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CENTERS OF QUASI–HOMOGENEOUS POLYNOMIAL DIFFERENTIAL EQUATIONS OF DEGREE THREE

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Abstract. We characterize the centers of the quasi–homogeneous planar polynomial differential systems of degree three. Such systems do not admit isochronous centers. At most one limit cycle can bifurcate from the periodic orbits of a center of a cubic homogeneous polynomial system using the averaging theory of first order.

1. Introduction and statement of the results

Poincaré in [25] was the first to introduce the notion of a center for a vector field defined on the real plane. So according to Poincaré a center is a singular point surrounded by a neighborhood filled of closed orbits with the unique exception of the singular point.

Since then the center–focus problem, i.e. the problem to distinguish when a singular point is either a focus or a center is one of the hardest problem in the qualitative theory of planar differential systems, see for instance [1] and the references quoted there. This paper deals mainly with the characterization of the centers problem for the class of quasi–homogeneous polynomial differential systems of degree 3.

In the literature we found classifications of polynomial differential systems having a center. For the quadratic systems we refer to the works of Dulac [10], Kapteyn [13, 14], Bautin [3] among others. In [28] Schlomiuk, Guckenheimer and Rand gave a brief history of the center problem for quadratic systems.

There are many partial results about centers for polynomial differential systems of degree greater than two. Some of them (closed to our work) are for instance, the classification by Malkin [19] and Vulpe and Sibirskii [28] about the centers for cubic polynomial differential systems of the form linear with homogeneous nonlinearities of degree three. Note that for polynomial differential systems of the form linear with homogeneous nonlinearities of degree $k > 3$ the centers are not classified. However, there are some results for $k = 4, 5$ see for instance the works by Chavarriga and Giné [7, 8]. It seems difficult for the moment to obtain a complete classification of the centers for the class of all polynomial differential systems of degree 3. Actually, there are some subclasses of cubic systems well studied like the ones of Rousseau and Schlomiuk [26] and the ones of Zołądek [31, 32]. Some centers for arbitrary degree polynomial differential systems have been studied in [18].

In what follows we denote by $\mathbb{R}[x, y]$ the ring of all polynomials in the variables x and y and coefficients in the real numbers $\mathbb{R}$. In this work we consider polynomial differential systems of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

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with $P, Q \in \mathbb{R}[x, y]$ and its corresponding vector field $\mathcal{X} = (P, Q)$. Here the dot denotes derivative with respect to the time t (independent variable). The *degree* of the differential polynomial system (1) is the maximum of the degrees of the polynomials P and Q.

System (1) is a *quasi–homogeneous polynomial differential system* if there exist natural numbers s_1, s_2, d such that for an arbitrary non–negative real number α it holds

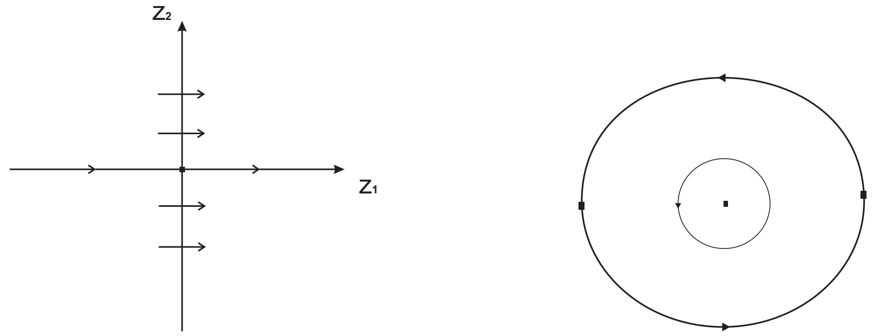
$$P(\alpha^{s_1}x,\ \alpha^{s_2}y) = \alpha^{s_1+d-1}P(x, y), \qquad Q(\alpha^{s_1}x,\ \alpha^{s_2}y) = \alpha^{s_2+d-1}Q(x, y). \tag{2}$$

The natural numbers s_1 and s_2 are the *weight exponents* of system (1) and d is the *weight degree* with respect to the weight exponents s_1 and s_2. When $s_1 = s_2 = s$ then we obtain the classical *homogeneous polynomial differential system* of degree $s + d - 1$.

It is well known that all quasi–homogeneous vector fields are integrable with a Liouvillian first integral [11, 12, 16].

From Theorem 2 of [17] we have that there are only two families of cubic polynomial differential homogeneous systems with a center.

In the next result we characterize all the centers of quasi–homogeneous polynomial differential systems.



(a) $(z_1, z_2) = (0, 0)$ on the $(U_1, F_1.)$ $\qquad$ (b) On the Poincaré disk.

FIGURE 1. (a) The local phase portrait at the origin in the local chart U_1. (b) Phase portrait of a cubic quasi–homogeneous non–homogeneous system (3) in the Poincaré disk. This system has a global center.

Theorem 1. *The following two statements hold.*

(a) *The unique cubic quasi–homogeneous non–homogeneous polynomial differential system* (1) *with* P *and* Q *coprime and* $s_1 > s_2$ *having a center after a rescaling of the variables can be written as*

$$\dot{x} = y(ax + by^2), \qquad \dot{y} = x + y^2, \tag{3}$$

with $(a-2)^2+8b < 0$. *For all* a *and* b *satisfying* $(a-2)^2+8b < 0$ *the phase portrait in the Poincaré disk of system* (3) *is topologically equivalent to the one given in Figure 1(b). Moreover, its parameter space* (a, b) *is described in Figure 2(a). Additionally, these centers are not isochronous.*

(b) *The unique cubic homogeneous polynomial differential systems having a center after a linear transformation and a rescaling of independent variable can be written in one of the following four forms:*

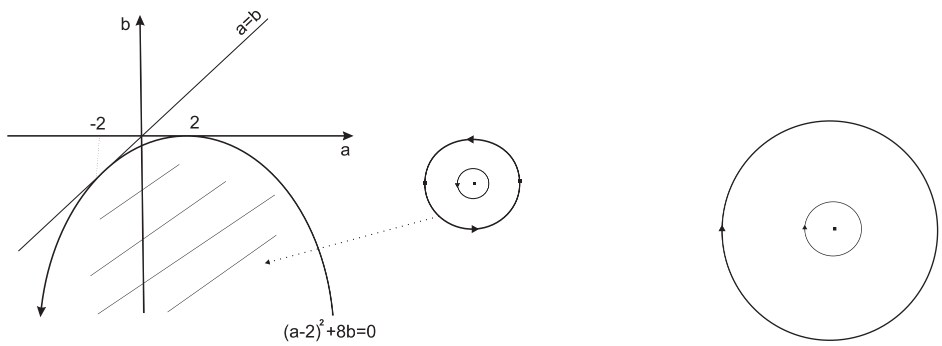
$$\dot{x} = -3\alpha\mu x^2 y - \alpha y^3 + P_3, \quad \dot{y} = \alpha x^3 + 3\alpha\mu x y^2 + Q_3, \tag{4}$$

where $\alpha = \pm 1$, $\mu > -1/3$ *and* $\mu \neq 1/3$*;*

$$\dot{x} = -\alpha x^2 y - \alpha y^3 + P_3, \quad \dot{y} = \alpha x^3 + \alpha x y^2 + Q_3, \tag{5}$$

with $\alpha = \pm 1$*. Here* $P_3 = p_1 x^3 + p_2 x^2 y - p_1 x y^2$ *and* $Q_3 = p_1 x^2 y + p_2 x y^2 - p_1 y^3$*. The phase portraits in the Poincaré disk of systems* (4) *and* (5) *are topologically equivalent to the ones of Figure 2(b). Moreover, these centers are not isochronous.*

The proof of Theorem 1 is given in section 3.



(a) cubic quasi–homogeneous (b) cubic homogeneous

Figure 2. (a) The parameter space (a, b) and the phase portrait of cubic quasi–homogeneous systems (3). (b) Cubic homogeneous systems (5) having a center, see also [5].

Additional to the classification of centers, another classical problem in the qualitative theory of planar differential systems is the study of their limit cycles. Recall that a *limit cycle* of a planar polynomial differential system is a periodic orbit of the system isolated in the set of all periodic orbits of the system. Thus in what follows we study, using the averaging theory of first order, the limit cycles which bifurcate from the periodic orbits of the centers (4) and (5) of Theorem 1 when these centers are perturbed inside the class of all cubic polynomial differential systems.

Theorem 2. *Consider the cubic homogeneous system* (4) *and* (5) *and its perturbation inside the class of all cubic polynomial differential systems. Then, for* $|\varepsilon| \neq 0$ *sufficiently small one limit cycle can bifurcate from the continuum of the periodic orbits of the center of systems* (4) *and* (5) *using averaging theory of first order.*

The proof of Theorem 2 is given in section 4.

In section 2 we provide the basic results that we shall need for proving Theorems 1 and 2.

2. SOME KNOWN RESULTS

2.1. **Classification of quasi–homogeneous non–homogeneous cubic polynomial differential systems.** For proving Theorem 1 we should need the following result.

Proposition 3. *A quasi–homogeneous non–homogeneous cubic polynomial differential systems (1) with P and Q coprime and $s_1 > s_2$ after a rescaling of the variables can be written as one of the following systems.*

(a) $\dot{x} = y(ax + by^2)$, $\dot{y} = x + y^2$, *with* $a \neq b$, *or* $\dot{x} = y(ax \pm y^2)$, $y' = x$, *and both with minimal weight vector* $(2, 1, 2)$.

(b) $\dot{x} = x^2 + y^3$, $\dot{y} = axy$, *with* $a \neq 0$*and minimal weight vector* $(3, 2, 4)$.

(c) $\dot{x} = y^3$, $\dot{y} = x^2$, *and minimal weight vector* $(4, 3, 6)$.

(d) $\dot{x} = x(x + ay^2)$, $\dot{y} = y(bx + y^2)$, *with* $(a,\, b) \neq (1,\, 1)$, *and minimal weight vector* $(2, 1, 3)$.

(e) $\dot{x} = axy^2$, $\dot{y} = \pm x^2 + y^2$, *with* $a \neq 0$*and minimal weight vector* $(3, 2, 5)$.

(f) $\dot{x} = axy^2$, $\dot{y} = x + y^3$, *with* $a \neq 0$ *and minimal weight vector* $(3, 1, 3)$.

(g) $\dot{x} = ax + y^3$, $\dot{y} = y$, *or* $\dot{x} = ax$, $\dot{y} = y$ *with* $a \neq 0$, *and minimal weight vector* $(3, 1, 1)$.

Proof. See [12]. □

2.2. **Nilpotent center–focus.** A singular point is *nilpotent* if both eigenvalues of its linear part are zero but its linear part is not identically zero. Andreev [2] was the first in characterizing the local phase portraits of the nilpotent singular points. In what follows we summarize the results of the local phase portraits of the nilpotent singular points that we need in this paper, for more details see Theorem 3.5 of [9].

Theorem 4. *Let* $(0, 0)$ *be an isolated singular point of the vector field* $\mathcal{X}$ *given by*

$$\dot{x} = y + A(x, y), \qquad \dot{y} = B(x, y),$$

where A *and* B *are analytic in a neighborhood of the point* $(0, 0)$ *starting with terms of second degree.*

Let $y = f(x)$ *be the solution of the equation* $y + A(x, y) = 0$ *in a neighborhood of the point* $(0, 0)$, *and consider* $F(x) = B(x, f(x))$ *and* $G(x) = (\partial A/\partial x + \partial B/\partial y)(x, f(x))$.

Then the origin can be a focus or a center if and only if one of the following statements holds:

(a) *If* $G(x) \equiv 0$ *and* $F(x) = ax^m + o(x^m)$ *for* $m \in \mathbb{N}$ *with* $m \geq 1$, m *odd and* $a < 0$ *then the origin of* $\mathcal{X}$ *is a center or a focus.*

(b) *If* $F(x) = \alpha x^m + o(x^m)$ *with* $\alpha < 0$, $m \in \mathbb{N}$, $m \geq 2$, m *odd, and* $G(x) = \beta x^n + o(x^n)$ *with* $\beta \neq 0$, $n \in \mathbb{N}$, $n \geq 1$ *and if either* $m < 2n+1$, *or* $m = 2n+1$ *and* $\beta^2 + 4\alpha(n+1) < 0$, *then the origin of* $\mathcal{X}$ *is a center or a focus.*

2.3. **Isochronicity.** The following result characterizes the isochronous centers.

Theorem 5. *A center of an analytic system is isochronous if and only if there exists an analytic change of coordinates of the form* $u = x + o(x, y)$ *and* $v = y + o(x, y)$ *changing the system to the linear isochronous system*

$$\dot{u} = -kv, \qquad \dot{v} = ku,$$

where k is a real constant.

For a proof of Theorem 5, see [20].

Assume that the origin is an isochronous center for system (1). Then Theorem 5 guarantees that there exists an analytic change of coordinates $u = x + o(x, y)$ and $v = y + o(x, y)$ such that $\dot{u} = -kv$, $\dot{v} = ku$. Then since $\ddot{u} + u = 0$, and $\ddot{v} + v = 0$, and doing a rescalling we can take $k = 1$.

2.4. **Poincaré compactification.** In order to plot the global phase portrait of the polynomial vector field (1) of degree m we should be able to control the orbits that come or escape at infinity. For this reason we consider the so called *Poincaré compactification* of the polynomial vector field $\mathcal{X}$.

Consider $\mathbb{R}^2$ as the plane in $\mathbb{R}^3$ defined by $(y_1, y_2, y_3) = (x_1, x_2, 1)$. We also consider the *Poincaré sphere* $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1 + y_2 + y_3 = 1\}$ (see also [24]) and we denote by $T_{(0,0,1)}\mathbb{S}^2$ the tangent space to $\mathbb{S}^2$ at the point $(0, 0, 1)$. The *Poincaré compactified vector field* $p(\mathcal{X})$ of $\mathcal{X}$ is an analytic vector field induced on $\mathbb{S}^2$ in the following way: We consider the central projection $f : T_{(0,0,1)} : \mathbb{S}^2 \to \mathbb{S}^2$. This map defines two copies of $\mathcal{X}$, one in the northern hemisphere $\{y \in \mathbb{S}^2 : y_3 > 0\}$ and the other in the southern hemisphere. We denote by $\tilde{\mathcal{X}}$ the vector field $Df \circ \mathcal{X}$ defined on $\mathbb{S}^2$ except on its equator. We notice that the points at infinity of $\mathbb{R}^2$ are in bijective correspondence with the points of the equator of $\mathbb{S}^2$, $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$ and so we identify $\mathbb{S}^1$ to be the infinity of $\mathbb{R}^2$.

Now we would like to extend the induced vector field $\tilde{\mathcal{X}}$ from $\mathbb{S}^2 \setminus \mathbb{S}^1$ to $\mathbb{S}^2$. It is possible that $\tilde{\mathcal{X}}$ does not stay bounded as we get close to $\mathbb{S}^1$. However, it turns out that if we multiply $\tilde{\mathcal{X}}$ by the factor y_3^{m-1}, namely, if we consider the vector field $y_3^{m-1}\tilde{\mathcal{X}}$ the extension is possible in the whole $\mathbb{S}^2$.

Note that on $\mathbb{S}^2 \setminus \mathbb{S}^1$ there are two symmetric copies of $\mathcal{X}$ and knowing the behavior of $p(\mathcal{X})$ around $\mathbb{S}^1$, we know the behavior of $\mathcal{X}$ at infinity. The *Poincaré disk* D^2 is the projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \longmapsto (y_1, y_2)$. Moreover, $\mathbb{S}^1$ is invariant under the flow of $p(\mathcal{X})$.

We also say that two polynomial vector fields $\mathcal{X}$ and $\mathcal{Y}$ on $\mathbb{R}^2$ are *topologically equivalent* if there exists a homeomorphism on $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying orbits of the flow induced by $p(\mathcal{X})$ into orbits of the flow induced by $p(\mathcal{Y})$. The homeomorphism should preserve or reverse simultaneously the sense of all orbits of the two compactified vector fields $p(\mathcal{X})$ and $p(\mathcal{Y})$.

Since $\mathbb{S}^2$ is a differentiable manifold we can consider the six local charts $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ for $i = 1, 2, 3$ and the diffeomorphisms $F_i : V_i \longrightarrow \mathbb{R}^2$ and $G_i : V_i \longrightarrow \mathbb{R}^2$ are the inverses of the central projections from the planes tangent at the points $(1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0), (0, 0, 1)$ and $(0, 0, -1)$ respectively. Now we denote by $z = (z_1, z_2)$ the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$. Then we obtain the following expressions of the compactified vector field $p(\mathcal{X})$ of $\mathcal{X}$ (for more details we refer to chapter V of [9] and references therein)

$$
\begin{aligned}
z_2^n\Delta(z)\left(Q\Big(\frac{1}{z_2},\frac{z_1}{z_2}\Big)-z_1P\Big(\frac{1}{z_2},\frac{z_1}{z_2}\Big),\,-z_2P\Big(\frac{1}{z_1},\frac{z_1}{z_2}\Big)\right) &\quad \text{in} \quad U_1,\\
z_2^n\Delta(z)\left(P\Big(\frac{z_1}{z_2},\frac{1}{z_2}\Big)-z_1Q\Big(\frac{z_1}{z_2},\frac{1}{z_2}\Big),\,-z_2Q\Big(\frac{z_1}{z_2},\frac{1}{z_2}\Big)\right) &\quad \text{in} \quad U_2,\\
\Delta(z)\big(P(z_1,z_2),Q(z_1,z_2)\big) &\quad \text{in} \quad U_3,
\end{aligned}
$$

where $\Delta(z)=(z_1^2+z_2^2+1)^{-\frac{1}{2(n-1)}}$. Note that in the two sets U_i and V_i the expressions of the vector field $p(\mathcal{X})$ are the same and only difference by the multiplicative factor $(-1)^{n-1}$. In these coordinates $z_2=0$ always denotes the points of $\mathbb{S}^1$. In what follows we omit the factor $\Delta(z)$ by rescaling the vector field $p(\mathcal{X})$ and so we obtain a polynomial vector field in each local chart.

2.5. **Separatrix configuration.** Let $p(\mathcal{X})$ be the Poincaré compactification of $\mathbb{S}^2$ of a polynomial vector field $\mathcal{X}$ in $\mathbb{R}^2$.

In what follows we consider the definition of parallel flows given by Markus [21] and Neumann in [22]. Let ϕ be a $\mathcal{C}^\omega$ local flow on the two dimensional manifold $\mathbb{R}^2$ or $\mathbb{R}^2\setminus\{0\}$. The flow (M,ϕ) is $\mathcal{C}^k$ *parallel* if it is $\mathcal{C}^\omega$-equivalent to one of the following ones:

strip: $(\mathbb{R}^2,\phi)$ with the flow ϕ defined by $\dot x=1,\dot y=0$;
annular: $(\mathbb{R}^2\setminus\{0\},\phi)$ with the flow ϕ defined (in polar coordinates) by $\dot r=0,\dot\theta=1$;
spiral: $(\mathbb{R}^2\setminus\{0\},\phi)$ with the flow ϕ defined by $\dot r=0,\dot\theta=1$.

It is known that the *separatrices* of the vector field $p(\mathcal{X})$ in the Poincaré disk D are

(i) all the orbits of $p(\mathcal{X})$ which are in the boundary $\mathbb{S}^1$ of the Poincaré disk (recall that $\mathbb{S}^1$ is the infinity of $\mathbb{R}^2$);
(ii) all the finite singular points of $p(\mathcal{X})$;
(iii) all the limit cycles of $p(\mathcal{X})$; and
(iv) all the separatrices of the hyperbolic sectors of the finite and infinite singular points of $p(\mathcal{X})$.

We denote by Σ the union of all separatrices of the flow (D,ϕ) defined by the compactified vector field $p(\mathcal{X})$ in the Poincaré disk D. Then Σ is a closed invariant subset of D. Every connected component of $D\setminus\Sigma$, with the restricted flow, is called a *canonical region* of ϕ.

For a proof of the following result see [15] and [22].

Theorem 6. *Let ϕ be a $\mathcal{C}^\omega$ flow in the Poincaré disk with finitely many separatrices, and let Σ be the union of all its separatrices. Then the flow restricted to every canonical region is $\mathcal{C}^\omega$ parallel.*

The *separatrix configuration* Σ_c of a flow (D,ϕ) is the union of all the separatrices Σ of the flow together with an orbit belonging to each canonical region. The separatrix configuration Σ_c of the flow (D,ϕ) is said to be *topologically equivalent* to the separatrix configuration $\tilde\Sigma_c$ of the flow $(D,\tilde\phi)$ if there exists a homeomorphism from D to D which transforms orbits of Σ_c into orbits of $\tilde\Sigma_c$, and orbits of Σ into orbits of $\tilde\Sigma$.

Theorem 7. *Let (D,ϕ) and $(D,\tilde\phi)$ be two compactified Poincaré flows with finitely many separatrices coming from two polynomial vector fields* (1). *Then they are topologically equivalent if and only if their separatrix configurations are topologically equivalent.*

For a proof of Theorem 7 see [21, 22, 23].

From Theorem 7 it follows that in order to classify the phase portraits in the Poincaré disk of a planar polynomial differential system having finitely many finite and infinite separatrices, it is enough to describe their separatrix configuration.

2.6. **Averaging theory and periodic solutions.** We consider the system

$$\mathbf{x}'(t) = F_0(t, \mathbf{x}), \tag{6}$$

with $F_0 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ a $\mathcal{C}^2$ function, T–periodic in the first variable and Ω is an open subset of $\mathbb{R}^n$. We assume that system (6) has a submanifold of periodic solutions.

Let ε be sufficiently small and we consider a perturbation of system (6) of the form

$$\mathbf{x}'(t) = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \tag{7}$$

with $F_1 : \mathbb{R} \times \omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T–periodic in the first variable and Ω is an open subset of $\mathbb{R}^n$. Averaging theory deals with the problem of the bifurcation of T–periodic solutions of system (7), see also for more information on the averaging theory [29, 30].

Let $\mathbf{x}(t, \mathbf{z})$ be the periodic solution of the unperturbed system (6) satisfying the initial condition $\mathbf{x}(0, \mathbf{z}) = \mathbf{z}$. Now we consider the linearization of system (6) along the solution $\mathbf{x}(t, \mathbf{z})$, namely

$$\mathbf{y}' = D_{\mathbf{x}} F_0(t, \mathbf{x}(t, \mathbf{z}))\mathbf{y},$$

and let $M_{\mathbf{z}}(t)$ be a fundamental matrix of this linear system satisfying that $M(0)$ is the identity matrix.

For a proof of the following theorem see [4].

Theorem 8 (Perturbations of an isochronous set)**.** *We assume that there exists an open bounded set V with $Cl(V) \subset \Omega$ such that for each $\mathbf{z} \in Cl(V)$, the solution $\mathbf{x}(t, \mathbf{z})$ is T–periodic, then we consider the function $\mathcal{F} : Cl(V) \to \mathbb{R}^n$*

$$\mathcal{F}(\mathbf{z}) = \int_0^T M_{\mathbf{z}}^{-1}(t, \mathbf{z}) F_1(t, \mathbf{x}(t, \mathbf{z}))dt. \tag{8}$$

If there exist $a \in V$ with $\mathcal{F}(a) = 0$ and $\det\left((d\mathcal{F}/d\mathbf{z})\,(a)\right) \neq 0$, then there exists a T–periodic solution $\phi(t, \varepsilon)$ of system (7) such that $\phi(0, \varepsilon) \to a$ as $\varepsilon \to 0$.

3. Proof of Theorem 1

All quasi–homogeneous non–homogeneous cubic polynomial differential systems are given by Proposition 3. Note that all those systems have the origin as the unique singular point.

Now we consider the first system of statement (a) of Proposition 3. This system admits the real first integral

$$\left(by^4 + (a-2)\,xy^2 - 2\,x^2\right)\left(\Delta\,x - 2\,by^2 - ax + 2\,x\right)^{\frac{3\,a+6+\Delta}{\Delta-a-2}}\left(2\,by^2 + ax - 2\,x + \Delta\,x\right),$$

with $(a-2)^2 + 8b \geq 0$ and $\Delta = \sqrt{(a-2)^2 + 8b}$. Note that the real invariant curve $2\,by^2 + ax - 2\,x + \Delta\,x = 0$ passes through the origin. Hence, the origin is not a center.

Now we consider the case where $(a-2)^2 + 8b < 0$. Under the change of coordinates $x \to Y$ $y \to X$ and after renaming (X, Y) by (x, y) we obtain

$$\dot{x} = y + x^2, \qquad \dot{y} = x(ay + bx^2). \tag{9}$$

Now we apply Theorem 4 to system (9). We have $A(x,y)=x^2$ and $B(x,y)=x(ay+bx^2)$ We have $F(x)=B(x,-x^2)=(b-a)x^3$ and $G(x)=(a+2)x$. Since $a\neq b$ we have that $F\not\equiv 0$. Following the notation of Theorem 4 we have $m=3$, $\alpha=b-a$, $n=1$ and $\beta=a+2$.

For $a=-2$ we have that $G(x)\equiv 0$ and $b<-2$. So $\alpha<0$ and by Theorem 4(a) the origin is a focus or a center. System (9) has the real first integral

$$H=\left(y-\left(-1+\frac{1}{2}\sqrt{2(2+b)}\right)x^2\right)\left(y-\left(-1-\frac{1}{2}\sqrt{2(2+b)}\right)x^2\right),$$

well defined at the origin and consequently the origin is a center.

For $a\neq -2$ we have $G(x)\not\equiv 0$. In order that the origin of system (9) can be a focus or a center, from Theorem 4(b), we need that $\alpha=b-a<0$ and $(a-2)^2+8b<0$. We notice that system (9) under these assumptions admits the real first integral

$$H(x,y)=\frac{\left(16\,y^2+16\,x^2y-8\,x^2ay+8\,x^4c^2+4\,x^4-4\,x^4a+x^4a^2\right)^{2c}}{\mathrm{e}^{\sqrt{2}(2+a)\arctan\left(\frac{\sqrt{2}}{4}\,\frac{-4\,y-2\,x^2+x^2a}{x^2c}\right)}},$$

with $c=\sqrt{-2((a-2)^2+8b)}/4$. Since this first integral is defined at the origin, the origin is a center.

The second family of systems of statement (a) of Proposition 3 admits the real invariant curves $\sqrt{a^2+8}x\pm 2\,y^2\pm ax=0$ which pass through the origin. So these systems have no centers.

Easy computations shows that systems (b), (c), (d),(e), (f) and (g) have real invariant curves passing through the origin. Therefore these systems have no centers.

In short, the quasi–homogeneous non–homogeneous cubic polynomial differential systems having a center are the system (3) satisfying either $a=-2$ and $b<-2$, or $a\neq-2$, $b-a<0$ and $(a-2)^2+8b<0$. An easy computation (see Figure 2) shows that these conditions for existence of the center in system (3) reduces to the unique condition $(a-2)^2+8b<0$.

Now we shall study the phase portrait in the Poincaré disk D and the parameter space of system (3). So, we study the infinite singular points of system (3) using subsection 2.4. On the local chart U_1 we obtain

$$\begin{aligned}\dot z_1 &= {z_2}^2+(1-a)\,{z_1}^2z_2-b{z_1}^4,\\ \dot z_2 &= -z_1\,z_2\left(az_2+b{z_1}^2\right).\end{aligned}\tag{10}$$

Since $8b+(a-2)^2<0$ we have that $(z_1,z_2)=(0,0)$ is the only infinite singular point in U_1 and it is linearly zero. In order to classify this infinite singular point we use the standard blow-up techniques, see for instance [9]. Then we obtain that the local phase portrait at the origin (0, 0) of system (10) is topologically equivalent to the one described in Figure 1(a). Additionally, note that in the chart (U_2,F_2) there are no infinite singular points. Hence, in the Poincaré disk the origin and $\mathbb{S}^1$ are the only separatrices. If we remove the origin and $\mathbb{S}^1$, then we have only one canonical region homomorphic to $\mathbb{R}^2\setminus\{\mathbf{0}\}$ and the flow is locally annular. According to Theorem 6 we obtain that the center is globally defined in $\mathbb{R}^2\setminus\{\mathbf{0}\}$. Hence, the phase portrait of the differential system (3) is topologically equivalent to the one of Figure 1(b).

The parameter space and phase portrait of system (3) is given in Figure 2(a).

Now we will study the isochronicity of the center of system (3). System (3) written in the polar coordinates is

$$\begin{aligned}\dot r &= P_1(\theta)r + P_2(\theta)r^2 + P_3(\theta)r^3,\\ \dot\theta &= Q_0(\theta) + Q_1(\theta)r + Q_2(\theta)r^2,\end{aligned}$$

with

$$\begin{array}{lll} P_1 = \cos\theta\sin\theta, & P_2 = (\sin^2\theta + a\cos^2\theta)\sin\theta, & P_3 = b\cos\theta\sin^3\theta,\\ Q_0 = \cos^2\theta, & Q_1 = -(a-1)\sin^2\theta\cos\theta, & Q_2 = -b\sin^4\theta.\end{array}$$

Consider the analytic function $H(r,\theta) = \sum_{n=1}^{\infty} H_n(\theta)r^n$ where $H_n(\theta)$ are trigonometric polynomials of degree n. If the condition

$$\ddot H + H = 0,$$

is satisfied then in the new variables $(H, -\dot H)$, system (3) could be transformed into the form

$$\dot u = -v, \qquad \dot v = u.$$

So system (3) could have an isochronous center at the origin.

If we expand $\ddot H + H = 0$ in power series of r we obtain a recursive system of differential equation. The coefficient of r^n for $n = 1, 2, \dots$ in this expansion is the differential equation of the form

$$\cos^4\theta H_n''(\theta) + 2(n-1)\sin\theta\,\cos\theta H_n'(\theta) + n\,\cos^2\theta H_n(\theta)\Big((n-1) - (n-2)\cos^2\theta\Big) + H_n(\theta) = 0,$$

and its general solution for $n = 1$ is

$$H_1(\theta) = \cos\theta\left(C_1\,\sin\left(\frac{\sin\theta}{\cos\theta}\right) + C_2\,\cos\left(\frac{\sin\theta}{\cos\theta}\right)\right).$$

For $n = 2, 3, \dots$ we have

$$H_n(\theta) = \big(\cos 2\theta + 1\big)^{\frac{n}{2}}\left(C_1\,\sin\left(\frac{\sin 2\theta}{\cos 2\theta + 1}\right) + C_2\,\cos\left(\frac{\sin 2\theta}{\cos 2\theta + 1}\right)\right).$$

Since these solutions $H_n(\theta)$ must be polynomials of trigonometric functions we have that $H_n \equiv 0$ for all n. Hence we have not an isochronous center and the proof of Theorem 1(a) is completed.

Now we are going to prove Theorem 1(b). The usual forms given in (5) for the cubic homogeneous polynomial differential systems having a center were obtained in Proposition 1 and Theorem 2 of [17]. The phase portrait were classified in [5]. See also Figure 2(b).

In order to study the isochronicity of systems (4) and (5) we can repeat the same mechanism used in the proof of statement (a). In polar coordinates system (5) takes the form

$$\dot r = P_3(\theta)r^3, \qquad \dot\theta = \alpha\, r^2$$

where $P_3 = p_1(\cos^2\theta - \sin^2\theta) + p_2\sin\theta\cos\theta$.

We can see that

$$H_1(\theta) = H_2(\theta) = H_3(\theta) = H_4(\theta) = 0,$$

and for $n \geq 5$ we have that

$$H_n(\theta) = -(\alpha^2\, H_{n-4}'' + 2(n-3)\,\alpha\, P_3\, H_{n-4}' + (n-4)\, H_{n-4}(\alpha\, P_3' + (n-2)\, P_3^2).$$

Clearly each $H_n \equiv 0$, for all n, so system of (5) is not an isochronous center.

System (4) can be written in polar coordinates as

$$\dot r = P_3(\theta) r^3, \qquad \dot\theta = \alpha\, Q_2(\theta)\, r^2,$$

where

$$P_3 = p_1(\cos^2\theta - \sin^2\theta) + p_2 \sin\theta\cos\theta,$$
$$Q_2 = \cos^4\theta + 6\,\mu\,\cos^2\theta\,\sin^2\theta + \sin^4\theta.$$

Again we obtain

$$H_1(\theta) = H_2(\theta) = H_3(\theta) = H_4(\theta) = 0,$$

and for $n \geq 5$ we have

$$H_n(\theta) = -(\alpha^2\, Q_2^2\, H''_{n-4} + \big(\alpha^2\, Q'^2\, Q_2 + 2\,(n-3)\,\alpha Q_2\, P_3\big)\, H'_{n-4} + (n-4)\, H_{n-4}(\alpha\, P'_3 Q_2 + (n-2)\, P_3^2).$$

Clearly each $H_n \equiv 0$, for all n and therefore system (4) is not an isochronous center. This completes the proof of Theorem 1.

4. Proof of Theorem 2

System (5) in polar coordinates can be written into the form

$$\dot r = r^3\left(p_1\,\cos^2\theta + p_2 \sin\theta\,\cos\theta - p_1\,\sin^2\theta\right), \qquad \dot\theta = \alpha r^2,$$

or equivalently

$$\frac{dr}{d\theta} = \frac{r}{\alpha}\left(p_1\,\cos^2\theta + \sin\theta p_2\,\cos\theta - p_1\,\sin^2\theta\right),$$

its solution satisfying the initial condition $r(0) = r_0$ is

$$\tilde r\,(\theta,\ r_0) = r_0 \exp\left((p_2 + 2p_1\,\sin(2\,\theta) - p_2\,\cos(2\,\theta))/(4\alpha)\right).$$

Now the fundamental matrix of the linearized equation evaluated on a closed orbit is

$$M_{r_0}(\theta) = M(\theta) = \exp\left(\big(p_2 + 2p_1\,\sin(2\,\theta) - p_2\,\cos(2\,\theta)\big)/(4\alpha)\right),$$

and satisfies the condition $M(0) = 1$.

Now we perturb system (5) inside the class of all cubic polynomial differential systems and we have

$$\begin{aligned} \dot x &= p_1 x^3 + (p_2-\alpha)x^2 y - p_1 x y^2 - \alpha y^3 + \varepsilon\left(\sum_{0\leq i+j\leq 3} a_{ij} x^i y^j\right), \\ \dot y &= \alpha x^3 + p_1 x^2 y + (p_2+\alpha) x y^2 - p_1 y^3 + \varepsilon\left(\sum_{0\leq i+j\leq 3} b_{ij} x^i y^j\right). \end{aligned}$$

The corresponding differential equation in polar coordinates becomes

$$\frac{d\,r}{d\,\theta} = F_0(\theta, r) + \varepsilon F_1(\theta, r) + O(\epsilon^2),$$

with

$$F_0(\theta, r) = \frac{r}{\alpha}\left(p_1\,(2\cos^2\theta - 1) + p_2 \sin\theta\cos\theta\right),$$
$$F_1(\theta, r) = \frac{1}{\alpha r^3}(B_4 r^4 + B_3 r^3 + B_2 r^2 + B_1 r),$$

where

$$\begin{aligned}
B_4 = &\ \frac{1}{\alpha}\left(B_{46}\cos^6\theta + B_{45}\sin\theta\cos^5\theta + B_{44}\cos^4\theta + B_{43}\sin\theta\cos^3\theta + B_{42}\cos^2\theta\right.\\
&\left. + B_{41}\sin\theta\cos\theta + B_{40}\right),\\
B_3 = &\ -\frac{1}{\alpha}\left(B_{35}\cos^5\theta + B_{34}\sin\theta\cos^4\theta + B_{33}\cos^3\theta + B_{32}\sin\theta\cos^2\theta + B_{31}\theta\cos\theta + B_{30}\sin\theta\right),\\
B_2 = &\ -\frac{1}{\alpha}\left(B_{24}\cos^4\theta + B_{23}\sin\theta\cos^3\theta + B_{22}\cos^2\theta + B_{21}\sin\theta\cos\theta + B_{20}\right),\\
B_1 = &\ -\frac{1}{\alpha}\left(B_{13}\cos^3\theta + B_{12}\sin\theta\cos^2\theta + B_{10}\sin\theta\right);
\end{aligned}$$

with

$$\begin{aligned}
B_{46} = &\ 2\,p_1\,a_{03} + 2\,p_1\,b_{12} - 2\,p_1\,a_{21} - 2\,p_1\,b_{30} + p_2\,a_{12} - p_2\,a_{30} + p_2\,b_{21} - p_2\,b_{03},\\
B_{45} = &\ -2\,p_1\,a_{12} + 2\,p_1\,a_{30} + p_2\,a_{03} - p_2\,b_{30} - 2\,p_1\,b_{21} - p_2\,a_{21} + p_2\,b_{12} + 2\,p_1\,b_{03},\\
B_{44} = &\ -5\,p_1\,a_{03} + 3\,p_1\,a_{21} - 3\,p_1\,b_{12} - b_{21}\alpha - a_{12}\alpha + a_{30}\alpha + p_2\,a_{30} - p_2\,b_{21} + p_1\,b_{30}\\
&+ b_{03}\alpha + 2\,p_2\,b_{03} - 2\,p_2\,a_{12},\\
B_{43} = &\ -p_2\,b_{12} + 3\,p_1\,a_{12} - a_{03}\alpha + p_2\,a_{21} + b_{30}\alpha - 3\,p_1\,b_{03} - p_1\,a_{30} - b_{12}\alpha + p_1\,b_{21}\\
&+ a_{21}\alpha - 2\,p_2\,a_{03},\\
B_{42} = &\ 4\,p_1\,a_{03} + p_1\,b_{12} - p_2\,b_{03} + b_{21}\alpha + a_{12}\alpha - 2\,b_{03}\alpha - p_1\,a_{21} + p_2\,a_{12},\\
B_{41} = &\ -p_1\,a_{12} + p_2\,a_{03} + b_{12}\alpha + a_{03}\alpha + p_1\,b_{03},\\
B_{40} = &\ b_{03}\alpha - p_1 a_{03},\\
B_{35} = &\ 2\,p_1\,b_{02} - 2\,p_1\,a_{11} - 2\,p_1\,b_{20} + p_2\,a_{02} + p_2\,b_{11} - p_2\,a_{20},\\
B_{34} = &\ -2\,p_1\,b_{11} - p_2\,b_{20} + 2\,p_1\,a_{20} - p_2\,a_{11} - 2\,a_{02}p_1 + p_2\,b_{02},\\
B_{33} = &\ -a_{02}\alpha + p_1\,b_{20} + a_{20}\alpha + p_2\,a_{20} - 2\,p_2\,a_{02} - p_2\,b_{11} - 3\,p_1\,b_{02} + 3\,p_1\,a_{11} - b_{11}\alpha,\\
B_{32} = &\ -b_{02}\alpha + p_2\,a_{11} - p_1\,a_{20} + b_{20}\alpha + p_1\,b_{11} + 3\,a_{02}p_1 + a_{11}\alpha - p_2\,b_{02},\\
B_{31} = &\ -p_1\,a_{11} + p_2\,a_{02} + b_{11}\alpha + a_{02}\alpha + p_1\,b_{02},\\
B_{30} = &\ -a_{02}p_1 + b_{02}\alpha,\\
B_{24} = &\ -2\,a_{01}p_1 - 2\,p_1\,b_{10} + p_2\,b_{01} - p_2\,a_{10},\\
B_{23} = &\ -2\,p_1\,b_{01} - p_2\,b_{10} - p_2\,a_{01} + 2\,p_1\,a_{10},\\
B_{22} = &\ 3\,a_{01}p_1 + a_{10}\alpha + \alpha\,b_{01} + p_2\,a_{10} + p_1\,b_{10} - p_2\,b_{01},\\
B_{21} = &\ p_1\,b_{01} - p_1\,a_{10} + b_{10}\alpha + a_{01}\alpha + p_2\,a_{01},\\
B_{20} = &\ -a_{01}p_1 + \alpha\,b_{01},\\
B_{13} = &\ -2\,p_1\,b_{00} - p_2\,a_{00},\\
B_{12} = &\ 2\,a_{00}p_1 - p_2\,b_{00},\\
B_{11} = &\ p_1\,b_{00} + p_2\,a_{00} + \alpha\,a_{00},\\
B_{10} = &\ b_{00}\alpha - a_{00}p_1.
\end{aligned}$$

Note that

$$\begin{aligned}\mathcal{F}(r_0) &= \int_0^{2\pi} M^{-1}(\theta)\, F_1(\theta, \tilde{r}(\theta, r_0))\, d\theta \\ &= \frac{1}{r_0} A_0 I_0 + \frac{2}{r_0^2}\left(A_1 I_1 + A_2 I_2 + A_3 I_3 + A_4 I_4 + \pi\, C_1 + \frac{3\pi}{4} C_2\right) \\ &\quad + 2 r_0 \pi(\alpha b_{03} - p_1 a_{03}) + \frac{5\,\pi}{8}\, C_3,\end{aligned}$$

where we have

$$I_0 = \int_0^{2\,\pi} E\; d\theta, \qquad I_1 = \int_0^{2\,\pi} E \cos\theta\;\sin\theta\; d\theta,$$

$$I_2 = \int_0^{2\,\pi} E\;\cos^2\theta\; d\theta, \qquad I_3 = \int_0^{2\,\pi} E \cos^3\theta\;\sin\theta\; d\theta,$$

$$I_4 = \int_0^{2\,\pi} E \cos^4\theta\; d\theta, \qquad E = \exp\left(-\frac{\sin\theta\,(2\,p_1\;\cos(\theta) + p_2 \sin\theta)}{\alpha}\right),$$

and

$$\begin{aligned}
A_0 &= -a_{01} p_1 + \alpha\, b_{01}, \\
A_1 &= -\frac{1}{2}\left((a_{10} - b_{01})\, p_1 - p_2\, a_{01} - \alpha\; (a_{01} + b_{1,0})\right) r_0, \\
A_2 &= \left(\left(\frac{3}{2}\, a_{01} + \frac{1}{2}\, b_{10}\right) p_1 + \frac{1}{2}\; (p_2 + \alpha)\, (a_{10} - b_{01})\right) r_0, \\
A_3 &= \left((a_{10} - b_{01})\, p_1 - \frac{1}{2}\, p_2\; (a_{01} + b_{10})\right) r_0, \\
A_4 &= \left((-a_{01} - b_{10})\, p_1 - \frac{1}{2}\, p_2\; (a_{10} - b_{01})\right) r_0, \\
C_1 &= \left(\left(2\, a_{03} + \frac{1}{2}\, b_{12} - \frac{1}{2}\, a_{21}\right) p_1 + \left(\frac{1}{2}\, a_{12} - \frac{1}{2}\, b_{03}\right) p_2 \right. \\
&\quad \left. - \left(-\frac{1}{2}\, a_{12} + b_{03} - \frac{1}{2}\, b_{21}\right) \alpha\right) {r_0}^3, \\
C_2 &= \left(\left(\frac{3}{2}\, a_{21} + \frac{1}{2}\, b_{30} - 5/2\, a_{03} - 3/2\, b_{12}\right) p_1 + \left(\frac{1}{2}\, a_{30} - a_{1,2} - \frac{1}{2}\, b_{21} + b_{03}\right) p_2 \right. \\
&\quad \left. + \frac{1}{2}\, \alpha\; (b_{03} + a_{30} - a_{12} - b_{21})\right) {r_0}^3, \\
C_3 &= 2\left((b_{12} + a_{03} - a_{21} - b_{30})\, p_1 - \frac{1}{2}\, p_2\; (b_{03} + a_{30} - a_{12} - b_{21})\right) r_0.
\end{aligned}$$

In short, the function $\mathcal{F}(r)$ of Theorem 8 is of the form

$$\mathcal{F}(r) = \frac{\alpha r^2 + \beta}{r},$$

so it has at most one real positive root given by $r = \sqrt{-\beta/\alpha}$. Moreover, we have that $\mathcal{F}'\left(\sqrt{-\beta/\alpha}\right) = 2\alpha$. So by Theorem 8 if $-\beta/\alpha > 0$ then there is one limit cycle bifurcating from a periodic orbit of the center of system (5). This completes the proof of Theorem 2 for system (5).

The rest of the proof of Theorem 2 for system (4) is completely analogous to the one done for system (5), only changes the computations, and we do not repeat it here.

4.1. **Examples.** First we give an example satisfying the result of Theorem 2 for system (5). We consider the system

$$\dot{x} = x^3 + 2\,x^2y - xy^2 - y^3, \qquad \dot{y} = x^3 + x^2y + 4\,xy^2 - y^3,$$

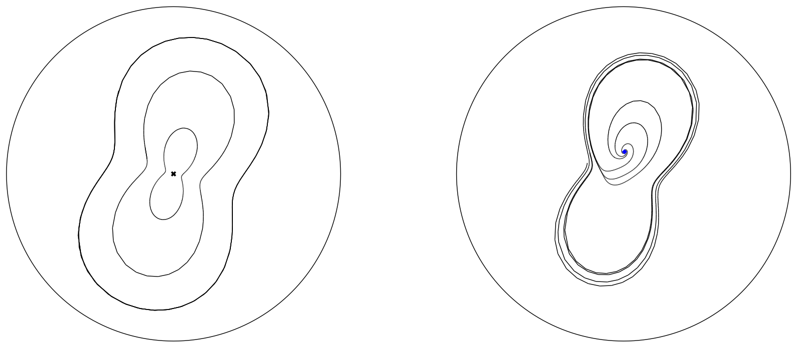
and its perturbation

$$\begin{aligned} \dot{x} = &\; x^3 + 2\,x^2y - xy^2 - y^3 \\ &+\varepsilon\left(4\,y^3 + 3\,xy^2 + 3\,x^2y + 5\,x^3 + 3\,y^2 + 3\,xy + 3\,x^2 - y - x + 2\right), \\ \dot{y} = &\; x^3 + x^2y + 4\,xy^2 - y^3 \\ &+\varepsilon\left(-3\,y^3 + xy^2 + x^2y + x^3 + 5\,y^2 + xy + x^2 + y + 2\,x + 1\right). \end{aligned} \tag{11}$$

Then

$$\mathcal{F}(r_0) = \frac{11.78097245r_0^2 - 4.108168642}{r_0}$$

and $\mathcal{F}(r) = 0$ gives $r = 0.5905185728$. So according to Theorem 8 at most one limit cycle can bifurcated from the origin, see also Figure 3.



$\varepsilon = 0$ $\qquad\qquad$ $\varepsilon = 0.01$

FIGURE 3. Phase portrait of system (11) in the Poincaré disk.

Example-2 Now we give an example satisfying the result of Theorem 2 for system (4). For $\varepsilon = 0$ the origin of the system

$$\begin{aligned} \dot{x} = &\; x^3 - 6\,x^2y - xy^2 - y^3 \\ &+\epsilon\left(2\,y^3 + 3\,xy^2 + 3\,x^2y - 5\,x^3 + 3\,y^2 + 10\,xy + 3\,x^2 - y - x - 20\right), \\ \dot{y} = &\; x^3 + x^2y + 12\,xy^2 - y^3 \\ &+\epsilon\left(-3\,y^3 + xy^2 - 10\,x^2y + x^3 + 5\,y^2 + xy + 1/5\,x^2 + y + 2\,x + 100\right), \end{aligned} \tag{12}$$

is a center and for $\varepsilon = 0.01$ one limit cycle is produced, see Figure 4.

$\varepsilon = 0$ $\varepsilon = 0.01$

FIGURE 4. Phase portrait of system (12) in the Poincaré disk.

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