$$\begin{aligned} H(x,y) &= \int P(x,y)\ dy + f(x) \\ &= axy + \frac{b}{2}y^2 + p_1x^3y + \frac{p_2 - 3\mu}{2}x^2y^2 + \frac{p_3}{3}xy^3 - \frac{1}{4}y^4 + f(x), \end{aligned}$$

for some real polynomial f. Then the derivative of H with respect to x is

$$H_x(x,y) = ay + 3p_1x^2y + (p_2 - 3\mu)xy^2 + \frac{p_3}{3}y^3 + f'(x),$$

where $f'(x)$ is the first derivative of the polynomial f. Equating H_x to $-Q$ we obtain the three equations

$$3p_1 = -p_1, \quad p_2 - 3\mu = -p_2 - 3\mu, \quad \frac{p_3}{3} = -p_3,$$

which hold if and only if $p_1 = p_2 = p_3 = 0$. $\square$

Remark 3. We note that when the parameters p_1, p_2 and p_3 are all zero, system (i) is not cubic. For this reason, we will restrict our attention to systems $(ii) - (x)$.

Proposition 4. *Finite singular points of Hamiltonian planar polynomial vector fields are either centers, or have a finite union of an even number of hyperbolic sectors.*

Proof. For analytic planar differential systems, it is known that any singular point is either a center, a focus, or has a finite union of hyperbolic, elliptic and parabolic sectors, for details see [10]. Moreover, if the system is Hamiltonian, its flow preserves area, see [1]. So a singular point of a Hamiltonian system cannot be a focus or have elliptic and parabolic sectors. Finally, since the index of a singular point formed by hyperbolic sectors is $1 - h/2$, with h being the number of its hyperbolic sectors, it follows that h is even. For more details about the index, see [10]. $\square$

Proposition 5. *The linear part of each of the ten classes of Hamiltonian cubic planar polynomial vector fields having only linear and cubic homogeneous terms which have a non-degenerate center at the origin can be chosen as*

$$\begin{aligned} \dot{x} &= ax + by, \\ \dot{y} &= -\frac{a^2 + \beta^2}{b}x - ay, \end{aligned}$$

where $a, b, \beta \in \mathbb{R}$ such that $b \neq 0$ and $\beta > 0$.

Proof. We will again give the proof only for system (x) as the remaining cases can be proved in the same way.

Let X be system (x) plus a linear part and let it be Hamiltonian. Then, by Proposition 2, X is

$$\begin{aligned}\dot{x} &= ax + by - 3\mu x^2 y - y^3,\\ \dot{y} &= cx + dy + x^3 + 3\mu x y^2,\end{aligned}$$

for some real constants a, b, c, d. The eigenvalues of the linear part of system X at the origin are

$$\lambda_{1,2} = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2}.$$

In order to have a non-degenerate center at the origin, these eigenvalues must be $\pm\beta i$, for some $\beta > 0$, see [10]. So we have

$$\frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2} = \pm\beta i. \tag{1}$$

From (1) we get that $a + d = 0$, and hence we obtain

$$a^2 + bc = -\beta^2.$$

We see that $b \neq 0$, otherwise the left hand side would be non-negative. Then we can solve for c and get $c = -(a^2 + \beta^2)/b$.

It can be easily shown that with this choice of parameters, X is Hamiltonian with the Hamiltonian

$$H(x, y) = -\frac{1}{4}(x^4 + y^4) - \frac{3\mu}{2} x^2 y^2 + \frac{a^2 + \beta^2}{2b} x^2 + \frac{b}{2} y^2 + axy.$$

Since X is Hamiltonian, by Proposition 4, the origin cannot be a focus, and hence it is a center. $\square$

Remark 6. In all of the nine vector fields $(ii) - (x)$ we are going to study, we can assume $\alpha = 1$ because the Hamiltonian systems with $\alpha = -1$ can be obtained from those with $\alpha = 1$ simply by the linear transformation $x \mapsto -x$.

In light of the above classification, propositions and remarks, the nine systems that we are going to study become

$$
\begin{array}{lll}
(ii') & \dot{x} = ax + by, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3, \\
(iii') & \dot{x} = ax + by - x^3, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + 3x^2y, \\
(iv') & \dot{x} = ax + by - 3x^2y, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + 3xy^2, \\
(v') & \dot{x} = ax + by - x^2y - y^3, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3 + xy^2, \\
(vi') & \dot{x} = ax + by - 3x^2y + y^3, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + 3xy^2, \\
(vii') & \dot{x} = ax + by - 3x^2y - y^3, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + 3xy^2, \\
(viii') & \dot{x} = ax + by - 3\mu x^2y + y^3, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3 + 3\mu xy^2, \\
(ix') & \dot{x} = ax + by - 3\mu x^2y - y^3, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3 + 3\mu xy^2, \\
(x') & \dot{x} = ax + by - 3\mu x^2y - y^3, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3 + 3\mu xy^2,
\end{array}
$$

where $\mu < -1/3$ for system (x'), whereas $\mu > -1/3$ but different from $1/3$ for system (ix'). Hence these last two systems are the same except for the intervals of the parameter μ. When $\mu = 1/3$, system (x') (or (ix')) becomes system (v'), and in the following proposition we will show that when $\mu = -1/3$, system (x') becomes system (iv').

Proposition 7. *When $\mu = -1/3$ system (x') becomes system (iv') via a linear transformation.*

Proof. Consider system (x') with $\mu = -1/3$:

$$\dot{x} = ax + by + x^2y - y^3, \quad \dot{y} = -\frac{a^2+\beta^2}{b}x - ay + x^3 - xy^2.$$

We introduce the new variables (X, Y) obtained by the linear transformation

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (x-y)/\sqrt{2} \\ (x+y)/\sqrt{2} \end{pmatrix}.$$

Hence we have

$$x = (X+Y)/\sqrt{2}, \quad y = (Y-X)/\sqrt{2}.$$

Then we obtain

$$
\begin{aligned}
\dot{X} =& \frac{a^2-b^2+\beta^2}{2b}X + \frac{(a+b)^2+\beta^2}{2b}Y - 2X^2Y, \\
\dot{Y} =& -\frac{(a-b)^2+\beta^2}{2b}X - \frac{a^2-b^2+\beta^2}{2b}Y + 2XY^2.
\end{aligned}
$$

Finally, after a time rescale $dT = 2/3\ dt$, and defining

$$A = 3\frac{a^2-b^2+\beta^2}{4b}, \quad B = 3\frac{(a+b)^2+\beta^2}{4b}$$

we get the system

$$\dot{X} = AX + BY - 3X^2Y, \quad \dot{Y} = -\frac{A^2+\beta^2}{B}X - AY + 3XY^2,$$

which is exactly system (iv'). $\square$

In short, Hamiltonian planar polynomial vector fields having only linear and cubic terms which have a non-degenerate center at the origin can be classified into the six vector fields $(I)-(VI)$ given in Section 1.

Remark 8. Because the right hand sides of each of the vector fields $(I)-(VI)$ are odd functions, they are symmetric with respect to the origin.

3. Poincaré compactification

In this section we summarize the Poincaré compactification that we shall use for describing the global phase portraits of our Hamiltonian systems. For more details on the Poincaré compactification see Chapter 5 of [10].

Let $\mathbb{S}^2$ be the set of points $(s_1, s_2, s_3) \in \mathbb{R}^3 : s_1^2 + s_2^2 + s_3^2 = 1$. We will call this set the Poincaré sphere. Given a polynomial vector field

$$X = (\dot{x}, \dot{y}) = (P(x,y), Q(x,y))$$

in $\mathbb{R}^2$, it can be extended analytically to the Poincaré sphere by projecting each point $x \in \mathbb{R}^2 = (x_1, x_2, 1) \in \mathbb{R}^3$ onto the Poincaré sphere using a straight line through x and the origin of $\mathbb{R}^3$. This way we obtain two copies of X: one on the northern hemisphere $\{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_3 > 0\}$ and another on the southern hemisphere $\{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_3 < 0\}$. The equator $\mathbb{S}^1 = \{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_3 = 0\}$ corresponds to the infinity of $\mathbb{R}^2$. The local charts needed for doing the calculations on the Poincaré sphere are

$$U_i = \{\mathbf{s} \in \mathbb{S}^2 : s_i > 0\}, \quad V_i = \{\mathbf{s} \in \mathbb{S}^2 : s_i < 0\},$$

where $\mathbf{s} = (s_1, s_2, s_3)$, with the corresponding local maps

$$\varphi_i(s) : U_i \to \mathbb{R}^2, \quad \psi_i(s) : V_i \to \mathbb{R}^2,$$

such that $\varphi_i(s) = -\psi_i(s) = \left(\dfrac{s_m}{s_i}, \dfrac{s_n}{s_i}\right)$ for $m < n$ and $m, n \neq i$, for $i = 1, 2, 3$. The expression for the corresponding vector field on $\mathbb{S}^2$ in the local chart U_1 is given by

$$\dot{u} = v^d \left[-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right)\right], \quad \dot{v} = -v^{d+1}P\left(\frac{1}{v}, \frac{u}{v}\right); \tag{2}$$

the expression for U_2 is

$$\dot{u} = v^d \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right)\right], \quad \dot{v} = -v^{d+1}Q\left(\frac{u}{v}, \frac{1}{v}\right); \tag{3}$$

and the expression for U_3 is just

$$\dot{u} = P(u,v), \quad \dot{v} = Q(u,v), \tag{4}$$

where d is the degree of the vector field X. The expressions for the charts V_i are those for the charts U_i multiplied by $(-1)^{d-1}$, for $i = 1, 2, 3$. Hence, to study the vector field X, it is enough to study its Poincaré compactification restricted to the northern hemisphere plus $\mathbb{S}^1$, which we will call the Poincaré disk. To draw the phase portraits, we will consider the projection of the Poincaré disk on to $\mathbb{R}^2$ by $\pi(s_1, s_2, s_3) = (s_1, s_3)$.

Finite singular points of X are the singular points of its compactification which are in $\mathbb{S}^2 \setminus \mathbb{S}^1$, and they can be studied using U_3. Infinite singular points of X, on the other hand, are the singular points of the corresponding vector field on the Poincaré disk lying on $\mathbb{S}^1$. Since $s \in \mathbb{S}^1$ is an infinite singular point whenever $-s \in \mathbb{S}^1$ is, and the local behavior of one is that of the other multiplied by $(-1)^{d-1}$, to study the infinite singular points it suffices to only look at $U_1|_{v=0}$ and the origin of U_2.

The next theorem by Neumann [17] gives a characterization of two topologically equivalent vector fields in the Poincaré sphere.

Theorem 9 (Neumann's Theorem)**.** *Two continuous flows in $\mathbb{S}^2$ with isolated singular points are topologically equivalent if and only if their separatrix configurations are equivalent.*

This theorem implies that once the separatrices of a vector field in the Poincaré sphere are determined, the global phase portrait of that vector field is obtained up to topological equivalence.

4. Global phase portraits of system (I)

System (I)

$$\dot{x} = ax + by, \quad \dot{y} = -\frac{a^2+\beta^2}{b}x - ay + x^3,$$

with the Hamiltonian

$$H_1(x,y) = -\frac{1}{4}x^4 + \frac{a^2+\beta^2}{2b}x^2 + \frac{b}{2}y^2 + axy.$$

We first investigate the infinite singular points of this system. Using (2), we see that in the local chart U_1 system (I) becomes

$$\begin{aligned}\dot{u} &= -v^2\left(bu^2 + 2au + \frac{a^2+\beta^2}{b}\right) + 1,\\ \dot{v} &= -v^3\,(bu + a)\,.\end{aligned}$$

When $v = 0$, there are no singular points on U_1.

Next we will check whether the origin of the local chart U_2 is a singular point. In U_2 we use (3) to get

$$\begin{aligned}\dot{u} &= v^2\left(\frac{a^2+\beta^2}{b}u^2 + 2au + b\right) - u^4,\\ \dot{v} &= v^3\left(\frac{a^2+\beta^2}{b}u + a\right) - u^3 v,\end{aligned} \tag{5}$$

and we see that the origin is a singular point and that its linear part is zero. We need to do blow-ups to understand the local behavior at this point. We perform the directional blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$ and have

$$\begin{aligned}\dot{u} =& u^2w^2\left(\frac{a^2+\beta^2}{b}u^2 + 2au + b\right) - u^4,\\ \dot{w} =& -uw^3(au + b).\end{aligned}$$

We eliminate the common factor u between $\dot{u}$ and $\dot{w}$, and get the vector field

$$\begin{aligned} \dot{u} &= uw^2\left(\frac{a^2+\beta^2}{b}u^2+2au+b\right)-u^3, \\ \dot{w} &= -w^3(au+b). \end{aligned} \tag{6}$$

When $u=0$, the only singular point of system (6) is the origin, whose linear part is again zero. Hence we do another blow-up $(u,w)\to(u,z)$ with $z=w/u$, eliminate the common factor u^2, and get the vector field

$$\begin{aligned} \dot{u} &= uz^2\left(\frac{a^2+\beta^2}{b}u^2+2au+b\right)-u, \\ \dot{z} &= -z^3\left(\frac{a^2+\beta^2}{b}u^2+3au+2b\right)+z. \end{aligned} \tag{7}$$

When $u=0$, system (7) has the singular points $(0,0)$, $\left(0,\pm\sqrt{1/2b}\right)$. The sign of the parameter b determines the existence of these points, hence we need to analyze these points in two cases. We note that the linear part of system (7) at any point $(0,z)$ on the (u,z) plane is

$$\begin{pmatrix} bz^2-1 & 0 \\ -3az^3 & -6bz^2+1 \end{pmatrix}.$$

When $b<0$, the points $\left(0,\pm\sqrt{1/2b}\right)$ are not real. Hence the only singular point is the origin which is a saddle because the eigenvalues of the linear part at the origin are ±1. Going back through the change of variables until system (5) as shown in Figure 2, we see that locally the origin of U_2 consists of two hyperbolic sectors.

When $b>0$, however, all three singular points are real. The points $\left(0,\pm\sqrt{1/2b}\right)$ are attracting nodes since the eigenvalues at these points are $-1/2$ and -2. Again, tracing back the change of variables to system (5), see Figure 3, we see that the origin of U_2 has two elliptic and two parabolic sectors.

We now look at the finite singular points of system (I). Using (4) we find that they are

$$(0,0) \quad \text{and} \quad \pm\left(-\frac{\mathrm{a}\beta}{\mathrm{b}^{3/2}},\frac{\beta}{\sqrt{\mathrm{b}}}\right).$$

We know that the origin is a center. When $b<0$, we do not have any other finite singular point, and we get the global phase portrait 1.1 in Figure 1.

On the other hand, when $b>0$, the remaining two finite singular points are saddles since the eigenvalues of the linear part of system (I) at each of these points are $\pm\sqrt{2}\beta$. At least one of the saddles must be on the boundary of the period annulus of the center at the origin, and by symmetry, we conclude that both saddles are on this boundary.

To determine the global phase portrait in this case, we observe that on the y-axis, the Hamiltonian H_1 is quadratic. This means that the y-axis and the separatrices of a saddle can have at most two intersection points since the Hamiltonian is constant on the separatrices and $H_1|_{x=0}=h$ can have at

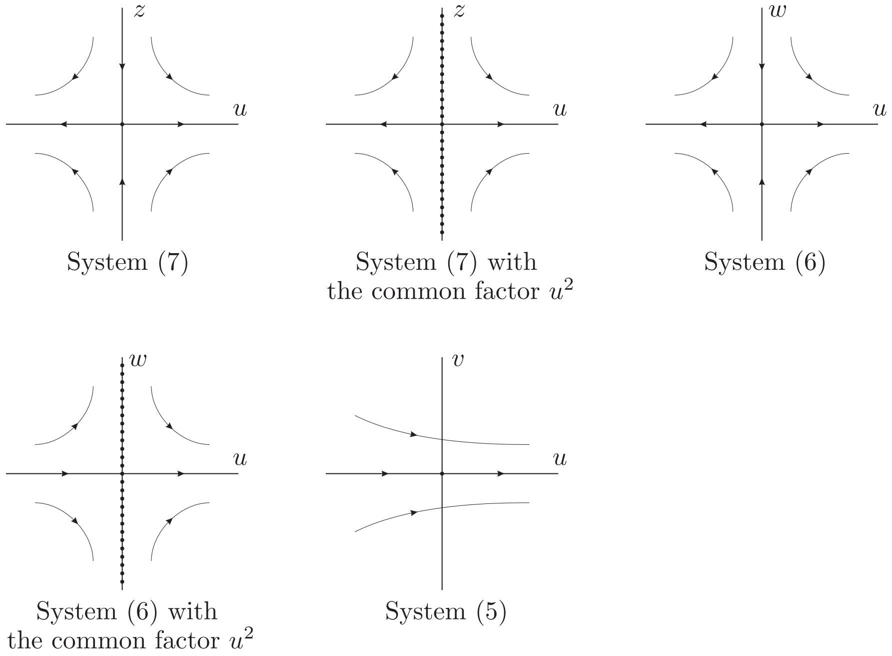

FIGURE 2. Blow-up of the origin of U_2 of system (I) when $b < 0$.

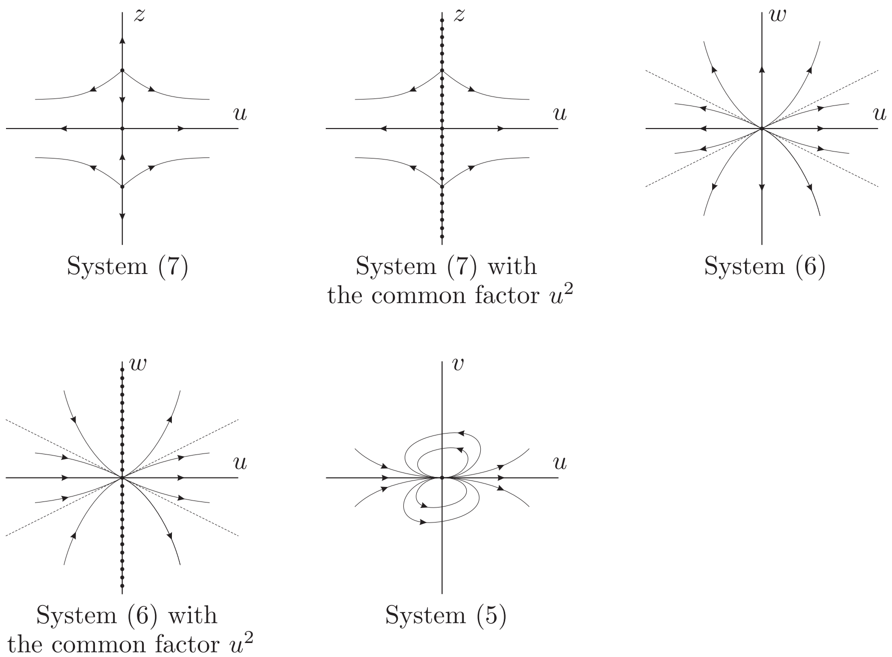

FIGURE 3. Blow-up of the origin of U_2 of system (I) when $b > 0$.

most two roots for any $h \in \mathbb{R}$. Hence the separatrices passing through the saddles can cross the y-axis exactly two times while forming the boundary of the period annulus of the center at the origin. Since there are no singular points on the y-axis other than the origin, the global phase portrait when $b>0$ is topologically equivalent to the portrait 1.2 of Figure 1.

5. Global phase portraits of system (II)

System (II)

$$\dot{x} = ax + by - x^3, \quad \dot{y} = -\frac{a^2+\beta^2}{b}x - ay + 3x^2y,$$

has the Hamiltonian

$$H_2(x,y) = -x^3y + \frac{a^2+\beta^2}{2b}x^2 + \frac{b}{2}y^2 + axy.$$

Note that we can assume $b>0$ because the linear change $y \mapsto -y$ gives exactly the same system with the opposite sign of the parameter b.

Again we will first find the infinite singular points. In the local chart U_1 system (II) is

$$\begin{aligned}\dot{u} &= -v^2\left(bu^2 + 2au + \frac{a^2+\beta^2}{b}\right) + 4u,\\ \dot{v} &= -v^3\left(bu+a\right) + v.\end{aligned}$$

When $v=0$, only the origin of U_1 is singular. The eigenvalues at this point are 4 and 1, meaning that it is a repelling node.

Next, we should check the origin of U_2, in which system (II) becomes

$$\begin{aligned}\dot{u} &= v^2\left(\frac{a^2+\beta^2}{b}u^2 + 2au + b\right) - 4u^3,\\ \dot{v} &= v^3\left(\frac{a^2+\beta^2}{b}u + a\right) - 3u^2v.\end{aligned} \tag{8}$$

We see that the origin is singular and its linear part is zero. We need blow-up to understand the local behavior at this point. Doing the blow-up $(u,v) \mapsto (u,w)$ with $w = v/u$ and eliminating the common factor u we get the system

$$\begin{aligned}\dot{u} &= uw^2\left(\frac{a^2+\beta^2}{b}u^2 + 2au + b\right) - 4u^2,\\ \dot{w} &= -w^3\left(au+b\right) - uw.\end{aligned} \tag{9}$$

When $u=0$, the only singular point of system (9) is the origin, whose linear part is again zero. So, we do another blow-up $(u,w) \mapsto (u,z)$ with $z = w/u$, eliminate the common factor u, and obtain

$$\begin{aligned}\dot{u} &= u^2z^2\left(\frac{a^2+\beta^2}{b}u^2 + 2au + b\right) - 4u,\\ \dot{z} &= -uz^3\left(\frac{a^2+\beta^2}{b}u^2 + 3au + 2b\right) + 5z.\end{aligned} \tag{10}$$

When $u=0$, the only singular point of system (10) is the origin, which is a saddle. We trace the change of variables back to system (8) as shown in Figure 4, and we find out that the origin of U_2 is an attracting node.

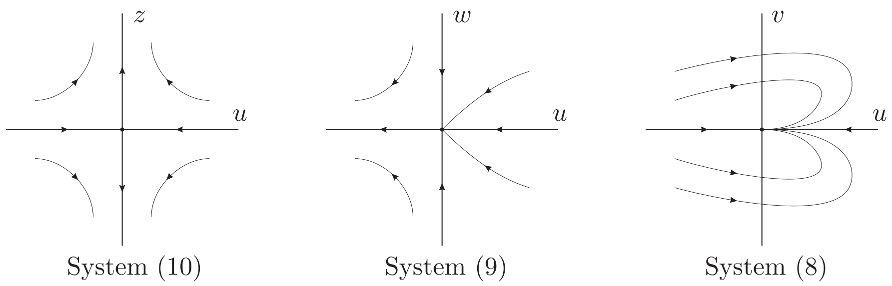

FIGURE 4. Blow-up of the origin of U_2 of system (II).

The finite singular points of system (II) other than the origin are

$$\pm\left(\frac{\sqrt{2a+\sqrt{4a^2+3\beta^2}}}{\sqrt{3}}, -\frac{(a-\sqrt{4a^2+3\beta^2})\sqrt{2a+\sqrt{4a^2+3\beta^2}}}{3\sqrt{3}b}\right).$$

The eigenvalues of the linear part of system (II) at these points are

$$\pm\frac{2\sqrt{4a^2+3\beta^2+2a\sqrt{4a^2+3\beta^2}}}{\sqrt{3}},$$

which means that they are saddles since $\beta > 0$.

Now we will determine the global phase portrait according to this local information. The two saddles must be on the boundary of the period annulus of the center at the origin due to the symmetry of the system. Also there are no singular points other than the origin on the axes, on either of which the Hamiltonian H_2 is quadratic. By the same argument used for system (I), this means that the separatrices passing through saddles cannot cross the axes anymore. Hence we obtain the global phase portrait 1.3 in Figure 1.

6. Global phase portraits of system (III)

System (III)

$$\dot{x} = ax + by - 3x^2y + y^3, \tag{11a}$$

$$\dot{y} = -\frac{a^2+\beta^2}{b}x - ay + 3xy^2, \tag{11b}$$

has the Hamiltonian

$$H_3(x,y) = \frac{y^4}{4} - \frac{3}{2}x^2y^2 + \frac{a^2+\beta^2}{2b}x^2 + \frac{b}{2}y^2 + axy.$$

In U_1 the system becomes

$$\begin{aligned}\dot{u} &= -v^2\left(bu^2 + 2au + \frac{a^2+\beta^2}{b}\right) - u^2(u^2-6),\\ \dot{v} &= -v^3\,(bu+a) - uv(u^2-3).\end{aligned} \tag{12}$$

When $v = 0$, there are three singular points on U_1: $(0,0), (\pm\sqrt{6}, 0)$. The linear part of system (12) is

$$\begin{pmatrix} -4u(u^2-3) & 0 \\ 0 & -u(u^2-3) \end{pmatrix}.$$

Hence the singular points $(\sqrt{6}, 0)$ and $(-\sqrt{6}, 0)$ are attracting and repelling nodes, respectively.

At the origin, however, the linear part is zero. Therefore to understand the local behavior we do the blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$. After eliminating the common factor u between $\dot{u}$ and $\dot{w}$, we obtain the system

$$\begin{aligned} \dot{u} &= -uw^2\left(bu^2 + 2au + \frac{a^2+\beta^2}{b}\right) - u(u^2-6), \\ \dot{w} &= w^3\left(au + \frac{a^2+\beta^2}{b}\right) - 3w. \end{aligned} \tag{13}$$

When $u = 0$, system (13) has the singular points $(0,0)$, $\left(0, \pm\sqrt{3b/(a^2+\beta^2)}\right)$. The linear part of system (13) at the points $(0, w)$ is

$$\begin{pmatrix} -\dfrac{a^2+\beta^2}{b}w^2 + 6 & 0 \\ aw^3 & 3\dfrac{a^2+\beta^2}{b}w^2 - 3 \end{pmatrix}.$$

When $b < 0$, we see that $\left(0, \pm\sqrt{3b/(a^2+\beta^2)}\right)$ are not real, hence the only singular point is the origin, which is a saddle. It is shown in Figure 5 that the origin of U_1 consists of two hyperbolic sectors.

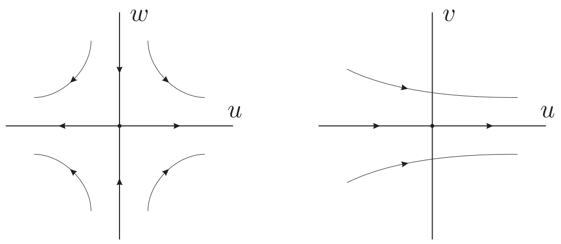

FIGURE 5. Blow-up of the origin of U_1 of system (III) when $b < 0$.

When $b > 0$, all three singular points are real. In addition to the saddle at the origin, the points $\left(0, \pm\sqrt{3b/(a^2+\beta^2)}\right)$ are repelling nodes. This time we see that locally the origin of U_1 has two elliptic sectors and two parabolic sectors, see Figure 6.

We now look at the origin of U_2, in which system (III) writes

$$\begin{aligned} \dot{u} &= v^2\left(\frac{a^2+\beta^2}{b}u^2 + 2au + b\right) - 6u^2 + 1, \\ \dot{v} &= v^3\left(\frac{a^2+\beta^2}{b}u + a\right) - 3uv. \end{aligned}$$

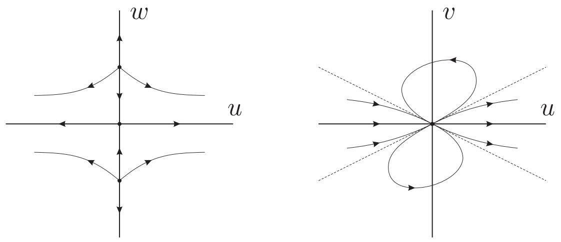

FIGURE 6. Blow-up of the origin of U_1 of system (III) when $b > 0$.

We see that the origin of U_2 is not a singular point. Hence all the infinite singular points are in U_1.

Now we analyze the finite singular points of system (III). The origin is already known to be center. Note that the linear part of system (III) is

$$\begin{pmatrix} a - 6xy & b - 3x^2 + 3y^2 \\ 3y^2 - \dfrac{a^2+\beta^2}{b} & -a + 6xy \end{pmatrix}$$

The explicit expressions for the finite singular points in terms of the parameters a, b, β are big, and therefore it is hard to study them numerically. For this reason we follow a different way. We first find the maximum number of finite singular points allowed by the system. To do this, we equate (11a) to 0, solve for x and get

$$x_{1,2} = \frac{a \pm \sqrt{a^2 + 12by^2 + 12y^2}}{6y}. \tag{14}$$

Note that when $y = 0$, (11b) is zero if and only if $x = 0$. But since we are looking for finite singular points other than the origin, we can assume that $y \neq 0$. Then, if we substitute (14) into (11b) we obtain

$$\dot{y}_{1,2} = -\frac{a^3 + 3aby^2 + a\beta^2 \pm (a^2 + \beta^2 - 3by^2)\sqrt{a^2 + 12by^2 + 12y^2}}{6by},$$

where $\dot{y}_1$ and $\dot{y}_2$ denote $\dot{y}$ with x_1 and x_2 substituted, respectively. Then the maximum number of roots of the product $\dot{y}_1\dot{y}_2$ will give us an upper bound for the number of finite singular points. So we multiply $\dot{y}_1$ and $\dot{y}_2$ and obtain

$$-3y^6 + \frac{2a^2 + 2\beta^2 - 3b^2}{b}y^4 - \frac{(a^2+\beta^2)(a^2+\beta^2-6b^2)}{3b^2}y^2 - \frac{\beta^2(a^2+\beta^2)}{3b}. \tag{15}$$

We see that (15) cannot be identically zero, so it has at most six real roots. This means that all the finite singular points of system (III) are isolated. In fact, if we multiply (15) by $3b^2$ and replace y^2 by z, we get the cubic polynomial

$$-9b^2z^3 + 3b(2a^2+2\beta^2-3b^2)z^2 - (a^2+\beta^2)(a^2+\beta^2-6b^2)z - b\beta^2(a^2+\beta^2). \tag{16}$$

In order that (15) has six real roots, (16) must have all of its roots positive. To find the maximum number of positive roots of the polynomial (16) we use *Descartes' rule of signs*:

Theorem 10. *The number of positive roots of a real polynomial is either equal to the number of sign differences between consecutive nonzero coefficients, or is less than it by a multiple of 2, when the terms of the polynomial are placed in descending or ascending order of exponents.*

The terms of (16) are already ordered correctly, so there must be three changes of sign between its coefficients to have all three roots positive. When $b > 0$, since the constant term and the coefficient of z^3 are both negative, there cannot be three changes of sign, hence neither three positive roots. When $b < 0$, the constant term is positive and the coefficient of z^3 is negative. Thus, to have three positive roots, the coefficients of z and z^2 must be negative and positive, respectively. So we must have

$$a^2 + \beta^2 - 6b^2 > 0 \quad \text{and} \quad 2a^2 + 2\beta^2 - 3b^2 < 0,$$

which is not possible since

$$2a^2 + 2\beta^2 - 3b^2 > a^2 + \beta^2 - 6b^2.$$

Therefore polynomial (16) cannot have three positive roots, and system (III) has at most four finite singular points other than the origin.

Next we count the indices of the known singular points, both finite and infinite. Since the infinite singular points depend on the sign of the parameter b, we need to investigate those two cases separately. We first state two theorems which will play essential roles in the process of determining the finite singular points of the vector fields. For details about these theorems, see Chapters 1 and 6 of [10]. The first one is the well known *Poincaré Formula* for the index of a singular point of a planar vector field, and the second one is the famous *Poincaré-Hopf Theorem* for vector fields in the 2-dimensional sphere.

Theorem 11 (Poincaré Formula)**.** *Let q be an isolated singular point having the finite sectorial decomposition property. Let $e, h,$ and p denote the number of elliptic, hyperbolic, and parabolic sectors of q, respectively. Then the index of q is $(e - h)/2 + 1$.*

Theorem 12 (Poincaré-Hopf Theorem)**.** *For every tangent vector field on $\mathbb{S}^2$ with a finite number of singular points, the sum of the indices of the singular points is* 2.

Corollary 13. *The index of a saddle, a center and a cusp are* -1*,* 1 *and* 0*, respectively.*

We now present a lemma that will also play a crucial role in determining the finite singular points. We define *energy levels* of a vector field as the curves on which its Hamiltonian is constant. We say that a singular point is *non-degenerate* if none of the eigenvalues of the linear part of the vector field at that point is zero, and *degenerate* otherwise. A degenerate singular point is called *linearly zero* if the linear part is identically zero, otherwise it is called *nilpotent*.

Remark 14. Nilpotent singular points of Hamiltonian planar polynomial vector fields are either saddles, centers, or cusps (for more details see chapters 2 and 3 of [10], specifically sections 2.6 and 3.5)

A hyperbolic saddle with a loop and a center inside the loop as in Figure 7 will be called a *center-loop.*

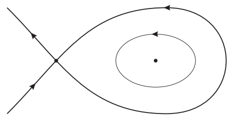

FIGURE 7. A center-loop.

Lemma 15. *Let X_ε be a real Hamiltonian planar polynomial vector field having only linear and cubic terms. Then X_ε can be written as*

$$\dot{x} = a_{10}x + a_{01}y + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3,$$
$$\dot{y} = b_{10}x - a_{10}y + b_{30}x^3 - 3a_{30}x^2y - a_{21}xy^2 - \frac{1}{3}a_{12}y^3 + \varepsilon x.$$

Suppose that p is an isolated singular point of X_ε different from the origin. If $a_{10}^2 + a_{01}b_{10} < 0$, i.e. if the origin of X_0 is a center, then the following statements hold:

(a) *If p is degenerate, then it is nilpotent.*
(b) *If p is a degenerate singular point of X_0, then it is a non-degenerate singular point of X_ε with $\varepsilon \neq 0$.*
(c) *If p is a cusp of X_0, then for $\varepsilon \neq 0$ small enough the local phase portrait of X_ε at p is a center-loop.*

Proof. Let X_ε be the vector field defined in the lemma. It is easy to check that X_ε is Hamiltonian with the Hamiltonian polynomial

$$H_\varepsilon = \frac{1}{4}(a_{03}y^4 - b_{30}x^4) + a_{30}x^3y + \frac{1}{2}a_{21}x^2y^2 + \frac{1}{3}a_{12}xy^3 + \frac{1}{2}(a_{01}y^2 - b_{10}x^2) + a_{10}xy.$$

Without loss of generality we can assume that $p = (0, y_0)$, otherwise doing a rotation of the coordinates we can get its x-coordinate to be zero.

Assume that $a_{10}^2 + a_{01}b_{10} < 0$. Note that this condition implies $a_{01} \neq 0$.

We first prove (a). At $(0, y_0)$ system X_ε becomes

$$\dot{x} = y_0(a_{01} + a_{03}y_0^2), \quad \dot{y} = -y_0(a_{10} + \frac{1}{3}a_{12}y_0^2),$$

whereas M_ε, the linear part of X_ε at a point, is

$$\begin{pmatrix} a_{10} + a_{12}y_0^2 & a_{01} + 3a_{03}y_0^2 \\ b_{10} - a_{21}y_0^2 + \varepsilon & -a_{10} - a_{12}y_0^2 \end{pmatrix}.$$

Since $y_0 \neq 0$, we have $(0, y_0)$ linearly zero only if

$$a_{01} + a_{03}y_0^2 = a_{01} + 3a_{03}y_0^2 = 0, \tag{17}$$

which requires $a_{03} = 0$. However, since $a_{01} \neq 0$, equation (17) cannot be satisfied. Therefore we conclude that a degenerate singular point of X_ε must be nilpotent.

Now we prove (b). Assume that $(0, y_0)$ is a degenerate singular point of X_0. We will prove that the eigenvalues of the linear part of X_ε, with $\varepsilon \neq 0$, are different from zero.

The characteristic polynomial at a singular point of X_ε is of the form $\lambda^2+\det(M_\varepsilon)$. So the eigenvalues λ of the linear part of X_ε at a singular point are $\pm\sqrt{-\det(M_\varepsilon)}$. Since we have already assumed that the eigenvalues of M_0 at $(0, y_0)$ are zero, the only nonzero terms in the determinant of M_ε at the same point are those having a factor of ε. Hence the eigenvalues of M_ε at $(0, y_0)$ are

$$\lambda = \pm\sqrt{\varepsilon(3a_{03}y_0^2 + a_{01})}, \tag{18}$$

where $\varepsilon \neq 0$. Then the eigenvalues are zero only if we have (17), but we have already shown that it is not possible. Therefore $(0, y_0)$ is a non-degenerate singular point of X_ε.

Finally we prove part (c). Assume that $(0, y_0)$ is a cusp of X_0. First we note that for $\varepsilon \neq 0$, due to (18), by a proper choice of sign of ε we can assume that $(0, y_0)$ is a saddle of X_ε. In addition, since $(0, y_0)$, which was a cusp with index zero, is now a saddle having index -1, new singular points must emerge in a neighborhood W_ε of $(0, y_0)$ to keep the total index of the vector field fixed. Because of the symmetry of the system, there can be at most three new singular points in W_ε so that the total number of finite singular points does not exceed 9. Since X_ε is Hamiltonian, these singular points can only be saddles, centers or cusps. Therefore in W_ε there are additionally to the saddle at $(0, y_0)$ either (i) one center, (ii) one center and one cusp, (iii) one center and two cusps, or (iv) two centers and one saddle. Our claim is that (i) is the only realizable case and that $(0, y_0)$ is the saddle of a center-loop.

Because of the continuity of system X_ε with respect to ε, the new separatrices of $(0, y_0)$ must be arbitrarily close to $(0, y_0)$ for small ε, therefore they cannot go to any other singular point outside W_ε. Note that in all the possibilities $(i) - (iv)$, there exists a center with $(0, y_0)$ on the boundary of its period annulus. Then we see that $(0, y_0)$ cannot be on the boundary of the period annulus of the center at the origin. Otherwise we could find a straight line l through the origin intersecting the boundary of the period annulus of the new center twice, which would, in fact, have at least three intersection points with the separatrices of $(0, y_0)$, the other being on the boundary of the period annulus of the center at the origin, see Figure 8. Then, due to the symmetry of the system with respect to the origin, there would be six points on l all of which are on the same energy level. Clearly this is not possible since the Hamiltonian H_ε is a quartic polynomial.

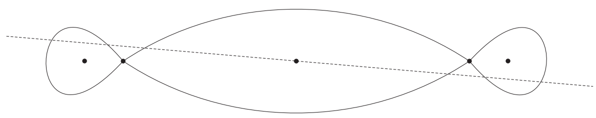

FIGURE 8. The straight line through the origin intersects the separatrices six times.

If $(0, y_0)$ is not on the boundary of the period annulus of the center at the origin, then there must be other saddles on that boundary. This means that system X_0 has at least five finite singular points. This immediately eliminates the possibilities (*iii*) and (*iv*), otherwise the number of finite singular points exceeds the maximum of 9. Furthermore, by the same arguments used for $(0, y_0)$, the cusp in case (*ii*) would also lead to the existence of more singular points. Therefore we dismiss case (*ii*) also, proving our claim. □

We continue determining the finite singular points of system (III). When $b < 0$, the infinite singular points on the Poincaré sphere are $(\pm\sqrt{6}, 0)$ and $(0, 0)$ in U_1, and also the corresponding points on V_1. The origins of U_1 and V_1 consist of two hyperbolic sectors, hence, by Theorem 11, have index 0. The other four infinite singular points are nodes, hence each has index 1, again due to Theorem 11. Among the finite singular points we only know that the origins of U_3 and V_3 are centers with index 1. Hence, the known singular points have total index 6 on the Poincaré sphere. By Theorem 12, the remaining finite singular points, if any, must have total index -4. Thus, on the Poincaré disk, the finite singular points other than the origin must have total index -2.

We claim that system (III) has at most two degenerate singular points. To prove it we need to show that (11a), (11b) and the determinant of the linear part of system (III) cannot simultaneously vanish at more than two points. We note that the linear part of system (III) is

$$M_3 = \begin{pmatrix} a - 6xy & b - 3x^2 + 3y^2 \\ 3y^2 - \dfrac{a^2 + \beta^2}{b} & -a + 6xy \end{pmatrix}.$$

We compute the Gröbner basis for these three equations and obtain a set of sixteen equations. Two of these equations are

$$\beta^4 \left(a^2 - 6b^2 - 9by^2 + \beta^2\right) = 0, \quad (19)$$

$$\beta^2\big(162ab^3x - y(4a^4 - 57a^2b^2 + 36b^4 + 8a^2\beta^2 + 24b^2\beta^2 + 4\beta^4)\big) = 0. \quad (20)$$

From equation (19) we get that there are at most two solutions for y. In addition, since equation (20) is only linear in x, we deduce that the determinant of M_3 can be zero at no more than two singular points of system (III). Then, by part (a) of Lemma 15, the unique degenerate finite singular points of system (III) are at most two nilpotent singular points.

Having established the above claim, we continue with determining the global phase portrait of system (III) in the case $b < 0$. By part (a) of Lemma 15, the remaining finite singular points are either non-degenerate or nilpotent, hence they are either saddles, centers or cusps. Recall that system (III) can have at most four finite singular points other than the origin. Then by Corollary 13, there must be either just two saddles, or two saddles and two cusps so that they have total index -2.

If there were two saddles and two cusps, due to part (c) of Lemma 15, after a small perturbation, there would be four saddles and two centers, which is more than the maximum number of finite singular points allowed by the system. Therefore, when $b < 0$, system (III) has only two saddles in the finite region other than the origin. By the symmetry of the vector field, the saddles are located on the boundary of the period annulus of the center at the origin. On the x-axis the Hamiltonian H_3 is quadratic, hence the separatrices through the saddles cannot cross the x-axis any more, therefore we have the global phase portrait 1.4 of Figure 1.

When $b > 0$, on the other hand, the infinite singular points at the origins of U_1 and V_1 consist of two elliptic and two parabolic sectors, hence each have index 2. Together with the remaining nodes at the infinity and the centers at the origins of U_3 and V_3, they have total index 10 on the Poincaré sphere. So, on the Poincaré disk, the remaining finite singular points must have total index -4. Then, by Corollary 13, there must be four other singular points which are all saddles. Then on the boundary of the period annulus of the center at the origin there are either two or four saddles.

Suppose first that all of the four saddles are on the boundary of the period annulus of the center at the origin. Since $b > 0$, the flow around the center at the origin is clockwise. Because the separatrices through the saddles cannot cross the x-axis anymore, the global phase portrait 1.5 shown in Figure 1 is obtained. We note that when the parameters $a = 0$ and $b = \beta = 1$, we get a global phase portrait topologically equivalent to the portrait 1.5 of Figure 1.

Now assume there are only two saddles on the boundary of the period annulus of the center. We claim that these saddles cannot be connected to any of the other saddles. If this were the case, that is if one of these saddles were connected to another saddle p which is not on the boundary of the period annulus of the center, then, on the quadrant of the xy-plane where p lies, a straight line l through the origin passing sufficiently close to p would have at least three intersection points with the separatrices on the same energy level as p (one with the boundary of the period annulus of the center and at least two with the separatrices of p), see Figure 9 for an example. Taking into account the symmetry of the vector field with respect to the origin, the straight line l would have six intersection points with the separatrices on the same energy level as p. This means that on the straight line l, which could be defined by $y = cx$ for some real number c, the equation $H_3 = H(p)$ would have six solutions. But this is not possible as H_3 is only a fourth degree polynomial. Therefore the saddles on the boundary of the period annulus of the center has to be connected with the infinite singular points. Due to the fact that the separatrices through these saddles cannot cross the x-axis anymore and the clockwise flow around the origin , we get the global phase portrait 1.6 shown in Figure 1. A phase portrait in this case is realized when $-a = b = \beta = 1$.

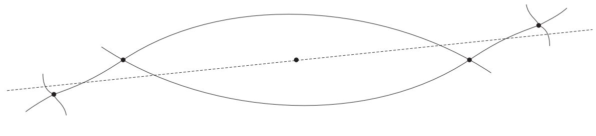

FIGURE 9. The straight line through the origin intersects the separatrices six times.

7. Global phase portraits of system (IV)

System (IV) is defined by the equations

$$\dot{x} = ax + by - 3x^2y - y^3, \tag{21a}$$

$$\dot{y} = -\frac{a^2+\beta^2}{b}x - ay + 3xy^2, \tag{21b}$$

and has the Hamiltonian

$$H_4(x,y) = -\frac{y^4}{4} - \frac{3}{2}x^2y^2 + \frac{a^2+\beta^2}{2b}x^2 + \frac{b}{2}y^2 + axy.$$

In the local chart U_1 system (IV) writes

$$\begin{aligned} \dot{u} &= -v^2\left(bu^2 + 2au + \frac{a^2+\beta^2}{b}\right) + u^2(u^2+6), \\ \dot{v} &= -v^3(bu+a) + uv(u^2+3). \end{aligned} \tag{22}$$

When $v=0$, the only singular point on U_1 is $(0,0)$, at which the linear part of the vector field is zero. Therefore to study the local behaviour at the origin of U_1 we do the blow-up $(u,v)\mapsto(u,w)$ with $w=v/u$. Eliminating the common factor u between $\dot{u}$ and $\dot{w}$, we obtain

$$\begin{aligned} \dot{u} &= -uw^2\left(bu^2 + 2au + \frac{a^2+\beta^2}{b}\right) + u(u^2+6), \\ \dot{w} &= w^3\left(au + \frac{a^2+\beta^2}{b}\right) - 3w. \end{aligned} \tag{23}$$

When $u=0$, system (23) has three singular points: $\left(0, \pm\sqrt{3b/(a^2+\beta^2)}\right)$, $(0,0)$. The linear part of system (23) at the points $(0,w)$ is

$$\begin{pmatrix} -\frac{a^2+\beta^2}{b}w^2+6 & 0 \\ aw^3 & 3\frac{a^2+\beta^2}{b}w^2-3 \end{pmatrix}.$$

We see that when $b<0$, $\left(0, \pm\sqrt{3b/(a^2+\beta^2)}\right)$ are not real, hence the only singular point in U_1 is the origin, which is a saddle. We see that the blow-up analysis gives the same result as in the case $b<0$ of system (III), hence the local behavior at the origin of U_1 consists of two hyperbolic sectors.

When $b>0$ all three singular points are real. In addition to the saddle at the origin, the points $\left(0, \pm\sqrt{3b/(a^2+\beta^2)}\right)$ are repelling nodes. The blow-up of the origin gives the same information as in the case $b>0$ of system (III), and we see that the local behavior at the origin of U_1 in this case consists of two elliptic sectors and two parabolic ones.

In U_2 system (IV) is expressed as

$$\begin{aligned} \dot{u} &= v^2\left(\frac{a^2+\beta^2}{b}u^2 + 2au + b\right) - 6u^2 - 1, \\ \dot{v} &= v^3\left(\frac{a^2+\beta^2}{b}u + a\right) - 3uv. \end{aligned}$$

The origin of U_2 is not a singular point. Hence the only infinite singular points are the origins of U_1 and V_1.

Now we investigate the finite singular points of this system. We know that the origin is a center and we look for other singular points, if there exists any. Again we first find the maximum number of such points allowed by the system. Equating (21a) to zero and solving for x gives

$$x_{1,2} = \frac{a \pm \sqrt{a^2 + 12by^2 - 12y^4}}{6y}. \tag{24}$$

Note that when $y = 0$ we have $x = 0$ due to (21b), so we can assume $y \neq 0$. Then we substitute (24) into (21b) and obtain

$$\dot{y}_{1,2} = -\frac{a^3 + 3aby^2 + a\beta^2 \pm (a^2 + \beta^2 - 3by^2)\sqrt{a^2 + 12by^2 + 12y^2}}{6by}.$$

Then the product $\dot{y}_1\dot{y}_2$ is

$$3y^6 - \frac{2a^2 + 2\beta^2 + 3b^2}{b}y^4 + \frac{(a^2 + \beta^2)(a^2 + \beta^2 + 6b^2)}{3b^2}y^2 - \frac{\beta^2(a^2 + \beta^2)}{3b}. \tag{25}$$

We see that (25) is not identically zero, hence has at most six real roots.

We cannot efficiently compute the finite singular points of system (IV) but we can show that the system has at most two degenerate singular points. The linear part M_4 of system (IV) is

$$M_4 = \begin{pmatrix} a - 6xy & b - 3x^2 - 3y^2 \\ 3y^2 - \dfrac{a^2 + \beta^2}{b} & -a + 6xy \end{pmatrix}.$$

The process is exactly the same as in system (III). We consider the system of three equations obtained by equating (21a), (21b) and the determinant of M_4 to zero. We pass to the Gröbner basis and see that two of the sixteen equations are

$$\beta^4 \left(a^2 + 6b^2 - 9by^2 + \beta^2\right) = 0,$$
$$\beta^2\big(162ab^3x - y(4a^4 + 57a^2b^2 + 36b^4 + 8a^2\beta^2 - 24b^2\beta^2 + 4\beta^4)\big) = 0.$$

These two equations yield that the determinant of M_4 can be zero at most at two points. Then, by part (a) of Lemma 15, we conclude that system (IV) has at most two nilpotent singular points.

We will now determine the finite singular points by considering the total index of system (IV) on the Poincaré sphere.

Case 1 $(b < 0)$: When $b < 0$, the infinite singular points on the Poincaré sphere are the origins of U_1 and V_1, each of which consists of two hyperbolic sectors and hence has index 0 due to Theorem 11. Considering the finite singular points, we know that the origins of U_3 and V_3 are centers, each having index 1. So, for the moment, all known finite and infinite singular points have total index 2. Then by Proposition 12, the remaining finite singular points on the Poincaré disk should have total index 0. Since the system has at most six such points, there are the following possibilities: (i) no more finite singular points, (ii) two cusps, (iii) two saddles and two centers, or (iv) two saddles, two centers and two cusps. Case (iv) cannot occur because by Lemma 15, after a small perturbation, each cusp will produce two singular points, exceeding the maximum number of possible finite singular points.

Consider case (*iii*). If we place the two saddles on the Poincaré disk, they will be symmetric with respect to the origin and also be on the boundary of the period annulus of the center at the origin. Moreover, these saddles cannot be on the x-axis because when $y = 0$, equation (21b) is zero only when $x = 0$, but the origin is known to be a center. Since the infinite singular points have only hyperbolic sectors, there are only two possible ways to construct the other two centers, see Figure 10:

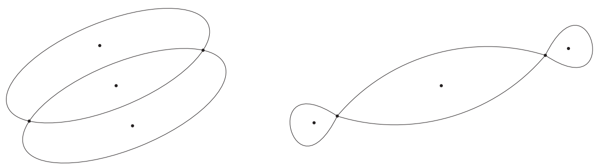

FIGURE 10. Two saddles forming the two centers.

If the first figure is the case, then the x-axis cuts the separatrices through these saddles four times, which is not possible since when $y = 0$ the Hamiltonian H_4 is only quadratic in x. If the second figure is the case, then a straight line through the origin passing sufficiently close to the saddles will have six intersection points with the separatrices through the saddles (see Figure 8). This also is not possible because H_4 is a quartic polynomial on this line, and consequently there can be at most four such points. So case (*iii*) of two saddles and two centers cannot exist.

Note that applying Lemma 15 in case (*ii*) produces a system in case (*ii*) and therefore case (*ii*) cannot exist either.

In short, when $b < 0$ there are no other finite singular points on the Poincaré disk except the origin, and the global phase portrait is topologically equivalent to the phase portrait 1.1 of Figure 1.

Case 2 ($b > 0$): In this case, the infinite singular points, which are the origins of U_1 and V_1, consist of two elliptic and two parabolic sectors. By Proposition 11, each has index 2. Together with the centers at the origins of U_3 and V_3, their total index on the Poincaré sphere is 6. This means that in the Poincaré disk, the remaining finite singular points must have total index -2. Then we have the following three possibilities: (*i*) two saddles, (*ii*) two saddles and two cusps, or (*iii*) four saddles and two centers.

Consider case (*i*). The two saddles are of course symmetric with respect to the origin and are on the boundary of the period annulus of the center at the origin. As in the case $b < 0$, they cannot be on the x-axis. Since H_4 is quadratic on the x-axis, the separatrices through these saddles cannot cross it, hence the saddles must be connected with the infinite singular points, and we obtain the global phase portrait which is topologically equivalent to the phase portrait 1.2 of Figure 1. For the values $2a = b = \beta = 1$ a topologically equivalent phase portrait is achieved.

Now consider case (*iii*). There may be either two or four saddles on the boundary of the period annulus of the center at the origin.

Assume first that the four saddles are on the boundary of the period annulus of the center at the origin. Because of the symmetry of the system, two of them will be above the x-axis and two will be below because the only singular point lying on the axes is the origin. Remember that the separatrices through the saddles cannot cross the x-axis anymore. So there must be a separatrix connecting the saddles which are on the same side of the x-axis. Moreover, the remaining separatrices of the two saddles which are on the same side of the x-axis must go to different infinite singular points. If they went to the same point, then the y-axis would intersect the separatrices of these saddles, which are all on the same energy level, six times (three below the x-axis and three above). However, this is not possible since H_4 is quartic in y on the y-axis. Therefore we get the global phase portrait 1.7 shown in Figure 1. Such a phase portrait is obtained if $a = 0$ and $b = \beta = 1$.

Assume now that only two of the saddles are on the boundary of the period annulus of the center at the origin. Just as in the case $b > 0$ of system (III), the saddles on this boundary cannot be connected to other saddles (see Figure 9). Then, taking into account that the separatrices of the saddles on this boundary cannot cross the x-axis anymore, we see that these saddles must be connected with the infinite singular points, and the separatrices of these saddles are as shown in Figure 11. We next claim that the centers must be inside the region enclosed by the separatrices connecting the saddles on the boundary of the period annulus of the center at the origin with the infinite singular points. Suppose this were not the case, i.e. suppose that a center were in the region outside the previously mentioned area. Because of the flow in these regions, one of the remaining saddles must also be in the same region. Moreover, the saddle must be on the boundary of the period annulus of this center. But then a straight line through the origin passing sufficiently close to the saddle and intersecting this boundary twice would also intersect another separatrix of the saddle because of the flow, see Figure 11. So, by the symmetry, this straight line would have six intersection points with the separatrices on the same energy level, which is impossible. This proves our claim about the location of the centers. Then again the remaining saddles must be on the boundaries of these centers. Therefore the global phase portrait is the one 1.8 of Figure 1. A phase portrait in this case is realized when $a = 2/5$ and $b = \beta = 1$.

Remark 16. In fact a center-loop may exist in any one of the vector fields $(I) - (VI)$ only if a straight line l_1 passing through the origin and the saddle of the center-loop intersects the separatrices of the saddle exactly at one point, the saddle itself. Otherwise one can find another straight line l_2 passing through the origin and sufficiently close to the saddle of the center-loop such that the number of intersection points is at least three, see Figure 12. Then, by the symmetry of these systems, l_2 would intersect the separatrices on the same energy level at six points, which is impossible since the systems are cubic.

Finally we consider case (ii). Due to Lemma 15, case (ii) will produce a system in case (iii) after a small perturbation. Consequently, the global phase portrait in this case is topologically equivalent to that 1.9 shown in

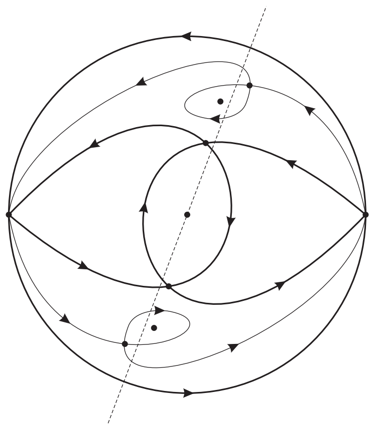

FIGURE 11. The center cannot be outside the region enclosed by the separatrices of the saddles.

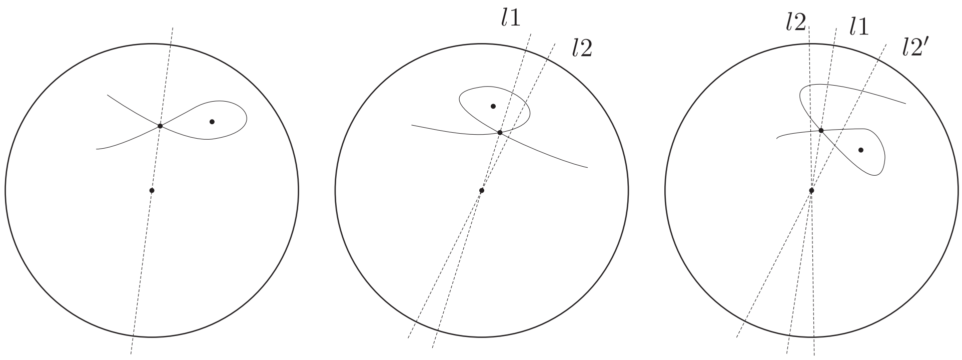

FIGURE 12. Center-loop configuration.

Figure 1. We note that as a result of the fact that system III attains the global phase portraits 1.2 when $a = 0.5$, $b = \beta = 1$ and 1.8 when $a = 0.4$, $b = \beta = 1$, we deduce that case (ii) exists when $b = \beta = 1$ and a is between 0.4 and 0.5.

8. Global phase portraits of system (V)

We remind that system (V) is

$$\dot{x} = ax + by - 3\mu x^2 y + y^3, \tag{26a}$$

$$\dot{y} = -\frac{a^2 + \beta^2}{b} x - ay + x^3 + 3\mu x y^2, \tag{26b}$$

with the Hamiltonian

$$H_5(x, y) = \frac{y^4 - x^4}{4} - \frac{3\mu}{2} x^2 y^2 + \frac{a^2 + \beta^2}{2b} x^2 + \frac{b}{2} y^2 + axy.$$

As in the previous systems, we first investigate the infinite singular points of system (V). On the local chart U_1 we have

$$\begin{aligned}\dot{u} =& -v^2\left(bu^2+2au+\frac{a^2+\beta^2}{b}\right)-u^4+6\mu u^2+1,\\ \dot{v} =& -v^3(bu+a)-vu(u^2-3\mu).\end{aligned} \tag{27}$$

When $v=0$, the u-coordinates of the singular points are $\pm\sqrt{3\mu\pm\sqrt{9\mu^2+1}}$. Of these four points, only $\left(\pm\sqrt{3\mu+\sqrt{9\mu^2+1}},0\right)$ are real. The linear part of (27) when $v=0$ is

$$\begin{pmatrix} -4u(u^2-3\mu) & 0 \\ 0 & -u(u^2-3\mu)\end{pmatrix},$$

so that the eigenvalues at those singular points are both negative if $u>0$, and positive if $u<0$. Hence the points $\left(\sqrt{3\mu+\sqrt{9\mu^2+1}},0\right)$ and $\left(-\sqrt{3\mu+\sqrt{9\mu^2+1}},0\right)$ are attracting and repelling nodes respectively.

In U_2 system (V) becomes

$$\begin{aligned}\dot{u} =& v^2\left(\frac{a^2+\beta^2}{b}u^2+2au+b\right)-u^4-6\mu u^2+1,\\ \dot{v} =& v^3\left(\frac{a^2+\beta^2}{b}u+a\right)-vu(u^2+3\mu),\end{aligned}$$

and we see that the origin is not a singular point. Hence system (V) has four infinite singular points, all of which are on U_1 and are nodes.

Now we shall discuss the finite singular points. By Bezout's theorem, the system has at most nine isolated finite singular points. To see if there are any non-isolated singularities we compute the resultant of (26a) and (26b) with respect to x and see that the numerator is

$$\begin{aligned}& b^2(1+9\mu^2)^2y^9+3b(1+9\mu^2)(b^2-2a^2\mu-2\beta^2\mu+3b^2\mu^2)y^7\\ &+3(b^4-4b^2\beta^2\mu+3a^4\mu^2+6b^4\mu^2+6a^2\beta^2\mu^2+3\beta^4\mu^2-18a^2b^2\mu^3-18b^2\beta^2\mu^3)y^5\\ &-b(a^4-b^4+a^2\beta^2-6a^2b^2\mu+6b^2\beta^2\mu-9a^2\beta^2\mu^2-9\beta^4\mu^2)y^3-a^2b^2\beta^2y,\end{aligned}$$

which cannot be identically zero since the coefficient of y^9 is always positive. Hence we conclude that all of the finite singular points, if any, are isolated.

We know that the origin is a center, which leaves us with eight more possible singularities. On the Poincaré sphere, the total index of the infinite singular points is 4. Together with the centers at the origins of U_3 and V_3, their total index becomes 6. Thus, on the Poincaré disk, we need to get a total index of -2 from the possible eight finite singular points.

We claim that system (V) has at most two nilpotent singularities in the finite region of the Poincaré disk. For the proof we consider the Gröbner basis of the polynomials (26a), (26b) and the determinant of the linear part of the system. Similar to the situation in system (III), we see that there exist two polynomials, one linear in x and another quadratic in y having no x variable. This proves the claim.

Having established the above claim, we see that the finite singular points of system (V) other than the origin can be (i) two saddles, (ii) two saddles and two cusps, (iii) four saddles and two centers, or (iv) four saddles, two centers and two cusps.

Case (iv) cannot occur because due to Lemma 15, it would require another case with two more finite singular points which is not possible.

Before studying the other cases we remark that without loss of generality we can assume $b > 0$. If we do the linear transformation $(x, y) \mapsto (-y, -x)$, system (V) becomes

$$
\begin{aligned}
-\dot{y} &= -ay - bx + 3\mu y^2 x - x^3,\\
-\dot{x} &= \frac{a^2+\beta^2}{b} y + ax - y^3 - 3\mu y x^2,
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
\dot{x} &= -ax - \frac{a^2+\beta^2}{b} y + 3\mu x^2 y + y^3,\\
\dot{y} &= bx + ay + x^3 - 3\mu x y^2.
\end{aligned}
\tag{28}
$$

After defining $\bar{a} = -a$, $\bar{\mu} - \mu$, and $\bar{b} = (a^2 + \beta^2)/b$, we see that system (28) is basically system (V) with $b \mapsto -b$. So, we assume $b > 0$.

Consider case (i). The two saddles must be on the boundary of the period annulus of the center at the origin. Their remaining separatrices cannot cross the straight lines passing through the origin and the infinite singular points, namely $y = \pm\sqrt{3\mu + \sqrt{9\mu^2 + 1}x}$, because the Hamiltonian H_5 is quadratic on these lines. Then, due to the flow at infinity, we get a global phase portrait which is topologically equivalent to 1.3 of Figure 1. We remark that this phase portrait is achieved for the values $a = b = \beta = 1$ and $\mu = 0$.

Similar to the previous systems, in case (iii) there are two possibilities.

Assume first that the four saddles are on the boundary of the period annulus of the center at the origin. Since the infinite singular points are nodes, the centers can only be created by connecting two adjacent saddles. Since we have $b > 0$, the flow around the origin clockwise. In addition, the remaining separatrices of any of the saddles must lie on different sides of the straight line passing through that saddle and the origin, otherwise there would exist another straight line through the origin intersecting both separatrices in six intersection points in the same energy level. Then the flow at infinity ensures that the remaining two centers are formed by connecting adjacent saddles which lie on the same side of the y-axis. Recalling from case (i) that the remaining separatrices of these saddles cannot cross the lines $y = \pm\sqrt{3\mu + \sqrt{9\mu^2 + 1}x}$ passing through the origin and the infinite singular points, we get the global phase portrait 1.10 in Figure 1. This phase portrait is actually realized when $a = 0$ and $b = \beta = \mu = 1$, for instance.

Assume now that only two of the saddles are on the boundary of the period annulus of the center at the origin. These saddles cannot be connected by a separatrix with each other since their separatrices cannot cross the lines

$y = \pm\sqrt{3\mu + \sqrt{9\mu^2+1}x}$ anymore. Their separatrices can neither return to the same saddles (see Figure 8) nor go to one of the other two saddles (see Figure 9). Therefore they must go to the infinite singular points like in case (i). Moreover, because of the symmetry of system (V) and the flow, the remaining finite singular points must be symmetric with respect to the origin, and also there must be a saddle on the boundary of the period annulus of each of the centers, creating a center-loop. Due to Remark 16 we made earlier, these center-loops can appear only in one of the regions indicated in Figure 13 (see also Figure 11). Therefore, up to topological equivalence, the global phase portrait 1.11 of Figure 1 is obtained. A realization of this phase portrait is achieved when $a = b = \beta = \mu = 1$.

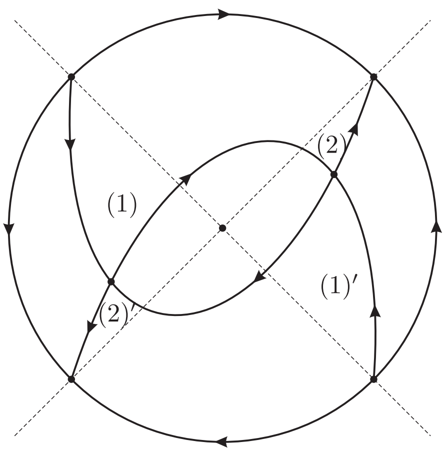

FIGURE 13. Location of the center-loop in system (V).

Lastly we investigate case (ii). Since, by Lemma 15, case (ii) leads to a system in case (iii) after a small perturbation, we conclude that the phase portrait is the one 1.12 shown in Figure 1. Due to the fact that the global phase portrait 1.3 is obtained when $a = b = \beta = 1$ and $\mu = 0$, and 1.11 if $a = b = \beta = \mu = 1$, a realization of the phase portrait 1.12 $a = b = \beta = 1$ for some μ between 0 and 1 is ensured.

9. Global phase portraits of system (VI)

System (VI)

$$\dot{x} = ax + by - 3\mu x^2 y - y^3, \tag{29a}$$

$$\dot{y} = -\frac{a^2+\beta^2}{b}x - ay + x^3 + 3\mu x y^2, \tag{29b}$$

has the Hamiltonian

$$H_6(x,y) = -\frac{y^4+x^4}{4} - \frac{3\mu}{2}x^2y^2 + \frac{a^2+\beta^2}{2b}x^2 + \frac{b}{2}y^2 + axy.$$

In the local chart U_1 system (I) writes

$$\begin{aligned} \dot u &= -v^2\left(bu^2+2au+\frac{a^2+\beta^2}{b}\right)+u^4+6u^2\mu+1,\\ \dot v &= -v^3(a+bu)+vu(u^2+3\mu). \end{aligned} \tag{30}$$

When $v=0$, the singular points of system (30) are $\left(\pm\sqrt{-3\mu\pm\sqrt{9\mu^2-1}},0\right)$. Therefore, in U_1 there are four singular points if $\mu<-1/3$, two if $\mu=-1/3$, and none if $\mu>-1/3$.

In U_2 system (VI) becomes

$$\begin{aligned} \dot u &= v^2\left(\frac{a^2+\beta^2}{b}u^2+2au+b\right)-u^4-6u^2\mu-1,\\ \dot v &= v^3\left(a+\frac{a^2+\beta^2}{b}u\right)-vu(u^2+3\mu), \end{aligned}$$

and we see that the origin is not a singular point. Hence all the infinite singular points are on the local charts U_1 and V_1. The existence of these singular points depend on the parameter μ, so we will investigate the phase portraits of system (VI) in corresponding subcases.

We first we make two remarks here. One is that just like in system (III), we can show using Gröbner basis that at most two of the finite singular points of system (VI) can be nilpotent. Secondly, system (VI) has non-isolated finite singular points only when $a=0$, $\mu=1/3$, and $b=\beta>0$. The proof is as follows: The numerator of the resultant of (29a) and (29b) with respect to x is

$$\begin{aligned} &b^2(3\mu-1)^2(1+3\mu)^2y^9-3b(3\mu-1)(1+3\mu)(2a^2\mu+2\beta^2\mu+3b^2\mu^2-b^2)y^7\\ &3\big(b^4-4b^2\beta^2\mu+(3a^4-6b^4+6a^2\beta^2+3\beta^4)\mu^2+(18a^2b^2+18b^2\beta^2)\mu^3\big)y^5\\ &-b(a^4+b^4+a^2\beta^2+6a^2b^2\mu-6b^2\beta^2\mu+9a^2\beta^2\mu^2+9\beta^4\mu^2)y^3+a^2b^2\beta^2y. \end{aligned} \tag{31}$$

For (31) to be identically zero, we need $a=0$ so that the coefficient of y is zero. Then (31) simplifies to

$$\begin{aligned} &-b^2(-1+3\mu)^2(1+3\mu)^2y^9+3b(-1+3\mu)(1+3\mu)(-b^2+2\beta^2\mu+3b^2\mu^2)y^7\\ &-3(b^2-3\mu\beta^2)(-b^2+\beta^2\mu+6b^2\mu^2)y^5+b(b^2-3\mu\beta^2)^2y^3. \end{aligned}$$

Since $b\neq 0$, coefficients of y^3 and y^9 imply that we must have $\mu=1/3$ and $b^2=\beta^2$. But when $a=0$ and $\mu=1/3$ (29a) becomes $y(b-x^2-y^2)$, meaning that b must be positive, hence $b=\beta$. This finishes the proof.

9.1. **The case $\mu<-1/3$.** In this case all of the four singular points $\big(\pm\sqrt{-3\mu\pm\sqrt{9\mu^2-1}},0\big)$ are real. The linear part of system (30) on $v=0$ is

$$\begin{pmatrix} 4u(u^2+3\mu) & 0\\ 0 & u(u^2+3\mu) \end{pmatrix}.$$

Hence, in U_1, $\left(\sqrt{-3\mu+\sqrt{9\mu^2-1}},0\right)$ and $\left(-\sqrt{-3\mu-\sqrt{9\mu^2-1}},0\right)$ are repelling nodes, whereas $\left(\sqrt{-3\mu-\sqrt{9\mu^2-1}},0\right)$ and $\left(-\sqrt{-3\mu+\sqrt{9\mu^2-1}},0\right)$

are attracting ones. The corresponding points on V_1 have the same properties since the degree of the vector field is odd.

We will now study the finite singular points. When $\mu < -1/3$, we claim that, system (VI) can have at most six finite singular points on U_3 other than the origin. For this we first show that without loss of generality we can assume $b > 0$. If we do the transformation $(x, y) \mapsto \big((x - y)/\sqrt{2}, (x + y)/\sqrt{2}\big) = (X, Y)$, i.e. rotation by $\pi/4$, then system (VI) becomes

$$\dot{X} = \frac{a^2 - b^2 + \beta^2}{2b} X + \frac{(a + b)^2 + \beta^2}{2b} Y - \frac{3 - 3\mu}{2} X^2 Y - \frac{1 + 3\mu}{2} Y^3,$$
$$\dot{Y} = -\frac{(a - b)^2 + \beta^2}{2b} X - \frac{a^2 - b^2 + \beta^2}{2b} Y + \frac{3 - 3\mu}{2} XY^2 + \frac{1 + 3\mu}{2} X^3.$$

After the rescale $dT = (1 + 3\mu)/2\ dt$, which is well defined since $\mu < -1/3$, we finally get the system

$$\begin{aligned} \dot{X} &= \frac{a^2 - b^2 + \beta^2}{b(1 + 3\mu)} X + \frac{(a + b)^2 + \beta^2}{b(1 + 3\mu)} Y - \frac{3 - 3\mu}{(1 + 3\mu)} X^2 Y - Y^3, \\ \dot{Y} &= -\frac{(a - b)^2 + \beta^2}{b(1 + 3\mu)} X - \frac{a^2 - b^2 + \beta^2}{b(1 + 3\mu)} Y + \frac{3 - 3\mu}{(1 + 3\mu)} XY^2 + X^3. \end{aligned} \tag{32}$$

If we define the variables

$$\bar{a} = \frac{a^2 - b^2 + \beta^2}{b(1 + 3\mu)}, \quad \bar{b} = \frac{(a + b)^2 + \beta^2}{b(1 + 3\mu)}, \quad \bar{\mu} = \frac{1 - \mu}{1 + 3\mu},$$

then system (32) can be rewritten as

$$\begin{aligned} \dot{X} &= \bar{a}X + \bar{b}Y - 3\bar{\mu}X^2 Y - Y^3, \\ \dot{Y} &= -\frac{\bar{a}^2 + \beta^2}{\bar{b}} X - \bar{a}Y + 3\bar{\mu}XY^2 + X^3. \end{aligned} \tag{33}$$

Note that $\bar{\mu}$ is monotone decreasing in μ, and that $\bar{\mu} < -1/3$. Hence system (33) is essentially system (VI) with $b\bar{b} < 0$, which proves our claim that we can assume $b > 0$. Now we will determine the maximum number of finite singular points.

First suppose that $a = 0$. Then system (VI) becomes

$$\dot{x} = by - 3\mu x^2 y - y^3, \quad \dot{y} = -\frac{\beta^2}{b} x + x^3 + 3\mu x y^2.$$

If we solve for y in the equation $\dot{x} = 0$, we get either $y = 0$ or $y = \pm\sqrt{b - 3\mu x^2}$. When we substitute either of the latter two values of y into $\dot{y}$ we get

$$\dot{y} = x \left((1 - 3\mu)(1 + 3\mu)x^2 + 3b\mu - \frac{\beta^2}{b} \right),$$

which is zero if and only if $x = 0$ since $b > 0$ and $\mu < -1/3$. Substituting $y = 0$ into $\dot{y}$, however, gives $\dot{y} = x(x^2 - \beta^2/b)$, which is zero when $x = 0$ or $x = \pm\sqrt{\beta^2/b}$. Therefore, when $a = 0$, there are at most four finite singular points other than the origin, namely $(0, \pm\sqrt{b - 3\mu x^2})$, $\quad (\pm\sqrt{\beta^2/b}, 0)$.

Now we consider $a \neq 0$. If we equate (29a) to zero and solve for x we obtain

$$x_{1,2} = \frac{a \pm \sqrt{a^2 + 12b\mu y^2 - 12\mu y^4}}{6\mu y}.$$

Note that when $y = 0$, (29a) becomes $\dot{x} = ax$ which is zero only if $x = 0$ because $a \neq 0$. So we can assume $y \neq 0$ because we are looking for points other than the origin. Since $\mu < -1/3$, both x_1 and x_2 are well defined. If we substitute these into (29b) we get

$$\begin{aligned}\dot{y}_{1,2} =& \frac{1}{216by^3\mu^3}\big(4a^3b - 36ay^2\mu(-b^2 + a^2\mu + \beta^2\mu) - 36aby^4\mu(1 + 3\mu^2)\\ &+ \sqrt{a^2 + 12by^2\mu - 12y^4\mu}\big(4a^2b + 12by^4\mu(-1 + 3\mu)(1 + 3\mu)\\ &+ 12y^2\mu(b^2 - 3a^2\mu - 3\beta^2\mu)\big)\big).\end{aligned}$$

Then the maximum number of roots of the product $\dot{y}_1\dot{y}_2$ will be the maximum number of finite singular points of system (VI) other than the origin. The numerator of the product $\dot{y}_1\dot{y}_2$ is

$$\begin{aligned}&b^2y^9(3\mu - 1)^2(1 + 3\mu)^2 - 3by^7(3\mu - 1)(1 + 3\mu)(2a^2\mu + 2\beta^2\mu + 3b^2\mu^2 - b^2)\\ &+3y^5\big(b^4 - 4b^2\beta^2\mu + \mu^2(3a^4 - 6b^4 + 6a^2\beta^2 + 3\beta^4) + \mu^3(18a^2b^2 + 18b^2\beta^2)\big)\\ &-by^3(a^4 + b^4 + a^2\beta^2 + 6a^2b^2\mu - 6b^2\beta^2\mu + 9a^2\beta^2\mu^2 + 9\beta^4\mu^2) + a^2b^2y\beta^2.\end{aligned} \tag{34}$$

Note that this is exactly the negative of the resultant (31). Since we are not interested in the solution $y = 0$, we can eliminate the common factor of y. Then (34) becomes an eighth order polynomial containing only even powers of y. So, with the change $z = y^2$, we can rewrite (34) as

$$\begin{aligned}&b^2(3\mu - 1)^2(1 + 3\mu)^2z^4 - 3b(3\mu - 1)(1 + 3\mu)(2a^2\mu + 2\beta^2\mu + 3b^2\mu^2 - b^2)z^3\\ &+3\big(b^4 - 4b^2\beta^2\mu + \mu^2(3a^4 - 6b^4 + 6a^2\beta^2 + 3\beta^4) + \mu^3(18a^2b^2 + 18b^2\beta^2)\big)z^2\\ &-b(a^4 + b^4 + a^2\beta^2 + 6a^2b^2\mu - 6b^2\beta^2\mu + 9a^2\beta^2\mu^2 + 9\beta^4\mu^2)z + a^2b^2\beta^2.\end{aligned} \tag{35}$$

Then the maximum number of roots of (34) is equal to the maximum number of positive roots of (35), which can be determined by the Descartes' rule of signs. First of all, the coefficient of z^4 is positive. So (35) can have four positive roots only if the coefficients of z^3 and z^2 are negative and positive, respectively. This can happen if and only if

$$A = -b^2 + 2a^2\mu + 2\beta^2\mu + 3b^2\mu^2 > 0, \tag{36}$$

$$B = -b^4 + 4b^2\beta^2\mu - 3\big((a^2 + \beta^2)^2 - 2b^4\big)\mu^2 - 18b^2(a^2 + \beta^2)\mu^3 < 0, \tag{37}$$

because we have $b > 0$ and $\mu < -1/3$. We see that A, when considered as a polynomial in μ, has one negative and one positive root. Then, since $\mu < -1/3$ and $\lim_{\mu\to-\infty} A = +\infty$, we have $A > 0$ when μ is less than the negative root

$$\mu_0 = -\frac{a^2 + \beta^2\sqrt{a^4 + 3b^4 + 2a^2\beta^2 + \beta^4}}{3b^2}.$$

On the other hand, if we apply Descartes' sign of rules to B, again when considered as a polynomial in μ, we see that it also has only one negative root, say μ_1. The limit $\lim_{\mu\to-\infty} B = +\infty$ also. So $B > 0$ when $\mu < \mu_1$. In addition, since $B = -b^4 < 0$ when $\mu = 0$, we have $B < 0$ when $\mu_1 < \mu < 0$. If we evaluate B at μ_0, we get

$$\frac{1}{3b^4}\big(6(a^2+\beta^2)^4 + b^4(19a^4+3b^4+34a^2\beta^2+15\beta^4) \\ + 2\big(3(a^2+\beta^2)^3 + b^4(5a^2+3\beta^2)\big)\sqrt{a^4+3b^4+2a^2\beta^2+\beta^4}\big) > 0.$$

This means that $\mu_0 < \mu_1$. But then, when $A > 0$, that is when $\mu < \mu_0$, we have $B > 0$, meaning that (36) and (37) cannot hold together. Hence (35) cannot have four positive roots, which in turn implies that system (VI) can have at most six, but not eight, finite singular points other than the origin.

We saw that when $\mu < -1/3$, system (VI) has eight nodes for infinite singular points on the Poincaré sphere. Together with the centers at the origins on the upper and lower hemispheres, their total index is 10. Hence, the total index of the remaining finite singular points on the Poincaré disk must be -4. Therefore there are only two possibilities: either (i) four saddles, or (ii) four saddles and two cusps. But by Lemma 15, case (ii) cannot exist. Hence we are only left with (i).

We first assume that there are only two saddles on the boundary of the period annulus of the center at the origin. As in system (III), these saddles cannot be connected with the other saddles, see Figure 9. So they must be connected directly with an infinite singular point because there are no other finite singular points. Moreover, the remaining separatrices of these saddles cannot cross the straight lines $y = \big(\pm\sqrt{-3\mu \pm \sqrt{9\mu^2-1}x}\big)$ which pass through the origin and the infinite singular points. This is due to the fact that on these lines the Hamiltonian H_6 becomes

$$\frac{x^2}{2}\left(\frac{a^2+\beta^2}{b} + b(-3\mu \pm \sqrt{9\mu^2-1}) \pm 2a\sqrt{-3\mu \pm \sqrt{9\mu^2-1}}\right),$$

which is quadratic so that $H_6 = c$ has at most two real roots for any $c = \mathrm{R}$. In addition, there cannot be any singular points on these lines. On $y = -\sqrt{-3\mu - \sqrt{9\mu^2-1}x}$ the finite singular points of system (VI) are given by the equations

$$\begin{aligned}\dot{x} = &\left(a + b\sqrt{-3\mu - \sqrt{9\mu^2-1}}\right)x \\ &+ \sqrt{9\mu^2-1}\sqrt{-3\mu-\sqrt{9\mu^2-1}x}^3 = 0\end{aligned}, \tag{38a}$$

$$\begin{aligned}\dot{y} = -&\left(\frac{a^2+\beta^2}{b} + a\sqrt{-3\mu-\sqrt{9\mu^2-1}}\right)x \\ &+ \sqrt{9\mu^2-1}(-3\mu - \sqrt{9\mu^2-1})x^3 = 0\end{aligned}, \tag{38b}$$

If we multiply (38a) by $\sqrt{-3\mu - \sqrt{9\mu^2-1}}$ and then subtract if from (38b) we get

$$-\frac{a^2+\beta^2}{b} + b(3\mu + \sqrt{9\mu^2-1}) = 0,$$

which holds only when $x = 0$ because the coefficient of x^2 is strictly negative due to the fact that we have $b > 0$ and $\mu < -1/3$. Similar calculations give the same result for the straight line $y = \sqrt{-3\mu - \sqrt{9\mu^2 - 1}}x$.

Having established the above properties, we get the global phase portrait 1.13 of Figure 1. This phase portrait is realized for the values $a = -b = \beta = -\mu = 1$.

Next we assume that all of the saddles are on the boundary of the period annulus of the center at the origin. Because there are no other finite singular points and the remaining separatrices of the saddles cannot cross the straight lines passing through the origin and the infinite singular points, all the saddles must go to the infinite singular points as shown in the global phase portrait 1.14 of Figure 1. If, for instance, $a = 0$ and $b = \beta = -\mu = 1$, one actually obtains a topologically equivalent phase portrait. This finishes the case $\mu < -1/3$.

9.2. **The case $\mu = -1/3$.** When $\mu = -1/3$ the linear part of system (30) at both of the singular points $(\pm 1, 0)$ is zero. So we need blow-ups to understand the local behavior at these points. We will do the computations for the point $(1, 0)$, and the other point $(-1, 0)$ can be studied in the same way.

First we move $(1, 0)$ to the origin by the shift $u \mapsto u + 1$, and get the system

$$\begin{aligned}
\dot{u} &= u^2(u+2)^2 - v^2\left(bu^2 + 2(a+b)u + 2a + b + \frac{a^2+\beta^2}{b}\right),\\
\dot{v} &= uv(u+1)(u+2) - v^3(bu + a + b).
\end{aligned}$$

Now if we do the blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$ and eliminate the common factor u, we get the system

$$\begin{aligned}
\dot{u} &= u(u+2)^2 - uw^2\left(bu^2 + 2(a+b)u + 2a + b + \frac{a^2+\beta^2}{b}\right),\\
\dot{w} &= -w(u+2) + w^3\left((a+b)u + 2a + b + \frac{a^2+\beta^2}{b}\right).
\end{aligned} \tag{39}$$

We see that in system (39), when $u = 0$ we have $\dot{w} = 0$ if and only if $w = 0$ or

$$w = \pm\frac{\sqrt{2}}{\sqrt{2a + b + \dfrac{a^2+\beta^2}{b}}} = \pm\frac{\sqrt{2b}}{\sqrt{(a+b)^2+\beta^2}}.$$

So, on the w-axis, system (39) has one singular point if $b < 0$, and three otherwise. The linear part of system (39) when $u = 0$ is

$$\begin{pmatrix} 4 - \left(2a + b + \dfrac{a^2+\beta^2}{b}\right)w^2 & 0 \\ -w + w^3(a+b) & -2 + 3\left(2a + b + \dfrac{a^2+\beta^2}{b}\right)w^2 \end{pmatrix}.$$

Hence the origin is a saddle, whereas the other two singular points, when they exist (depending on the sign of the parameter b), are repelling nodes.

Therefore the local phase portrait at the singular point $(1,0)$ of system (30), similar to that of system (III) at the origin of U_1 (see Figure 5 and Figure 6), consists of two hyperbolic sectors when $b<0$, and two parabolic and two elliptic ones when $b>0$.

Performing the same procedure for the point $(-1,0)$ reveals that the local behavior around this point is the same as that of $(1,0)$ with the direction of the flow reversed.

Now we analyze the finite singular points. We will show that in this case also system (VI) can have at most six finite singular points other than the origin. When $y \neq 0$, the number of finite singular points is given by the number of roots of (34), which is at most four since $\mu = -1/3$. When $y = 0$, on the other hand, system (VI) becomes

$$\dot{x} = ax,$$
$$\dot{y} = x\left(x^2 - \frac{a^2+\beta^2}{b}\right),$$

which has at most two finite singular points other than the origin, proving the claim.

Next we count the indices of the singular points. We note that in this case we have to distinguish the phase portraits when $b<0$ and when $b>0$.

Case 1 $(b<0)$: In this case the infinite singular points and the centers at the origins of U_1 and V_1 have a total index of 2 on the Poincaré sphere. Hence, in the Poincaré disk, the total index of the remaining possible six finite singular points must be 0. Then, other than the origin, there are either (i) no more singular points, (ii) two cusps, (iii) 2 saddles and 2 centers, (iv) 2 saddles, 2 centers and 2 cusps.

We can immediately eliminate case (iv) since by Lemma 15, it would require the existence of a case with eight such points which is not possible. The global phase portrait in case (i) is also immediate, see the phase portrait 1.15 in Figure 1.

Consider case (iii). The two saddles must be on the boundary of the period annulus of the center at the origin. Note that they cannot be located on the straight lines $y=\pm x$ which pass through the origin and the infinite singular points. On $y=x$, for instance, system (VI) becomes

$$\dot{x} = (a+b)x,$$
$$\dot{y} = -\left(\frac{a^2+\beta^2}{b} + a\right)x,$$

and both polynomials are zero only when $x=0$, which implies that the only finite singular point on the straight line $y=x$ is the origin. The same holds for the straight line $y=-x$. Moreover, the remaining separatrices of these saddles neither can return the same point, see Figure 8, nor can cross $y=\pm x$ because on the lines $y=\pm x$ the Hamiltonian H_6 becomes

$$H_6 = \frac{(a\pm b)^2+\beta^2}{2b}x^2,$$

meaning that it can have the same fixed value at most at two points. This means that those separatrices must go to infinite singular points, yet the infinite singular points only have hyperbolic sectors. Therefore case (*iii*) is eliminated too.

Since case (*ii*) cannot exist without case (*iii*), the only global phase portrait of system (VI) when $\mu = -1/3$ and $b < 0$ is the one in case (*i*).

Case 2 ($b > 0$): When $b > 0$ the total index of the infinite singular points and the finite centers on the Poincaré sphere is 10. Hence the total index of the remaining finite singular points on the Poincaré disk must be -4. Since there are at most six such points, by Lemma 15 there has to be four saddles.

We first consider the possibility that only two of the saddles are on the boundary of period annulus of the center at the origin. As always, these saddles cannot be connected with the remaining two. So, they must go to the infinite singular points. Remember that these saddles are not on the straight lines $y = \pm x$, and that their separatrices cannot cross these lines anymore. Then, in accordance with the fact that the infinite singular points have elliptic sectors, the global phase portrait follows, see 1.16 in Figure 1. Setting $a = b = \beta = 1$ provides a realization of such a phase portrait.

If all of the saddles are on the boundary of the center at the origin, then due to the same reasons in the previous case, the global phase portrait 1.17 of Figure 1 is obtained. This phase portrait is achieved for $a = 0$ and $b = \beta = 1$.

9.3. **The case $\mu > -1/3$.** Finally, when $\mu > -1/3$, system (VI) has no infinite singular points since $-3\mu \pm \sqrt{9\mu^2 - 1}$ is not real when $-1/3 < \mu < 1/3$, and is negative when $\mu \geq 1/3$.

As for the finite singularities, since we already have the centers at the origins of U_3 and V_3 on the Poincaré sphere, the total index of the remaining possible eight finite singular points on the Poincaré disk must be 0. Hence we have the following possibilities: (*i*) no singular points, (*ii*) 2 cusps, (*iii*) two saddles and two centers, (*iv*) two saddles, two centers, and two cusps, (*v*) four saddles and four centers. Of course if $\mu = 1/3$, there are non-isolated singular points when $a = 0$ and $b = \beta$, so we will study this case separately. Note that case (*ii*) is immediately discarded by the proof of statement (*c*) of Lemma 15 showing that there cannot be a cusp on the boundary of the period annulus of the center at the origin.

Case (*i*) can occur only if $b < 0$ so that the flow around the origin and the infinity match. Then we easily get the phase portrait 1.18 of Figure 1. A realization of this phase portrait is attained when $a = -b = \beta = \mu = 1$.

In case (*iii*) both of the saddles must be on the boundary of the period annulus of the center at the origin, as usual. Then, the centers can only be created if different saddles are connected once again, to avoid having six points on a straight line through the origin which are on the same energy level, see Figure 8. Hence the phase portrait turns out to be topologically equivalent to 1.19 as shown in Figure 1. One actually obtains such a phase portrait if $a = b = \beta = 1$ and $\mu = 0$.

Next we consider case (*v*). Assume first that all four saddles are on the boundary of the period annulus of the center at the origin. The remaining separatrices of any of the saddles must be on different sides of the straight line passing through that sadle and the origin, or else one could find a straight line l through the origin, passing close enough to the saddle and intersecting three of the separatrices of the saddle, which would, taking into account the symmetry, lead to the existence of six points on l which are all in the same energy level. Consequently we obtain the global phase portrait 1.20 of Figure 1. If we take $a = 0$, $b = \beta = 1$ and $\mu = -1/4$, we get the phase portrait 1.20 up to topological equivalence.

Now assume that only two of the saddles are on the boundary of the period annulus of the center at the origin. These saddles cannot be connected with the other saddles, see Figure 9. Since there are no infinite singular points and Figure 8 is not allowed, their remaining separatrices must coincide symmetrically with respect to the origin. For the remaining two saddles we have two possibilities on their locations: they are either outside the region enclosed by the separatrices of the saddles which are on the boundary of the period annulus of the center at the origin, or inside. Assume that they are outside. If each of them had two separatrices that returned to the same point, there would be a total of six centers in the Poincaré disk, two more than the actual number. So, these saddles must be connected with each other at least once. Then, just like the saddles around the origin, their remaining separatrices must also coincide in order to avoid Figure 8. Then, because of the counterclockwise flow at infinity, we must have $b < 0$. This means that on the positive y-axis (29a) becomes $\dot{x} = by - y3 < 0$ which contradicts the phase portrait. Hence the assumption that these saddles are outside the region enclosed by the separatrices of the saddles around the origin is false. Therefore they must be inside, and we have the global phase portrait 1.21 of Figure 1. The values $a = b = \beta = 1$ and $\mu = -1/4$ provides a realization of this phase portrait.

Consider case (iv). By Lemma 15 it would lead to the global phase portrait in case (v). Therefore the only possibility is that the global portrait in this case must be the one 1.22 of Figure 1. The facts that the global phase portrait 1.19 is obtained when $a = b = \beta = 1$ and $\mu = 0$, and that 1.21 is achieved if $a = b = \beta = 1$ and $\mu = -1/4$ lead to the conclusion that the phase portrait 1.22 is realized when $a = b = \beta = 1$ for some μ between 0 and -1/4.

Finally we consider the case $\mu = 1/3$, $a = 0$ and $b = \beta$. In this case system (VI) becomes

$$\begin{aligned}\dot{x} &= y(\beta - x^2 - y^2),\\ \dot{y} &= x(-\beta + x^2 + y^2).\end{aligned}$$

We see that other than the origin, the circle $x^2 + y^2 = \beta$ is a set of non-isolated singular points. Then the phase portrait 1.23 of Figure 1 is easily obtained.

Acknowledgments

The first author has been supported by AGAUR FI-DGR 2010. The second author has been supported by the grants MICINN/FEDER MTM

2009-03437, AGAUR 2009SGR 410, ICREA Academia and FP7-PEOPLE-2012-IRSES-316338. The third author has been supported by AGAUR PIV-DGR-2010, FCT grant PTDC/MAT/117106/2010 and through CAMGSD.

References

[1] R. Abraham and J.E. Marsden, "*Foundations of Mechanics*", Second Edition, Revised and Enlarged, Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, Mass., 1978.

[2] N.N. Bautin, "*On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type*", Mat. Sb. **30** (1952), 181-196; Mer. Math. Soc. Transl. **100** (1954) 1-19.

[3] J. Chavarriga and J. Giné, "*Integrability of a linear center perturbed by a fourth degree homogeneous polynomial*", Publ. Mat. **40** (1996), 21-39.

[4] J. Chavarriga and J. Giné, "*Integrability of a linear center perturbed by a fifth degree homogeneous polynomial*", Publ. Mat. **41** (1997), 335-356.

[5] J. Chavarriga, H. Giacomini, J. Giné and J. Llibre, "*Local analytic integrability for nilpotent centers*", Ergod. Th. & Dynam. Sys. **23** (2003), 417-428.

[6] J. Chazy, "*Sur la thèorie de centres*", C. R. Acad. Sci. Paris **221** (1947), 7-10.

[7] F. Chen, C. Li, J. Llibre and Z. Zhang "*A unified proof on the weak Hilbert 16th problem for n=2*", J. Differential Equations **221** (2006), 309-342.

[8] A. Cima and J. Llibre, "*Algebraic and topological classification of the homogeneous cubic vector fields in the plane*", J. Math. Anal. and Appl. **147** (1990), 420-448.

[9] H. Dulac, "*Détermination et integration d'une certaine classe d'équations différentielle ayant par point singulier un centre,*", Bull. Sci. Math. Sér (2) **32** (1908), 230-252.

[10] F. Dumortier, J. Llibre and J. C. Artés, "*Qualitative Theory of Planar Differential Systems*", UniversiText, Springer-verlag, New York, 2006.

[11] H. Giacomini, J. Giné and J. Llibre, "*The problem of distinguishing between a center and a focus for nilpotent and degenerate analytic systems*", J. Differential Equations **227** (2006), 406-426; J. Differential Equations **232** (2007), 702.

[12] W. Kapteyn, "*On the midpoints of integral curves of differential equations of the first Degree*", Nederl. Akad. Wetensch. Verslag Afd. Natuurk. Konikl. Nederland (1911), 1446-1457 (in Dutch).

[13] W. Kapteyn, "*New investigations on the midpoints of integrals of differential equations of the first degree*", Nederl. Akad. Wetensch. Verslag Afd. Natuurk. **20** (1912), 1354-1365; Nederl. Akad. Wetensch. Verslag Afd. Natuurk. **21** (1913) 27-33 (in Dutch).

[14] M.A. Liapunov, "*Problème général de la stabilité du mouvement*", (Ann. Math. Stud., 17) Princeton University Press, 1947.

[15] K.E. Malkin, "*Criteria for the center for a certain differential equation*", Vols. Mat. Sb. Vyp. **2** (1964), 87-91 (in Russian).

[16] R. Moussu, "*Une démonstration d'un théorème de Lyapunov-Poincaré*", Astérisque **98-99** (1982), 216-223.

[17] D. Neumann, "*Classification of continuous flows on 2-manifolds*", Proc. Amer. Math. Soc. **48** (1975), 73-81.

[18] H. Poincaré, "*Mémoire sur les courbes définies par les équations différentielles*", Oeuvreus de Henri Poincaré, vol. 1, Gauthier-Villars, Paris, 1951, pp. 95-114.

[19] H. Poincaré, "*Mémoire sur les courbes définies par les équations différentielles*", J. Mathématiques **37** (1881), 375-422; Oeuvreus de Henri Poincaré, vol. 1, Gauthier-Villars, Paris, 1951, pp. 3-84.

[20] D. Schlomiuk, "*Algebraic particular integrals, integrability and the problem of the center*", Trans. Amer. Math. Soc. **338** (1993), 799-841.

[21] C. Rousseau and D. Schlomiuk, "*Cubic vector fields symmetric with respect to a center*", J. Differential Equations **123** (1995), 388-436.

[22] N.I. Vulpe and K.S. Sibirskii, "*Centro-affine invariant conditions for the existence of a center of a differential system with cubic nonlinearities*", Dokl. Akad. Nauk.

SSSR **301** (1988), 1297-1301 (in Russian); translation in: Soviet Math. Dokl. **38** (1989) 198-201.

[23] H. Zoladek, "*Quadratic systems with center and their perturbations*", J. Differential Equations **109** (1994), 223-273.

[24] H. Zoladek, "*The classification of reversible cubic systems with center*", Topol. Methods Nonlinear Anal. **4** (1994), 79-136.

[25] H. Zoladek, "*Remarks on: 'The classification of reversible cubic systems with center' [Topol. Methods Nonlinear Anal.* **4** *(1994), 79-136]*", Topol. Methods Nonlinear Anal. **8** (1996), 335-342.

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: ilkercolak@mat.uab.cat

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: jllibre@mat.uab.cat

Departamento de Matemática, Instituto Superior Técnico, Universidade Técnica de Lisboa, 1049-001 Lisboa, Portugal

E-mail address: cvalls@math.ist.utl.pt