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SIMULTANEOUS BIFURCATION OF LIMIT CYCLES FROM A LINEAR CENTER WITH EXTRA SINGULAR POINTS

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ABSTRACT. The period annuli of the planar vector field $x' = -yF(x,y), y' = xF(x,y)$, where the set $\{F(x,y) = 0\}$ consists of k different isolated points, is defined by $k+1$ concentric annuli. In this paper we perturb it with polynomials of degree n and we study how many limit cycles bifurcate, up to a first order analysis, from all the period annuli simultaneously in terms of k and n. Additionally, we prove that the associated Abelian integral is piecewise rational and, when $k = 1$, the provided upper bound is reached. Finally, the case $k = 2$ is also treated.

1. INTRODUCTION

Let H, f, g be polynomials in x, y such that $\gamma_h \subseteq \{H(x,y) = h\}$, with $h \in (h_0, h_1)$, are simple closed curves around the point $(x_0, y_0) = \gamma_{h_0}$. Then the system

$$\begin{cases} \dot{x} &=& \dfrac{\partial H(x,y)}{\partial y} + \varepsilon f(x,y), \\ \dot{y} &=& -\dfrac{\partial H(x,y)}{\partial x} + \varepsilon g(x,y), \end{cases} \tag{1}$$

has a center at (x_0, y_0) when $\varepsilon = 0$. V. I. Arnold, see [1, 2], states the *weak Hilbert 16th problem* asking for the maximum number of isolated zeros of the Abelian integral, associated to system (1),

$$I(h) = \oint_{\gamma_h} \left(f(x,y)dy - g(x,y)dx\right). \tag{2}$$

For this case we have that $I(h)$ is the first order approximation of the Poincaré map. Then, each simple zero of I, h^*, corresponds to a limit cycle of (1) bifurcating from γ_{h^*} for ε small enough, see [17, 18]. This function is also known as the *Poincaré-Pontrjagin-Melnikov function* of system (1).

Usually, the centers studied up to now have only one period annulus or, when they have more than one, the study is restricted to one of them. There are not so many papers focused on the study of simultaneous bifurcation of limit cycles from centers with different period annuli. Some of them are [5, 8] that deal with the simultaneity in two different regions, or [7] where three separated period annuli appear. A study of the bifurcation of limit cycles from different period annuli of polynomial Hamiltonian systems to obtain lower bounds for the Hilbert number is done in [4] and, more recently, in [14]. In this paper, we consider a center with nested period annuli. The main goals are that we obtain an explicit expression for the Abelian integral and that we can study the number of zeros in all regions simultaneously.

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More concretely, the aim of this paper is to study the number of limit cycles that bifurcate, for ε small enough, from system

$$\begin{cases} \dot{x} &= \quad y\,K(x,y) + \varepsilon P(x,y), \\ \dot{y} &= -x\,K(x,y) + \varepsilon Q(x,y), \end{cases} \tag{3}$$

where P and Q are arbitrary real polynomials of degree n and $K(x,y)$ is an specific kind of polynomial. There are several papers where different $K(x,y)$ are considered. In [12, 19, 20] the set $\{K(x,y)=0\}$ represents a straight line of simple or multiple singular points. In [3, 11, 13], the problem when $K(x,y)$ are some concrete quadratic polynomials is considered. When the set $\{K(x,y)=0\}$ represents a collection of straight lines parallel to one or two orthogonal directions is studied in [9]. Here K is defined by a collection of k different points. We refer them as the singularities of system (3) or the extra singularities in order to distinguish them from the origin. This work can be a considered as a continuation of [10], where only the first period annulus is studied. In [11], among other general conics, the case of one singularity is also done for both period annuli, but only under cubic perturbations. Our aim is to obtain the maximum number of limit cycles that bifurcate from the periodic orbits of all period annuli simultaneously, for a fixed collection of singularities, up to a first order study.

Where $K(x,y)$ does not vanish, after a time rescaling, system (3) is equivalent to

$$\begin{cases} \dot{x} &= \quad y + \varepsilon \dfrac{P(x,y)}{K(x,y)}, \\ \dot{y} &= -x + \varepsilon \dfrac{Q(x,y)}{K(x,y)}. \end{cases} \tag{4}$$

The above system corresponds to a rational perturbation of the harmonic oscillator, and the level curves of the unperturbed system are the circles $\gamma_r = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=r^2\}$. Then, as we have mentioned above, the number of perturbed limit cycles that equation (4) can have, up to a first order analysis, can also be bounded below by the number of simple zeros of

$$I(r) = \oint_{\gamma_r} \frac{P(x,y)dy - Q(x,y)dx}{K(x,y)}. \tag{5}$$

In fact, this is the problem that we consider along this paper: the study of the number of simultaneous zeros of (5) for

$$K(x,y) = \prod_{j=1}^{k} \left((x-a_j)^2 + (y-b_j)^2\right) \tag{6}$$

where (a_j,b_j) are k different points in $\mathbb{R}^2$. We consider only the generic situation where these points are isolated and nonaligned with the origin. More concretely, they satisfy $0 < \tilde{r}_1 < \ldots < \tilde{r}_k$ if $\tilde{r}_j = \sqrt{a_j^2+b_j^2}$, $j=1,\ldots,k$. As the associated Abelian integral is not well defined for each r, we restrict our results to the level curves, γ_r, completely contained in

$$\mathcal{R}_j = \left\{(x,y) \in \mathbb{R}^2 : \tilde{r}_j^2 = a_j^2 + b_j^2 < x^2+y^2 < a_{j+1}^2 + b_{j+1}^2 = \tilde{r}_{j+1}^2\right\},$$

for $j=0,\ldots,k-1$, considering $(a_0,b_0)=(0,0)$ and $\mathcal{R}_k = \{(x,y) \in \mathbb{R}^2 : a_k^2 + b_k^2 < x^2+y^2\}$. See Figure 1 for a particular configuration with three singularities. The above regions are also indicated.

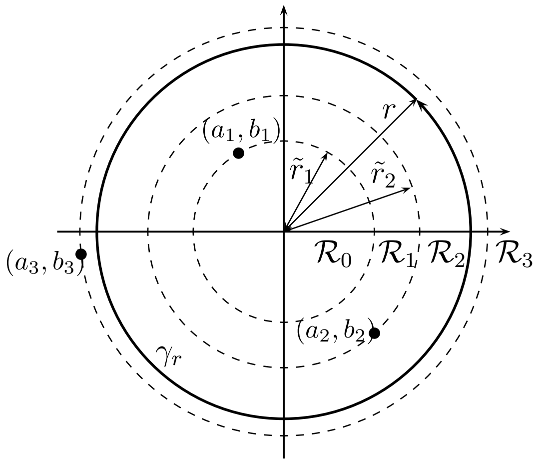

FIGURE 1. Example of location of the singularities (a_j, b_j) and the regions $\mathcal{R}_j$ for $k = 3$

For the general case, we prove the following result.

Theorem 1. *Let k and n be a pair of natural numbers. Given system (4) with $K(x, y)$ defined in (6) and P and Q polynomials of degree n, then the Abelian integral $I(r)$, defined in (5), is a piecewise rational function in r^2. In fact, we can identify I with the vector $(I_0, \ldots, I_k)$ where $I_j(r) = I(r)$ if $\gamma_r \subset \mathcal{R}_j$ for $j = 0, \ldots, k$. So, moreover, the number or zeros of I_j is lower than or equal to $N = [\max\{n+1, (2k-1)(k-1)\}/2] + n + 1 + (2k-1)(k-1)$ for $j = 1, \ldots, k-1$ and lower than or equal to $N - \min\{n + 1, (2k-1)(k-1)\}$ for $j = k$. We have denoted by $[\cdot]$ the integer part function.*

When we fix the value of k we improve the upper bounds.

Theorem 2. *Under the conditions stated in Theorem 1, the number of simultaneous zeros of system (3) for $k = 1$, up to a first order analysis, is at most $n + [(n+1)/2]$. Additionally, given a pair (i, j) of natural numbers such that $i \le n$ and $j \le [(n+1)/2]$, there exist P, Q polynomials of degree n such that system (3) has at least i and j bifurcated limit cycles in $\mathcal{R}_0$ and $\mathcal{R}_1$, respectively. We have denoted by $[\cdot]$ the integer part function.*

The above result generalizes a result of [11] for the case of two complex straight lines intersecting in a real point. Although, we consider every value of n instead of a perturbation in the cubic class.

Theorem 3. *Under the assumptions of Theorem 1, the number of zeros of I_0, I_1 and I_2, for $k = 2$, is lower than or equal to $n + 4$, $n + 5$ and $[(n+1)/2] + 4$, respectively. Hence the total number of zeros of I is bounded by $2n + [(n+1)/2] + 13$. In particular, if $a_1/b_1 = a_2/b_2$, the number of zeros of I_0, I_1 and I_2 is bounded by $n + 1$, $n + 2$ and $[(n+1)/2] + 1$, respectively. We have denoted by $[\cdot]$ the integer part function.*

This paper is organized as follows. In Section 2, we obtain the expression of the Abelian integral, (5), and we prove the first part of Theorem 1. Section 3 provides the upper bounds for the number of zeros of the Abelian integrals given in the second part of Theorem 1. Section 4 deals with the case of a unique singularity, $k = 1$. In particular, we compute an explicit expression for its Abelian integral. Finally, Section 5 is devoted to study the case $k = 2$.

2. Rationality of the Abelian integral

This section is devoted to prove the rationality of the Abelian integral, $I(r)$, defined in (5). Next technical lemma ensures that locating a singular point at $(1,0)$ is not restrictive, for example the closest to the origin.

Lemma 4. *System* (3) *is invariant under dilations and rotations with the center at the origin.*

Proof. Rotations and dilations are linear transformation of the space. So, they do not change the form of system (3) nor the degree of the polynomials. Moreover, the transformed system has also the inner structure of being a polynomial perturbation of a linear center multiplied by a function like $K(x,y)$ with the zeros located in other place but with the same properties of the original one. □

Although there is not a unique way to express $1/K(x,y)$, for K given in (6), as a sum of partial fractions, the following lemma gives us a possible decomposition for every value of k.

Lemma 5. *For all k,*

$$\frac{1}{\prod_{j=1}^{k} F_j(r,\cos t,\sin t)} = \sum_{j=1}^{k} \frac{A_j(r^2, r\cos t, r\sin t)}{D(r^2)F_j(r,\cos t,\sin t)},$$

where $F_j(r,\cos t,\sin t) = r^2 - 2a_j r\cos t - 2b_j r\sin t + a_j^2 + b_j^2$, A_j and D, for $j=1,\dots,k$, are polynomials of degree $(2k-1)(k-1)$ and $2k(k-1)$, respectively.

Proof. We consider the conjugate of each term in the denominator of the expression of the statement, i.e. the polynomial $G_j(r,\cos t,\sin t) = r^2 - 2a_j r\cos t + 2b_j r\sin t + a_j^2 + b_j^2$ for $j=1,\dots,k$. Thus

$$\begin{aligned} F_jG_j =& 4(a_j^2+b_j^2)r^2\cos^2 t - 4a_j(a_j^2+b_j^2+r^2)r\cos t \\ &+ (r^2+a_j^2-2rb_j+b_j^2)(r^2+a_j^2+2rb_j+b_j^2). \end{aligned}$$

This polynomial, seen as a one variable polynomial with respect to $r\cos t$, has a negative discriminant, $-16(a_j^2+b_j^2-r^2)^2 b_j^2 < 0$. Therefore, by the one variable Partial Fraction Decomposition Theorem, there exist polynomials, B_j, with degree one such that for each k, we have

$$\left(\prod_{j=1}^{k} F_jG_j\right)^{-1} = \sum_{j=1}^{k} \frac{B_j(r\cos t)}{F_jG_j}.$$

In fact $B_j(r\cos t) = \rho_j(r^2)r\cos t + \xi_j(r^2)$, with ρ_j and ξ_j rational functions. The expressions of ρ_j and ξ_j can be found solving a linear system with $2k$ equations and $2k$ variables. This system comes from the equation

$$1 = \sum_{i=1}^{k}\left(\prod_{\substack{j=1\\ j\neq i}}^{k} F_jG_j\right)\left(\rho_i(r^2)r\cos t + \xi_i(r^2)\right).$$

We just study the degrees of the polynomials involved in the matrix of the above system. Let us denote by Δ_l a non-fixed polynomial of degree l with respect to r^2.

Consequently the system can be written as

$$\begin{pmatrix} 0 & \cdots & 0 & \Delta_{2k-2} & \cdots & \Delta_{2k-2} \\ \Delta_{2k-2} & \cdots & \Delta_{2k-2} & \Delta_{2k-3} & \cdots & \Delta_{2k-3} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \Delta_1 & \cdots & \Delta_1 & \Delta_0 & \cdots & \Delta_0 \\ \Delta_0 & \cdots & \Delta_0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_k \\ \xi_1 \\ \vdots \\ \xi_k \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{7}$$

Therefore, by the Cramer's rule, the solutions ρ_j and ξ_j, as products of one term from each different row and column of the matrix of the linear system of equations (7), M, have the following rational form

$$\rho_j = \frac{\Delta_{-k+\ell}}{\Delta_{k-1+\ell}} = \frac{\Delta_{2(k-1)^2-1}}{\Delta_{2k(k-1)}}, \quad \text{and} \quad \xi_j = \frac{\Delta_{-(k-1)+\ell}}{\Delta_{k-1+\ell}} = \frac{\Delta_{2(k-1)^2}}{\Delta_{2k(k-1)}},$$

where $\ell = \sum_{l=1}^{2(k-1)} l$. In fact both expressions have the same denominator, the determinant of M.

Now we can consider

$$\frac{1}{\prod_{j=1}^{k} F_j} = \frac{\prod_{l=1}^{k} G_l}{\prod_{j=1}^{k} F_j G_j} = \sum_{j=1}^{k} \frac{B_j(\cos t)\prod_{l=1}^{k} G_l}{F_j G_j} = \sum_{j=1}^{k} \frac{B_j(\cos t)\prod_{\substack{l=1\\ l\neq j}}^{k} G_l}{F_j}.$$

Hence we just need to take $A_j(r^2, r\cos t, r\sin t)$ as the numerator of $B_j(\cos t)\prod_{\substack{l=1\\ l\neq j}}^{k} G_l$ and $D(r^2)$ as the determinant of M, so D is a polynomial of degree $2k(k-1)$.

Additionally, by the definition of G_l,

$$\prod_{\substack{l=1\\ l\neq j}}^{k} G_l = \Delta_0(r^2)\Delta_{k-1}(r\cos t, r\sin t) + \cdots + \Delta_{k-2}(r^2)\Delta_1(r\cos t, r\sin t) + \Delta_{k-1}(r^2),$$

and it follows that

$$A_j = \left(\Delta_{2(k-1)^2-1}(r^2) r\cos t + \Delta_{2(k-1)^2}(r^2)\right)\prod_{\substack{l=1\\ l\neq j}}^{k} G_l = \Delta_{(2k-1)(k-1)}(r^2, r\cos t, r\sin t).$$

This completes the proof. □

In order to obtain a general expression of $I(r)$ additionally to the previous results we need the following one. It is a generalization of [10, Lemma 2.2]. Our proof is considered as a corollary of that one, instead of an adapted one. The statement here is given in terms of Chebyshev's polynomials, see more information about them in [6].

Lemma 6. *For all $l \in \mathbb{N}$, we have that*

$$I_l^c(r) = \int_0^{2\pi} \frac{\cos(lt)}{r^2+a^2+b^2-2ar\cos t-2br\sin t}dt$$
$$= \begin{cases} \dfrac{2\pi}{(a^2+b^2)^{l/2}}\, T_l\left(\dfrac{a}{\sqrt{a^2+b^2}}\right)\dfrac{r^l}{a^2+b^2-r^2} & 0 \le r < \sqrt{a^2+b^2}, \\ 2\pi(a^2+b^2)^{l/2}\, T_l\left(\dfrac{a}{\sqrt{a^2+b^2}}\right)\dfrac{r^{-l}}{r^2-(a^2+b^2)} & r > \sqrt{a^2+b^2}, \end{cases}$$

$$I_l^s(r) = \int_0^{2\pi} \frac{\sin(lt)}{r^2+a^2+b^2-2ar\cos t-2br\sin t}dt$$
$$= \begin{cases} \dfrac{2\pi\, b}{(a^2+b^2)^{(l+1)/2}}\, U_{l-1}\left(\dfrac{a}{\sqrt{a^2+b^2}}\right)\dfrac{r^l}{a^2+b^2-r^2} & 0 \le r < \sqrt{a^2+b^2}, \\ 2\pi\, b\,(a^2+b^2)^{(l-1)/2}\, U_{l-1}\left(\dfrac{a}{\sqrt{a^2+b^2}}\right)\dfrac{r^{-l}}{r^2-(a^2+b^2)} & r > \sqrt{a^2+b^2}, \end{cases}$$

where T_l and U_l are the Chebyshev's polynomials of first and second kind of degree l, respectively.

Proof. We consider the change $t = \tau + \theta$ taking the unique value of $\theta \in (0, 2\pi)$ such that $\sin\theta = b/\sqrt{a^2+b^2}$ and $\cos\theta = a/\sqrt{a^2+b^2}$, so

$$\begin{aligned} I_l^c(r) &= \int_0^{2\pi} \frac{\cos(l(\tau+\theta))}{r^2+a^2+b^2-2ar\cos(\tau+\theta)-2br\sin(\tau+\theta)}d\tau \\ &= \int_0^{2\pi} \frac{\cos(l\tau)\cos(l\theta)-\sin(l\tau)\sin(l\theta)}{r^2+a^2+b^2-2r(a\cos\theta+b\sin\theta)\cos\tau-2r(b\cos\theta-a\sin\theta)\sin\tau}d\tau \qquad (8) \\ &= \int_0^{2\pi} \frac{\cos(l\theta)\cos(l\tau)-\sin(l\theta)\sin(l\tau)}{r^2+a^2+b^2-2r\sqrt{a^2+b^2}\cos\tau}d\tau. \end{aligned}$$

The change of variable $r = \rho\sqrt{a^2+b^2}$ transforms equation (8) to the form given on the hypothesis of [10, Lemma 2.2]. Therefore,

$$I_l^c(r) = \frac{\cos(l\theta)}{a^2+b^2}\int_0^{2\pi}\frac{\cos(l\tau)}{\rho^2+1-2\rho\cos\tau}d\tau + \frac{\sin(l\theta)}{a^2+b^2}\int_0^{2\pi}\frac{\sin(l\tau)}{\rho^2+1-2\rho\cos\tau}d\tau$$
$$= \begin{cases} 2\pi\dfrac{\cos(l\theta)}{a^2+b^2}\dfrac{\rho^l}{1-\rho^2} & 0 \le \rho < 1, \\ 2\pi\dfrac{\cos(l\theta)}{a^2+b^2}\dfrac{1}{(\rho^2-1)\rho^l} & \rho > 1. \end{cases}$$

Now, if we undo the change in r, we have that

$$I_l^c(r) = \begin{cases} 2\pi\dfrac{\cos(l\theta)}{a^2+b^2-r^2}\dfrac{r^l}{\sqrt{a^2+b^2}^{\,l}} & 0 \le r < \sqrt{a^2+b^2}, \\ 2\pi\dfrac{\cos(l\theta)}{r^2-(a^2+b^2)}\dfrac{\sqrt{a^2+b^2}^{\,l}}{r^l} & r > \sqrt{a^2+b^2}. \end{cases}$$

By the definition of θ and the properties of the Chebyshev's polynomials T_l, see [16], we obtain

$$\cos(l\theta) = T_l(\cos\theta) = T_l\left(\frac{a}{\sqrt{a^2+b^2}}\right).$$

This gives us the expression of the statement of the lemma.

The proof finishes doing analogous computations for

$$I_l^s(r) = \begin{cases} 2\pi \dfrac{\sin(l\theta)}{a^2+b^2-r^2} \dfrac{r^l}{\sqrt{a^2+b^2}^l} & 0 \le r < \sqrt{a^2+b^2}, \\ 2\pi \dfrac{\sin(l\theta)}{r^2-(a^2+b^2)} \dfrac{\sqrt{a^2+b^2}^l}{r^l} & r > \sqrt{a^2+b^2} \end{cases}$$

and

$$\sin(l\theta) = \sin\theta U_{l-1}(\cos\theta) = \frac{b}{\sqrt{a^2+b^2}} U_{l-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right).$$

□

Now we have the tools to prove the first part of Theorem 1, that is the rationality of (5).

Proposition 7. *Let k and n be a pair of natural numbers. Given system* (4) *with $K(x,y) = \prod_{j=1}^{k}\left((x-a_j)^2+(y-b_j)^2\right)$ and P and Q been polynomials of degree n. Then the Abelian integral $I(r)$, defined in* (5)*, is a piecewise rational function in r^2. Moreover, the expression of $I(r)$ depends on the position of γ_r with respect to the period annuli $\mathcal{R}_j$, so we can identify I with the vector $(I_0,\dots,I_k)$ where $I_j(r) = I(r)$ if $\gamma_r \subset \mathcal{R}_j$ for $j = 0,\dots,k$.*

Proof. After changing to polar coordinates and using the usual trigonometric identities, the expression of $I(r)$ at (5) writes as

$$\begin{aligned} I(r) &= \int_0^{2\pi} \frac{Q(r\cos t, r\sin t) r\sin t + P(r\cos t, r\sin t) r\cos t}{\prod_{j=1}^{k}\left(r^2 - 2ra_j\cos t - 2rb_j\sin t + a_j^2 + b_j^2\right)} dt \\ &= \sum_{m=1}^{n+1} r^m \int_0^{2\pi} \frac{\sum_{l=0}^{m}(\alpha_{l\,m}\cos(lt) + \beta_{l\,m}\sin(lt))}{\prod_{j=1}^{k}(r^2 - 2ra_j\cos t - 2rb_j\sin t + a_j^2 + b_j^2)} dt, \end{aligned} \tag{9}$$

where $\alpha_{l\,m}$ and $\beta_{l\,m}$ are independent real coefficients such that if l and m are not of the same parity then $\alpha_{l\,m} = \beta_{l\,m} = 0$. Finally, ordering and gathering coefficients,

$$\begin{aligned} I(r) =& \sum_{l=0}^{n+1} r^l R_l(r^2) \int_0^{2\pi} \frac{\cos(lt)}{\prod_{j=1}^{k}(r^2 - 2ra_j\cos t - 2rb_j\sin t + a_j^2 + b_j^2)} dt \\ &+ \sum_{l=1}^{n+1} r^l S_l(r^2) \int_0^{2\pi} \frac{\sin(lt)}{\prod_{j=1}^{k}(r^2 - 2ra_j\cos t - 2rb_j\sin t + a_j^2 + b_j^2)} dt, \end{aligned} \tag{10}$$

where $R_l(r^2)$ and $S_l(r^2)$ are polynomials of degree at most $[(n+1-l)/2]$ with arbitrary coefficients for all $l = 0,\dots,n+1$.

Lemma 5 allows us to break up the rational functions in the integrand of (10) as a sum of more simple functions and Lemma 6 gives us an expression for the integral of each term. As every of that expressions are rational piecewise functions of r^2, we can assure that the sum of them is also a rational piecewise function of r^2. □

3. Upper bounds for the number of zeros of the Abelian integral

Proposition 7 provides a vectorial notation for the Abelian integral (5), then we extend it to our problem. Hence, we consider the following definitions.

Definition 8. *Fixed a natural number k, a vector $\mathcal{Z} = (z_0, \ldots, z_k) \in \mathbb{N}^{k+1}$ is called a configuration of simultaneous zeros of $I = (I_0, \ldots, I_k)$ if there exists a collection of points (a_i, b_i) for $i = 1, \ldots, k$ such that $I_j(r)$ has exactly z_j zeros contained in $(\widetilde{r}_j, \widetilde{r}_{j+1})$ for every $j = 0, \ldots, k$ where $\widetilde{r}_{k+1} = \infty$.*

Since the set of configurations of zeros of I is contained in $\mathbb{N}^{k+1}$, we can induce an structure on it. Consequently, we can define a partial order relation (known as the product order) and a norm (the norm of the sum) on the set of configurations.

Definition 9. *Given $\mathcal{Z} = (z_0, \ldots, z_k)$ and $\mathcal{W} = (w_1, \ldots, w_k)$ two configurations of simultaneous zeros of I, we say that $\mathcal{Z} \leq \mathcal{W}$ if $z_j \leq w_j$ for every $j = 0, \ldots, k$. Moreover, we define the norm of $\mathcal{Z}$ as $|\mathcal{Z}| = z_0 + \ldots + z_k$.*

From this partial order on the set of configurations and the rationality of I, Theorem 1, the next result is obtained.

Corollary 10. *The maximum of the configurations of simultaneous zeros of I, $\mathcal{Z}^M$, exists.*

Additionally, as the set of configurations of I is bounded, it should be finite. This fact gives the existence of maximal configurations. A configuration $\mathcal{Z}$ is called *maximal* if there not exist any other configuration of simultaneous zeros of I greater than $\mathcal{Z}$. We remark that maximal configurations of I are, in general, not unique. See a detailed study, among others, of this fact for $k = 2$ and $n = 1$ in [15].

Although the expressions of the corresponding I_j are unique, the bounds on its degrees that can be achieve depend on the Partial Fraction Decomposition of $1/K(x, y)$, that is not unique. For fixed values of k the above result can be improved. In next section we present better explicit values of upper bounds for $k = 1$.

Proposition 11. *Under the hypotheses of Theorem 1, we have*

$$\mathcal{Z}^M \leq (N, \ldots, N, N - \min\{n+1, (2k-1)(k-1)\}),$$

where $N = \left[\dfrac{\max\{n+1, (2k-1)(k-1)\}}{2}\right] + n + 1 + (2k-1)(k-1)$. *So,*

$$|\mathcal{Z}^M| \leq kn + (k+3)\left[\frac{\max\{n+1, (2k-1)(k-1)\}}{2}\right] + k(2k^2 - 3k + 2).$$

Where $[\cdot]$ denotes the integer part function.

Proof. From the proof of Proposition 7, the numerator of $I(r)$ can be studied from the numerator of the integrand of (10), that is

$$R_0(r^2) + \sum_{l=1}^{n+1} r^l \big(R_l(r^2)\cos(lt) + S_l(r^2)\sin(lt)\big).$$

The numerators that appear applying Lemma 5 are

$$A_j(r^2, r\cos t, r\sin t) = \sum_{i=0}^{\kappa}\sum_{m=i}^{\kappa} \alpha_{i,m}(r^2) r^m \cos^i t \sin^{m-i} t$$

for every $j = 1, \ldots, k$, where $\kappa = (2k-1)(k-1)$ and $\alpha_{i,m}$ are polynomials of degree $\kappa - m$.

We can write $\cos(lt)\cos^i t \sin^{m-i} t$ as a real combination of the functions

$$\cos((l+m)t), \cos((l+m-2)t), \cos((l+m-4)t), \ldots \text{ if } m-i \text{ is even, and}$$
$$\sin((l+m)t), \sin((l+m-2)t), \sin((l+m-4)t), \ldots \text{ if } m-i \text{ is odd,}$$

for every $i = 1, \ldots, \kappa$ and $m = i, \ldots, \kappa$. Likewise, function $\sin(lt)\cos^i t \sin^{m-i} t$ writes as a real combination of

$$\sin((l+m)t), \sin((l+m-2)t), \sin((l+m-4)t), \ldots \text{ if } m-i \text{ is even, and}$$
$$\cos((l+m)t), \cos((l+m-2)t), \cos((l+m-4)t), \ldots \text{ if } m-i \text{ is odd,}$$

for every $i = 1, \ldots, \kappa$ and $m = i, \ldots, \kappa$.

Therefore the numerator of the integrand for each term in the Abelian integral can be written as

$$\widetilde{R}_0(r^2) + \sum_{\lambda=1}^{n+1+\kappa} r^\lambda \big(\widetilde{R}_\lambda(r^2)\cos(\lambda t) + \widetilde{S}_\lambda(r^2)\sin(\lambda t)\big). \tag{11}$$

Here $\widetilde{R}_\lambda(r^2)$ and $\widetilde{S}_\lambda(r^2)$ are polynomials of degree

$$\max\left\{ \left[\frac{n+1-l}{2}\right] + \kappa - m : \begin{array}{c} l = \max\{0, \lambda-\kappa\}, \ldots, n+1 \\ m = \max\{0, \lambda-(n+1)\}, \ldots, \kappa \end{array} \right\}$$

for every $\lambda = 0, \ldots, n+1+\kappa$ and $[\cdot]$ denotes the integer part function. That is

$$\deg \widetilde{R}_\lambda = \deg \widetilde{S}_\lambda = \begin{cases} [(n+1)/2] + \kappa & \text{if } \lambda \le \min\{n+1, \kappa\}, \\ [(n+1)/2] + \kappa - \lambda + n + 1 & \text{if } n+1 < \lambda \le \kappa, \\ [(n+1+\kappa-\lambda)/2] + \kappa & \text{if } \kappa < \lambda \le n+1, \\ [(n+1+\kappa-\lambda)/2] + \kappa - \lambda + n + 1 & \text{if } \max\{n+1, \kappa\} \le \lambda. \end{cases}$$

Then, using (11), the degree of the numerator of the Abelian integral, $\widehat{I}(r)$, depends on the region where r is considered. That is

$$\deg \widehat{I}(r) = \begin{cases} \deg \widehat{I}_j(r) = \max\limits_{\lambda=0,\ldots,n+1+\kappa} \{\deg \widetilde{R}_\lambda + \lambda\}, & \text{if } r^2 < a_{j+1}^2 + b_{j+1}^2, j = 0, \ldots, k-1, \\ \deg I_k(r) = \max\limits_{\lambda=0,\ldots,n+1+\kappa} \{\deg \widetilde{R}_\lambda\} & \text{if } r^2 > a_k^2 + b_k^2, \end{cases}$$

where $\widehat{I}_j(r)$ denotes the numerator of $I_j(r)$ for $j = 0, \ldots, k$. Hence,

$$\begin{aligned} \deg \widehat{I}_j(r) =& 2\max\left\{ \left[\frac{n+1}{2}\right] + \kappa + \min\{n+1, \kappa\}, \left[\frac{n+1}{2}\right] + \kappa, \left[\frac{\kappa}{2}\right] + \kappa + n + 1, \right. \\ & \left. \left[\frac{\max\{n+1,\kappa\}}{2}\right] + \kappa + n + 1 \right\} = 2\left(\left[\frac{\max\{n+1,\kappa\}}{2}\right] + \kappa + n + 1 \right) \end{aligned}$$

for $j = 0, \ldots, k-1$ and

$$\begin{aligned} \deg \widehat{I}_k(r) =& 2\max\left\{ \left[\frac{n+1}{2}\right] + \kappa, \left[\frac{\max\{n+1,\kappa\}}{2}\right] + \max\{n+1, \kappa\} \right\} \\ =& 2\left(\left[\frac{\max\{n+1,\kappa\}}{2}\right] + \max\{n+1, \kappa\} \right). \end{aligned}$$

Finally, the total number of zeros, except the origin, is bounded by the sum of all the degrees. So, the bound is given by

$$\begin{aligned}|\mathcal{Z}^{\mathcal{M}}| &\leq k\,(n+1+\kappa) + (k+1)\,[\max\{n+1,\kappa\}/2] + \max\{n+1,\kappa\} - 1\\ &\leq kn + (k+3)\,[\max\{n+1,\kappa\}/2] + k(2k^2-3k+2).\end{aligned}$$

□

The proof of the second part of Theorem 1 follows directly from the latter result.

4. Simultaneity of zeros for one singularity

The aim of this section is to improve the general bound for the number of zeros of I provided by Theorem 1 when we have only one extra singularity, that is the case $k=1$.

Proposition 12. *Let (a,b) be the singularity of system (3) with $k=1$. Then the Abelian integral $I(r)$, defined in (5), is the piecewise rational function*

$$I(r) = \begin{cases} \dfrac{r^2\Phi(r^2)}{a^2+b^2-r^2} & \text{if } \; 0<r<\sqrt{a^2+b^2},\\[2ex] \dfrac{\Psi(r^2)}{a^2+b^2-r^2} & \text{if } \; \sqrt{a^2+b^2}<r, \end{cases} \tag{12}$$

where Φ and Ψ are polynomials of degree less than or equal to n and $[(n+1)/2]$, respectively. So, $\left|\mathcal{Z}^{M}\right| \leq n + [(n+1)/2]$, where $[\cdot]$ denotes the integer part function.

Proof. After a rescaling and a rotation, see Lemma 4, the singularity can be located at $(1,0)$.

Fixed P and Q in (3) we have

$$I(r) = \int_0^{2\pi} \frac{Q(r\cos t, r\sin t)r\sin t + P(r\cos t, r\sin t)r\cos t}{r^2+1-2r\cos t}\,dt.$$

Following the procedure in the proof of Proposition 7, $I(r)$ can be written as

$$I(r) = \sum_{l=0}^{n+1} r^l R_l(r^2)\int_0^{2\pi}\frac{\cos(lt)}{r^2+1-2r\cos t}\,dt + \sum_{l=1}^{n+1} r^l S_l(r^2)\int_0^{2\pi}\frac{\sin(lt)}{r^2+1-2r\cos t}\,dt.$$

By Lemma 6, the terms corresponding to the second summation vanish because $b=0$. Then, we only need to consider the first one taking into account the relative position of the point $(1,0)$ and the circle γ_r. When $0<r<1$ we have

$$I(r) = I_0(r) = 2\pi\frac{1}{1-r^2}\sum_{l=0}^{n+1} r^{2l}R_l(r^2) = 2\pi\frac{r^2}{1-r^2}\widehat{\Phi}(r^2),$$

where

$$\widehat{\Phi}(s) = \sum_{l=0}^{n+1} s^{l-1}R_l(s) = \sum_{i=0}^{n}\phi_i s^i. \tag{13}$$

And, when $1<r$,

$$I(r) = I_1(r) = 2\pi\frac{1}{r^2-1}\sum_{l=0}^{n+1} R_l(r^2) = 2\pi\frac{1}{1-r^2}\widehat{\Psi}(r^2),$$

where

$$\widehat{\Psi}(s) = -\sum_{l=0}^{n+1} R_l(s) = \sum_{j=0}^{[(n+1)/2]}\psi_j s^j. \tag{14}$$

Moreover, the coefficients $\{\phi_i\}$ and $\{\psi_j\}$ are linear combinations of the coefficients $\{\alpha_{l,m}\}$ given by (9) in the proof of Proposition 7.

The proof finishes recovering the original radius, before the rescaling considered at the beginning. Consequently, the functions Φ and Ψ of the statement are also polynomials of the same degree than $\widehat{\Phi}$ and $\widehat{\Psi}$, respectively. □

Proposition 13. *Let n be a natural number, $k = 1$ and $(a, b) = (1, 0)$. Given every pair i and j such that $i \le n$ and $j \le [(n + 1)/2]$ and given $x_1, \dots, x_i \in (0, 1)$, $x_{i+1} \dots, x_n \notin (0, 1)$, $y_1, \dots, y_j \in (1, \infty)$ and $y_{j+1}, \dots, y_{[(n+1)/2]} \notin (1, \infty)$, then there exist P and Q of degree n such that $I(r)$, defined in (5), has exactly i zeros in $(0, 1)$ located in $\{x_k\}$ and j zeros in $(1, \infty)$ located in $\{y_l\}$.*

Proof. When $n = 0$, Φ and Ψ are constants. Hence there are no isolated zeros and the statement is proved.

For the other cases, $n \ge 1$, we can choose P and Q, from the proof of Proposition 7, such that the values of $\{\alpha_{l,m}\}$ are arbitrary. Then the proof follows showing that the polynomials $\Phi(s)$ and $\Psi(s)$, from (13) and (14), have exactly the configuration of zeros given in the statement. The main problem is that the values of $\{\phi_i\}$ and $\{\psi_j\}$ are not independent. This is due to the fact that, from expressions (13) and (14),

$$\Phi(1) + \Psi(1) = 0. \tag{15}$$

We use this relation to determine the coefficient of the leading term of Ψ, so

$$\psi_{[(n+1)/2]} = -\sum_{i=0}^{n} \phi_i - \sum_{j=0}^{[(n+1)/2]-1} \psi_j.$$

First we show, when n is even, that the coefficients $\{\alpha_{l,m}\}$ can be given in terms of $\{\phi_i\}$ and $\{\psi_j\}$. The case when n is odd follows similarly. The system of equations given by the expressions of $\{\phi_i\}$ and $\{\psi_j\}$ with respect to $\{\alpha_{l,m}\}$, (9), is solvable if we take out one equation. In fact, if we eliminate the equation relative to the expression of $\psi_{[(n+1)/2]}$, the solution $\{\alpha_{l,m}\}$ can be written as

$$\alpha_{l,m} = \begin{cases} \sum_{i=0}^{m/2-1} (\phi_i + \phi_{n-i}) + \sum_{j=0}^{m/2-1} \psi_j & \text{if } l = 0, m = 2, 4, \dots, n, \\ -\phi_n - \psi_0 & \text{if } l = 1, m = 1, \\ -\sum_{i=0}^{(m+1)/2-2} \phi_i - \sum_{i=0}^{(m+1)/2-1} \phi_{n-i} - \sum_{j=0}^{(m+1)/2-1} \psi_j & \text{if } l = 1, m = 3, 5, \dots, n-1, \\ \phi_{n/2+(l-1)/2} & \text{if } l = 1, 3, \dots, n+1, m = n+1 \\ 0 & \text{in other cases.} \end{cases} \tag{16}$$

Secondly, we compute the values $\{\phi_i\}$ and $\{\psi_j\}$ for a fixed location of the corresponding zeros, taking into account (15). Given $x_i \ne 1$, $i = 1, \dots, n$ and $y_j \ne 1$, $j = 1, \dots, [(n + 1)/2]$, there exists a unique collection of numbers, $\widetilde{\phi}_i$ for $i = 0, \dots, n-1$ and $\widetilde{\psi}_j$ for $j = 0, \dots, [(n + 1)/2] - 1$, such that

$$s^n + \sum_{i=0}^{n-1} \widetilde{\phi}_i s^i = \prod_{i=0}^{n-1} (s - x_i) := \widetilde{\Phi}(s)$$

and

$$s^{[(n+1)/2]}+\sum_{j=0}^{[(n+1)/2]-1}\widetilde{\psi}_j s^j=\prod_{j=0}^{[(n+1)/2]-1}(s-y_j):=\widetilde{\Psi}(s)$$

with $\widetilde{\Phi}(1)\neq 0$ and $\widetilde{\Psi}(1)\neq 0$. Then choosing $\phi_n=1$ and $\psi_{[(n+1)/2]}=-\widetilde{\Phi}(1)/\widetilde{\Psi}(1)$ we can define $\Phi(s)=\phi_n\widetilde{\Phi}(s)$ and $\Psi(s)=\psi_n\widetilde{\Psi}(s)$. These two functions satisfy the condition (15) and the solution (16) provides the existence of a perturbation where the functions $\Phi(s)$ and $\Psi(s)$ defined in (12) are done. The arbitrariness choosing the zeros, x_i and y_j, finishes the proof. □

Finally, we conclude this section proving Theorem 2.

Proof of Theorem 2. Propositions 12 and 13 and Lemma 4 allow us to choose the perturbation in (3) that has an associated Abelian integral with a fixed configuration of simple zeros. So, see [18], there exist perturbations with at least every configuration of limit cycles less than or equal to $(n,[(n+1)/2])$. □

5. The case of two singularities

This section deals with the case $k=2$, for which Theorem 3 gets a better upper bound of the maximum configuration. Before to prove it, we introduce some technical results. First of all, Lemma 14 transform (3) into another one which simplifies the computations. Then, Lemma 15 gives a better rational decomposition than Lemma 5. And Proposition 16 provides an explicit expression for $I(r)$. Therefore, in the proof of Theorem 3, we study the degree of the numerator of each piece of $I(r)$. At the end of this section we compute these degrees for the first values of n.

Lemma 14. *Assuming the hypotheses of Theorem 1 with $k=2$. If (a_1,b_1), (a_2,b_2) and the origin are not collinear, equivalently $a_1/b_1\neq a_2/b_2$, then there exists a change of variables that restricts the study of (3) to the case $a_1=a_2=a>0$. Moreover, the value $a_j^2+b_j^2$ for $j=1,2$ remains unchanged.*

Proof. Let us consider, in the plane (x,y), the rotation with respect to the origin of angle $\theta=\arctan\left((a_1-a_2)/(b_1-b_2)\right)$. As it is a linear transformation, it does not modify the structure of the system, nor the degree of the perturbation (P,Q). Then, the singularities are moved to $(\widetilde{a}_1,\widetilde{b}_1)$ and $(\widetilde{a}_2,\widetilde{b}_2)$ with

$$\widetilde{a}_1=\widetilde{a}_2=\widetilde{a}=\operatorname{sign}(b_1-b_2)\frac{-b_1a_2+a_1b_2}{\sqrt{(a_1-a_2)^2+(b_1-b_2)^2}}.$$

If $\widetilde{a}$ is negative then we take a rotation of angle $\theta+\pi$ instead of θ. □

The following result provides a new partial fraction decomposition for $k=2$, different from the given in Lemma 5, such that the numerator of each addend has degree one. It requires that $a_1=a_2>0$, but this is not so restrictive using the latter lemma.

Lemma 15. *Under the hypotheses of Theorem 1 for $k=2$, $a_1=a_2=a>0$ and $b_1\neq b_2$, we have that:*

$$\frac{1}{\prod_{j=1}^{2}F_j(r,\cos t,r\sin t)}=A(r^2)+\sum_{j=1}^{2}\frac{B_j(r^2)(r\cos t+1)+C_j(r^2)r\sin t}{F_j(r,\cos t,r\sin t)},\qquad(17)$$

where

$$
\begin{aligned}
A(r^2) &= \frac{r^4 + 2(a^2 - b_1 b_2)r^2 + (a^2 + b_1^2)(a^2 + b_2^2) - 4a^2}{D_1(r^2)D_2(r^2)E(r^2)},\\
B_1(r^2) &= \frac{4ar^2 + 2a(a^2 + b_1^2)(b_1 - b_2)}{(b_1 - b_2)D_1(r^2)E(r^2)},\\
B_2(r^2) &= \frac{-4ar^2 + 2(a^2 + b_2^2)(b_1 - b_2)}{(b_1 - b_2)D_2(r^2)E(r^2)},\\
C_1(r^2) &= \frac{-2(a^2 - b_1^2)r^2 + 2(a^2 + b_1^2)(a^2 + 2a + b_1 b_2)}{(b_1 - b_2)D_1(r^2)E(r^2)},\\
C_2(r^2) &= \frac{2(a^2 - b_2^2)r^2 + 2(a^2 + b_2^2)(a^2 + 2a + b_1 b_2)}{(b_1 - b_2)D_2(r^2)E(r^2)},\\
D_j(r^2) &= r^2 + a^2 + 2a + b_j^2,\\
E(r^2) &= r^4 - 2\,(a^2 + b_1 b_2)r^2 + (a^2 + b_1^2)(a^2 + b_2^2),\\
F_j(r, \cos t, r \sin t) &= r^2 - \;2ar\cos t \;-\; 2b_j r \sin t \;+\; a^2 + b_j^2,
\end{aligned}
$$

for $j = 1, 2$.

Proof. The values of A, B_j, C_j, D_j and E, for $j = 1, 2$, follow by straightforward computations solving the system obtained matching the coefficients of the trigonometric functions of the numerator in both sides of equation (17). □

Proposition 16. *Let A, D_j, E be the functions defined in Lemma 15. Under the same hypotheses, the Abelian integral $I(r)$ writes as*

$$
I(r) = 2\pi A(r^2)R_0(r^2) + 2\pi \sum_{j=1}^{2}\sum_{l=0}^{n+2} \frac{\phi_{j,l}(r^2)r^{l(1+(-1)^{\delta_j})}}{(-1)^{\delta_j}(a^2 + b_j^2 - r^2)}, \tag{18}
$$

where $\delta_j = \begin{cases} 0 & \text{if } r^2 < a^2 + b_j^2,\\ 1 & \text{if } r^2 > a^2 + b_j^2, \end{cases}$ *for $j = 1, 2$. Moreover, taking $s = r^2$, $R_0(s)$ is a polynomial of degree $[(n+1)/2]$ and the $\phi_{j,l}(s)$ are rational functions depending on a, b_1, b_2 and n. The numerator of each $\phi_{j,l}$, for $j = 1, 2$ and $l = 0, \dots, n+2$, has degree lower than or equal to $2 + [(n-l)/2]$ and the denominator is the polynomial $D_j(s)E(s)$, which has degree 3.*

Proof. The proof follows as the proof of Proposition 7, applying Lemma 15 instead of Lemma 5. □

Proof of Theorem 3. Under the notation introduced in Section 3, the statement is proved if we see that

$$
\mathcal{Z}^M(n,2) \le \begin{cases} \left(n+4, n+5, \left[\dfrac{n+1}{2}\right] + 4\right) & \text{if } a_1/b_1 \ne a_2/b_2,\\[2ex] \left(n+1, n+2, \left[\dfrac{n+1}{2}\right] + 1\right) & \text{if } a_1/b_1 = a_2/b_2. \end{cases}
$$

We distinguish two cases: when the singularities (a_j, b_j) are collinear with the origin and when they are not. For the non collinear case, Lemma 14 and Proposition 16 provide explicit expressions for I on each period annulus. The collinear case, $a_1/b_1 = a_2/b_2$, follows similarly but using the decomposition introduced in [10].

Finally, a careful study on the degree of the numerator of each I_j, as the one developed in the proof of Proposition 11, gives the upper bounds of the number of zeros described in the statement. □

Although the different decompositions given in Lemmas 5 and 15, the expression of $I(r)$ is unique. For concrete values of n and k, straightforward computations show that, the bounds provided by Theorem 3 are larger than the explicit degrees. Table 1 summarizes the values of the maximum configuration, $\mathcal{Z}^M$, for $k=2$ and $n \leq 6$. These computations have been done using the algebraic manipulator MAPLE. In fact the degree of I_0, I_1 and I_2 decreases in 2, 4 and 2 units, respectively. These differences are due to some simplifications that appear during the explicit computations. Therefore, this phenomenon is still an open question. The computations involved in obtaining these values are not difficult but the memory requirements are too high to provide more values.

n	$\mathcal{Z}^M$=($\mathcal{Z}_0$, $\mathcal{Z}_1$, $\mathcal{Z}_2$)	$\lvert\mathcal{Z}^M\rvert$	$\widetilde{\mathcal{Z}}$
0	(1,1,1)	3	13
1	(3,2,3)	8	16
2	(4,3,3)	10	18
3	(5,4,4)	13	21
4	(6,5,4)	15	23
5	(7,6,5)	18	26
6	(8,7,5)	20	28

Table 1. Degree of the numerators of I in each region. $\widetilde{\mathcal{Z}}$ shows the value given by Theorem 3

References

[1] V. I. Arnol'd. Loss of stability of self-induced oscillations near resonance, and versal deformations of equivariant vector fields. *Funkcional. Anal. i Priložen.*, 11(2):1–10, 95, 1977.

[2] V. I. Arnol'd. *Geometrical methods in the theory of ordinary differential equations*, volume 250 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science]*. Springer-Verlag, New York, 1983.

[3] A. Buică and J. Llibre. Limit cycles of a perturbed cubic polynomial differential center. *Chaos Solitons Fractals*, 32(3):1059–1069, 2007.

[4] C. Christopher and N. G. Lloyd. Polynomial systems: a lower bound for the Hilbert numbers. *Proc. Roy. Soc. London Ser. A*, 450(1938):219–224, 1995.

[5] B. Coll, C. Li, and R. Prohens. Quadratic perturbations of a class of quadratic reversible systems with two centers. *Discrete Contin. Dyn. Syst.*, 24(3):699–729, 2009.

[6] S. D. Conte. *Elementary numerical analysis: An algorithmic approach.* McGraw-Hill Book Co., New York, 1965.

[7] F. Dumortier and C. Li. Perturbation from an elliptic Hamiltonian of degree four. IV. Figure eight-loop. *J. Differential Equations*, 188(2):512–554, 2003.

[8] A. Garijo, A. Gasull, and X. Jarque. Simultaneous bifurcation of limit cycles from two nests of periodic orbits. *J. Math. Anal. Appl.*, 341(2):813–824, 2008.

[9] A. Gasull, T. Lazaro, and J. Torregrosa. Upper bounds for the number of zeroes for some Abelian integrals. *Nonlinear Anal.*, 75:5169–5179, 2012.

[10] A. Gasull, R. Prohens, and J. Torregrosa. Bifurcation of limit cycles from a polynomial non-global center. *J. Dynam. Differential Equations*, 20(4):945–960, 2008.

[11] J. Giné and J. Llibre. Limit cycles of cubic polynomial vector fields via the averaging theory. *Nonlinear Anal.*, 66(8):1707–1721, 2007.

[12] C. Li, W. Li, J. Llibre, and Z. Zhang. Linear estimate for the number of zeros of Abelian integrals for quadratic isochronous centres. *Nonlinearity*, 13(5):1775–1800, 2000.
[13] C. Li, W. Li, J. Llibre, and Z. Zhang. Linear estimation of the number of zeros of Abelian integrals for some cubic isochronous centers. *J. Differential Equations*, 180(2):307–333, 2002.
[14] J. Li. Hilbert's 16th problem and bifurcations of planar polynomial vector fields. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 13(1):47–106, 2003.
[15] S. Pérez-González and J. Torregrosa. Bifurcation diagram for the number of limit cycles on a linear center with two singular points under perturbations of degree one. Preprint, 2013.
[16] G. M. Phillips. *Interpolation and approximation by polynomials*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 14. Springer-Verlag, New York, 2003.
[17] H. Poincaré. Sur le problème des trois corps et les équations de la dynamique. *Acta Mathematica*, 13:1–270, 1890.
[18] L. S. Pontrjagin. Über autoschwingungssysteme, die den Hamiltonshen nahe liegen. *Physikalische Zeitschrift der Sowjetunion*, 6(1-2):25–28, 1934.
[19] G. Xiang and M. Han. Global bifurcation of limit cycles in a family of multiparameter system. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 14(9):3325–3335, 2004.
[20] G. Xiang and M. Han. Global bifurcation of limit cycles in a family of polynomial systems. *J. Math. Anal. Appl.*, 295(2):633–644, 2004.

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