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Central configurations of the 4–body problem with masses $m_1 = m_2 > m_3 = m_4 = m > 0$ and m small

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Abstract

In this paper we give a complete description of the families of central configurations of the planar 4–body problem with two pairs of equals masses and two equal masses sufficiently small. In particular, we give an analytical proof that this particular 4–body problem has exactly 34 different classes of central configurations. Moreover for this problem we prove the following two conjectures: *There is a unique convex planar central configuration of the 4–body problem for each ordering of the masses in the boundary of its convex hull*, which appears in [3]. We also prove the conjecture: *There is a unique convex planar central configuration having two pairs of equal masses located at the adjacent vertices of the configuration and it is an isosceles trapezoid.* Finally, the families of central configurations of this 4–body problem are numerically continued to the 4–body problem with four equal masses.

Keywords: 4–body problem, central configurations, two–small masses, convex central configurations, trapezoidal central configurations
2000 MSC: 70F10, 70F15

1. Introduction and statement of the main results

We consider the planar N–body problem

$$m_k\,\ddot{\mathbf{q}}_k = -\sum_{\substack{j\,=\,1\\ j\,\neq\,k}}^{N} G\,m_k\,m_j\,\frac{\mathbf{q}_k-\mathbf{q}_j}{|\mathbf{q}_k-\mathbf{q}_j|^3}\ ,$$

$k=1,\ldots,N$, where $\mathbf{q}_k\in\mathbb{R}^2$ is the position vector of the punctual mass m_k in an inertial coordinate system and G is the gravitational constant which can be taken equal to one by choosing conveniently the unit of time. The *configuration space* of the planar N–body problem is

$$\mathcal{E}=\{(\mathbf{q}_1,\ldots,\mathbf{q}_N)\in\mathbb{R}^{2N}\ :\ \mathbf{q}_k\neq\mathbf{q}_j,\ \text{for}\ k\neq j\}.$$

Given $m_1,\ldots,m_N$ a configuration $(\mathbf{q}_1,\ldots,\mathbf{q}_N)\in\mathcal{E}$ is *central* if the acceleration vector for each body is a common scalar multiple of its position vector (with respect to the center of mass). That is, if there exists a positive constant λ such that

$$\ddot{\mathbf{q}}_k=-\lambda\,(\mathbf{q}_k-\mathbf{cm})\ ,$$

for $k=1,\ldots,N$, where $\mathbf{cm}$ is the position vector of the center of mass of the system, which is given by

$$\mathbf{cm}=\frac{\sum_{k=1}^{N}m_k\mathbf{q}_k}{\sum_{k=1}^{N}m_k}.$$

The configuration $(\mathbf{q}_1,\ldots,\mathbf{q}_N)\in\mathcal{E}$ of the N–body problem with positive masses $m_1,\ldots,m_N$ is central if the exists λ such that $(\lambda,\mathbf{q}_1,\ldots,\mathbf{q}_N)$ is a solution of the system

$$\lambda\,(\mathbf{q}_k-\mathbf{cm})=\sum_{\substack{j\,=\,1\\ j\,\neq\,k}}^{N} m_j\,\frac{\mathbf{q}_k-\mathbf{q}_j}{|\mathbf{q}_k-\mathbf{q}_j|^3}\ , \tag{1}$$

for $k=1,\ldots,N$.

We say that two planar central configurations *belong to the same class* if there exists a rotation of $SO(2)$ and a homothecy of $\mathbb{R}^2$ with respect to the center of mass which transform one into the other.

The set of planar central configurations of the N–body problem is completely known only for $N=2,3$. For $N=2$ there is a unique class of central configurations. For $N = 3$ there are exactly five classes of central configurations for each choice of three positive masses, the three classes of collinear central configurations found in 1767 by Euler [11] and the two classes of equilateral triangle central configurations found in 1772 by Lagrange [14].

The are some partial results on the problem of finding the exact number of classes of central configurations of the N–body problem when $N > 3$. In 1910 Moulton [19] showed that there exists exactly $n!/2$ classes of collinear central configurations for a given set of positive masses, one for each possible ordering of the masses. Palmore in [20] obtained a lower bound of the number of planar non–collinear central configurations. Pedersen [21] numerically and Gannaway [12] and Arenstorf [7] numerically and analytically obtained the number of central configurations of the 4–body problem when one of the masses is sufficiently small. Later on Barros and Leandro in [8] and [9] completed the study of the central configurations of the 4–body problem when one of the masses is sufficiently small showing that in the triangle of masses there is a simple closed bifurcation curve such that outside it there is 8 classes of central configurations, on the bifurcation curve 9 and in the region limited by this curve 10. Xia in [24] studied the number of central configurations for all N when some of the masses are sufficiently small.

Simó in [23] gave a numerical study for the number of central configurations for $N = 4$ and arbitrary masses. Hampton and Moeckel [13], by a computer assisted proof, proved the finiteness of the number of central configurations for $N = 4$ and any choice of the masses. Albouy and Kaloshin [5] proved analytically the finiteness of the number of classes of central configurations for $N = 4$ for any choice of the masses and for $N = 5$ for almost all choice of the masses. The question about the finiteness of the number of classes of central configurations remains open for $N > 4$.

Although the set of all planar central configurations of the 4–body problem is not completely known, we can find in the literature several papers that provide the existence and classification of central configurations of the 4–body problem in some particular cases.

Definition 1. *Assume that* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4)$ *is a central configuration of the planar* 4*–body problem.*

(i) $\mathbf{q}$ *is* convex *if none of the bodies is located in the interior of the triangle formed by the others,*

(ii) $\mathbf{q}$ *is* concave *if one of the bodies is in the interior of the triangle formed by the others,*

(iii) **q** *is a* kite *central configuration if it has an axis of symmetry passing through two non–adjacent bodies,*

(iv) **q** *is a* rhombus *if it is convex and the four exterior edges are equal to each other.*

Under the assumption that every central configuration of the 4–body problem has an axis of symmetry when the four masses are equal, Llibre in [16] characterized the planar central configurations of the 4–body problem with equal masses by studying the intersection points of two planar curves. Later on Albouy in [1, 2] provided a complete analytic proof of the central configurations of the 4–body problem with equal masses.

Bernat et al. in [10] characterized the kite planar non–collinear classes of central configurations having some symmetry for the 4–body problem with three equal masses, see also Leandro [15]. The characterization of the convex central configurations with an axis of symmetry and the concave central configurations of the 4–body problem when the masses satisfy that $m_1 = m_2 \neq m_3 = m_4$ is done in Álvarez and Llibre [6].

MacMillan and Bartky in [18] proved that for any four positive masses and any assigned order, there is a convex planar central configuration of the 4–body problem with that order (see Xia [25] for a simpler proof). Albouy and Fu in [3] (see also [18, 22]) stated the following conjecture, well known in the central configuration community.

Conjecture 1. *There is a unique convex planar central configuration of the 4–body problem for each ordering of the masses in the boundary of its convex hull.*

MacMillan and Bartky also proved that there is a unique isosceles trapezoid central configuration of the 4–body when two pairs of equal masses are located at adjacent vertices. This result has been reproved recently by Xie in [26].

The following subconjecture of Conjecture 1 is well known between people working on central configurations.

Conjecture 2. *There is a unique convex planar central configuration having two pairs of equal masses located at the adjacent vertices of the configuration and it is an isosceles trapezoid.*

Long and Sun in [17] proved that any convex non–collinear central configurations with two equal masses $m_1 = m_2 < m_3 = m_4$ located at the opposite vertices of a quadrilateral and such that the diagonal corresponding to the mass m_1 is not shorter than the one corresponding to the mass m_3, must posses a symmetry and therefore must be a rhombus. Pérez–Chavela and Santoprete in [22] extended this result to the case where two of the masses are equal and at most, only one of the remaining mass is larger than the equal masses. In particular, they proved that there exist exactly one convex non–collinear central configuration when the opposite masses are equal and it is a rhombus. Albouy et. al. in [4] proved that in the planar 4–body problem a convex central configuration is symmetric with respect to one diagonal if and only if the masses of the two particles on the other diagonal are equal. If these two masses are unequal, then the less massive one is closer to the former diagonal.

In this paper we give a complete description of the central configurations of the 4–body problem when $m_1 = m_2 > m_3 = m_4 = m > 0$ and m is sufficiently small. In particular, we prove Conjectures 1 and 2 under these assumptions on the masses.

The existence of the central configurations of the 4–body problem when $m_1 = m_2 > m_3 = m_4 = m > 0$ and m sufficiently small is established analytically by Xia in [24]. More precisely, Xia shows that the five relative equilibria of the restricted 3–body problem (i.e. the two equilateral triangle solutions and the three collinear solutions), can be continued to 5×4 classes of central configurations of the 4–body problem with two small masses which are away from each other and to $2 \times 4 + 3 \times 2 = 14$ classes of central configurations with two small masses close to each other. We note that in Xia results the two small masses do not need to be equal. Xia results agree with the ones obtained numerically by Simó in [23].

The work of Xia does not provide the geometrical shape of the central configurations, which is our main objective.

Theorem 2. *Let $m_1 = m_2 = 1$, $m_3 = m_4 = m$, $\mathbf{q}_1 = (-1, 0)$, $\mathbf{q}_2 = (1, 0)$, $\mathbf{q}_3 = (x_3, y_3)$ and $\mathbf{q}_4 = (x_4, y_4)$ be the positions of the masses m_1, m_2, m_3 and m_4 respectively. Let $\mathbf{s} = (x_3, y_3, x_4, y_4)$. Without loss of generality we assume that $x_3, y_3 \geqslant 0$, and that two planar central configurations are equivalent if one can be obtained from the other by doing either a rotation in dimension three or by interchanging the names of the masses m_3 and m_4. Then the following statements hold.*

(a) For $m = 0$ we have the following classes of non–equivalent planar central configurations.

(a.1) Five different non–equivalent classes of non–collision central con-

figurations given by the positions $\mathbf{s}_1 = (0, \sqrt{3}, 0, 0)$, $\mathbf{s}_2 = (0, \sqrt{3}, 0, -\sqrt{3})$, $\mathbf{s}_3 = (\alpha, 0, 0, 0)$, $\mathbf{s}_4 = (\alpha, 0, -\alpha, 0)$, *and* $\mathbf{s}_5 = (0, \sqrt{3}, \alpha, 0)$ *where* $\alpha = 2.39681\ldots$ *is the unique real root of the equation* $x^5 - 2x^3 - 8x^2 + x - 8 = 0$. *See Figure 1. We note that the central configurations given by* $\mathbf{s}_3$ *and* $\mathbf{s}_4$ *are collinear.*

(a.2) *Three different classes of non–equivalent collision central configurations given by the positions* $\mathbf{sc}_1 = (0, 0, 0, 0)$, $\mathbf{sc}_2 = (0, \sqrt{3}, 0, \sqrt{3})$, *and* $\mathbf{sc}_3 = (\alpha, 0, \alpha, 0)$. *See Figure 2.*

(b) *The central configuration for* $m = 0$ *given by* $\mathbf{s}_1 = (0, \sqrt{3}, 0, 0)$ *can be continued to a unique family* $(x_3(m), y_3(m), x_4(m), y_4(m))$ *of concave kite central configurations for* $m > 0$ *small where*

$$\begin{aligned} x_3(m) &= x_4(m) = 0, \\ y_3(m) &= \sqrt{3} + \frac{16\left(1 - 3\sqrt{3}\right)}{27} m + O(m^2), \\ y_4(m) &= \frac{8 - 3\sqrt{3}}{42} m + O(m^2). \end{aligned}$$

(c) *The central configuration for* $m = 0$ *given by* $\mathbf{s}_2 = (0, \sqrt{3}, 0, -\sqrt{3})$ *can be continued to a unique family* $(x_3(m), y_3(m), x_4(m), y_4(m))$ *of convex kite central configurations for* $m > 0$ *small where*

$$\begin{aligned} x_3(m) &= x_4(m) = 0, \\ y_3(m) &= \sqrt{3} + \frac{4}{27}\left(1 - 3\sqrt{3}\right) m + O(m^2), \\ y_4(m) &= -y_3(m). \end{aligned}$$

(d) *The central configuration for* $m = 0$ *given by* $\mathbf{s}_3 = (\alpha, 0, 0, 0)$ *can be continued to a unique family* $(x_3(m), y_3(m), x_4(m), y_4(m))$ *of collinear central configurations for* $m > 0$ *small where*

$$\begin{aligned} x_3(m) &= \alpha - \frac{4(\alpha^2 - 1)\left(\alpha^7 - 2\alpha^5 - 4\alpha^4 + \alpha^3 + \alpha^2 - 1\right)}{\alpha^2\left(\alpha^6 - 3\alpha^4 + 16\alpha^3 + 3\alpha^2 + 48\alpha - 1\right)} m + O(m^2) \\ &= 2.39681\cdots - 1.36514\ldots\, m + O(m^2), \\ x_4(m) &= \frac{4\left(3\alpha^2 - 1\right)}{17\alpha^2\left(\alpha^2 - 1\right)^2} m + O(m^2) = 0.0295360\ldots\, m + O(m^2), \\ y_3(m) &= y_4(m) = 0. \end{aligned}$$

(e) The central configuration for $m = 0$ given by $\mathbf{s}_4 = (\alpha, 0, -\alpha, 0)$ can be continued to a unique family $(x_3(m), y_3(m), x_4(m), y_4(m))$ of collinear central configurations for $m > 0$ small where

$$
\begin{aligned}
x_3(m) &= \alpha + \frac{(\alpha^2 - 1)\left(17\alpha^4 - 2\alpha^2 + 1\right)}{\alpha^2\left(\alpha^6 - 3\alpha^4 + 16\alpha^3 + 3\alpha^2 + 48\alpha - 1\right)}\, m + O(m^2) \\
&= 2.39681\cdots + 1.02836\ldots\ m + O(m^2), \\
x_4(m) &= -x_3(m), \\
y_3(m) &= y_4(m) = 0.
\end{aligned}
$$

(f) The central configuration for $m = 0$ given by $\mathbf{s}_5 = (0, \sqrt{3}, \alpha, 0)$ can be continued to a unique family $(x_3(m), y_3(m), x_4(m), y_4(m))$ of non–symmetric central configurations for $m > 0$ small where

$$
\begin{aligned}
x_3(m) &= \frac{16}{3}\left(\frac{\alpha^2+1}{(\alpha^2-1)^2} - \frac{\alpha}{(\alpha^2+3)^{3/2}}\right) m + O(m^2) \\
&= 1.10354\ldots\ m + O(m^2), \\
x_4(m) &= \alpha + \frac{\frac{\alpha}{(\alpha^2+3)^{3/2}} - \frac{\alpha}{8} + \frac{3\alpha^2+1}{(\alpha^2-1)^2}}{\frac{4\alpha(\alpha^2+3)}{(\alpha^2-1)^3} + \frac{1}{4}}\, m + O(m^2) \\
&= 2.39681\cdots + 0.582716\ldots\ m + O(m^2), \\
y_3(m) &= \sqrt{3} + \frac{16}{3\sqrt{3}}\left(\frac{2\alpha}{(\alpha^2-1)^2} + \frac{1}{(\alpha^2+3)^{3/2}}\right) m + O(m^2) \\
&= 1.73205\cdots + 0.774741\ldots\ m + O(m^2), \\
y_4(m) &= -\frac{\sqrt{3}\left(\frac{1}{8} - \frac{1}{(\alpha^2+3)^{3/2}}\right)}{\frac{2\alpha(\alpha^2+3)}{(\alpha^2-1)^3} - \frac{1}{4}}\, m + O(m^2) \\
&= -1.04970\ldots\ m + O(m^2).
\end{aligned}
$$

(g) The central configuration for $m = 0$ given by $\mathbf{sc}_1 = (0, 0, 0, 0)$ can be continued to a unique family $(x_3(m), y_3(m), x_4(m), y_4(m))$ of collinear central configurations for $m > 0$ small where

$$
\begin{aligned}
x_3(m) &= \frac{1}{17^{1/3}}\, m^{1/3} - \frac{32}{867}\, m + O(m^{4/3}), \\
x_4(m) &= -x_3(m), \\
y_3(m) &= y_4(m) = 0.
\end{aligned}
$$

(h) The central configuration for $m=0$ given by $\mathbf{sc}_2=(0,\sqrt{3},0,\sqrt{3})$ can be continued to

(h.1) a unique family $(x_3(m),y_3(m),x_4(m),y_4(m))$ of concave kite central configurations for $m>0$ small where

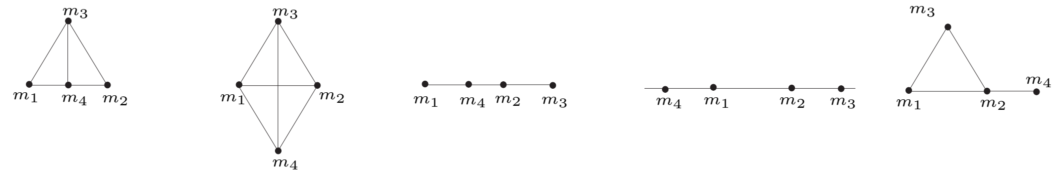
$$\begin{aligned}
x_3(m) &= x_4(m)=0,\\
y_3(m) &= \sqrt{3}+\frac{2^{2/3}}{3^{2/3}}\,m^{1/3}+\frac{1}{12^{5/6}}\,m^{2/3}+\frac{1}{81}\,m+O(m^{4/3}),\\
y_4(m) &= \sqrt{3}-\frac{2^{2/3}}{3^{2/3}}\,m^{1/3}+\frac{1}{12^{5/6}}\,m^{2/3}-\frac{1}{81}\,m+O(m^{4/3}).
\end{aligned}$$

(h.2) a unique family $(x_3(m),y_3(m),x_4(m),y_4(m))$ of isosceles trapezoid central configurations for $m>0$ small where

$$\begin{aligned}
x_3(m) &= \frac{2^{2/3}}{3^{1/3}}\,m^{1/3}+\frac{5}{27}\,m+O(m^{4/3}),\\
x_4(m) &= -x_3(m),\\
y_3(m) &= \sqrt{3}+\frac{1}{2^{5/3}\,3^{7/6}}\,m^{2/3}+O(m^{4/3}),\\
y_4(m) &= y_3(m).
\end{aligned}$$

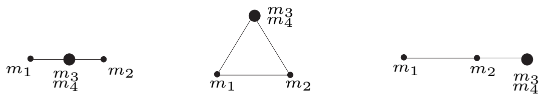
(i) The central configuration for $m=0$ given by $\mathbf{sc}_3=(\alpha,0,\alpha,0)$ can be continued to a unique family $(x_3(m),y_3(m),x_4(m),y_4(m))$ of collinear central configurations for $m>0$ small where

$$\begin{aligned}
x_3(m) &= \alpha+\overline{x_{31}}\,m^{1/3}+\overline{x_{32}}\,m^{2/3}+\overline{x_{33}}\,m+O(m^{4/3})\\
&= 2.39681\cdots+0.622799\ldots m^{1/3}+0.303818\ldots m^{2/3}+\\
&\quad 1.60489\ldots m+O(m^{4/3}),\\
x_4(m) &= \alpha-\overline{x_{31}}\,m^{1/3}+\overline{x_{32}}\,m^{2/3}+\overline{x_{43}}\,m+O(m^{4/3})\\
&= 2.39681\cdots-0.622799\ldots m^{1/3}+0.303818\ldots m^{2/3}+\\
&\quad 1.52572\ldots m+O(m^{4/3}),\\
y_3(m) &= y_4(m)=0,
\end{aligned}$$



$\mathbf{s}_1 = (0, \sqrt{3}, 0, 0)$ $\mathbf{s}_2 = (0, \sqrt{3}, 0, -\sqrt{3})$ $\mathbf{s}_3 = (\alpha, 0, 0, 0)$ $\mathbf{s}_4 = (\alpha, 0, -\alpha, 0)$ $\mathbf{s}_5 = (0, \sqrt{3}, \alpha, 0)$

Figure 1: The classes of non–equivalent non–collision planar central configurations for $m = 0$.



$\mathbf{sc}_1 = (0, 0, 0, 0)$ $\mathbf{sc}_2 = (0, \sqrt{3}, 0, \sqrt{3})$ $\mathbf{sc}_3 = (\alpha, 0, \alpha, 0)$

Figure 2: The classes of non–equivalent collision planar central configurations for $m = 0$.

and

$$
\begin{aligned}
\overline{x_{31}} &= \frac{\alpha^2 - 1}{\sqrt[3]{\alpha^6 - 3\alpha^4 + 16\alpha^3 + 3\alpha^2 + 48\alpha - 1}}, \\
\overline{x_{32}} &= \frac{24\left(\alpha^6 + 5\alpha^4 - 5\alpha^2 - 1\right)}{\left(\alpha^6 - 3\alpha^4 + 16\alpha^3 + 3\alpha^2 + 48\alpha - 1\right)^{5/3}}, \\
\overline{x_{33}} &= \frac{8}{3}\left(9\alpha^{16} - 60\alpha^{14} + 284\alpha^{13} + 168\alpha^{12} - 216\alpha^{11} + 2132\alpha^{10} - \right. \\
&\quad 1708\alpha^9 + 13314\alpha^8 + 3312\alpha^7 + 13004\alpha^6 - 1788\alpha^5 - \\
&\quad \left. 20896\alpha^4 - 152\alpha^3 - 7524\alpha^2 + 268\alpha - 147\right) / \left(\alpha^6 - 3\alpha^4 + \right. \\
&\quad \left. 16\alpha^3 + 3\alpha^2 + 48\alpha - 1\right)^3, \\
\overline{x_{43}} &= -\overline{x_{33}} + \frac{16\left(3\alpha^4 - 2\alpha^2 - 1\right)}{\alpha^6 - 3\alpha^4 + 16\alpha^3 + 3\alpha^2 + 48\alpha - 1}.
\end{aligned}
$$

(j) The central configurations described in statements (b)–(i) are all the families of non–equivalent central configurations defined for $m > 0$ sufficiently small.

Note that Theorem 2 provides all classes of equivalent central configurations of the 4–body problem with two pairs of equal masses and two equal masses sufficiently small. Recall that two planar central configurations are equivalent if one can be obtained from the other by doing either a rotation in dimension three or by interchanging the names of the masses m_3 and m_4. If we do not take into account this equivalence relation, then Theorem 2 provides that the 34 classes of central configurations predicted in [24] and [23] are the unique central configuration classes for the 4–body problem here studied. In particular, Theorem 2 describes the geometrical shape of these 34 classes of central configurations. See for more details Figure 3.

From Theorem 2 we get the following result.

Corollary 3. *The following statements hold for the 4–body problem with two pairs of equal masses and two equal masses sufficiently small.*

(a) It has exactly 34 classes of central configurations.

(b) It has exactly one convex central configuration for each ordering of the masses in the boundary of its convex hull (i.e. Conjecture 1 holds).

(c) It has exactly one convex central configuration having two pairs of equal masses located at the adjacent vertices of the configuration and it is an isosceles trapezoid (i.e. Conjecture 2 holds).

2. Equations for the central configurations

The center of mass of the central configurations studied in Theorem 2 is

$$\mathbf{cm} = \left(\frac{m(x_3 + x_4)}{2(m+1)}, \frac{m(y_3 + y_4)}{2(m+1)} \right),$$

and equations (1) become

$$e_i = 0, \qquad \text{for } i = 1, \dots, 8, \tag{2}$$

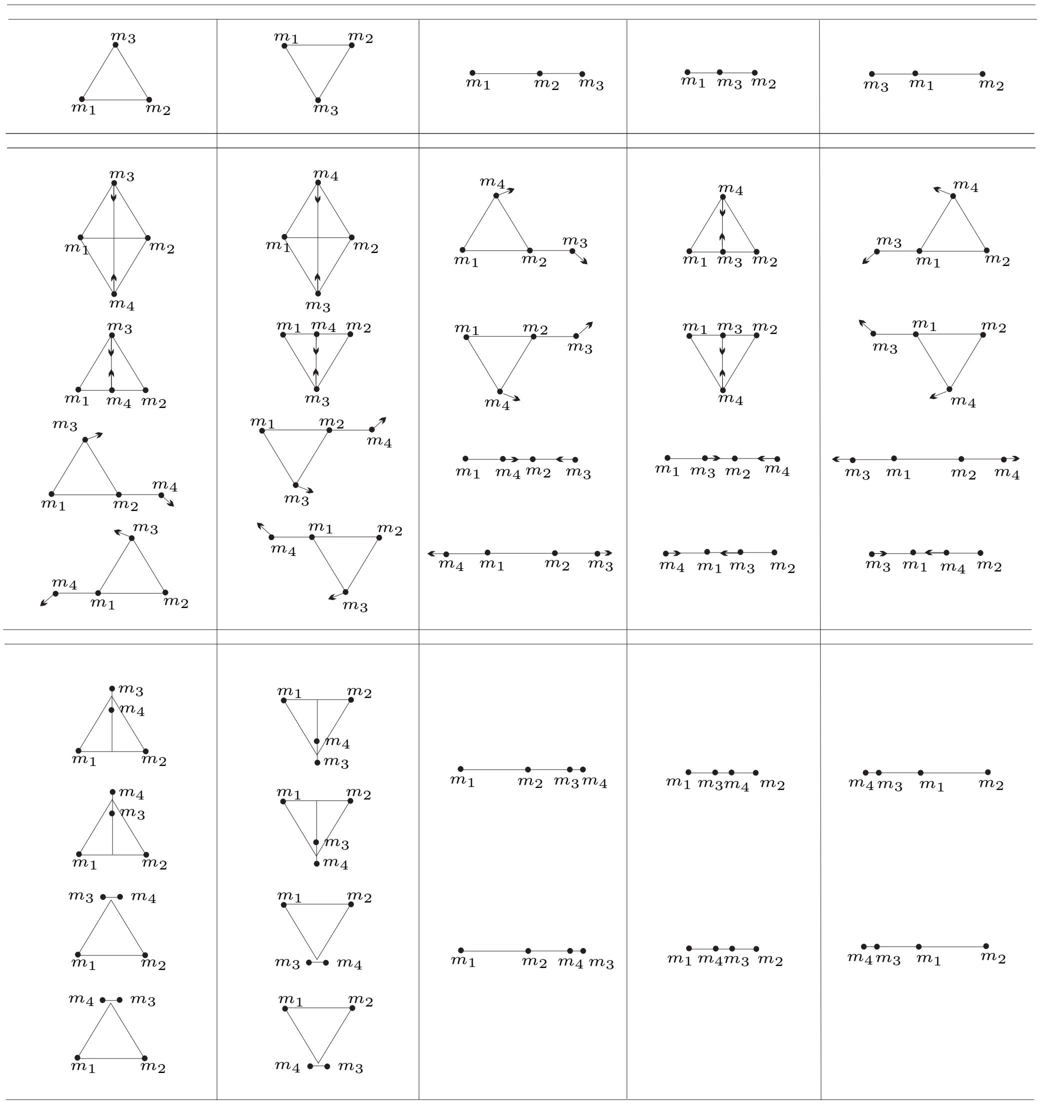


Figure 3: The classes of planar central configurations that emanate from the five central configurations of the restricted 3–body problem to the four body problem with $m_1 = m_2$ and $m_3 = m_4$ small. The direction of the arrows indicates how the position of the masses m_3 and m_4 changes when $m_3 = m_4 > 0$ and small.

where

$$
\begin{aligned}
e_1 &= -\frac{1}{4} - \frac{m(x_3+1)}{r_{13}^3} - \frac{m(x_4+1)}{r_{14}^3} + \left(1 + \frac{m(x_3+x_4)}{2(m+1)}\right)\lambda, \\
e_2 &= \frac{1}{4} - \frac{m(x_3-1)}{r_{23}^3} - \frac{m(x_4-1)}{r_{24}^3} - \left(1 - \frac{m(x_3+x_4)}{2(m+1)}\right)\lambda, \\
e_3 &= \frac{x_3+1}{r_{13}^3} + \frac{x_3-1}{r_{23}^3} + \frac{m(x_3-x_4)}{r_{34}^3} - \left(x_3 - \frac{m(x_3+x_4)}{2(m+1)}\right)\lambda, \\
e_4 &= \frac{x_4+1}{r_{14}^3} + \frac{x_4-1}{r_{24}^3} + \frac{m(x_4-x_3)}{r_{34}^3} - \left(x_4 - \frac{m(x_3+x_4)}{2(m+1)}\right)\lambda, \\
e_5 &= m\left(-\frac{y_3}{r_{13}^3} - \frac{y_4}{r_{14}^3} + \frac{\lambda(y_3+y_4)}{2(m+1)}\right), \\
e_6 &= m\left(-\frac{y_3}{r_{23}^3} - \frac{y_4}{r_{24}^3} + \frac{\lambda(y_3+y_4)}{2(m+1)}\right), \\
e_7 &= \frac{y_3}{r_{13}^3} + \frac{y_3}{r_{23}^3} + \frac{m(y_3-y_4)}{r_{34}^3} - \left(y_3 - \frac{m(y_3+y_4)}{2(m+1)}\right)\lambda, \\
e_8 &= \frac{y_4}{r_{14}^3} + \frac{y_4}{r_{24}^3} + \frac{m(y_4-y_3)}{r_{34}^3} - \left(y_4 - \frac{m(y_3+y_4)}{2(m+1)}\right)\lambda,
\end{aligned}
$$

and

$$
\begin{array}{ll}
r_{13} = \sqrt{(x_3+1)^2 + y_3^2}, & r_{14} = \sqrt{(x_4+1)^2 + y_4^2}, \\
r_{23} = \sqrt{(x_3-1)^2 + y_3^2}, & r_{24} = \sqrt{(x_4-1)^2 + y_4^2} \\
r_{34} = \sqrt{(x_3-x_4)^2 + (y_3-y_4)^2}. &
\end{array}
$$

Notice that equations (2) are not defined at the binary colúlisions between the masses. That is, when either $r_{13} = 0$, $r_{14} = 0$, $r_{23} = 0$, $r_{24} = 0$ or $r_{34} = 0$.

Clearly the eight equations (2) are not all independent. It is not difficult to prove that

$$
\begin{aligned}
e_1 + e_2 + m\,e_3 + m\,e_4 = 0, \\
e_5 + e_6 + m\,e_7 + m\,e_8 = 0.
\end{aligned} \tag{3}
$$

By defining

$$
\begin{array}{lll}
E_1 = e_1 - e_2, & E_2 = e_3 - e_2, & E_3 = e_4 - e_2, \\
E_4 = e_5 - e_6, & E_5 = e_7 - e_6, & E_6 = e_8 - e_6,
\end{array}
$$

system (2) taking into account (3) is equivalent to system

$$
E_i = 0, \qquad \text{for } i = 1, \ldots, 6. \tag{4}
$$

By isolating λ from equation $E_1 = 0$ and substituting it into the other equations of (4) we get system

$$
F_i = 0, \qquad \text{for } i = 1, \ldots, 5, \tag{5}
$$

where

$$
\begin{aligned}
F_1 &= \frac{x_3 - 1}{r_{23}^3} - \frac{x_3}{4} + \frac{x_3 + 1}{r_{13}^3} + m\left(-\frac{x_3^2 - 1}{2r_{13}^3} + \frac{x_3^2 - 1}{2r_{23}^3} + \frac{x_3 - x_4}{r_{34}^3} + \right. \\
&\quad \left. \frac{(x_3 + 1)(x_4 - 1)}{2r_{24}^3} - \frac{(x_3 - 1)(x_4 + 1)}{2r_{14}^3}\right), \\
F_2 &= \frac{x_4 - 1}{r_{24}^3} - \frac{x_4}{4} + \frac{x_4 + 1}{r_{14}^3} + m\left(-\frac{(x_3 + 1)(x_4 - 1)}{2r_{13}^3} + \right. \\
&\quad \left. \frac{(x_3 - 1)(x_4 + 1)}{2r_{23}^3} + \frac{x_4 - x_3}{r_{34}^3} - \frac{x_4^2 - 1}{2r_{14}^3} + \frac{x_4^2 - 1}{2r_{24}^3}\right), \\
F_3 &= m\left(-\frac{y_3}{r_{13}^3} + \frac{y_3}{r_{23}^3} + \frac{y_4}{r_{24}^3} - \frac{y_4}{r_{14}^3}\right), \\
F_4 &= \frac{y_3}{r_{13}^3} + \frac{y_3}{r_{23}^3} - \frac{y_3}{4} + m\left(-\frac{(x_3 + 1)y_3}{2r_{13}^3} + \frac{(x_3 + 1)y_3}{2r_{23}^3} - \frac{(x_4 + 1)y_3}{2r_{14}^3} + \right. \\
&\quad \left. \frac{y_3 - y_4}{r_{34}^3} + \frac{(x_4 - 1)y_3 + 2y_4}{2r_{24}^3}\right), \\
F_5 &= \frac{y_4}{r_{14}^3} + \frac{y_4}{r_{24}^3} - \frac{y_4}{4} + m\left(-\frac{(x_3 + 1)y_4}{2r_{13}^3} - \frac{(x_4 + 1)y_4}{2r_{14}^3} + \frac{(x_4 + 1)y_4}{2r_{24}^3} + \right. \\
&\quad \left. \frac{y_4 - y_3}{r_{34}^3} + \frac{2y_3 + (x_3 - 1)y_4}{2r_{23}^3}\right).
\end{aligned}
$$

3. Central configurations with $m = 0$

When $m = 0$ system (5) is equivalent to system

$$\begin{aligned} G(x_3, y_3) = 0, \quad G(x_4, y_4) = 0, \\ H(x_3, y_3) = 0, \quad H(x_4, y_4) = 0, \end{aligned} \tag{6}$$

where

$$\begin{aligned} G(x, y) &= \frac{x-1}{((x-1)^2 + y^2)^{3/2}} + \frac{x+1}{((x+1)^2 + y^2)^{3/2}} - \frac{x}{4}, \\ H(x, y) &= \frac{y}{((x-1)^2 + y^2)^{3/2}} + \frac{y}{((x+1)^2 + y^2)^{3/2}} - \frac{y}{4}. \end{aligned}$$

Clearly (x_3, y_3, x_4, y_4) is a solution of (6) if and only if (x_3, y_3) (respectively, (x_4, y_4)) is a solution of

$$G(x, y) = 0, \qquad H(x, y) = 0. \tag{7}$$

Solving system (7) we find the following solutions

$$\begin{array}{lll} (x, y) = (0, 0), & (x, y) = (0, \sqrt{3}), & (x, y) = (0, -\sqrt{3}), \\ (x, y) = (-\alpha, 0), & (x, y) = (\alpha, 0), & \end{array}$$

where $\alpha = 2.39681\ldots$ is the unique real root of the equation $x^5 - 2x^3 - 8x^2 + x - 8 = 0$.

We note that the five solutions of (7) that we have found correspond to the five relative equilibria of the restricted 3–body problem; the two equilateral triangle solutions and the three collinear solutions.

Since we have assumed that $x_3, y_3 \geqslant 0$, the solutions of (6) satisfying these conditions are

$$
\begin{array}{lll}
C_1: & (x_3,y_3)=(0,0), & (x_4,y_4)=(0,0),\\
C_2: & (x_3,y_3)=(0,0), & (x_4,y_4)=(0,\sqrt{3}),\\
C_3: & (x_3,y_3)=(0,0), & (x_4,y_4)=(0,-\sqrt{3}),\\
C_4: & (x_3,y_3)=(0,0), & (x_4,y_4)=(-\alpha,0),\\
C_5: & (x_3,y_3)=(0,0), & (x_4,y_4)=(\alpha,0),\\
C_6: & (x_3,y_3)=(0,\sqrt{3}), & (x_4,y_4)=(0,0),\\
C_7: & (x_3,y_3)=(0,\sqrt{3}), & (x_4,y_4)=(0,\sqrt{3}),\\
C_8: & (x_3,y_3)=(0,\sqrt{3}), & (x_4,y_4)=(0,-\sqrt{3}),\\
C_9: & (x_3,y_3)=(0,\sqrt{3}), & (x_4,y_4)=(-\alpha,0),\\
C_{10}: & (x_3,y_3)=(0,\sqrt{3}), & (x_4,y_4)=(\alpha,0),\\
C_{11}: & (x_3,y_3)=(\alpha,0), & (x_4,y_4)=(0,0),\\
C_{12}: & (x_3,y_3)=(\alpha,0), & (x_4,y_4)=(0,\sqrt{3}),\\
C_{13}: & (x_3,y_3)=(\alpha,0), & (x_4,y_4)=(0,-\sqrt{3}),\\
C_{14}: & (x_3,y_3)=(\alpha,0), & (x_4,y_4)=(-\alpha,0),\\
C_{15}: & (x_3,y_3)=(\alpha,0), & (x_4,y_4)=(\alpha,0).
\end{array}
$$

Notice that the solutions C_1, C_7, and C_{15} correspond to central configurations where m_3 and m_4 are colliding.

The central configuration given by C_3 can be obtained from the one given by C_2 after doing a rotation of 180 degrees around the x–axis. The central configuration given by C_4 (respectively C_9) can be obtained from the one given by C_5 (respectively C_{10}) after doing a rotation of 180 degrees around the y–axis. The central configurations given by C_2, C_5 and C_{12} can be obtained from the ones given by C_6, C_{11} and C_{10}, respectively, after interchanging the names of the masses m_3 and m_4. The central configuration given by C_{13} can be obtained from the one given by C_{10} after doing a rotation of 180 degrees around the x–axis and interchanging the names of the masses m_3 and m_4.

Assuming that two different central configurations are equivalent if one can be obtained from the other one by doing either a rotation in dimension three or by interchanging the names of the masses m_3 and m_4, we have that for $m = 0$ there are five non–equivalent classes of non–collision central configurations C_6, C_8, C_{10}, C_{11} and C_{14}, and three non–equivalent classes of collision central configurations C_1, C_7 and C_{15}. This proves statement (a) of Theorem 2.

4. Central configurations with $x_3 = 0$ and $x_4 = 0$ for $m > 0$ small

In this section we consider the kite central configurations; i.e, central configurations such that $x_3 = 0$ and $x_4 = 0$. More precisely, we will find the analytic expression of the kite central configurations of the 4–body problem when $m_1 = m_2 = 1$ and $m_3 = m_4 = m > 0$ small that emanate from the central configurations with $m = 0$ and $x_3 = x_4 = 0$.

Without loss of generality we can assume that $y_3 \geqslant 0$ and $y_3 \geqslant y_4$. Under these conditions the first three equations of (5) are always satisfied and the last two equations become

$$\begin{aligned} \widetilde{F}_4 &= m\left(\frac{y_4 - y_3}{\left(y_4^2+1\right)^{3/2}} + \frac{1}{(y_3-y_4)^2}\right) + \frac{2y_3}{\left(y_3^2+1\right)^{3/2}} - \frac{y_3}{4} = 0, \\ \widetilde{F}_5 &= m\left(\frac{y_3 - y_4}{\left(y_3^2+1\right)^{3/2}} - \frac{1}{(y_3-y_4)^2}\right) + \frac{2y_4}{\left(y_4^2+1\right)^{3/2}} - \frac{y_4}{4} = 0. \end{aligned} \tag{8}$$

Let $\mathbf{t} = (y_3, y_4)$. The solutions of (8) that provide non–equivalent non–collision kite central configurations with $m = 0$ are $\mathbf{t}_1 = (\sqrt{3}, 0)$ and $\mathbf{t}_2 = (\sqrt{3}, -\sqrt{3})$. They correspond to the components y_3 and y_4 of the solutions $\mathbf{s}_1$ and $\mathbf{s}_2$ given in Theorem 2(a.1). The solutions that provide non–equivalent collision kite central configurations with $m = 0$ are $\mathbf{tc}_1 = (0, 0)$ and $\mathbf{tc}_2 = (\sqrt{3}, \sqrt{3})$. They correspond to the components y_3 and y_4 of the solutions $\mathbf{sc}_1$ and $\mathbf{sc}_2$ given in Theorem 2(a.2).

In our analysis the central configurations with $x_3 = x_4 = 0$ and $y_4 = -y_3$ will play an important role. So first we analyze them.

4.1. Central configurations with $x_3 = x_4 = 0$ and $y_4 = -y_3$

When $y_4 = -y_3$ system (8) is equivalent to equation

$$\frac{2y_3}{\left(y_3^2+1\right)^{3/2}} - \frac{y_3}{4} + m\left(\frac{1}{4y_3^2} - \frac{2y_3}{\left(y_3^2+1\right)^{3/2}}\right) = 0.$$

By solving this equation with respect to m we get

$$m = f(y_3) = -\frac{\frac{2y_3}{\left(y_3^2+1\right)^{3/2}} - \frac{y_3}{4}}{\frac{1}{4y_3^2} - \frac{2y_3}{\left(y_3^2+1\right)^{3/2}}}.$$

It is not difficult to see that the numerator of m equals zero when $y_3 = 0$ and $y_3 = \pm\sqrt{3}$, and the denominator of m equals zero when $y_3 = 1/\sqrt{3}$.

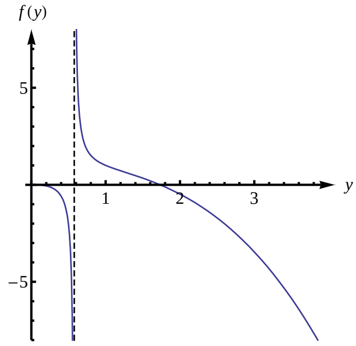


Figure 4: The graph of the function $f(y)$ for $y \geqslant 0$.

Then analyzing the sign of m for $y_3 \geqslant 0$ we have that $m > 0$ for $y_3 \in (1/\sqrt{3}, \sqrt{3})$, $m = 0$ for $y_3 = 0$ and $y_3 = \sqrt{3}$, and $m < 0$ for $y_3 \in (0, 1/\sqrt{3}) \cup (\sqrt{3}, +\infty)$. Furthermore we can prove easily that $f(y_3)$ is decreasing for all $y_3 \in (1/\sqrt{3}, \sqrt{3})$, $f(\sqrt{3}) = 0$ and $\lim_{y_3 \to (1/\sqrt{3})^+} = +\infty$ (see Figure 4). This proves the following lemma.

Lemma 4. *There exists a unique family of kite central configurations with $y_4 = -y_3$ and $y_3 \geqslant 0$ defined for all $m \geqslant 0$. This family is given by $y_3 = y_3(m) = f^{-1}(m)$, and it satisfies that $y_3 \in (1/\sqrt{3}, \sqrt{3}]$, and that $y_3 \to \sqrt{3}$ when $m \to 0$, and $y_3 \to 1/\sqrt{3}$ when $m \to +\infty$.*

4.2. Central configurations with $m > 0$ small emanating from the solutions $\mathbf{t}_1$ and $\mathbf{t}_2$

Notice that system (8) is analytic with respect to all its variables except at the points corresponding to binary collisions between the masses. Therefore it is analytic in a neighborhood of the solutions $\mathbf{t}_1$ and $\mathbf{t}_2$.

Let

$$D = \begin{vmatrix} \dfrac{\partial \widetilde{F}_4}{\partial y_3} & \dfrac{\partial \widetilde{F}_4}{\partial y_4} \\ \dfrac{\partial \widetilde{F}_5}{\partial y_3} & \dfrac{\partial \widetilde{F}_5}{\partial y_4} \end{vmatrix}.$$

It is easy to check that

$$D|_{m=0,\mathbf{t}=\mathbf{t}_1} = -\frac{63}{64} \neq 0, \qquad D|_{m=0,\mathbf{t}=\mathbf{t}_2} = \frac{81}{256} \neq 0.$$

Therefore from the Implicit Function Theorem we can find unique analytic functions $y_3^i(m)$ and $y_4^i(m)$ satisfying system (8) and $(y_3^i(0), y_4^i(0)) = \mathbf{t}_i$ for $i = 1, 2$ which are defined in a sufficiently small neighborhood U of $m = 0$.

Next we analyze the functions $y_3^i(m)$ and $y_4^i(m)$. Let $\mathbf{t}^i(m) = (y_3^i(m), y_4^i(m))$ with

$$y_3^i(m) = \sum_{k=0}^{\infty} y_{3k}^i\, m^k, \quad y_4^i(m) = \sum_{k=0}^{\infty} y_{4k}^i\, m^k,$$

and $(y_{30}^i, y_{40}^i) = \mathbf{t}_i$; and let

$$\widetilde{F}_4\Big|_{\mathbf{t}=\mathbf{t}^i(m)} = \sum_{k=0}^{\infty} d_{4k}^i\, m^k, \qquad \widetilde{F}_5\Big|_{\mathbf{t}=\mathbf{t}^i(m)} = \sum_{k=0}^{\infty} d_{5k}^i\, m^k, \tag{9}$$

be the expansion in power series of m of the functions $\widetilde{F}_4$ and $\widetilde{F}_5$ evaluated at $\mathbf{t} = \mathbf{t}^i(m)$. Clearly $\mathbf{t}^i(m)$ is a solution of system (8) if and only if $d_{4k}^i = 0$ and $d_{5k}^i = 0$ for all $k \in \mathbb{N} \cup \{0\}$. Moreover since (y_{30}^i, y_{40}^i) is a solution of system (8) for $m = 0$, the terms of order 0 of the power series expansions (9) are zero; that is, $d_{40}^i = 0$ and $d_{50}^i = 0$.

Case $i = 1$**.** By computing the terms of order 1 of the power series expansions (9) we get

$$d_{41}^1 = \frac{1}{48}\left(-27\, y_{31} - 48\sqrt{3} + 16\right), \qquad d_{51}^1 = \frac{1}{24}\left(42\, y_{41} + 3\sqrt{3} - 8\right).$$

We equate these terms to zero and we obtain

$$y_{31} = \frac{16\left(1 - 3\sqrt{3}\right)}{27}, \qquad y_{41} = \frac{8 - 3\sqrt{3}}{42}.$$

We substitute the values of y_{31} and y_{41} into the expression of $\mathbf{t}^1(m)$, and then we compute the terms of order 2 of the power series expansions (9) obtaining

$$d_{42}^1 = \frac{-15309 y_{32} + 62824\sqrt{3} + 7920}{27216}, \quad d_{52}^1 = \frac{47628 y_{42} + 10235\sqrt{3} - 34128}{27216}.$$

By equating these terms to zero we get

$$y_{32} = \frac{8\left(990 + 7853\sqrt{3}\right)}{15309}, \quad y_{42} = \frac{34128 - 10235\sqrt{3}}{47628}.$$

In short,

$$\begin{aligned} y_3^1(m) &= \sqrt{3} + \frac{16\left(1-3\sqrt{3}\right)}{27}\, m + \frac{8\left(990+7853\sqrt{3}\right)}{15309}\, m^2 + O(m^3), \\ y_4^1(m) &= \frac{8-3\sqrt{3}}{42}\, m + \frac{34128-10235\sqrt{3}}{47628}\, m^2 + O(m^3). \end{aligned} \tag{10}$$

Case $i = 2$**.** Proceeding as in the case $i = 1$ we get

$$\begin{aligned} y_3^2(m) &= \sqrt{3} + \frac{4}{27}\left(1-3\sqrt{3}\right) m + \frac{4\left(90-101\sqrt{3}\right)}{2187}\, m^2 + O(m^3), \\ y_4^2(m) &= -\sqrt{3} - \frac{4}{27}\left(1-3\sqrt{3}\right) m - \frac{4\left(90-101\sqrt{3}\right)}{2187}\, m^2 + O(m^3). \end{aligned} \tag{11}$$

By observing the first terms of the expansion of $y_3^2(m)$ and $y_4^2(m)$ in power series of m we claim that the solution $\mathbf{t}^2(m)$ satisfies that $y_4^2(m) = -y_3^2(m)$. The proof of the claim is an immediate consequence of the uniqueness of the solution $\mathbf{t}^2(m) = (y_3^2(m), y_4^2(m))$ together with Lemma 4, which assures the existence of a solution of system (8) with $y_4 = -y_3$ satisfying that $y_3 \to \sqrt{3}$ when $m \to 0$. In short we have proved the following result.

Proposition 5. *The following statements hold.*

(a) *There exists a unique family* $\mathbf{t}^1(m) = (y_3^1(m), y_4^1(m))$*, with* $y_3^1(m)$ *and* $y_4^1(m)$ *given by* (10)*, of kite central configurations emanating from the central configuration with* $m = 0$*,* $x_3 = x_4 = 0$ *and* $(y_3, y_4) = (\sqrt{3}, 0)$*.*

(b) *There exists a unique family* $\mathbf{t}^2(m) = (y_3^2(m), y_4^2(m))$*, with* $y_4^2(m) = -y_3^2(m)$ *and* $y_3^2(m)$ *given by* (11)*,of kite central configurations emanating from the central configuration with* $m = 0$*,* $x_3 = x_4 = 0$ *and* $(y_3, y_4) = (\sqrt{3}, -\sqrt{3})$*.*

4.3. Central configurations with $m > 0$ *small emanating from the solutions* $\mathbf{tc}_1$ *and* $\mathbf{tc}_2$

Next we analyze the existence of families of central configurations with $x_3 = 0$ and $x_4 = 0$ emanating from the collision solutions $\mathbf{tc}_1$ and $\mathbf{tc}_2$.

We define two new equations in the following way

$$\begin{aligned} \overline{F} = \widetilde{F}_4 + \widetilde{F}_5 &= m\left(\frac{y_3-y_4}{\left(y_3^2+1\right)^{3/2}} - \frac{y_3-y_4}{\left(y_4^2+1\right)^{3/2}}\right) + \frac{2y_3}{\left(y_3^2+1\right)^{3/2}} \\ &\quad -\frac{y_3}{4} + \frac{2y_4}{\left(y_4^2+1\right)^{3/2}} - \frac{y_4}{4} = 0, \\ \overline{G} &= (y_3-y_4)^2\, \widetilde{F}_4 = 0. \end{aligned} \tag{12}$$

Obviously, a solution of system (8) is also a solution of (12). Furthermore the functions $\overline{F}$ and $\overline{G}$ are analytic with respect to all its variables. So we shall work with system (12) instead of (8).

Let now

$$D = \begin{vmatrix} \dfrac{\partial \overline{F}}{\partial m} & \dfrac{\partial \overline{F}}{\partial y_4} \\ \dfrac{\partial \overline{G}}{\partial m} & \dfrac{\partial \overline{G}}{\partial y_4} \end{vmatrix}.$$

It is easy to check that

$$D|_{m=0,\mathbf{t}=\mathbf{tc}_1} = -\frac{7}{4} \neq 0, \qquad D|_{m=0,\mathbf{t}=\mathbf{tc}_2} = \frac{9}{16} \neq 0.$$

Therefore, from the Implicit Function Theorem, we can find unique analytic functions $m^i(y_3)$ and $y_4^i(y_3)$ satisfying system (12) and $m^1(0) = 0$, $m^2(\sqrt{3}) = 0$, $y_4^1(0) = 0$ and $y_4^2(\sqrt{3}) = \sqrt{3}$ which are defined in a sufficiently small neighborhood V of $y_3 = y_{30}^i = y_3|_{\mathbf{t}=\mathbf{tc}_i}$ for $i = 1, 2$.

Next we analyze the functions $m^i(y_3)$ and $y_4^i(y_3)$ by proceeding in a similar way than in Subsection 4.2. Let $Y_3 = y_3 - y_{30}^i$, $\tau^i(Y_3) = (m^i(Y_3), y_4^i(Y_3))$, let

$$m^i(Y_3) = \sum_{k=0}^{\infty} m_k^i\, Y_3^k, \quad y_4^i(Y_3) = \sum_{k=0}^{\infty} y_{4k}^i\, Y_3^k,$$

where $m_0 = 0$, $y_{40}^i = y_4|_{\mathbf{t}=\mathbf{tc}_i}$; and let

$$\overline{F}\big|_{(m,y_4)=\tau^i(Y_3)} = \sum_{k=0}^{\infty} \overline{f}_k^i\, Y_3^k, \qquad \overline{G}\big|_{(m,y_4)=\tau^i(Y_3)} = \sum_{k=0}^{\infty} \overline{g}_k^i\, Y_3^k, \tag{13}$$

be the expansion in power series of Y_3 of the functions $\overline{F}$ and $\overline{G}$ evaluated at $m = m^i(Y_3), y_4 = y_4^i(Y_3)$.

The terms of order 0 of the power series expansions (13), $\overline{F}_0^i = 0$ and $\overline{G}_0^i = 0$, are zero because $m = m_0^i$ and $y_4 = y_{40}^i$ is a solution of system (12) for $Y_3 = 0$. Next we analyze higher order terms of this power series expansions.

Case $i = 1$. After some computations we see that the terms of order 1 of the power series expansions (13) are

$$\overline{f}_1^1 = \frac{7y_{41}}{4} + \frac{7}{4}, \qquad \overline{g}_1^1 = m_1.$$

By equating these terms to zero we get $m_1 = 0$ and $y_{41} = -1$. We substitute them into the expressions of $m^i(Y_3)$ and $y_4^i(Y_3)$, and then we compute the terms of order 2 of the power series expansions (13) obtaining

$$\overline{f}_2^1 = \frac{7y_{42}}{4}, \qquad \overline{g}_2^1 = m_2.$$

We equate these terms to zero and we get $m_2 = 0$ and $y_{42} = 0$. By computing the terms of order 3 of the power series expansions (13) we get

$$\overline{f}_3^1 = \frac{7y_{43}}{4}, \qquad \overline{g}_3^1 = m_3 + 7.$$

So $m_3 = -7$ and $y_{43} = 0$. By computing the terms of order 4 of the power series expansions (13) we get

$$\overline{f}_4^1 = \frac{7y_{44}}{4}, \qquad \overline{g}_4^1 = m_4,$$

therefore $m_4 = 0$ and $y_{44} = 0$. By computing the terms of order 5 of the power series expansions (13) we get

$$\overline{f}_5^1 = \frac{7y_{45}}{4}, \qquad \overline{g}_5^1 = m_5 - 12,$$

hence $m_5 = 12$ and $y_{45} = 0$. In short,

$$m^1(Y_3) = -7\,Y_3^3 + 12\,Y_3^5 + O(Y_3^6), \qquad y_4^1(Y_3) = -Y_3 + O(Y_3^6).$$

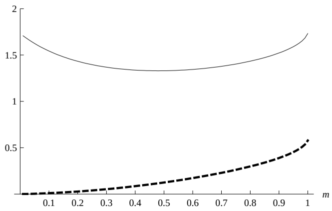
By observing the first terms of the power series expansions of $m^1(Y_3)$ and $y_4^1(Y_3)$ we claim that $y_4^1(Y_3) = -Y3$. Indeed, we have proved that $m^1(Y_3)$ and $y_4^1(Y_3)$ are the unique functions satisfying (12), and consequently satisfying (8). Moreover in Section 4.1 we have proved that there exists a family of solutions of (8) with $y_4 = -y_3$ which is defined in a neighborhood of $y_3 = 0$. Therefore we can conclude that $y_4^1(Y_3) = -Y_3$ which proves the claim. This solution does not provide a family of central configurations with $x_3 = x_4 = 0$ because on this family $m < 0$ (see Figure 4).

Case $i = 2$**.** Proceeding as in the case $i = 1$ we get

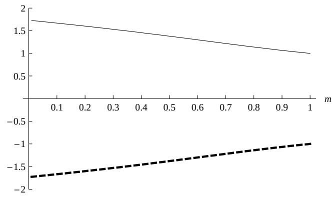
$$\begin{aligned} m^2(Y_3) &= \frac{9\,Y_3^3}{4} - \frac{27\sqrt{3}\,Y_3^4}{32} + \frac{195\,Y_3^5}{256} + O(Y_3^6), \\ y_4^2(Y_3) &= \sqrt{3} - Y_3 + \frac{\sqrt{3}\,Y_3^2}{4} - \frac{3\,Y_3^3}{16} - \\ &\quad \frac{7\sqrt{3}\,Y_3^4}{768} + \frac{3(31-512\sqrt{3})\,Y_3^5}{1024} + O(Y_3^6), \end{aligned} \tag{14}$$

where $Y_3 = y_3 - \sqrt{3}$.

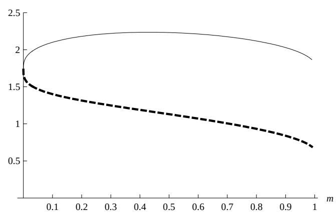
In short we have proved the following result.



a) The family $\mathbf{t}^1$



b) The family $\mathbf{t}^2$



c) The family $\mathbf{tc}^2$

Figure 5: The graph of the functions $y_3(m)$ (continuous line) and $y_4(m)$ (dashed line), for $m \in (0, 1]$, on the families of central configurations given by Propositions 5 and 6.

Proposition 6. *The following statements hold.*

(a) There is no family of kite central configurations emanating from the collision central configuration with $m = 0$, $x_3 = x_4 = 0$ *and* $(y_3, y_4) = (0, 0)$.

(b) There exists a unique family $\mathbf{tc}^2$ *of kite central configurations emanating from the collision central configuration with* $m = 0$, $x_3 = x_4 = 0$ *and* $(y_3, y_4) = (\sqrt{3}, \sqrt{3})$. *This family is given by* (14).

4.4. Numerical study of the families of central configurations with $x_3 = x_4 = 0$

With the help of Mathematica, we have followed the families of central configurations $\mathbf{t}^1$, $\mathbf{t}^2$ and $\mathbf{tc}^2$ given by Propositions 5 and 6 respectively from $m = 0$ to $m = 1$. The results that we have obtained are plotted in Figure 5.

It is well known that there are three different classes of planar non–collinear central configurations of the four–body problem with equal masses:

the square, an equilateral triangle with a mass at its center, and an isosceles triangle with one mass on its axis of symmetry (see [1]).

By computing the solutions of system (8) when $m = 1$ we find exactly three real solutions satisfying $y_3 \geqslant 0$ and $y_3 \geqslant y_4$,

(i) the solution $y_3 = \sqrt{3}$, $y_4 = 1/\sqrt{3}$ which belongs to the family $\mathbf{t}^1$ and provides an equilateral triangle with the mass m_4 at its center

(ii) the solution $y_3 = 1$, $y_4 = -1$ which belongs to the family $\mathbf{t}^2$ and provides a square

(iii) the solution $y_3 = 1.81723\ldots$, $y_4 = 0.650378\ldots$ which belongs to the family $\mathbf{tc}^2$ and provides an isosceles triangle with the masses m_3 and m_4 on its axis of symmetry.

5. Central configurations with $y_3 = 0$ and $y_4 = 0$ for $m > 0$ small

In this section we consider the collinear central configurations; i.e. central configurations such that that $y_3 = 0$ and $y_4 = 0$. Without loss of generality we can assume that $x_3 \geqslant 0$ and $x_3 \geqslant x_4$. Under these conditions the last three equations of (5) are always satisfied, and the first two equations become

$$\widetilde{F}_1 = 0, \qquad \widetilde{F}_2 = 0, \tag{15}$$

with

$$\begin{aligned}
\widetilde{F}_1 &= -\frac{x_3}{4} + \frac{x_3 - 1}{|x_3 - 1|^3} + \frac{1}{(x_3 + 1)^2} + m\left(-\frac{x_3 - 1}{2(x_3 + 1)^2} + \right. \\
&\left. \frac{(x_4 - 1)(x_3 + 1)}{2|x_4 - 1|^3} + \frac{x_3^2 - 1}{2|x_3 - 1|^3} - \frac{(x_3 - 1)(x_4 + 1)}{2|x_4 + 1|^3} + \frac{1}{(x_3 - x_4)^2}\right), \\
\widetilde{F}_2 &= -\frac{x_4}{4} + \frac{x_4 - 1}{|x_4 - 1|^3} + \frac{x_4 + 1}{|x_4 + 1|^3} + m\left(-\frac{(x_4 - 1)}{2(x_3 + 1)^2} - \right. \\
&\left. \frac{x_4^2 - 1}{2|x_4 + 1|^3} + \frac{(x_3 - 1)(x_4 + 1)}{2|x_3 - 1|^3} + \frac{x_4^2 - 1}{2|x_4 - 1|^3} - \frac{1}{(x_3 - x_4)^2}\right).
\end{aligned}$$

Let $\mathbf{r} = (x_3, x_4)$. The solutions of (15) that provide non–equivalent non–collision collinear central configurations with $m = 0$ are $\mathbf{r}_1 = (\alpha, 0)$ and $\mathbf{r}_2 = (\alpha, -\alpha)$, where $\alpha = 2.39681\ldots$ is the unique real root of the equation $x^5 - 2x^3 - 8x^2 + x - 8 = 0$. These solutions correspond to the components x_3 and x_4 of the solutions $\mathbf{s}_3$ and $\mathbf{s}_4$ given in Theorem 2(a.1). The solutions of (15) that provide non–equivalent collision collinear central configurations

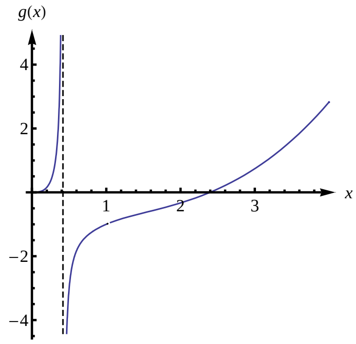


Figure 6: The graph of the function $g(x)$ for $x \geqslant 0$.

with with $m = 0$ are $\mathbf{rc}_1 = (0, 0)$ and $\mathbf{rc}_2 = (\alpha, \alpha)$. They correspond to the components x_3 and x_4 of the solutions $\mathbf{sc}_1$ and $\mathbf{sc}_3$ given in Theorem 2(a.2)).

We start analyzing the collinear central configurations with $x_4 = -x_3$, which play an important role in our study.

5.1. Central configurations with $y_3 = y_4 = 0$ and $x_4 = -x_3$

When $x_4 = -x_3$ system (15) is equivalent to equation

$$-\frac{x_3}{4} + \frac{x_3 - 1}{|x_3 - 1|^3} + \frac{1}{(x_3 + 1)^2} + m\left(\frac{1}{4x_3^2} + \frac{(x_3 - 1)\,x_3}{|x_3 - 1|^3} - \frac{x_3}{(x_3 + 1)^2}\right) = 0.$$

By solving this equation with respect to m we get

$$m = g(x_3) = -\frac{-\frac{x_3}{4} + \frac{x_3-1}{|x_3-1|^3} + \frac{1}{(x_3+1)^2}}{\frac{1}{4x_3^2} + \frac{(x_3-1)\,x_3}{|x_3-1|^3} - \frac{x_3}{(x_3+1)^2}}.$$

It is not difficult to prove that the numerator of m equals zero when $x_3 = 0$ and $x_3 = \alpha$, and the denominator of m equals zero when $x_3 = \beta = 0.417220\ldots$, where β is the unique real root of equation $8x^5 - x^4 + 8x^3 + 2x^2 - 1 = 0$. Analyzing the sign of m when $x_3 \geqslant 0$ we have that $m > 0$ for $x_3 \in (0, \beta) \cup (\alpha, +\infty)$, $m = 0$ when $x_3 = 0$ and $x_3 = \alpha$, and $m < 0$ when $x_3 \in (\beta, \alpha)$. Moreover, the function $g(x_3)$ is increasing in $[0, \beta) \cup (\beta, +\infty)$, $g(0) = 0$, $g(\alpha) = 0$ and $\lim_{x_3 \to \beta^-} = +\infty$ and $\lim_{x_3 \to +\infty} = +\infty$ (see Figure 6). This proves the following lemma.

Lemma 7. *For all $m \geqslant 0$ there exist two families of collinear central configurations with $x_4 = -x_3$ and $x_3 \geqslant 0$, one for each branch of $x_3(m) = g^{-1}(m)$.*

(a) A family $x_3(m) \in (0, \beta)$ for $m > 0$ satisfying that $x_3(m) \to 0$ when $m \to 0$, and $x_3(m) \to \beta$ when $m \to +\infty$.

(b) A family $x_3(m) \in (\alpha, +\infty)$ for $m > 0$ satisfying that $x_3(m) \to \alpha$ when $m \to 0$, and $x_3(m) \to +\infty$ when $m \to +\infty$.

5.2. Central configurations with $m > 0$ small emanating from the solutions $\mathbf{r}_1$ and $\mathbf{r}_2$

Notice that system (15) is analytic with respect to all its variables except at the points corresponding to binary collisions between the masses. Therefore it is analytic in a neighborhood of the solutions $\mathbf{r}_1$ and $\mathbf{r}_2$.

Let

$$\widetilde{D} = \begin{vmatrix} \dfrac{\partial \widetilde{F}_1}{\partial x_3} & \dfrac{\partial \widetilde{F}_1}{\partial x_4} \\ \dfrac{\partial \widetilde{F}_2}{\partial x_3} & \dfrac{\partial \widetilde{F}_2}{\partial x_4} \end{vmatrix}.$$

It is easy to check that

$$\begin{aligned} D|_{m=0,\mathbf{r}=\mathbf{r}_1} &= \frac{17\left(\alpha^6 - 3\alpha^4 + 16\alpha^3 + 3\alpha^2 + 48\alpha - 1\right)}{16\left(\alpha^2 - 1\right)^3} = 4.39829\cdots \neq 0, \\ D|_{m=0,\mathbf{r}=\mathbf{r}_2} &= \frac{\left(\alpha^6 - 3\alpha^4 + 16\alpha^3 + 3\alpha^2 + 48\alpha - 1\right)^2}{16\left(\alpha^2 - 1\right)^6} = 1.07100\cdots \neq 0. \end{aligned}$$

Therefore from the Implicit Function Theorem we can find unique analytic functions $x_3^i(m)$ and $x_4^i(m)$ satisfying system (15) and $(x_3^i(0), x_4^i(0)) = \mathbf{r}_i$ for $i = 1, 2$ which are defined in a sufficiently small neighborhood U of $m = 0$.

Next we analyze the functions $x_3^i(m)$ and $x_4^i(m)$ by proceeding as in Section 4.2. Let $\mathbf{r}^i(m) = (x_3^i(m), x_4^i(m))$ with

$$x_3^i(m) = \sum_{k=0}^{\infty} x_{3k}^i\, m^k, \quad x_4^i(m) = \sum_{k=0}^{\infty} x_{4k}^i\, m^k,$$

where $(x_{30}^i, x_{40}^i) = \mathbf{r}_i$. We expand the functions $\widetilde{F}_1$ and $\widetilde{F}_2$ evaluated at $\mathbf{r} = \mathbf{r}^i(m)$ in power series of m. By computing the first terms of these power series expansions and equating them to zero we get

$$
\begin{aligned}
x_3^1(m) &= \alpha - \frac{4(\alpha^2-1)\left(\alpha^7-2\alpha^5-4\alpha^4+\alpha^3+\alpha^2-1\right)}{\alpha^2\left(\alpha^6-3\alpha^4+16\alpha^3+3\alpha^2+48\alpha-1\right)}\, m + O(m^2) \\
&= 2.39681\cdots - 1.36514\ldots\, m + O(m^2), \\
x_4^1(m) &= \frac{4\left(3\alpha^2-1\right)}{17\alpha^2\left(\alpha^2-1\right)^2}\, m + O(m^2) = 0.0295360\ldots\, m + O(m^2),
\end{aligned}
\tag{16}
$$

and

$$
\begin{aligned}
x_3^2(m) &= \alpha + \frac{(\alpha^2-1)\left(17\alpha^4-2\alpha^2+1\right)}{\alpha^2\left(\alpha^6-3\alpha^4+16\alpha^3+3\alpha^2+48\alpha-1\right)}\, m + O(m^2) \\
&= 2.39681\cdots + 1.02836\ldots\, m + O(m^2), \\
x_4^2(m) &= -\alpha - \frac{(\alpha^2-1)\left(17\alpha^4-2\alpha^2+1\right)}{\alpha^2\left(\alpha^6-3\alpha^4+16\alpha^3+3\alpha^2+48\alpha-1\right)}\, m + O(m^2) \\
&= -2.39681\cdots - 1.02836\ldots\, m + O(m^2).
\end{aligned}
\tag{17}
$$

By observing the first terms of the expansions of $x_3^2(m)$ and $x_4^2(m)$ in power series of m we claim that the solution $\mathbf{r}^2(m)$ satisfies that $x_4^2(m) = -x_3^2(m)$. The proof of the claim is an immediate consequence of the uniqueness of the solution $\mathbf{r}^2(m) = (x_3^2(m), x_4^2(m))$ together with Lemma 7, which assures the existence of a solution of system (8) with $x_4 = -x_3$ satisfying that $x_3 \to \alpha$ when $m \to 0$. In short we have proved the following result.

Proposition 8. *The following statements hold.*

(a) There exists a unique family $\mathbf{r}^1(m) = \left(x_3^1(m), x_4^1(m)\right)$, with $x_3^1(m)$ and $x_4^1(m)$ given by (16)*, of collinear central configurations emanating from the central configuration with $m = 0$, $y_3 = y_4 = 0$ and $(x_3, x_4) = (\alpha, 0)$.*

(b) There exists a unique family $\mathbf{r}^2(m) = \left(x_3^2(m), -x_3^2(m)\right)$, with $x_3^2(m)$ is given by (17)*, of collinear central configurations emanating from the central configuration with $m = 0$, $y_3 = y_4 = 0$ $(x_3, x_4) = (\alpha, -\alpha)$.*

5.3. Central configurations with $m > 0$ small emanating from the solutions $\mathbf{rc}_1$ and $\mathbf{rc}_2$

Next we analyze the existence of families of central configurations with $y_3 = 0$ and $y_4 = 0$ emanating from the collision solutions $\mathbf{rc}_1$ and $\mathbf{rc}_2$.

We define two new equations

$$\overline{F} = \widetilde{F}_1 + \widetilde{F}_2 = 0, \qquad \overline{G} = (x_3 - x_4)^2\, \widetilde{F}_1 = 0. \tag{18}$$

Obviously, a solution of system (8) is also a solution of (18) and the functions $\overline{F}$ and $\overline{G}$ are analytic with respect to all its variables except when $x_3 = 1$ and $x_4 = \pm 1$ (remember that we have considered only solutions with $x_3 \geqslant 0$); i.e. at the binary collisions between m_3 and m_2, m_4 and m_1, and m_4 and m_2. Therefore $\overline{F}$ and $\overline{G}$ are analytic in a neighborhood of $\mathbf{rc}_1$ and $\mathbf{rc}_2$.

We shall work with system (18) instead of (8). Let now

$$D = \begin{vmatrix} \dfrac{\partial \overline{F}}{\partial m} & \dfrac{\partial \overline{F}}{\partial y_4} \\ \dfrac{\partial \overline{G}}{\partial m} & \dfrac{\partial \overline{G}}{\partial y_4} \end{vmatrix}.$$

It is easy to check that

$$\begin{aligned} D|_{m=0,\mathbf{r}=\mathbf{rc}_1} &= \frac{17}{4} \neq 0, \\ D|_{m=0,\mathbf{r}=\mathbf{rc}_2} &= \frac{\alpha^6 - 3\alpha^4 + 16\alpha^3 + 3\alpha^2 + 48\alpha - 1}{4\,(\alpha^2-1)^3} = 1.03489\cdots \neq 0. \end{aligned}$$

Therefore from the Implicit Function Theorem we can find unique analytic functions $m^i(x_3)$ and $x_4^i(x_3)$ satisfying system (18) and $m^1(0) = 0$, $m^2(\alpha) = 0$, $x_4^1(0) = 0$ and $x_4^2(\alpha) = \alpha$ which are defined in a sufficiently small neighborhood V of $x_3 = x_{30}^i = x_3|_{\mathbf{r}=\mathbf{rc}_i}$ with $i = 1, 2$.

Next we analyze the functions $m^i(x_3)$ and $x_4^i(x_3)$. Let $X_3 = x_3 - x_{30}^i$, $\rho^i(X_3) = (m^i(X_3), x_4^i(X_3))$ with

$$m^i(X_3) = \sum_{k=0}^{\infty} m_k^i\, X_3^k, \quad x_4^i(X_3) = \sum_{k=0}^{\infty} x_{4k}^i\, X_3^k,$$

and $m_0 = 0$, $x_{40}^i = x_4|_{\mathbf{r}=\mathbf{rc}_i}$. By proceeding as in Subsection 4.3 for $i = 1$ we get

$$\begin{aligned} m^1(X_3) &= 17\,X_3^3 + 32\,X_3^5 + O(X_3^6), \\ x_4^1(X_3) &= -X_3 + O(X_3^6), \end{aligned} \tag{19}$$

where $X_3 = x_3$. We claim that $x_4^1(X_3) = -X_3$. The proof of the claim is an immediate consequence of the fact that $m^1(X_3)$ and $x_4^1(X_3)$ are the unique functions satisfying (18), $m^1(0) = 0$ and $x_4^1(0) = 0$ together with the fact that there exists a family of of collinear central configurations with $x_4 = -x_3$ defined in a neighborhood of $x_3 = 0$ (see Lemma 7(a)).

For $i = 2$ we get

$$
\begin{aligned}
m^2(X_3) &= m_3\, X_3^3 + O(X_3^4), \\
x_4^2(X_3) &= \alpha - X_3 + x_{42}^2\, X_3^2 + x_{43}^2\, X_3^3 + O(X_3^4),
\end{aligned}
\tag{20}
$$

where $X_3 = x_3 - \alpha$ and

$$
\begin{aligned}
m_3 &= -\left(\alpha^2-1\right)^{-3}\left(\alpha^6 - 3\alpha^4 + 16\alpha^3 + 3\alpha^2 + 48\alpha - 1\right)^{-1} \\
&\quad \big(47\alpha^{12} - 170\alpha^{10} - 752\alpha^9 + 209\alpha^8 + 704\alpha^7 + 2356\alpha^6 + \\
&\quad 736\alpha^5 + 2913\alpha^4 - 576\alpha^3 + 150\alpha^2 - 112\alpha + 639\big) = 4.13957\ldots, \\
x_{42}^2 &= \frac{16\left(3\alpha^7 - 5\alpha^5 - 21\alpha^4 + \alpha^3 - 14\alpha^2 + \alpha - 5\right)}{(\alpha^2-1)\,(\alpha^6 - 3\alpha^4 + 16\alpha^3 + 3\alpha^2 + 48\alpha - 1)} = 1.56656\ldots, \\
x_{43}^2 &= -16\left(\alpha^2-1\right)^{-2}\left(\alpha^6 - 3\alpha^4 + 16\alpha^3 + 3\alpha^2 + 48\alpha - 1\right)^{-2} \\
&\quad \big(141\alpha^{14} - 463\alpha^{12} - 2112\alpha^{11} + 457\alpha^{10} + 1984\alpha^9 + 6269\alpha^8 + \\
&\quad 1664\alpha^7 + 4375\alpha^6 - 896\alpha^5 - 1917\alpha^4 - 576\alpha^3 - 45\alpha^2 - \\
&\quad 64\alpha + 399) = 10.5053\ldots.
\end{aligned}
$$

In short, we have proved the following result.

Proposition 9. *The following statements hold.*

(a) *There exists a unique family* $\mathbf{rc}^1$ *of collinear central configurations emanating from the collision collinear central configuration with* $m = 0$, $x_3 = x_4 = 0$ *and* $y_3 = y_4 = 0$. *This family satisfies that* $x_4 = -x_3$ *and it given by* (19).

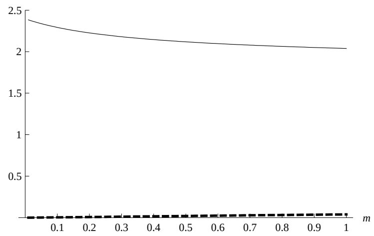
(b) *There exists a unique family* $\mathbf{rc}^2$ *of collinear central configurations emanating from the collinear collision central configuration with* $m = 0$, $x_3 = x_4 = \alpha$ *and* $y_3 = y_4 = 0$. *This family is given by* (20).

5.4. Numerical study of the families of central configurations with $y_3 = y_4 = 0$

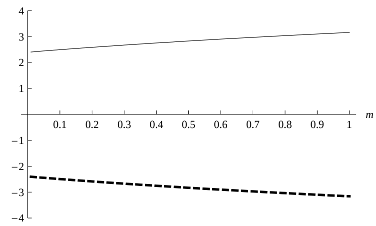
We have followed the families of central configurations $\mathbf{r}^1$, $\mathbf{r}^2$, $\mathbf{rc}^1$ and $\mathbf{rc}^2$ given by Propositions 8 and 9 respectively from $m = 0$ to $m = 1$. The results that we have obtained are plotted in Figure 7.

We have computed the solutions of system (15) when $m = 1$ and we have found exactly four real solutions satisfying $x_3 \geqslant 0$ and $x_3 \geqslant x_4$,

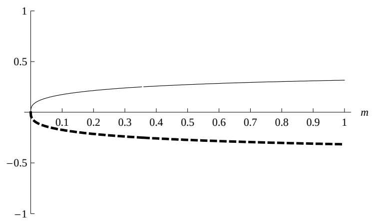
(i) the solution $(x_3, y_3) = (2.03895\ldots, 0.0389514\ldots)$ which belongs to the family $\mathbf{r}^1$,



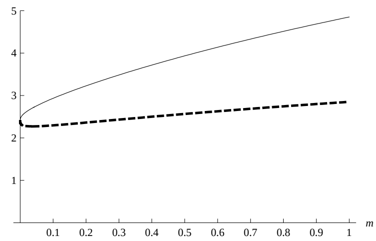
a) The family $\mathbf{r}^1$



b) The family $\mathbf{r}^2$



c) The family $\mathbf{rc}^1$



d) The family $\mathbf{rc}^2$

Figure 7: The graphs of the functions $x_3(m)$ (continuous line) and $x_4(m)$ (dashed line), for $m \in (0, 1]$, on the families of collinear central configurations given by Propositions 8 and 9.

(ii) the solution $(x_3, y_3) = (3.16212\ldots, -3.16212\ldots)$ which belongs to the family $\mathbf{r}^2$,

(iii) the solution $(x_3, y_3) = (0.316243\ldots, -0.316243\ldots)$ which belongs to the family $\mathbf{rc}^1$,

(iv) the solution $(x_3, y_3) = (4.85003\ldots, 2.85003\ldots)$ which belongs to the family $\mathbf{rc}^2$.

6. Central configurations for $m > 0$ sufficiently small emanating from non–collision central configurations for $m = 0$

Let $\mathbf{s} = (x_3, y_3, x_4, y_4)$, and let $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$, $\mathbf{s}_4$, and $\mathbf{s}_5$ be the solutions of (6) for $m = 0$ given by Theorem 2(a).

System (5) is analytic with respect to all its variables except at the points $\mathbf{s}$ corresponding to binary collisions between the masses. Therefore it is analytic in a neighborhood of the solutions $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$, $\mathbf{s}_4$, and $\mathbf{s}_5$.

Let

$$D = \begin{vmatrix} \dfrac{\partial F_1}{\partial x_3} & \dfrac{\partial F_1}{\partial y_3} & \dfrac{\partial F_1}{\partial x_4} & \dfrac{\partial F_1}{\partial y_4} \\ \dfrac{\partial F_2}{\partial x_3} & \dfrac{\partial F_2}{\partial y_3} & \dfrac{\partial F_2}{\partial x_4} & \dfrac{\partial F_2}{\partial y_4} \\ \dfrac{\partial F_4}{\partial x_3} & \dfrac{\partial F_4}{\partial y_3} & \dfrac{\partial F_4}{\partial x_4} & \dfrac{\partial F_4}{\partial y_4} \\ \dfrac{\partial F_5}{\partial x_3} & \dfrac{\partial F_5}{\partial y_3} & \dfrac{\partial F_5}{\partial x_4} & \dfrac{\partial F_5}{\partial y_4} \end{vmatrix}.$$

Let

$$A = \frac{1}{(\alpha+1)^3} + \frac{1}{(\alpha-1)^3}.$$

It is not difficult to check that

$$\begin{aligned} D|_{m=0,\mathbf{s}=\mathbf{s}_1} &= \frac{3213}{4096} \neq 0, \\ D|_{m=0,\mathbf{s}=\mathbf{s}_2} &= -\frac{729}{65536} \neq 0, \\ D|_{m=0,\mathbf{s}=\mathbf{s}_3} &= \frac{119}{16}\left(\frac{1}{4} + 2\,A\right)\left(\frac{1}{4} - A\right) = -1.09641\cdots \neq 0, \\ D|_{m=0,\mathbf{s}=\mathbf{s}_4} &= -\left(\frac{1}{4} - A\right)^2\left(\frac{1}{4} + 2\,A\right)^2 = -0.0217317\cdots \neq 0, \\ D|_{m=0,\mathbf{s}=\mathbf{s}_5} &= -\frac{27}{256}\left(\frac{1}{4} + 2\,A\right)\left(\frac{1}{4} - A\right) = 0.0155478\cdots \neq 0. \end{aligned}$$

Therefore from the Implicit Function Theorem we can find unique analytic functions $x_3^i(m)$, $y_3^i(m)$, $x_4^i(m)$, and $y_4^i(m)$ satisfying system $F_1 = 0$, $F_2 = 0$, $F_4 = 0$ and $F_5 = 0$ and $(x_3^i(0), y_3^i(0), x_4^i(0), y_4^i(0)) = \mathbf{s}_i$ for $i = 1, \ldots, 5$ which are defined in a sufficiently small neighborhood U of $m = 0$.

Xia in [24] proves that for each $i = 1, \ldots, 5$ the central configuration $\mathbf{s}_i$ can be continued to a family of central configurations with $m > 0$ small. Therefore the solution $\mathbf{s}^i(m) = (x_3^i(m), y_3^i(m), x_4^i(m), y_4^i(m))$ of system $F_1 = 0$, $F_2 = 0$, $F_4 = 0$ and $F_5 = 0$ provides a family of central configurations for $m > 0$ small and consequently it satisfies also equation $F_3 = 0$. Here we shall give the analytical expression of the families of central configurations given by the solutions $\mathbf{s}^i(m) = (x_3^i(m), y_3^i(m), x_4^i(m), y_4^i(m))$.

By uniqueness, the family of central configurations given by the solution $\mathbf{s}^i(m) = (x_3^i(m), y_3^i(m), x_4^i(m), y_4^i(m))$ with $i = 1$ coincides with the family given by Proposition 5(a), the one with $i = 2$ coincides with the one given by Proposition 5(b), the one with $i = 3$ coincides with the one given by Proposition 8(a), and the one with $i = 4$ coincides with the one given by Proposition 8(b). This proves statements (b), (c), (d) and (e) of Theorem 2.

Next we analyze the family of central configurations given by the solution $\mathbf{s}^5(m)$ by finding the analytic expression of the functions $x_3^5(m)$, $y_3^5(m)$, $x_4^5(m)$ and $y_4^5(m)$ as in Section 4.2. Let

$$x_3^5(m) = \sum_{k=0}^{\infty} x_{3k}^5\, m^k, \quad x_4^5(m) = \sum_{k=0}^{\infty} x_{4k}^5\, m^k,$$
$$y_3^5(m) = \sum_{k=0}^{\infty} y_{3k}^5\, m^k, \quad y_4^5(m) = \sum_{k=0}^{\infty} y_{4k}^5\, m^k,$$

where $(x_{30}^5, y_{30}^5, x_{40}^5, y_{40}^5) = \mathbf{s}_5$. We expand the functions F_1, F_2, F_4 and F_5 evaluated at $\mathbf{s} = \mathbf{s}^5(m)$ in power series of m. By computing the first terms of these power series expansions and equating them to zero we get

$$\begin{aligned} x_3^5(m) &= \frac{16}{3}\left(\frac{\alpha^2+1}{(\alpha^2-1)^2} - \frac{\alpha}{(\alpha^2+3)^{3/2}}\right) m + O(m^2) \\ &= 1.10354\ldots\, m + O(m^2), \end{aligned}$$

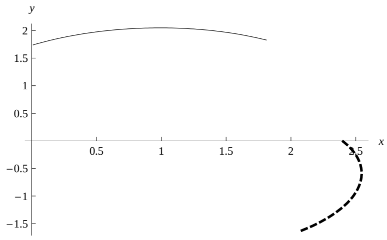


Figure 8: The graph of the points $(x_3^5(m), y_3^5(m))$ (continuous line) and the points $(x_4^5(m), y_4^5(m))$ (dashed line), for $m \in (0, 1]$, on the family of solutions $\mathbf{s}^5(m)$.

$$
\begin{aligned}
y_3^5(m) &= \sqrt{3} + \frac{16}{3\sqrt{3}}\left(\frac{2\alpha}{(\alpha^2-1)^2} + \frac{1}{(\alpha^2+3)^{3/2}}\right) m + O(m^2) \\
&= 1.73205\cdots + 0.774741\ldots\ m + O(m^2), \\
x_4^5(m) &= \alpha + \frac{\frac{\alpha}{(\alpha^2+3)^{3/2}} - \frac{\alpha}{8} + \frac{3\alpha^2+1}{(\alpha^2-1)^2}}{\frac{4\alpha(\alpha^2+3)}{(\alpha^2-1)^3} + \frac{1}{4}} m + O(m^2) \\
&= 2.39681\cdots + 0.582716\ldots\ m + O(m^2), \\
y_4^5(m) &= -\frac{\sqrt{3}\left(\frac{1}{8} - \frac{1}{(\alpha^2+3)^{3/2}}\right)}{\frac{2\alpha(\alpha^2+3)}{(\alpha^2-1)^3} - \frac{1}{4}} m + O(m^2) = -1.04970\ldots\ m + O(m^2).
\end{aligned}
$$

This completes the proof of statement (f) of Theorem 2.

6.1. Numerical study of the family of central configurations $\mathbf{s}^5(m)$

We have followed numerically the family of non–symmetric central configurations $\mathbf{s}^5(m)$ from $m = 0$ to $m = 1$. The solutions that we have obtained are plotted in Figure 8.

We note that when $m = 1$ the configuration $\mathbf{s}^5$ is given by $(x_3, y_3) = (1.81097\ldots, 1.82819\ldots)$ and $(x_4, y_4) = (2.06662\ldots, -1.64001\ldots)$, and it becomes an isosceles triangle with the masses m_2 and m_4 on its axis of symmetry.

7. Central configurations for $m > 0$ sufficiently small emanating from collision central configurations for $m = 0$

System (5) is not defined when $(x_3, y_3) = (x_4, y_4)$. Inspired in the work of Xia [24], we transform system (5) into a new system that is well defined, and in fact analytic, in a neighborhood of $(x_3, y_3) = (x_4, y_4)$ in the following way. First we consider the system of equations

$$
\begin{array}{lll}
G_1 = F_1 + F_2 = 0, & G_2 = F_4 + F_5 = 0, & G_3 = F_3, \\
G_4 = F_2 - F_1 = 0, & G_5 = F_5 - F_4 = 0, &
\end{array}
\tag{21}
$$

which is equivalent to system (5). It is easy to see that the first three equations of (21) are analytic with respect to all its variables in a neighborhood of $(x_3, y_3) = (x_4, y_4)$ and $m = 0$. The last two equations of (21) are not analytic at these points because they contain the term

$$
\frac{m}{r_{34}^3} = \frac{m}{((x_3 - x_4)^2 + (y_3 - y_4)^2)^{3/2}}.
$$

This term is well defined when $m \to 0$ if $(x_3, y_3) - (x_4, y_4) = O(m^\beta)$ with $\beta \leqslant 1/3$. Let $\mu = m^{1/3}$, then by doing the the change of variables defined by $(x_4, y_4) = (x_3, y_3) + \mu(X_4, Y_4)$, we obtain a new system of equations which is analytic in a neighborhood of the point (x_3, y_3), $\mu = 0$ and $(X_4, Y_4) \neq 0$ where

$$
\begin{aligned}
G_1 &= \frac{2(x_3 - 1)}{r_{23}^3} + \frac{2(x_3 + 1)}{r_{13}^3} - \frac{x_3}{2} + O(\mu) \\
G_2 &= \frac{2y_3}{r_{23}^3} + \frac{2y_3}{r_{13}^3} - \frac{y_3}{2} + O(\mu), \\
G_3 &= 2y_3 \left(\frac{1}{r_{23}^3} - \frac{1}{r_{13}^3} \right) \mu^3 + O(\mu^4), \\
G_4 &= \left(\frac{2X_4}{\left(X_4^2 + Y_4^2\right)^{3/2}} + \frac{X_4\left(-2x_3^2 + 4x_3 + y_3^2 - 2\right) - 3(x_3 - 1)y_3 Y_4}{r_{23}^5} \right. \\
&\quad \left. + \frac{X_4\left(-2x_3^2 - 4x_3 + y_3^2 - 2\right) - 3(x_3 + 1)y_3 Y_4}{r_{13}^5} - \frac{X_4}{4} \right) \mu + O(\mu^2), \\
G_5 &= \left(\frac{2Y_4}{\left(X_4^2 + Y_4^2\right)^{3/2}} + \frac{Y_4\left(x_3^2 - 2x_3 - 2y_3^2 + 1\right) - 3(x_3 - 1)X_4 y_3}{r_{23}^5} \right. \\
&\quad \left. + \frac{Y_4\left(x_3^2 + 2x_3 - 2y_3^2 + 1\right) - 3(x_3 + 1)X_4 y_3}{r_{13}^5} - \frac{Y_4}{4} \right) \mu + O(\mu^2).
\end{aligned}
$$

Consider now the system of equations

$$
\begin{array}{lll}
\overline{G}_1 = G_1 = 0, & \overline{G}_2 = G_2 = 0, & \overline{G}_3 = G_3/\mu^3 = 0, \\
\overline{G}_4 = G_4/\mu = 0, & \overline{G}_5 = G_5/\mu = 0, &
\end{array}
\tag{22}
$$

which is also analytic with respect to all its variables in a neighborhood of (x_3, y_3), $\mu = 0$ and $(X_4, Y_4) \neq 0$.

First we compute the solutions of (22) with $\mu = 0$. When $\mu = 0$ the third equation of (22) is always satisfied and the first two equations of (22) become $G(x_3, y_3) = 0$ and $H(x_3, y_3) = 0$ (see (7)). Therefore the solutions of $\overline{G}_1 = 0$, $\overline{G}_2 = 0$ and $G_3 = 0$ with $x_3, y_3 \geqslant 0$ are $(x_3, y_3) = (0, 0)$, $(x_3, y_3) = (0, \sqrt{3})$ and $(x_3, y_3) = (\alpha, 0)$. We substitute these solutions into the last two equations of (22), then by solving the resultant system of equations we get that when $\mu = 0$ system (22) has 8 different real solutions with $x_3, y_3 \geqslant 0$, they are given by

$$
\begin{aligned}
\mathbf{sc}_{11} &= \left(0, 0, \frac{2}{17^{1/3}}, 0\right), & \mathbf{sc}_{12} &= \left(0, 0, -\frac{2}{17^{1/3}}, 0\right), \\
\mathbf{sc}_{21} &= \left(0, \sqrt{3}, \frac{2^{5/3}}{3^{1/3}}, 0\right), & \mathbf{sc}_{22} &= \left(0, \sqrt{3}, -\frac{2^{5/3}}{3^{1/3}}, 0\right), \\
\mathbf{sc}_{23} &= \left(0, \sqrt{3}, 0, \frac{2^{5/3}}{3^{2/3}}\right), & \mathbf{sc}_{24} &= \left(0, \sqrt{3}, 0, -\frac{2^{5/3}}{3^{2/3}}\right), \\
\mathbf{sc}_{31} &= \left(\alpha, 0, \overline{X}, 0\right), & \mathbf{sc}_{32} &= \left(\alpha, 0, -\overline{X}, 0\right),
\end{aligned}
$$

where

$$
\overline{X} = \frac{2(\alpha^2 - 1)}{\sqrt[3]{\alpha^6 - 3\alpha^4 + 16\alpha^3 + 3\alpha^2 + 48\alpha - 1}} = 1.245598\ldots.
$$

Here the components of $\mathbf{sc}_{ij}$ are (x_3, y_3, X_4, Y_4). We note that system (22) has no solutions with $(X_4, Y_4) \to (0, 0)$ as $\mu \to 0$, because either $\overline{G}_4$ or $\overline{G}_5$ tend to $\pm\infty$ when $\mu \to 0$ and $(x_3, y_3) = (0, 0)$, $(x_3, y_3) = (0, \sqrt{3})$ or $(x_3, y_3) = (\alpha, 0)$.

Next we continue the solutions of system (22) with $\mu = 0$ to $\mu > 0$ small by applying the Implicit Function Theorem as in Section 6. Clearly system (22) is analytic with respect to all its variables in a neighborhood of the points $\mathbf{sc}_{1j}$, $\mathbf{sc}_{2j}$, $\mathbf{sc}_{2k}$ and $\mathbf{sc}_{3j}$ with $j = 1, 2$, and $k = 3, 4$.

Let

$$
\overline{D} = \begin{vmatrix}
\dfrac{\partial \overline{G}_1}{\partial x_3} & \dfrac{\partial \overline{G}_1}{\partial y_3} & \dfrac{\partial \overline{G}_1}{\partial X_4} & \dfrac{\partial \overline{G}_1}{\partial Y_4} \\
\dfrac{\partial \overline{G}_2}{\partial x_3} & \dfrac{\partial \overline{G}_2}{\partial y_3} & \dfrac{\partial \overline{G}_2}{\partial X_4} & \dfrac{\partial \overline{G}_2}{\partial Y_4} \\
\dfrac{\partial \overline{G}_4}{\partial x_3} & \dfrac{\partial \overline{G}_4}{\partial y_3} & \dfrac{\partial \overline{G}_4}{\partial X_4} & \dfrac{\partial \overline{G}_4}{\partial Y_4} \\
\dfrac{\partial \overline{G}_5}{\partial x_3} & \dfrac{\partial \overline{G}_5}{\partial y_3} & \dfrac{\partial \overline{G}_5}{\partial X_4} & \dfrac{\partial \overline{G}_5}{\partial Y_4}
\end{vmatrix},
$$

and let $\overline{\mathbf{s}} = (x_3, y_3, X_4, Y_4)$. It is not difficult to check that

$$\begin{aligned}
\overline{D}\big|_{\mu=0,\overline{\mathbf{s}}=\mathbf{sc}_{1j}} &= \frac{18207}{8} \neq 0, \\
\overline{D}\big|_{\mu=0,\overline{\mathbf{s}}=\mathbf{sc}_{2j}} &= \frac{729}{8192} \neq 0, \\
\overline{D}\big|_{\mu=0,\overline{\mathbf{s}}=\mathbf{sc}_{2k}} &= -\frac{2187}{8192} \neq 0, \\
\overline{D}\big|_{\mu=0,\overline{\mathbf{s}}=\mathbf{sc}_{3j}} &= -\frac{9\alpha\left(\alpha^2+3\right)}{8\left(\alpha^2-1\right)^{12}}\left(\alpha^6 - 3\alpha^4 - 8\alpha^3 + 3\alpha^2 - 24\alpha - 1\right) \\
&\quad \left(\alpha^6 - 3\alpha^4 + 16\alpha^3 + 3\alpha^2 + 48\alpha - 1\right)^2 = 2.15539\cdots \neq 0,
\end{aligned}$$

for all $j = 1, 2$ and $k = 3, 4$. Therefore from the Implicit Function Theorem we can find unique analytic functions $x_3^i(\mu)$, $y_3^i(\mu)$, $X_4^i(\mu)$, and $Y_4^i(\mu)$, defined in a sufficiently small neighborhood U of $\mu = 0$, satisfying system $\overline{G}_1 = 0$, $\overline{G}_2 = 0$, $\overline{G}_4 = 0$ and $\overline{G}_5 = 0$ and such that $(x_3^i(0), y_3^i(0), X_4^i(0), Y_4^i(0)) = \mathbf{sc}_{ij}$ for all $j = 1, 2$ when $i = 1, 3$; and $j = 1, \ldots, 4$ when $i = 2$. Next we will give the analytical expression of these functions.

Let $\overline{\mathbf{s}} = \overline{\mathbf{s}}(\mu) = (x_3(\mu), y_3(\mu), X_4(\mu), Y_4(\mu))$ be the solutions of system $\overline{G}_1 = 0$, $\overline{G}_2 = 0$, $\overline{G}_4 = 0$ and $\overline{G}_5 = 0$ with

$$\begin{aligned}
x_3(\mu) &= \sum_{k=0}^{\infty} x_{3k}\,\mu^k, \quad X_4(\mu) = \sum_{k=0}^{\infty} X_{4k}\,\mu^k, \\
y_3(\mu) &= \sum_{k=0}^{\infty} y_{3k}\,\mu^k, \quad Y_4(\mu) = \sum_{k=0}^{\infty} y_{4k}\,\mu^k.
\end{aligned}$$

We expand the functions $\overline{G}_1$, $\overline{G}_2$, $\overline{G}_4$ and $\overline{G}_5$ evaluated at $\overline{\mathbf{s}} = \overline{\mathbf{s}}(\mu)$ with $(x_{30}, y_{30}, X_{40}, Y_{40}) = \mathbf{sc}_{ij}$ for $j = 1, 2$ when $i = 1, 3$; and $j = 1, \ldots, 4$ when $i = 2$. By computing the first terms of these power series expansions and equating them to zero we get the following.

If $(x_{30}, y_{30}, X_{40}, Y_{40}) = \mathbf{sc}_{11}$, then

$$\begin{aligned}
x_3(\mu) &= -\frac{\mu}{\sqrt[3]{17}} + \frac{32\mu^3}{867} + O(\mu^4), & y_3(\mu) &= 0 + O(\mu^4), \\
X_4(\mu) &= \frac{2}{\sqrt[3]{17}} - \frac{64\mu^2}{867} - \frac{16\mu^3}{51\sqrt[3]{17}} + O(\mu^4), & Y_4(\mu) &= 0 + O(\mu^4).
\end{aligned}$$

This solution does not provide solutions of (5) with $x_3, y_3 \geqslant 0$.

If $(x_{30}, y_{30}, X_{40}, Y_{40}) = \mathbf{sc}_{12}$, then

$$x_3(\mu) = \frac{\mu}{\sqrt[3]{17}} - \frac{32\mu^3}{867} + O(\mu^4), \qquad y_3(\mu) = 0 + O(\mu^4),$$
$$X_4(\mu) = -\frac{2}{\sqrt[3]{17}} + \frac{64\mu^2}{867} + \frac{16\mu^3}{51\sqrt[3]{17}} + O(\mu^4), \quad Y_4(\mu) = 0 + O(\mu^4).$$

We undo the change of variables $(x_4, y_4) = (x_3, y_3) + \mu(X_4, Y_4)$ and we have

$$x_3(\mu) = \frac{1}{17^{1/3}}\,\mu - \frac{32}{867}\,\mu^3 + O(\mu^4), \qquad y_3(\mu) = 0 + O(\mu^4),$$
$$x_4(\mu) = -\frac{1}{17^{1/3}}\,\mu + \frac{32}{867}\,\mu^3 + O(\mu^4), \qquad y_4(\mu) = 0 + O(\mu^4).$$

By observing the first terms of these power series expansions we see that this solution must provide the family of collinear central configurations given by Proposition 9(a). This proves statement (g) of Theorem 2.

If $(x_{30}, y_{30}, X_{40}, Y_{40}) = \mathbf{sc}_{21}$, then

$$x_3(\mu) = -\frac{2^{2/3}\mu}{3^{1/3}} - \frac{5\mu^3}{27} + O\left(\mu^4\right), \qquad y_3(\mu) = \sqrt{3} + \frac{\mu^2}{2^{5/3}3^{7/6}} + O\left(\mu^4\right),$$
$$X_4(\mu) = \frac{2^{5/3}}{3^{1/3}} + \frac{10\mu^2}{27} - \frac{2^{11/3}\mu^3}{3^{7/3}} + O\left(\mu^4\right), \quad Y_4(\mu) = 0 + O(\mu^4).$$

This solution does not provide solutions of (5) with $x_3, y_3 \geqslant 0$.

If $(x_{30}, y_{30}, X_{40}, Y_{40}) = \mathbf{sc}_{22}$, then

$$x_3(\mu) = \frac{2^{2/3}\mu}{3^{1/3}} + \frac{5\mu^3}{27} + O\left(\mu^4\right), \qquad y_3(\mu) = \sqrt{3} + \frac{\mu^2}{2^{5/3}3^{7/6}} + O\left(\mu^4\right),$$
$$X_4(\mu) = -\frac{2^{5/3}}{3^{1/3}} - \frac{10\mu^2}{27} + \frac{2^{11/3}\mu^3}{3^{7/3}} + O\left(\mu^4\right), \quad Y_4(\mu) = 0 + O(\mu^4).$$

By undoing the change of variables $(x_4, y_4) = (x_3, y_3) + \mu(X_4, Y_4)$ we have

$$x_3(\mu) = \frac{2^{2/3}}{3^{1/3}}\,\mu + \frac{5}{27}\,\mu^3 + O(\mu^4), \qquad y_3(\mu) = \sqrt{3} + \frac{1}{2^{5/3}\,3^{7/6}}\,\mu^2 + O(\mu^4),$$
$$x_4(\mu) = -\frac{2^{2/3}}{3^{1/3}}\,\mu - \frac{5}{27}\,\mu^3 + O(\mu^4), \quad y_4(\mu) = \sqrt{3} + \frac{1}{2^{5/3}\,3^{7/6}}\,\mu^2 + O(\mu^4).$$

From the first terms of these power series expansions it seems that the solution $\mathbf{s}(\mu)$ satisfies that $x_4 = -x_3$ and $y_4 = y_3$, so it could be an isosceles trapezoid. From [26] we know the existence of a unique family of isosceles trapezoid central configurations defined for all $m > 0$ that tends to the equilateral triangle central configuration $\mathbf{sc}_2$ when $m \to 0$. Therefore the family of solutions $\mathbf{s}(\mu)$ must provide the family of isosceles trapezoid central configurations. This proves statement (h.2) of Theorem 2.

By proceeding in a similar way we see that the family of solutions of $\overline{G}_1 = 0$, $\overline{G}_2 = 0$, $\overline{G}_4 = 0$ and $\overline{G}_5 = 0$ with $(x_{30}, y_{30}, X_{40}, Y_{40}) = \mathbf{sc}_{23}$ provides a family of solutions of (5) with

$$x_3(\mu) = 0 + O(\mu^4), \qquad y_3(\mu) = \sqrt{3} + \frac{2^{2/3}}{3^{2/3}}\,\mu + \frac{1}{12^{5/6}}\,\mu^2 + \frac{1}{81}\,\mu^3 + O(\mu^4),$$
$$x_4(\mu) = 0 + O(\mu^4), \qquad y_4(\mu) = \sqrt{3} - \frac{2^{2/3}}{3^{2/3}}\,\mu + \frac{1}{12^{5/6}}\,\mu^2 - \frac{1}{81}\,\mu^3 + O(\mu^4).$$

This family must be the family of kite central configurations given by Proposition 6(b). This proves statement (h.1) of Theorem 2.

Since without loss of generality we can assume that $y_3 \geqslant y_4$, we can check that the family of solutions of $\overline{G}_1 = 0$, $\overline{G}_2 = 0$, $\overline{G}_4 = 0$ and $\overline{G}_5 = 0$ with $(x_{30}, y_{30}, X_{40}, Y_{40}) = \mathbf{sc}_{24}$ does not provide solutions of (5) with $x_3, y_3 \geqslant 0$ and $y_3 \geqslant y_4$.

Without loss of generality we also can assume that $x_3 \geqslant x_4$. Then we can see that the family of solutions of $\overline{G}_1 = 0$, $\overline{G}_2 = 0$, $\overline{G}_4 = 0$ and $\overline{G}_5 = 0$ with $(x_{30}, y_{30}, X_{40}, Y_{40}) = \mathbf{sc}_{31}$ does not provide a family of solutions of (5) with $x_3, x_4 \geqslant 0$ and $x_3 \geqslant x_4$.

The family of solutions of $\overline{G}_1 = 0$, $\overline{G}_2 = 0$, $\overline{G}_4 = 0$ and $\overline{G}_5 = 0$ with $(x_{30}, y_{30}, X_{40}, Y_{40}) = \mathbf{sc}_{32}$ provides the family of solutions of (5) given by

$$x_3(\mu) = \alpha + \overline{x_{31}}\,\mu + \overline{x_{32}}\,\mu^2 + \overline{x_{33}}\,\mu^3 + O(\mu^4), \qquad y_3(\mu) = O(\mu^4),$$
$$x_4(\mu) = \alpha - \overline{x_{31}}\,\mu + \overline{x_{32}}\,\mu^2 + \overline{x_{43}}\,\mu^3 + O(\mu^4), \qquad y_4(\mu) = O(\mu^4),$$

where $\overline{x_{31}} = \overline{X}/2 = 0.622799\ldots$, (see the definition of $\mathbf{sc}_{32}$) and

$$\begin{aligned}
\overline{x_{32}} &= \frac{24\left(\alpha^6 + 5\alpha^4 - 5\alpha^2 - 1\right)}{(\alpha^6 - 3\alpha^4 + 16\alpha^3 + 3\alpha^2 + 48\alpha - 1)^{5/3}} = 0.303818\ldots,\\
\overline{x_{33}} &= \frac{8}{3}\big(9\alpha^{16} - 60\alpha^{14} + 284\alpha^{13} + 168\alpha^{12} - 216\alpha^{11} + 2132\alpha^{10} -\\
&\quad 1708\alpha^9 + 13314\alpha^8 + 3312\alpha^7 + 13004\alpha^6 - 1788\alpha^5 -\\
&\quad 20896\alpha^4 - 152\alpha^3 - 7524\alpha^2 + 268\alpha - 147\big) \big/ \big(\alpha^6 - 3\alpha^4 + 16\alpha^3 +\\
&\quad 3\alpha^2 + 48\alpha - 1\big)^3 = 1.60489\ldots
\end{aligned}$$

and

$$\overline{x_{43}} = -\overline{x_{33}} + \frac{16\left(3\alpha^4 - 2\alpha^2 - 1\right)}{\alpha^6 - 3\alpha^4 + 16\alpha^3 + 3\alpha^2 + 48\alpha - 1} = 1.52572\ldots.$$

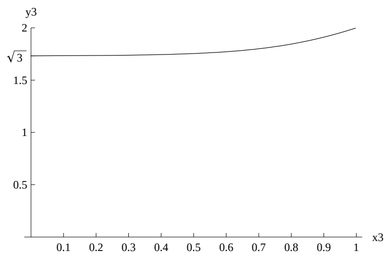


Figure 9: The graph of the points $(x_3(m), y_3(m))$ for $m \in (0, 1]$ on the family of solutions of (5) with $x_4 = -x_3$ and $y_4 = y_3 \neq 0$.

We note that these solutions must provide the family of collinear central configurations given by Proposition 9(b). This proves statement (i) of Theorem 2.

7.1. Numerical study of the family of isosceles trapezoid central configurations

We have followed numerically the family of isosceles trapezoid central configurations from $m = 0$ to $m = 1$, the solutions that we have obtained are plotted in Figure 9. We note that if $m \to 0$, then $(x_3, y_3) \to (0, \sqrt{3})$, and if $m = 1$ then the configurations tends to the square with $(x_3, y_3) \to (1, 2)$.

8. Central configurations for $m > 0$ sufficiently small that do not emanate from central configurations with $m = 0$

The families of solutions of system (5) for $m > 0$ small can come either from the solutions for $m = 0$, from the singularities of the equations (5) (which correspond to collision between the masses), or from infinity.

Up to here we have found all the families of non–equivalent central configurations of the planar four–body problem emanating from central configurations with $m = 0$. In this section we prove that there are no families of central configurations for $m > 0$ sufficiently small with one of the small masses near collision with either m_1 and m_2, and that there are no families of central configurations with one of the masses coming from infinity. This proves statement (j) of Theorem 2.

8.1. Central configurations for $m > 0$ sufficiently small with one small mass near collision with either m_1 and m_2

Without loss of generality we can assume that m_3 tends to collision with m_2 when $m \to 0$. Since each central configurations is a solution of system

	$\lim\limits_{\substack{(x,y)\to(0,0)\\(x,y)\in\gamma_{\theta_0}}} h_1(x,y)$	$\lim\limits_{\substack{(x,y)\to(0,0)\\(x,y)\in\gamma_{\theta_0}}} h_2(x,y)$
$\theta_0 \neq k\,\pi/2,\ k=0,1,2,3$	∞	∞
$\theta = 0$ or $\theta = \pi$	∞	0
$\theta = \pi/2$ or $\theta = -\pi/2$	0	∞
$\theta \xrightarrow{r\to 0} \theta_0$ with $\theta_0 = k\,\pi/2,\ k=0,1,2,3,$ and $\theta \neq \theta_0$	an arbitrary $a \in \mathbb{R}$	an arbitrary $b \in \mathbb{R}$

Table 1: The values of $\lim_{(x,y)\to(0,0)} h_1(x,y)$ and $\lim_{(x,y)\to(0,0)} h_2(x,y)$ along the paths γ_{θ_0} depending on the values of θ_0. In this work ∞ would mean the unsigned infinity, it could refer to either $+\infty$ or $-\infty$ depending on the context.

(5), a necessary condition in order to have a family of central configurations with $m_3 \to m_2$ as $m \to 0$ is that $F_1 \to 0$, $F_2 \to 0$, $F_3 \to 0$, $F_4 \to 0$ and $F_5 \to 0$ when $(x_3, y_3) \to (1, 0)$ and $m \to 0$. We will see that not all the functions F_i tend to 0 as $(x_3, y_3) \to (1, 0)$ and $m \to 0$, which implies that there are no solutions of (5) with $m_3 \to m_2$ as $m \to 0$.

In order to analyze the limits of F_i when $(x_3, y_3) \to (1, 0)$ we need the following lemma.

Lemma 10. *Let*

$$h_1(x,y) = \frac{x}{(x^2+y^2)^{3/2}}, \qquad h_2(x,y) = \frac{y}{(x^2+y^2)^{3/2}}.$$

We introduce polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. If γ_{θ_0} denotes an arbitrary path that approaches the origin along the direction of the ray $\theta = \theta_0$; i.e. $\theta \to \theta_0$ when $r \to 0$ along the path, then the following statements hold.

(a) The values of $\lim_{(x,y)\to(0,0)} h_1(x,y)$ and $\lim_{(x,y)\to(0,0)} h_2(x,y)$ along the path γ_{θ_0} depend on the values of θ_0 and they are summarized in Table 1.

(b) If $\theta_0 \neq \pm\pi/2$, then $\lim_{(x,y)\to(0,0),\ (x,y)\in\gamma_{\theta_0}} h_1(x,y)$ is infinity of order $1/r^2$ when $r \to 0$; i.e. $\lim_{r\to 0^+} r^2\, h_1(r\cos\theta_0, r\sin\theta_0) = \ell$ with $\ell \neq 0$ and $\ell \neq \infty$.

(c) If $\theta_0 \neq 0, \pi$, then $\lim_{(x,y)\to(0,0),\ (x,y)\in\gamma_{\theta_0}} h_2(x,y)$ is infinity of order $1/r^2$ when $r \to 0$; i.e. $\lim_{r\to 0^+} r^2\, h_2(r\cos\theta_0, r\sin\theta_0) = \ell$ with $\ell \neq 0$ and $\ell \neq \infty$.

PROOF. The proof is an immediate consequence of the fact that the expressions of h_1 and h_2 in polar coordinates are

$$h_1(r\,\cos\theta, r\,\sin\theta) = \frac{\cos\theta}{r^2}, \qquad h_2(r\,\cos\theta, r\,\sin\theta) = \frac{\sin\theta}{r^2}.$$

Indeed, if $\cos\theta \neq 0$, then $\lim_{r\to 0^+} h_1(r\,\cos\theta, r\,\sin\theta) = \infty$. If $\theta = \pm\pi/2$ is constant along the path, then $\lim_{r\to 0^+} h_1(r\,\cos\theta, r\,\sin\theta) = 0$. And finally if $\theta \to \pm\pi/2$ as $r \to 0$ but $\theta \neq \pm\pi/2$ along the path, then the limit will depend on the path that we choose in order to approach the origin on the direction of the rays $\theta = \pm\pi/2$. For instance, if we approach the origin along paths of the form $x = a\,y^3$ with $a \in \mathbb{R}$ arbitrary then

$$\lim_{y\to 0} h_1(a\,y^3, y) = \lim_{y\to 0} \frac{a\,y^3}{(a^2\,y^6 + y^2)^{3/2}} = a.$$

The limit $\lim_{(x,y)\to(0,0)} h_2(x,y)$ along the paths γ_{θ_0} depending on the values of θ_0 can be analyzed in a similar way.

In what follows we use the notation $\lim_{(x,y)\xrightarrow{\gamma_{\theta_0}}(0,0)} h(x,y)$ to denote the limit $\lim_{(x,y)\to(0,0)} h(x,y)$ along the path γ_{θ_0}.

By applying the properties of limits and after some computations we get

$$\begin{aligned}\lim_{(x_3,y_3)\to(1,0)} F_1 \;=\; & (1+m)\cdot \lim_{(x_3,y_3)\to(1,0)} \frac{x_3-1}{r_{23}^3} + m\cdot \lim_{(x_3,y_3)\to(1,0)} \frac{x_3-x_4}{r_{34}^3} + \\ & m\cdot\frac{x_4-1}{r_{24}^3} - m\cdot\frac{x_4+1}{r_{14}^3}\cdot \lim_{x_3\to 1}\frac{x_3-1}{2}. \end{aligned} \tag{23}$$

The limit $\lim_{(x_3,y_3)\to(1,0)}(x_3-1)/r_{23}^3$ depends on the path that we choose to approach the point $(1,0)$, see Lemma 10. We consider polar coordinates $x_3 = 1 + r\,\cos\theta$ and $y_3 = r\,\sin\theta$ and we denote by γ_{θ_0} an arbitrary path that approaches the point $(x_3,y_3) = (1,0)$ along the direction of the ray $\theta = \theta_0$, then

$$L_1 = \lim_{(x_3,y_3)\xrightarrow{\gamma_{\theta_0}}(1,0)} \frac{x_3-1}{((x_3-1)^2+y_3^2)^{3/2}} = \begin{cases} \infty & \text{if } \theta_0 \neq \pm\pi/2, \\ 0 & \text{if } \theta = \pm\pi/2, \\ a\in\mathbb{R} & \text{if } \theta \xrightarrow{r\to 0} \pm\pi/2. \end{cases} \tag{24}$$

Since we only are interested in solutions with $x_3, y_3 \geqslant 0$, we assume that $\theta \in [0, \pi/2]$.

We note that the second, the third and the fourth summands in (23) could tend to infinity when $m_4 \to m_2$ or $m_4 \to m_1$ as $m \to 0$. So the limit of F_1 when $(x_3,y_3) \to (1,0)$ and $m \to 0$ depends on wether $m_4 \to m_2$ as $m \to 0$, $m_4 \to m_1$ as $m \to 0$, or m_4 is far from collision with m_1 and m_2 as $m \to 0$.

8.1.1. Case m_4 far from collision with either m_1 or m_2 when $m \to 0$.

From (23) and (24), if m_4 is far from collision with either m_1 or m_2 when $m \to 0$ then

$$\lim_{(x_3,y_3)\xrightarrow{\gamma_{\theta_0}}(1,0)} F_1 = (1+m)\,L_1.$$

Since we need that $F_1 \to 0$ as $(x_3, y_3) \to (1, 0)$ and $m \to 0$, $\theta_0 = \pi/2$. On the other hand, if m_4 is far from collision with m_1 and m_2 when $m \to 0$, then it is easy to check that

$$\lim_{(x_3,y_3)\to(1,0)} F_4 = (1+m) \lim_{(x_3,y_3)\to(1,0)} \frac{y_3}{((x_3-1)^2+y_3^2)^{3/2}}.$$

By Lemma 10 this limit becomes ∞ when we approach the point $(x_3, y_3) = (1, 0)$ along an arbitrary path γ_{θ_0} with $\theta_0 = \pi/2$. Therefore there are no solutions of (5) in this case.

8.1.2. Case m_4 tending to collision with m_1 when $m \to 0$.

We define L_1 as in (24). We introduce polar coordinates $x_4 = -1 + R\cos\varphi$ and $y_4 = R\sin\varphi$ and we denote by γ_{φ_0} an arbitrary path that approaches the point $(x_4, y_4) = (-1, 0)$ along the direction of the ray $\varphi = \varphi_0$. Then we define

$$\overline{L}_2 = \lim_{(x_4,y_4)\xrightarrow{\gamma_{\varphi_0}}(-1,0)} \frac{x_4+1}{((x_4+1)^2+y_4^2)^{3/2}} = \begin{cases} \infty & \text{if } \varphi_0 \neq \pm\pi/2, \\ 0 & \text{if } \varphi = \pm\pi/2, \\ b \in \mathbb{R} & \text{if } \varphi \xrightarrow{r\to 0} \pm\pi/2, \end{cases} \tag{25}$$

see Lemma 10. From (23), (24), and (25) we get

$$\lim_{\substack{(x_3,y_3)\xrightarrow{\gamma_{\theta_0}}(1,0)\\ (x_4,y_4)\xrightarrow{\gamma_{\varphi_0}}(-1,0)}} F_1 = (1+m)\cdot L_1 - m\cdot \overline{L}_2 \cdot \lim_{x_3\to 1} \frac{x_3-1}{2}. \tag{26}$$

Next we analyze the values of (26) and the possible solutions of (5) depending on the values of L_1 and $\overline{L}_2$.

Case $L_1 = \infty$ and $\overline{L}_2 = \infty$. From Lemma 10, if $\theta_0 \neq \pi/2$ and $\varphi_0 \neq \pm\pi/2$, then L_1 is infinity of order $1/r^2$ as $r \to 0$ and $\overline{L}_2$ is infinity of order $1/R^2$ as $R \to 0$. Moreover $(x_3 - 1)$ is an infinitesimal of order r as $r \to 0$. Therefore if $F_1 \to 0$, then the mass m has order R^2/r^3 as $r, R \to 0$ (see (26)).

On the other hand, it is easy to see that

$$\lim_{\substack{(x_3,y_3) \xrightarrow{\gamma_{\theta_0}} (1,0) \\ (x_4,y_4) \xrightarrow{\gamma_{\varphi_0}} (-1,0)}} F_2 = (1+m)\cdot \overline{L}_2 - m\cdot L_1 \cdot \lim_{x_4\to -1} \frac{x_4+1}{2}. \qquad (27)$$

In order that $F_2 \to 0$ the mass m must have order r^2/R^3 as $r, R \to 0$. Therefore R^2/r^3 and r^2/R^3 must have the same order as $r, R \to 0$. This implies that R and r have the same order which is not possible because $m \to 0$ as $r, R \to 0$. So there are no solutions of (5) in this case.

If either $\theta_0 = \pi/2$ or $\varphi_0 = \pm\pi/2$, then L_1 is infinity of order $1/r^\alpha$ as $r \to 0$ and $\overline{L}_2$ is infinity of order $1/r^\beta$ as $r \to 0$ for some $\alpha, \beta > 0$. Moreover (x_4+1) is an infinitesimal of order r^γ as $r \to 0$ for some $\gamma > 0$. Therefore if $F_1 \to 0$, then the mass m has order $r^{\beta-\alpha-1}$ as $r \to 0$ (see (26)). On the other hand, in order that $F_2 \to 0$ the mass m must have order $r^{\alpha-\beta-\gamma}$ as $r \to 0$ (see (27)). Therefore $\beta - \alpha - 1 = \alpha - \beta - \gamma$. This implies that m has order $r^{-(\gamma+1)/2}$ as $r \to 0$ which is impossible because $\gamma > 0$ and $m \to 0$ as $r \to 0$. There are no solutions of (5) in this case.

Case $L_1 = a \neq \infty$ and $\overline{L}_2 = \infty$. There are no solutions of (5) when $L_1 = a \neq \infty$ and $\overline{L}_2 = \infty$ because F_2 tends to ∞ (see (27)).

Case $L_1 = \infty$ and $\overline{L}_2 = b \neq \infty$. In this case F_1 tends to ∞, so system (5) is not satisfied.

Case $L_1 = a \neq \pm\infty$ and $\overline{L}_2 = b \neq \pm\infty$. Under these assumptions F_1 tends to a, so a must be zero (see (26)). This means that $\theta_0 = \pi/2$. It is easy to check that

$$\lim_{\substack{(x_3,y_3) \xrightarrow{\gamma_{\theta_0}} (1,0) \\ (x_4,y_4) \xrightarrow{\gamma_{\varphi_0}} (-1,0)}} F_4 = (1+m)\cdot L_3 - m\cdot \overline{L}_2 \cdot \lim_{y_3\to 0} \frac{y_3}{2},$$

where

$$L_3 = \lim_{(x_3,y_3) \xrightarrow{\gamma_{\theta_0}} (1,0)} \frac{y_3}{((x_3-1)^2+y_3^2)^{3/2}} = \begin{cases} \infty & \text{if } \theta_0 \neq 0, \pi, \\ 0 & \text{if } \theta = 0, \pi, \\ c \in \mathbb{R} & \text{if } \theta \xrightarrow{r\to 0} 0, \pi, \end{cases} \qquad (28)$$

see Lemma 10. Since $\theta_0 = \pi/2$ and $\overline{L}_2 = b \neq \pm\infty$, F_4 tends to ∞. Therefore there are no solutions of (5) in this case.

8.1.3. Case m_4 tends to collision with m_2 when $m \to 0$.

We define L_1 and L_3 as in (24) and (28) respectively. We introduce polar coordinates $x_4 = 1 + R\cos\varphi$ and $y_4 = R\sin\varphi$ and we denote by γ_{φ_0} an arbitrary path that approaches the point $(x_4, y_4) = (1, 0)$ along the direction of the ray $\varphi = \varphi_0$. Then we define

$$
\begin{aligned}
L_2 &= \lim_{(x_4,y_4)\xrightarrow{\gamma_{\theta_0}}(1,0)} \frac{x_4 - 1}{((x_4-1)^2 + y_4^2)^{3/2}} = \begin{cases} \infty & \text{if } \varphi_0 \neq \pm\pi/2, \\ 0 & \text{if } \varphi = \pm\pi/2, \\ b \in \mathbb{R} & \text{if } \varphi \xrightarrow{r\to 0} \pm\pi/2, \end{cases} \\
L_4 &= \lim_{(x_4,y_4)\xrightarrow{\gamma_{\varphi_0}}(1,0)} \frac{y_4}{((x_4-1)^2 + y_4^2)^{3/2}} = \begin{cases} \infty & \text{if } \varphi_0 \neq 0, \pi, \\ 0 & \text{if } \varphi = 0, \pi, \\ d \in \mathbb{R} & \text{if } \varphi \xrightarrow{r\to 0} 0, \pi, \end{cases}
\end{aligned}
$$

see Lemma 10. And we define

$$
H_1 = \lim_{\substack{(x_3,y_3)\xrightarrow{\gamma_{\theta_0}}(1,0)\\(x_4,y_4)\xrightarrow{\gamma_{\varphi_0}}(1,0)}} \overline{H}_1, \qquad H_2 = \lim_{\substack{(x_3,y_3)\xrightarrow{\gamma_{\theta_0}}(1,0)\\(x_4,y_4)\xrightarrow{\gamma_{\varphi_0}}(1,0)}} \overline{H}_2,
$$

where

$$
\begin{aligned}
\overline{H}_1 &= \frac{x_3 - x_4}{((x_3-x_4)^2 + (y_3-y_4)^2)^{3/2}} = \frac{r\cos\theta - R\cos\varphi}{(r^2 + R^2 - 2\,r\,R\cos(\theta-\varphi))^{3/2}}, \\
\overline{H}_2 &= \frac{y_3 - y_4}{((x_3-x_4)^2 + (y_3-y_4)^2)^{3/2}} = \frac{r\sin\theta - R\sin\varphi}{(r^2 + R^2 - 2\,r\,R\cos(\theta-\varphi))^{3/2}}.
\end{aligned}
$$

By applying the properties of limits and after some computations we get

$$
\begin{aligned}
&\lim_{\substack{(x_3,y_3)\xrightarrow{\gamma_{\theta_0}}(1,0)\\(x_4,y_4)\xrightarrow{\gamma_{\varphi_0}}(1,0)}} F_1 = (1+m)\cdot L_1 + m\cdot L_2 + m\cdot H_1, \\
&\lim_{\substack{(x_3,y_3)\xrightarrow{\gamma_{\theta_0}}(1,0)\\(x_4,y_4)\xrightarrow{\gamma_{\varphi_0}}(1,0)}} F_2 = (1+m)\cdot L_2 + m\cdot L_1 - m\cdot H_1, \\
&\lim_{\substack{(x_3,y_3)\xrightarrow{\gamma_{\theta_0}}(1,0)\\(x_4,y_4)\xrightarrow{\gamma_{\varphi_0}}(1,0)}} F_4 = (1+m)\cdot L_3 + m\cdot L_2 \cdot \lim_{y_3\to 0}\frac{y_3}{2} + m\cdot L_4 + m\cdot H_2, \\
&\lim_{\substack{(x_3,y_3)\xrightarrow{\gamma_{\theta_0}}(1,0)\\(x_4,y_4)\xrightarrow{\gamma_{\varphi_0}}(1,0)}} F_5 = (1+m)\cdot L_4 + m\cdot L_1 \cdot \lim_{y_4\to 0}\frac{y_4}{2} + m\cdot L_3 - m\cdot H_2.
\end{aligned}
$$

We consider also the limits

$$
\begin{aligned}
\lim_{\substack{(x_3,y_3)\xrightarrow{\gamma_{\theta_0}}(1,0)\\ (x_4,y_4)\xrightarrow{\gamma_{\varphi_0}}(1,0)}} (F_1+F_2) &= (1+2\,m)\cdot(L_1+L_2),\\
\lim_{\substack{(x_3,y_3)\xrightarrow{\gamma_{\theta_0}}(1,0)\\ (x_4,y_4)\xrightarrow{\gamma_{\varphi_0}}(1,0)}} (F_4+F_5) &= (1+2\,m)\cdot(L_3+L_4)+\\
&\quad m\cdot L_1\cdot\lim_{y_4\to 0}\frac{y_4}{2}+m\cdot L_2\cdot\lim_{y_3\to 0}\frac{y_3}{2}.
\end{aligned}
\tag{29}
$$

Clearly the solutions of (5) are also solutions of $F_1+F_2=0$ and $F_4+F_5=0$. Next we analyze the possible solutions of (5) depending on the values of L_1 and L_2.

Case $L_1=\infty$, $L_2=\infty$. If L_1 and L_2 are infinity of different order, then F_1+F_2 tends to infinity (see (29)). Assume now that L_1 and L_2 are infinity of the same order. It is easy to see that if equation $F_1+F_2=0$ is satisfied then $\cos\theta_0=-\cos\varphi_0$; that is, $\varphi_0=\pi\pm\theta_0$. If $\theta_0\neq\pi/2$, then L_1 and L_2 are infinity of order $1/r^2$ as $r\to 0$, see Lemma 10. Moreover

$$
\overline{H}_1=\frac{\cos\theta_0-\cos\varphi_0}{r^2(2-2\,\cos(\theta_0-\varphi_0))^{3/2}}=\begin{cases}\dfrac{\cos\theta_0}{4r^2(\cos^2\theta_0)^{3/2}} & \text{if } \varphi_0=\pi-\theta_0,\\[2ex] \dfrac{\cos\theta_0}{4r^2} & \text{if } \varphi_0=\pi+\theta_0.\end{cases}
$$

Thus H_1 is infinity of order $1/r^2$ as $r\to 0$. In short, if $\theta_0\neq\pi/2$, then F_1 tends to infinity and system (5) cannot be satisfied.

If $\theta_0=\pi/2$ and consequently $\varphi_0=\pm\pi/2$, then that L_3 is infinity of order $1/r^2$ as $r\to 0$ and L_4 is infinity of order $1/R^2$ as $R\to 0$, see Lemma 10. Thus, if r and R have different orders then F_4+F_5 tends to infinity and there are no solutions of (5). If r and R have the same order, then it is easy to see that if equation $F_4+F_5=0$ is satisfied, then $\sin\theta_0=-\sin\varphi_0$. Thus $\theta_0=\pi/2$ and $\varphi_0=-\pi/2$ and

$$
\overline{H}_2=\frac{r\,\sin(\pi/2)-r\,\sin(-\pi/2)}{(r^2+r^2-2\,r^2\,\cos(\pi))^{3/2}}=\frac{1}{4\,r^2}.
$$

So H_2 is infinity of order $1/r^2$ as $r\to 0$. Therefore F_4 is infinity of order $1/r^2$ as $r\to 0$ which implies that there are no solutions of (5) in this case.

Cases $L_1=\infty$, $L_2=b\neq\infty$ and $L_1=a\neq\infty$, $L_2=\infty$. These cases cannot provide solutions of (5) because F_1+F_2 becomes infinity.

Case $L_1 = a \neq \infty$**,** $L_2 = b \neq \infty$. From (29), in order to have a solution of $F_1 + F_2 = 0$, we need that $a = -b$. Since $L_1 = a \neq \infty$, $L_2 = b \neq \infty$, from Lemma 10, we have that $\theta_0 = \pi/2$ and $\varphi_0 = \pm\pi/2$. Proceeding as in the case $L_1 = \infty$ and $L_2 = \infty$ with $\theta_0 = \pi/2$ we prove that the case $L_1 = a \neq \infty$, $L_2 = b \neq \infty$ cannot provide solutions of system (5).

8.2. Central configurations for $m > 0$ sufficiently small with one small mass coming from infinity

Without loss of generality we can assume that the small mass coming from infinity is m_3. We introduce polar coordinates $(x_3, y_3) = (r\cos\theta, r\sin\theta)$. So the mass m_3 comes from infinity when $r \to +\infty$ by following the direction of the ray $\theta = \theta_0$ with $\theta_0 \in [0, \pi/2]$ (remember that we have assumed that $x_3, y_3 \geqslant 0$. Next we will prove that there are no solutions of (5) such that $r \to +\infty$ as $m \to 0$.

8.2.1. Case m_4 comes from infinity when $m \to 0$

After some computations we can see easily that

$$\lim_{r\to+\infty} \frac{x_3+1}{r_{13}^3} = 0, \qquad \lim_{r\to+\infty} \frac{x_3-1}{r_{23}^3} = 0, \qquad \lim_{r\to+\infty} \frac{x_3^2-1}{r_{13}^3} = 0,$$
$$\lim_{r\to+\infty} \frac{x_3^2-1}{r_{23}^3} = 0, \qquad \lim_{r\to+\infty} \frac{y_3}{r_{13}^3} = 0, \qquad \lim_{r\to+\infty} \frac{y_3}{r_{23}^3} = 0,$$
$$\lim_{r\to+\infty} \frac{(x_3+1)\,y_3}{r_{13}^3} = 0, \qquad \lim_{r\to+\infty} \frac{(x_3+1)\,y_3}{r_{23}^3} = 0. \tag{30}$$

We introduce polar coordinates $(x_4, y_4) = (R\cos\varphi, R\sin\varphi)$. So the mass m_4 comes from infinity when $R \to +\infty$ by following the direction of the ray $\varphi = \varphi_0$. In a similar way than in (30) we get

$$\lim_{R\to+\infty} \frac{x_4+1}{r_{14}^3} = 0, \qquad \lim_{R\to+\infty} \frac{x_4-1}{r_{24}^3} = 0, \qquad \lim_{R\to+\infty} \frac{x_4^2-1}{r_{14}^3} = 0,$$
$$\lim_{R\to+\infty} \frac{x_4^2-1}{r_{24}^3} = 0, \qquad \lim_{R\to+\infty} \frac{y_4}{r_{14}^3} = 0, \qquad \lim_{R\to+\infty} \frac{y_4}{r_{24}^3} = 0,$$
$$\lim_{R\to+\infty} \frac{(x_4+1)\,y_4}{r_{14}^3} = 0, \qquad \lim_{R\to+\infty} \frac{(x_4+1)\,y_4}{r_{24}^3} = 0. \tag{31}$$

Moreover

$$
\begin{aligned}
\ell_1 &= \frac{x_3 - x_4}{r_{34}^3} = \frac{r\cos\theta - R\cos\varphi}{(r^2 + R^2 - 2rR\cos(\theta-\varphi))^{3/2}},\\
\ell_2 &= \frac{(x_3+1)(x_4-1)}{r_{24}^3} = \frac{-1 - r\cos\theta + R\cos\varphi + rR\cos\theta\cos\varphi}{(R^2 - 2R\cos\varphi + 1)^{3/2}},\\
\ell_3 &= \frac{(x_3-1)(x_4+1)}{r_{14}^3} = \frac{-1 + r\cos\theta - R\cos\varphi + rR\cos\theta\cos\varphi}{(R^2 + 2R\cos\varphi + 1)^{3/2}},\\
\ell_4 &= \frac{(x_3+1)(x_4-1)}{r_{13}^3} = \frac{-1 - r\cos\theta + R\cos\varphi + rR\cos\theta\cos\varphi}{(r^2 + 2r\cos\varphi + 1)^{3/2}},\\
\ell_5 &= \frac{(x_3-1)(x_4+1)}{r_{23}^3} = \frac{-1 + r\cos\theta - R\cos\varphi + rR\cos\theta\cos\varphi}{(r^2 - 2r\cos\varphi + 1)^{3/2}},
\end{aligned}
\tag{32}
$$

and

$$
\begin{aligned}
\ell_6 &= \frac{y_3 - y_4}{r_{34}^3} = \frac{r\sin\theta - R\sin\varphi}{(r^2 + R^2 - 2rR\cos(\theta-\varphi))^{3/2}},\\
\ell_7 &= \frac{y_3(x_4-1)}{r_{24}^3} = \frac{-r\sin\theta + rR\sin\theta\cos\varphi}{(R^2 - 2R\cos\varphi + 1)^{3/2}},\\
\ell_8 &= \frac{y_3(x_4+1)}{r_{14}^3} = \frac{r\sin\theta + rR\sin\theta\cos\varphi}{(R^2 + 2R\cos\varphi + 1)^{3/2}},\\
\ell_9 &= \frac{(x_3+1)y_4}{r_{13}^3} = \frac{R\sin\varphi + rR\cos\theta\sin\varphi}{(r^2 + 2r\cos\varphi + 1)^{3/2}},\\
\ell_{10} &= \frac{(x_3-1)y_4}{r_{23}^3} = \frac{-R\sin\varphi + rR\cos\theta\sin\varphi}{(r^2 - 2r\cos\varphi + 1)^{3/2}}.
\end{aligned}
\tag{33}
$$

From (30), (31), (32) and (33) the limit of F_1 as $r, R \to +\infty$ can be reduced to the limit of

$$m(\ell_1 + \ell_2 - \ell_3) - \frac{x_3}{4}. \tag{34}$$

Next we analyze this limit depending on the orders of r and R.

Case R and r are infinities of different orders. We assume that $R = r^\beta$ with $\beta > 0$ and $\beta \neq 1$ and that $\ell_2 - \ell_3$ has order r^γ for some $\gamma \in \mathbb{R}$ as $r \to +\infty$. It is easy to see that $\ell_1 \to 0$ as $r, R \to +\infty$. Moreover it is easy to see that the order of $\ell_2 - \ell_3$ is smaller than the order of $r^{1-2\beta}$; that is, $\gamma < 1 - 2\beta$. In order to have solutions of equation $F_1 = 0$ the order of m must be equal to the order of $r/r^\gamma = r^{1-\gamma}$. Since $\gamma < 1 - 2\beta$, $1 - \gamma > 2\beta$. Thus the order of m is bigger than the order of $r^{2\beta}$ which is impossible because $\beta > 0$ and $m \to 0$ as $r, R \to +\infty$. Therefore there are no solutions of (5) in this case.

Case R and r are infinities of the same orders. It is easy to see that if R and r have the same order, then ℓ_2, ℓ_3, ℓ_4 and ℓ_5 tend to 0 as $r \to +\infty$. Moreover the limit of $F_1 + F_2$ when $r \to +\infty$ is equivalent to the limit of $-x_3/4 - x_4/4$ when $r \to +\infty$. In order to have a solution of $F_1 + F_2 = 0$ we need that $\cos\theta = -\cos\varphi$. On the other hand, ℓ_7, ℓ_8, ℓ_9 and ℓ_{10} tend to 0 as $r \to +\infty$ and the limit of $F_4 + F_5$ when $r \to +\infty$ is equivalent to $-y_3/4 - y_4/4$ when $r \to +\infty$. In order to have a solution of $F_4 + F_5 = 0$ we need that $\sin\theta = -\sin\varphi$. Therefore we only can have solutions of $F_1 + F_2 = 0$ and $F_4 + F_5 = 0$ when either $\theta = 0$ and $\varphi = \pi$ or $\theta = \pi/2$ and $\varphi = 3\pi/2$. If $\theta = 0$ and $\varphi = \pi$, then $\ell_1 \to 0$ as $r \to +\infty$, so F_1 tends to ∞ as $r \to +\infty$, see (34). If $\theta = \pi/2$ and $\varphi = 3\pi/2$, then $\ell_6 \to 0$ as $r \to +\infty$, so by proceeding in a similar way wee see F_4 tends ∞ as $r \to +\infty$. In short there are no solutions of (5) in this case.

8.2.2. Case m_4 tends to m_1 when $m \to 0$

We introduce polar coordinates $(x_4, y_4) = (-1 + R\cos\varphi, R\sin\varphi)$. This means that if $R \to 0$, then the mass m_4 tends to m_1 following the direction of the ray $\varphi = \varphi_0$. We can see easily that

$$\frac{x_3 - x_4}{r_{34}^3} \to 0 \qquad \text{as } r \to +\infty \text{ and } R \to 0.$$

We define $\overline{L}_2$ as in (25). If $\overline{L}_2 = b \neq \pm\infty$, then it is easy to see from (30) that F_1 tends to ∞ as $r \to +\infty$ and $R \to 0$. Otherwise $\overline{L}_2$ is infinity of order $1/R^\alpha$ as $R \to 0$ for some $\alpha > 0$ and consequently F_2 is an infinity of order $1/R^\alpha$ as $R \to 0$. Therefore there are no solutions of (5) in this case.

8.2.3. Case m_4 tends to m_2 when $m \to 0$

By proceeding as in the previous case we can prove that there are no solutions of (5) with m_3 coming from infinity and m_4 tending to m_2 as $m \to 0$.

8.2.4. None of the above cases

If m_4 is far from either infinity, or m_1 and m_2 when $m \to 0$, then F_1 is infinity of order r as $r \to +\infty$. Therefore there are no solutions of (5) in this case.

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