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ZERO-HOPF BIFURCATION FOR A CLASS OF LORENZ-TYPE SYSTEMS

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Abstract. We apply the averaging theory to a class of three-dimensional autonomous quadratic polynomial differential systems known as Lorenz-type systems, and we prove the existence of a small amplitude periodic orbit bifurcating from a degenerate zero-Hopf equilibrium of these systems.

1. Introduction and statement of the main results

A generic Hopf bifurcation is a local bifurcation where a limit cycle bifurcates from an equilibrium point when one pair of complex eigenvalues cross the imaginary axis. There are many others non-generic or degenerate Hopf bifurcations. One of them is called the zero-Hopf bifurcation. *A zero-Hopf equilibrium* is an equilibrium point of a three-dimensional autonomous differential system, which has a zero eigenvalue and a pair of purely imaginary eigenvalues. Many different kind of bifurcations can take place in a zero-Hopf equilibrium. When a periodic orbit bifurcates from a zero-Hopf equilibrium we will say that the system exhibits a *zero-Hopf bifurcation*. It has been studied by many authors among them Guckenheimer, Holmes, Scheurle, Han and Kuznetsov in [2, 3, 4, 6].

In [5] the authors study three-dimensional autonomous quadratic polynomial differential systems having the four basic qualitative properties of the classical Lorenz system, that is:

- Symmetry with respect to the z-axis.
- Dissipation and existence of an attractor: As $t \to \infty$ all solutions of the differential system approach to a set of zero volume, i.e.,

$$\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} < 0.$$

- Pitchfork bifurcation. For certain choice of the parameters, the origin is an equilibrium point and from it bifurcate other two new equilibria. The equilibrium point at the origin changes its stability at the bifurcating point.
- The system can also exhibit Hopf bifurcations, homoclinic or heteroclinic orbits, and chaotic attractor.

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The quadratic systems having the above properties are called simply *Lorenz-type systems*, in general these kind of differential systems can have more than one chaotic attractor. Starting with a general 3–dimensional quadratic polynomial system of ordinary differential equations, the authors of [5] obtain necessary conditions to have a Lorenz-type system. More precisely, in order to have a Lorenz-type system, it should be written in the form:

$$\begin{aligned} \dot{X} =& AX + zBX, \\ \dot{z} =& a_1 z + X^T CX, \end{aligned} \tag{1}$$

where A is any 2×2 real matrix, B and C are non-zero 2×2 real matrices, the trace $\mathrm{tr}(B) = 0$, and $a_1 < -\mathrm{tr}(A)$ with $a_1 \neq 0$.

In this paper we are interested in studying a Hopf-bifurcation at a zero-Hopf equilibrium point, so in this way we consider the particular choice:

$$A = \begin{pmatrix} \alpha\varepsilon^2 & -\beta \\ \beta & \alpha\varepsilon^2 \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad C = \begin{pmatrix} d & e \\ g & -d \end{pmatrix}, \quad a_1 = \gamma\varepsilon^2, \quad \beta \neq 0.$$

In our case the condition $a_1 < -\mathrm{tr}(A)$ is equivalent to $\gamma < -2\alpha$ and in our analysis this condition is not necessary in order to have a Lorenz-type differential systems with a zero-Hopf bifurcation as we will show ahead in this paper.

In short, with the above choice of matrices and parameters the Lorenz-type differential system (1) becomes

$$\begin{aligned} \dot{x} =& \varepsilon^2 \alpha x - \beta y + axz + byz, \\ \dot{y} =& \beta x + \varepsilon^2 \alpha y + cxz - ayz, \\ \dot{z} =& \varepsilon^2 \gamma z + dx^2 + (e+g)xy - dy^2. \end{aligned} \tag{2}$$

Now we can write the main result of this paper.

Theorem 1. *For $\varepsilon > 0$ sufficiently small the class of Lorenz-type systems* (2) *with $\alpha, \beta, \gamma, a, b, c, d, e, g$ arbitrary parameters such that $4\alpha + \gamma \neq 0$ and $\alpha\beta[a(e+g) - d(b+c)] > 0$, has a small amplitude periodic orbit bifurcating from the equilibrium point located at the origin when $\varepsilon = 0$.*

In order to have a self-contained paper we have included, without proofs, some basic results on the second order averaging method, these are contained in section 2. In section 3 we prove Theorem 1.

2. Results from averaging theory

It is well known the importance of averaging theory in mathematics, it is particularly important when we are looking for periodic orbits of differential systems (see for instance [1], [3], [7], [8], [9], [10] and references therein).

The next theorem provides a second order approximation for the limit cycles of a periodic system when its average vanishes at first order. For a more complete statement see [7], for a proof see Theorem 3.5.1 of Sanders and Verhulst [9], or [1].

Consider the differential system

$$\dot{x} = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 Q(t,x,\varepsilon), \tag{3}$$

where $F_1, F_2 \colon \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $Q\colon [0,\infty] \times \Omega \times (0,\varepsilon_0] \to \mathbb{R}^n$ where Ω is an open subset of $\mathbb{R}^n$, such that F_1, F_2 and Q are T-periodic in the first variable. We set

$$\begin{aligned} F_{10}(x) &= \frac{1}{T}\int_0^T F_1(t,x)\,dt, \qquad (4)\\ F_{20}(x) &= \frac{1}{T}\int_0^T [D_x F_1(t,x)\cdot y_1(t,x) + F_2(t,x)]\,dt, \end{aligned}$$

where

$$y_1(t,x) = \int_0^t F_1(s,x)ds.$$

Theorem 2. *Assume that:*

(i) $F_{10} = 0$*;* $\partial F_1/\partial x$*,* F_2 *are locally Lipschitz in* x *and* Q *is twice differentiable with respect to* ε*.*

(ii) *For* $V \subset \Omega$ *and open and bounded set, and for each* $\varepsilon \in (-\varepsilon_f, \varepsilon_f)\setminus\{0\}$*, there exists* $a_\varepsilon \in V$ *such that* $F_{10}(a_\varepsilon) + \varepsilon F_{20}(a_\varepsilon) = 0$ *and* $d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0$*.*

Then for $|\varepsilon| > 0$ *sufficiently small there exists a* T*–periodic solution* $\varphi(t,\varepsilon)$ *of the system such that* $\varphi(0,\varepsilon) = a_\varepsilon$*.*

The expression $d_B(F_{10}+\varepsilon F_{20}, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $F_{10} : V \to \mathbb{R}^n$ at the fixed point a_ε is not zero. A sufficient condition for the inequality holds is that the Jacobian of the function $F_{10} + \varepsilon F_{20}$ (if exists) at a_ε be not zero.

If F_{10} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20}$ are essentially the zeros of F_{10} for ε sufficiently small. In this case we have the *averaging theory of first order*. When F_{10} is identically zero but F_{20} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20}$ are the zeros of F_{20}. In this case we have the *averaging theory of second order*. This last case is the result that we will use in this paper.

3. Proof of Theorem 1

It is easy to verify that system (2) has a zero-Hopf equilibrium at the origin for $\varepsilon = 0$, i.e., its eigenvalues are 0, $\pm\beta i$. For $0 < \varepsilon \ll 1$, we introduce cylindrical coordinates in order to look for periodic orbits

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z,\ r > 0.$$

Then system (2) becomes

$$\begin{aligned} \dot{r} &= \varepsilon^2\alpha r + \frac{1}{2} rz\,[2a\cos(2\theta) + (b+c)\sin(2\theta)],\\ \dot{\theta} &= (cz+\beta)\cos^2\theta - az\sin(2\theta) + (\beta - bz)\sin^2\theta, \qquad (5)\\ \dot{z} &= \varepsilon^2\gamma z + \frac{r^2}{2}[2d\,\cos(2\theta) + (e+g)\,\sin(2\theta)]. \end{aligned}$$

Now, taking θ as the new independent variable system (5) takes the form

$$\begin{aligned}
\frac{dr}{d\theta} &= \frac{\varepsilon^2 \alpha r + \dfrac{rz}{2}\left[2a\cos(2\theta) + (b+c)\sin(2\theta)\right]}{(cz+\beta)\cos^2\theta - az\sin(2\theta) + (\beta - bz)\sin^2\theta}, \\
\frac{dz}{d\theta} &= \frac{\varepsilon^2 \gamma z + \dfrac{r^2}{2}\left[2d\cos(2\theta) + (e+g)\sin(2\theta)\right]}{(cz+\beta)\cos^2\theta - az\sin(2\theta) + (\beta - bz)\sin^2\theta}.
\end{aligned} \tag{6}$$

The key idea for applying the averaging theory and to show the existence of a small amplitude periodic orbit bifurcating from the origin of system (2) when $\varepsilon = 0$ is to do the transformation

$$r = \varepsilon R, \quad z = \varepsilon Z. \tag{7}$$

We shall see that we can apply the averaging theory to the differential system ($dR/d\theta$, $dZ/d\theta$) and the periodic orbits of this system going back through the re-scaling (7) will be periodic orbits of system (6) tending to the origin when $\varepsilon \to 0$. So these periodic orbits in the coordinates (x, y, z) also will tend to the origin when $\varepsilon \to 0$, and they will provide the small amplitude periodic orbits borning at the origin when $\varepsilon = 0$.

Doing the change of variables (7) into system (6) and expanding it in power series of ε, we get

$$\begin{aligned}
\frac{dR}{d\theta} &= \varepsilon \frac{RZ}{2\beta}\left[2a\cos(2\theta) + (b+c)\sin(2\theta)\right] + \varepsilon^2 \frac{R}{8\beta^2}\big[8\alpha\beta \\
&\quad + Z^2\big(4a(b-c)\cos(2\theta) - 4a(b+c)\cos(4\theta) + 2(b^2-c^2)\sin(2\theta) \\
&\quad + (4a^2 - (b+c)^2)\sin(4\theta)\big)\big] + O(\varepsilon^3) \\
&= \varepsilon F_{11}(\theta, R, Z) + \varepsilon^2 F_{21}(\theta, R, Z) + Q_1(\theta, R, Z, \varepsilon), \\
\frac{dZ}{d\theta} &= \varepsilon \frac{R^2}{2\beta}\left[2d\cos(2\theta) + (e+g)\sin(2\theta)\right] + \varepsilon^2 \frac{Z}{8\beta^2}\big[2\left(a(e+g)\right. \\
&\quad \left. - d(b+c)\right)R^2 + 8\beta\gamma + R^2\left(4d(b-c)\cos(2\theta)\right. \\
&\quad - 2\left(d(b+c) + a(e+g)\right)\cos(4\theta) \\
&\quad \left. + 2(b-c)(e+g)\sin(2\theta) + (4ad - (b+c)(e+g))\sin(4\theta)\right)\big] + O(\varepsilon^3), \\
&= \varepsilon F_{12}(\theta, R, Z) + \varepsilon^2 F_{22}(\theta, R, Z) + Q_2(\theta, R, Z, \varepsilon).
\end{aligned} \tag{8}$$

Now we can apply the averaging method. The functions $F_{ij}(\theta, R, Z)$ and $Q_i(\theta, R, Z, \varepsilon)$ for $i, j = 1, 2$, defined in system (8) are 2π-periodic with respect to θ, and according to (3), we have $t = \theta$, $x = (R, Z)$,

$$F_1(\theta, R, Z) = \begin{pmatrix} F_{11} \\ F_{12} \end{pmatrix}, \quad F_2(\theta, R, Z) = \begin{pmatrix} F_{21} \\ F_{22} \end{pmatrix}, \quad Q(\theta, R, Z, \varepsilon) = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}.$$

For $\varepsilon > 0$ sufficiently small and in a neighborhood of the origin the functions which appear in equation (8) are analytic and therefore locally Lipschitz, and obviously $Q(\theta, R, Z, \varepsilon)$ is twice differentiable with respect to ε. So the assumption (i) of the averaging Theorem 2 holds. According with the assumption (ii) of that theorem we must compute the function F_{10}.

The averaging function F_{10} defined in (4) in our case is

$$F_{10}(R,Z) = \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta,R,Z)\,d\theta = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{9}$$

So according with section 2 we must compute now the function F_{20}. Thus we compute

$$\begin{aligned} M(\theta,R,Z) &= \frac{\partial\, F_1(\theta,R,Z)}{\partial(R,Z)} \\ &= \begin{pmatrix} \dfrac{Z\big(2a\cos 2\theta + (b+c)\sin 2\theta\big)}{2\beta} & \dfrac{R\big(2a\cos 2\theta + (b+c)\sin 2\theta\big)}{2\beta} \\ \dfrac{R(2d\cos 2\theta + (e+g)R\sin 2\theta}{\beta} & 0 \end{pmatrix}, \\ y_1(\theta,R,Z) &= \int_0^{\theta} F_1(t,R,Z)dt \\ &= \begin{pmatrix} \dfrac{RZ\sin\theta\big(2a\cos\theta + (b+c)\sin\theta\big)}{2\beta} \\ \dfrac{R^2\sin\theta\big(2d\cos\theta + (e+g)\sin\theta\big)}{2\beta} \end{pmatrix}. \end{aligned}$$

Then (4) implies

$$\begin{aligned} F_{20}(R,Z) &= \frac{1}{2\pi}\int_0^{2\pi} [M(s,R,Z)y_1(s,R,Z) + F_2(s,R,Z)]\,ds, \\ &= \begin{pmatrix} \dfrac{R\big(8\alpha\beta + R^2(d(b+c) - a(e+g))\big)}{8\beta^2} \\ \dfrac{Z\big(2\beta\gamma + R^2(a(e+g) - d(b+c))\big)}{2\beta^2} \end{pmatrix}. \end{aligned} \tag{10}$$

Since we are using polar coordinates, $R > 0$, thenfrom (10) it follows that the unique zero of F_{20} is

$$R^* = 2\sqrt{\frac{2\alpha\beta}{a(e+g) - d(b+c)}}, \quad Z^* = 0. \tag{11}$$

After straightforward computations it follows from (10) and (11) that

$$\det\left(\frac{\partial(F_{20})}{\partial(R,Z)}(R^*,Z^*)\right) = -\frac{2\alpha(4\alpha+\gamma)}{\beta^2}.$$

Finally, by Theorem 2 we obtain the existence of a 2π periodic solution $(R(\theta,\varepsilon),\ Z(\theta,\varepsilon))$ of system (8) for $\varepsilon > 0$ and sufficiently small, such that $(R(0,\varepsilon),\ R(0,\varepsilon)) \to (R^*, Z^*)$ when $\varepsilon \to 0$. Because of the change of variables (7), in the plane $(r(\theta), z(\theta))$ the system (6) also has the periodic solution $(\varepsilon R(\theta,\varepsilon), \varepsilon Z(\theta,\varepsilon))$ for every $\varepsilon > 0$ and sufficiently small. As a consequence of the above, this family of periodic solutions tend to the origin when $\varepsilon \to 0$. Of course, the same apply for the system (2) in coordinates $(x(t), y(t), z(t))$. This concludes the proof that Lorenz-type system (2) for $\varepsilon > 0$ sufficiently small has a small amplitud periodic orbit bifurcating from the equilibrium point localized at the origin of coordinates when $\varepsilon = 0$.

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