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ON NON–SMOOTH PERTURBATIONS OF DEGENERATE OR NON–DEGENERATE PLANAR CENTERS

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Abstract. We provide sufficient conditions for the existence of limit cycles of non–smooth perturbed planar centers, when the set of discontinuity is an algebraic variety. It is introduced a mechanism which allows us to deal with such system, even in higher dimension. The main tool used in this paper is the averaging method. Two applications are given in careful detail.

1. Introduction

The theory of discontinuous system has been developing at a very fast pace in recent years and it has become certainly an important frontiers between Mathematics, Physics, Engineering and other fields of science. The study of this kind of system is motivated by various applications. For instance, we may cite some problems in control theory [3], nonlinear oscillations [1, 20], non–smooth mechanics [6], economics[12, 16], biology [4], and others.

By the other hand, the knowledge of the existence or not of periodic solutions is very important for understanding the dynamics of the differential systems. One of the useful tools to detect such solutions is the averaging theory, which is a classical and matured tool that provides means to study the behavior of nonlinear smooth dynamical systems. We refer to the book of Sanders and Verhulst [21] and to the book of Verhulst [22] for a general introduction of this subject.

Buica and Llibre [7] generalize the averaging theory for studying periodic solutions of continuous differential systems using mainly the Brouwer degree. More recently, Llibre, Novaes and Teixeira [18], extended the averaging theory for studying periodic solutions of a class of discontinuous piecewise differential system, see Theorem 2. In what follows, we introduce the class of piecewise discontinuous system studied in [18].

Let D be an open subset of $\mathbb{R}^n$. Let $X, Y : \mathbb{R} \times D \to \mathbb{R}^n$ be two continuous vector fields and $h : \mathbb{R} \times D \to \mathbb{R}$ be a C^1 function. We Assume that the functions

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h, X and Y are T–periodic in the variable t. The set of discontinuity $h^{-1}(0)$ is denoted by M.

A *Discontinuous Piecewise Differential System* is defined as

$$x'(t) = Z(t,x) = \begin{cases} X(t,x) & \text{if} \quad h(t,x) > 0, \\ Y(t,x) & \text{if} \quad h(t,x) < 0. \end{cases} \tag{1}$$

We denote $Z = (X, Y)$.

Let the sign function be defined in $R \setminus \{0\}$ as

$$\text{sign}(u) = \begin{cases} 1 & \text{if} \quad u > 0, \\ -1 & \text{if} \quad u < 0. \end{cases}$$

The piecewise discontinuous system (1) can be written, more conveniently, as

$$x'(t) = Z(t,x) = Z_1(t,x) + \text{sign}(h(t,x))Z_2(t,x), \tag{2}$$

where

$$Z_1(t,x) = \frac{1}{2}\left(X(t,x) + Y(t,x)\right) \quad \text{and} \quad Z_2(t,x) = \frac{1}{2}\left(X(t,x) - Y(t,x)\right).$$

In [18], conditions for the existence of periodic solutions when the set of discontinuity M is a regular manifold are exhibited (see Theorem 2). However, many applications deal with discontinuous system having the set of discontinuity as an algebraic variety, see for instance the book of Andronov, Vitt, and Khaikin [1] and the book of Barbashin [3]. In this paper, motivated by these problems, we develop a technique that allows us to extend the averaging method for studying the periodic solutions of a class of discontinuous piecewise differential system when the set of discontinuity is an algebraic variety.

In few words, our main result deals with discontinuous perturbation of degenerate and non–degenerate planar centers, where the set of discontinuity M is an algebraic variety. Moreover, conditions for the existence of periodic solutions of such perturbed systems are presented.

We also provide two applications with careful details. The first one generalizes the problem of an *m–piecewise discontinuous Liénard polynomial differential equation of degree n* proposed by Llibre and Teixeira [19]; the second application deals with a plane divided in a mesh, where each piece admits one of the two vector fields. For these system, the existence of periodic solutions is studied.

The paper is organized as follows. In Section 2 we present the main theorem of this paper and some other results. In Section 3 we present the averaging method that we shall use in this paper. In Section 4 we present the proofs of the results presented in Section 2. In Section 5 we present two applications.

2. Statement of the Main Result

Let D be an open subset of $\mathbb{R}^2$. We consider the following planar discontinuous differential system

$$
\begin{aligned}
x'(t) &= X(x,y) + \varepsilon F_1(x,y),\\
y'(t) &= Y(x,y) + \varepsilon F_2(x,y),
\end{aligned}
\tag{3}
$$

with

$$F_i(x,y) = F_{i,1}(x,y) + \operatorname{sign}(h(x,y))\, F_{i,2}(x,y),$$

where $X, Y, F_{i,j} : D \to \mathbb{R}^2$ for $i,j = 1,2$ are continuous functions and $h : \mathbb{R}^2 \to \mathbb{R}$ is a a C^1 function.

Usually, 0 is assumed to be a regular value of the function h which implies that $M = h^{-1}(0)$ is a regular manifold, see for instance Theorem 2 of this paper. Here, we assume that

(H1) the set of non–regular points in $M = h^{-1}(0)$ is bounded. In other words, if

$$N = \{(x,y) \in M : \nabla h(x,y) = (0,0)\},$$

we can choose $\delta > 0$ such that $N \subset \overline{B_\delta(0,0)}$. Here, $\overline{B_\delta(0,0)} \subset \mathbb{R}^2$ is the closed ball with radius δ centered at $(0,0)$.

The idea of the proof of our main result consist in defining conveniently change of variables which drives some restriction of system (3) to a system whose the set of discontinuity is a regular manifold. For this goal, taking $\tilde{D} = \mathbb{S}^1 \times \mathbb{R}^+$, and $\delta \geq 0$ (chosen in H1), we define the function $\Psi_\delta : \tilde{D} \to \mathbb{R}^2$ as

$$\Psi_\delta(\theta, r) = ((r+\delta)\cos(\theta), (r+\delta)\sin(\theta)). \tag{4}$$

Clearly, this function is a diffeomorphism on its image $\Psi_\delta(\tilde{D}) = D \backslash \overline{B_\delta(0,0)}$.

We use in this paper the *Pullback* notation for the change of variables. Given a function $H : \Psi_\delta(\tilde{D}) \to \mathbb{R}$ and $\delta > 0$, let $\Psi_\delta^* H : \tilde{D} \to \mathbb{R}^2$ be defined as

$$\Psi_\delta^* H(\theta, r) = H \circ \Psi_\delta(\theta, r).$$

For simplicity, we denote $\delta^* H(\theta, r) = \Psi_\delta^* H(\theta, r)$.

Remark 1. *Consider $h(x,y) = (x^2-1)(y^2-1)$ and denote $M = h^{-1}(0)$. The set M is represented by the bold lines in the Figure 1.*

Proceeding with the change of variables, defined above, the set $\tilde{\mathcal{M}} = (\delta^* h)^{-1}(0)$, represented by the bold lines in the Figure 2, becomes a regular sub–manifold of $\tilde{D} = \mathbb{S}^1 \times \mathbb{R}^+$.

The procedure can be replied for other kinds of system, even in higher dimension, by finding a conveniently change of variables.

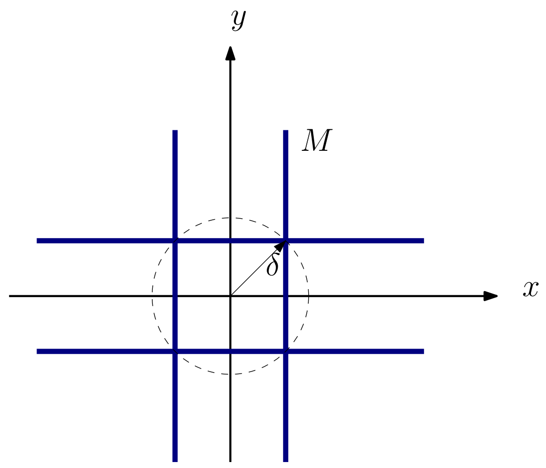

FIGURE 1. $M = h^{-1}(0) \subset D$.

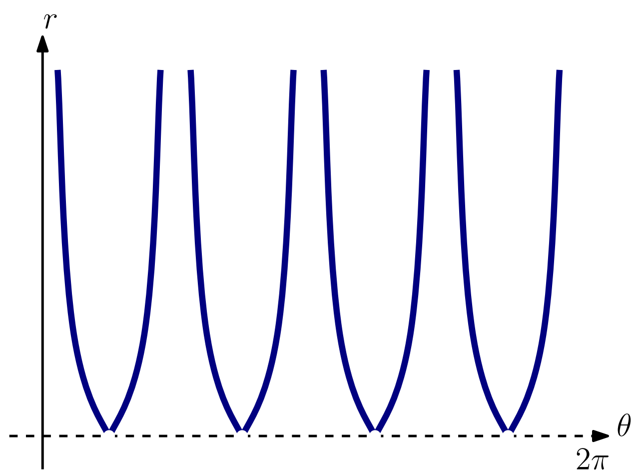

FIGURE 2. $\tilde{\mathcal{M}} = \tilde{h}^{-1}(0) \subset \tilde{D}$.

Now, we define the averaged function $f : \mathbb{R}^+ \to \mathbb{R}$ as

$$f(r) = \int_0^{2\pi} \left(\frac{\delta^* Y(\theta,r)\delta^* F_{1,1}(\theta,r) - \delta^* X(\theta,r)\delta^* F_{2,1}(\theta,r)}{(\cos(\theta)\delta^* Y(\theta,r) - \sin(\theta)\delta^* X(\theta,r))^2} d\theta \right.$$
$$\left. + \operatorname{sign}(\delta^* h(\theta,r)) \frac{\delta^* Y(\theta,r)\delta^* F_{1,2}(\theta,r) - \delta^* X(\theta,r)\delta^* F_{2,2}(\theta,r)}{(\cos(\theta)\delta^* Y(\theta,r) - \sin(\theta)\delta^* X(\theta,r))^2} \right) d\theta. \tag{5}$$

We denote $\Sigma_0 = D \backslash \left(M \cup \overline{B_\delta(0,0)} \right)$ and $\mathcal{M} = M \backslash \overline{B_\delta(0,0)}$. Note that $\mathcal{M}$ is an embedded sub–manifold in $D \subset \mathbb{R}^2$. Moreover, assume in addition that:

(H2) the functions X and Y satisfies

$$\cos(\theta)\delta^*X(\theta,r)+\sin(\theta)\delta^*Y(\theta,r)=0;$$

(H3) $\langle\nabla h(x,y),(-y,x)\rangle\neq 0$ for all $(x,y)\in\overline{\mathcal{M}}$;

(H4) $F_{i,j}$, for $i,j=1,2$, and h are locally Lipschitz with respect to any $(x,y)\in D$;

(H5) for some $a\in\Sigma_0$ with $f(|a|-\delta)=0$, there exist a neighborhood V of a such that $f(|z|-\delta)\neq 0$ for all $z\in\overline{V}\backslash\{a\}$ and $d_B(f,|V|-\delta,0)\neq 0$. Here, $|V|-\delta=\{|z|-\delta;\ z\in V\}$;

Observe that the Hypothesis H3 is equivalent to $(-y,x)\notin T_{(x,y)}\mathcal{M}$.

The following proposition assures that the hypothesis H2 is not empty.

Proposition 1. *Consider the functions*

$$X(x,y)=\sum_{m=1}^{M}f_m(x,y)\quad\text{and}\quad Y(x,y)=\sum_{m=1}^{M}g_m(x,y),$$

where

$$f_m(x,y)=\sum_{i=0}^{m}a_{m,i}x^{m-i}y^i\quad\text{and}\quad g_m(x,y)=\sum_{i=0}^{m}b_{m,i}x^{m-i}y^i.$$

Therefore, if $a_{m,0}=b_{m,m}=0$ and $b_{m,i}=-a_{m,i+1}$ for $i=0,2,\cdots,m-1$ and for $m=1,2,\cdots,M$, then hypothesis H2 holds for the function X and Y.

Hypothesis H2 implies that the unperturbed system (3), i.e. $\varepsilon=0$, consist in a degenerate or non–degenerate (linear) planar center. We emphasize here that the unperturbed system does not cover all non–degenerate center.

Our main result, which provides conditions for the existence of periodic solutions of the small perturbation of a degenerate or non–degenerate planar center satisfying the hypothesis H2, is the following.

Theorem A. *If H1 – H5 hold, then for $|\varepsilon|>0$ sufficiently small, there exists a periodic solution $(x(t,\varepsilon),y(t,\varepsilon))$ of system (3) such that $|(x(t,\varepsilon),y(t,\varepsilon))|\to|a|$ as $\varepsilon\to 0$, for every $t\in\mathbb{R}$.*

The following corollary deals with perturbation of non–degenerate (linear) planar centers.

Corollary B. *We consider the linear planar center $(X(x,y),Y(x,y))=(y,-x)$. The averaged function $f:\mathbb{R}^+\to\mathbb{R}$ is defined as*

$$\begin{aligned} f(r)= &\int_0^{2\pi}\delta^*F_{1,1}(\theta,r)\cos(\theta)+\delta^*F_{2,1}(\theta,r)\sin(\theta)d\theta \\ &+\int_0^{2\pi}\text{sign}\left(\delta^*h(\theta,r)\right)\big(\delta^*F_{1,2}(\theta,r)\cos(\theta)+\delta^*F_{2,2}(\theta,r)\sin(\theta)\big)d\theta. \end{aligned}\tag{6}$$

If H1, H3 – H5 hold, then for $|\varepsilon| > 0$ sufficiently small, there exists a periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ of system (3) such that $|(x(t,\varepsilon), y(t,\varepsilon))| \to |a|$ as $\varepsilon \to 0$, for every $t \in \mathbb{R}$.

3. Basic results on averaging theory for piecewise continuous systems

In this section we present the basic result needed for proving the main results of this paper.

Theorem 2. *We consider the following discontinuous differential system*

$$x'(t) = \varepsilon F(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{7}$$

with

$$F(t,x) = F_1(t,x) + \mathrm{sign}(h(t,x))F_2(t,x),$$
$$R(t,x,\varepsilon) = R_1(t,x,\varepsilon) + \mathrm{sign}(h(t,x))R_2(t,x,\varepsilon),$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $R_1, R_2 : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ and $h : \mathbb{R} \times D \to \mathbb{R}$ are continuous functions, T–periodic in the variable t and D is an open subset of $\mathbb{R}^n$. We also suppose that h is a C^1 function having 0 as a regular value. Denote $M = h^{-1}(0)$, $\Omega = \{0\} \times D \nsubseteq M$, and $\Omega_0 = \Sigma \backslash M \neq \varnothing$. Consider the identification $z \equiv (0,z) \notin M$.

The averaged function $f : D \to \mathbb{R}^n$ is defined as

$$f(x) = \int_0^T F(t,x)dt. \tag{8}$$

We assume the following conditions:

(i) *F_1, F_2, R_1, R_2 and h are locally L–Lipschitz with respect to x;*
(ii) *for $a \in \Omega_0$ with $f(a) = 0$, there exist a neighbourhood V of a such that $f(z) \neq 0$ for all $z \in \overline{V} \backslash \{a\}$ and $d_B(f, V, 0) \neq 0$.*
(iii) *$\partial h / \partial t \neq 0$, for all $p \in M$.*

Then, for $|\varepsilon| > 0$ sufficiently small, there exists a T–periodic solution $x(\cdot, \varepsilon)$ of system (7) such that $x(0,\varepsilon) \to a$ as $\varepsilon \to 0$.

For a proof of Theorem 2 see [18].

4. Proofs of Proposition 1, Theorem A, and Corollary B

• *Proof of Proposition 1.* We denote $S(\theta,r) = \cos(\theta)\delta^*X(\theta,r) + \sin(\theta)\delta^*Y(\theta,r)$. To prove the proposition we must show that $S \equiv 0$. So

$$\begin{aligned} S(\theta,r) = & \cos(\theta)\delta^*X(\theta,r) + \sin(\theta)\delta^*Y(\theta,r), \\ = & \cos(\theta)\delta^*\left(\sum_{m=1}^{M} f_m(\theta,r)\right) + \sin(\theta)\delta^*\left(\sum_{m=1}^{M} g_m(\theta,r)\right), \\ = & \sum_{m=1}^{M}\left(\cos(\theta)\delta^* f_m(\theta,r) + \sin(\theta)\delta^* g_m(\theta,r)\right), \\ = & \sum_{m=1}^{M}\sigma_m(\theta,r), \end{aligned}$$

where

$$\sigma_m(\theta,r) = \cos(\theta)\delta^* f_m(\theta,r) + \sin(\theta)\delta^* g_m(\theta,r)$$

for $m = 1, 2, \cdots, M$.

Now, if we assume that $b_{m,m} = a_{m,0} = 0$ and $b_{m,i} = -a_{m,i+1}$, for $i = 0, 2, \cdots, m-1$ and for $m = 1, 2, \cdots, M$, then $\sigma_m \equiv 0$. Indeed,

$$\begin{aligned} \sigma_m(\theta,r) = & \cos(\theta)\delta^* f_m(\theta,r) + \sin(\theta)\delta^* g_m(\theta,r), \\ = & \cos(\theta)\sum_{i=0}^{m} a_{m,i}(r+\delta)^{m-i}\cos^{m-i}(\theta)(r+\delta)^i\sin^i(\theta) \\ & +\sin(\theta)\sum_{i=0}^{m} b_{m,i}(r+\delta)^{m-i}\cos^{m-i}(\theta)(r+\delta)^i\sin^i(\theta), \\ = & (r+\delta)^m\left(\sum_{i=0}^{m} a_{m,i}\cos^{m-i+1}(\theta)\sin^i(\theta) + \sum_{i=0}^{m} b_{m,i}\cos^{m-i}(\theta)\sin^{i+1}(\theta)\right), \\ = & (r+\delta)^m\left(\sum_{i=1}^{m} a_{m,i}\cos^{m-i+1}(\theta)\sin^i(\theta) + \sum_{i=0}^{m-1} b_{m,i}\cos^{m-i}(\theta)\sin^{i+1}(\theta)\right) \\ & +(r+\delta)^m\left(a_{m,0}\cos^{m+1}(\theta) + b_{m,m}\sin^{m+1}(\theta)\right), \\ = & (r+\delta)^m\left(\sum_{i=0}^{m-1}(a_{m,i+1} + b_{m,i})\cos^{m-i}(\theta)\sin^{i+1}(\theta)\right), \\ = & 0. \end{aligned}$$

Hence $\sigma_m \equiv 0$ for each $m = 1, 2, \cdots, M$. Therefore $S \equiv 0$. □

• *Proof of Theorem A.* We consider the system (1) restricted to $\Psi_\delta(\tilde{D})$, i.e.

$$(9) \qquad (\dot{x}(t)\,,\,\dot{y}(t)) = Z(x,y) = \left(x + \varepsilon F^1(x,y), -y + \varepsilon F^2(x,y)\right)\Big|_{\Psi(\tilde{D})}.$$

Since $\Psi_\delta : D \to \Psi_\delta(\tilde{D})$ is a diffeomorphism, thus the pullback $\delta^*Z(\theta,r) : \tilde{D} \to \mathbb{R}^2$ is well defined and the differential system

$$\left(\dot{\theta}(t), \dot{r}(t)\right) = \delta^*Z(\theta,r), \tag{10}$$

is equivalent to (9). Moreover,

$$\begin{aligned} \dot{\theta}(t) = & \ \frac{\delta^*Y(\theta,r)\cos(\theta) - \delta^*X(\theta,r)\sin(\theta)}{r+\delta} \\ & + \varepsilon\frac{\delta^*F_2(\theta,r)\cos(\theta) - \delta^*F_1(\theta,r)\sin(\theta)}{r+\delta}, \\ \dot{r}(t) = & \ \varepsilon\left(\delta^*F_1(\theta,r)\cos(\theta) + \delta^*F_2(\theta,r)\sin(\theta)\right), \end{aligned} \tag{11}$$

since $\delta^*X(\theta,r)\cos(\theta) + \delta^*Y(r,\theta)\sin(\theta) = 0$.

We note that

$$\frac{\dot{r}(t)}{\dot{\theta}(t)} = -(r+\delta)\frac{\delta^*F_1(\theta,r)\cos(\theta) + \delta^*F_2(\theta,r)\sin(\theta)}{\delta^*F_2(\theta,r)\cos(\theta) - \delta^*F_1(\theta,r)\sin(\theta)}\left(\frac{z(\theta,r,\varepsilon)}{1 - z(\theta,r,\varepsilon)}\right), \tag{12}$$

where

$$z(\theta,r,\varepsilon) = \varepsilon\frac{\delta^*F_1(\theta,r)\sin(\theta) - \delta^*F_2(\theta,r)\cos(\theta)}{\delta^*Y(\theta,r)\cos(\theta) - \delta^*X(\theta,r)\sin(\theta)}.$$

Now, taking θ as the new independent variable of (11), we obtain the expression of $dr(\theta)/d\theta$ by expanding $\dot{r}(t)/\dot{\theta}(t)$ in *Taylor Series* around $\varepsilon = 0$ as

$$\frac{dr}{d\theta}(\theta) = \varepsilon\frac{\delta^*F_1(\theta,r)\cos(\theta) - \delta^*F_2(\theta,r)\sin(\theta)}{(\delta^*Y(\theta,r)\cos(\theta) - \delta^*X(\theta,r)\sin(\theta))^2} + \varepsilon^2 R(\theta,r,\varepsilon). \tag{13}$$

It is easy to see that hypothesis H3 implies that

$$\delta^*Y(\theta,r)\cos(\theta) - \delta^*X(\theta,r)\sin(\theta) \neq 0,$$

for $(\theta,r) \in \tilde{D}$. So expression (13) is well defined for every $(\theta,r) \in \tilde{D}$.

Moreover, given $\delta > 0$ and $r_0 > 0$, there exists $\varepsilon(r_0) > 0$ sufficiently small such that $|z(\theta,r,\varepsilon)| < 1$ for all $(\theta,r) \in \mathbb{S}^1 \times (0,r_0]$ and $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$. Observe that we can take r_0 sufficiently big, such that all periodic solutions of the system (13) have their amplitudes smaller than r_0. Therefore, expanding the expression (12), we may write

$$\varepsilon^2 R(\theta,r,\varepsilon) = -(r+\delta)\frac{\delta^*F_1(\theta,r)\cos(\theta) + \delta^*F_2(\theta,r)\sin(\theta)}{\delta^*F_2(\theta,r)\cos(\theta) - \delta^*F_1(\theta,r)\sin(\theta)}\sum_{n=2}^{\infty} z(\theta,r,\varepsilon)^n,$$

which implies the following claim:

Claim 1. *The function $R(\theta,r,\varepsilon)$ satisfies the hypotheses of Theorem 2.*

To prove Claim 1 we have to find continuous functions

$$R_1, R_2 : \mathbb{S}^1 \times (0, r_0] \times (-\varepsilon(r_0), \varepsilon(r_0)) \to \mathbb{R}^2,$$

2π–periodic in the variable θ and locally Lipschitz with respect to r, such that

$$R(\theta, r, \varepsilon) = R_1(\theta, r, \varepsilon) + \operatorname{sign}(\delta^* h(\theta, r))\, R_2(\theta, r, \varepsilon).$$

We note that

$$R(\theta, r, \varepsilon) = -(r + \delta)\left(\delta^* F_1 \cos(\theta) + \delta^* F_2 \sin(\theta)\right) \sum_{n=2}^{\infty} \varepsilon^{n-2} G_n(\theta, r), \tag{14}$$

where

$$G_n(\theta, r) = \frac{\left(\delta^* F_2 \cos(\theta) - \delta^* F_1 \sin(\theta)\right)^{n-1}}{\left(\delta^* Y \cos(\theta) - \delta^* X \sin(\theta)\right)^{n}}.$$

For simplicity, we are omitting the point (θ, r).

Applying the *Binomial Formula*, expression (14) becomes

$$\begin{aligned} R(\theta, r, \varepsilon) = & \sum_{n=2}^{\infty} \sum_{k=0}^{n-1} \varepsilon^{n-2} C_k^n(r) \cos^{n-k}\theta \sin^k\theta \frac{(\delta^* F_1)^{k+1} (\delta^* F_2)^{n-k-1}}{(\delta^* Y \cos\theta - \delta^* X \sin\theta)^n} \\ & + \sum_{n=2}^{\infty} \sum_{k=0}^{n-1} \varepsilon^{n-2} C_k^n(r) \cos^{n-k-1}\theta \sin^{k+1}\theta \frac{(\delta^* F_1)^{k} (\delta^* F_2)^{n-k}}{(\delta^* Y \cos\theta - \delta^* X \sin\theta)^n}, \end{aligned}$$

with

$$C_k^n(r) = \frac{(-1)^{k+1}}{(r + \delta)} \binom{n-1}{k}.$$

Again, applying the *Binomial Formula* in $(\delta^* F_i)^a$, for $i = 1, 2$ and $a \in \mathbb{N}$, we obtain

$$\begin{aligned} (\delta^* F_i)^a = & \underbrace{\sum_{l=0}^{\lfloor a/2 \rfloor} \binom{a}{2l} (\delta^* F_{i,1})^{a-2l} (\delta^* F_{i,2})^{2l}}_{P_i^a} \\ & + \operatorname{sign}(\delta^* h) \underbrace{\sum_{l=0}^{\lceil a/2 \rceil - 1} \binom{a}{2l+1} (\delta^* F_{i,1})^{a-2l-1} (\delta^* F_{i_2})^{2l+1}}_{Q_i^a}. \end{aligned}$$

Here, $\lfloor u \rfloor$ denotes as usual the greatest integer less than or equal to u; and $\lceil u \rceil$ denotes as usual the smallest integer greater than or equal to u.

Since

$$(\delta^* F_1)^a (\delta^* F_2)^b = P_1^a P_2^b + Q_1^a Q_2^b + \operatorname{sign}(\delta^* h) \left(P_1^a Q_2^b + P_2^b Q_1^a\right),$$

it follows that $R(\theta,r,\varepsilon)=R_1(\theta,r,\varepsilon)+\mathrm{sign}\left(\delta^*h(\theta,r)\right)R_2(\theta,r,\varepsilon)$, where

$$\begin{aligned}R_1(\theta,r,\varepsilon)=&\sum_{n=2}^{\infty}\sum_{k=0}^{n-1}\varepsilon^{n-2}C_k^n(r)\cos^{n-k}(\theta)\sin^k(\theta)\frac{P_{k+1}^1P_{n-k-1}^2+Q_{k+1}^1Q_{n-k-1}^2}{(\delta^*Y\cos\theta-\delta^*X\sin\theta)^n}\\&+\sum_{n=2}^{\infty}\sum_{k=0}^{n-1}\varepsilon^{n-2}C_k^n(r)\cos^{n-k-1}(\theta)\sin^{k+1}(\theta)\frac{P_k^1P_{n-k}^2+Q_k^1Q_{n-k}^2}{(\delta^*Y\cos\theta-\delta^*X\sin\theta)^n},\end{aligned}$$

and

$$\begin{aligned}R_2(\theta,r,\varepsilon)=&\sum_{n=2}^{\infty}\sum_{k=0}^{n-1}\varepsilon^{n-2}C_k^n(r)\cos^{n-k}(\theta)\sin^k(\theta)\frac{P_{k+1}^1Q_{n-k-1}^2+P_{n-k-1}^2Q_{k+1}^1}{(\delta^*Y\cos\theta-\delta^*X\sin\theta)^n}\\&+\sum_{n=2}^{\infty}\sum_{k=0}^{n-1}\varepsilon^{n-2}C_k^n(r)\cos^{n-k-1}(\theta)\sin^{k+1}(\theta)\frac{P_k^1Q_{n-k}^2+P_{n-k}^2Q_k^1}{(\delta^*Y\cos\theta-\delta^*X\sin\theta)^n}.\end{aligned}$$

Moreover, it is easy to see that the function R_1 and R_2 are locally Lipschitz in the variable r.

Now, rewriting the system 13, by making explicit the sign function, we obtain

$$\frac{dr}{d\theta}(\theta)=\varepsilon\left(G^1(\theta,r)+\mathrm{sign}\left(\delta^*h(\theta,r)\right)G^2(\theta,r)\right)+\varepsilon^2R(\theta,r,\varepsilon),\tag{15}$$

where

$$G^1(\theta,r)=\frac{\delta^*Y(\theta,r)\delta^*F_{1,1}(\theta,r)-\delta^*X(\theta,r)\delta^*F_{2,1}(\theta,r)\sin(\theta)}{\left(\delta^*Y(\theta,r)\cos(\theta)-\delta^*X(\theta,r)\sin(\theta)\right)^2},$$

and

$$G^2(\theta,r)=\frac{\delta^*Y(\theta,r)\delta^*F_{1,2}(\theta,r)-\delta^*X(\theta,r)\delta^*F_{2,2}(\theta,r)\sin(\theta)}{\left(\delta^*Y(\theta,r)\cos(\theta)-\delta^*X(\theta,r)\sin(\theta)\right)^2}.$$

In order, to apply Theorem 2 in the system (15) we shall verify hypothesis (iii). For this, we prove the following claim:

Claim 2. *If $\tilde{\mathcal{M}}=(\delta^*h)^{-1}(0)$, then $(\partial(\delta^*h)/\partial\theta)(\theta,r)\neq0$ for all $(\theta,r)\in\tilde{\mathcal{M}}$.*

Observe that $\tilde{\mathcal{M}}=\{(\theta,r)\in\tilde{D}:\Psi_\delta(\theta,r)\in\mathcal{M}\}$. We take $(\theta,r)\in\tilde{\mathcal{M}}$, and denote $(\tilde{x},\tilde{y})=\Psi_\delta(\theta,r)\in\mathcal{M}$. So

$$\begin{aligned}\frac{\partial}{\partial\theta}\delta^*h(\theta,r)&=\frac{\partial}{\partial\theta}(h\circ\Psi_\delta)(\theta,r),\\&=\langle\nabla h(\Psi_\delta(\theta,r)),(-(r+\delta)\sin(\theta),(r+\delta)\cos(\theta))\rangle,\\&=\langle\nabla h(\tilde{x},\tilde{y}),(-\tilde{y},\tilde{x})\rangle\neq0.\end{aligned}$$

Therefore, by Claims 1 and 2, the hypotheses (i) and (iii) of Theorem 2 hold for the system (15). Clearly, the hypothesis H2 of Theorem A implies the hypothesis (ii) of Theorem 2. Hence, applying Theorem 2, we conclude that for $|\varepsilon|>0$ sufficiently small, there exists a 2π–periodic solution $\theta\mapsto r(\theta,\varepsilon)$ of system (13) such that $r(0,\varepsilon)\to|a|$ as $\varepsilon\to0$. Which implies that for $|\varepsilon|>0$ sufficiently small, there exists a periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ of system (1) such that $|(x(t,\varepsilon), y(t,\varepsilon))| \to |a|$ as $\varepsilon \to 0$ for every $t \in \mathbb{R}$. □

• *Proof of Corollary B.* Corollary B is an immediate consequence of Proposition 1 and Theorem A. □

5. Applications

5.1. **Application 1.** In [19], Llibre and Teixeira have introduced the following *m–piecewise discontinuous Liénard polynomial differential equation of degree n*:

$$\begin{aligned} \dot{x} &= y + \operatorname{sgn}(g_m(x,y))F_n(x), \\ \dot{y} &= -x, \end{aligned} \tag{16}$$

where $F(x) = c_0 + c_1x + \ldots + c_nx^n$ and the zero set of the function $\operatorname{sgn}(g_m(x,y))$ with $m = 2,4,6,\ldots$ is the product of $m/2$ straight lines passing through the origin of coordinates dividing the plane in sectors of angle $2\pi/m$.

Here, we shall study an generalization of this problem.

Given $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m) \in \mathbb{T}^m$ (m–Torus), with $m = 2,4,6,\ldots$, such that $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_m \le 2\pi$, we consider a function $h_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\delta^* h_\alpha(\theta, r) = (\theta - \alpha_1)(\theta - \alpha_2)\cdots(\theta - \alpha_m). \tag{17}$$

Thus the set of discontinuity $M = {h_\alpha}^{-1}(0)$ is represented, partially, by the bold lines in the Figure 3.

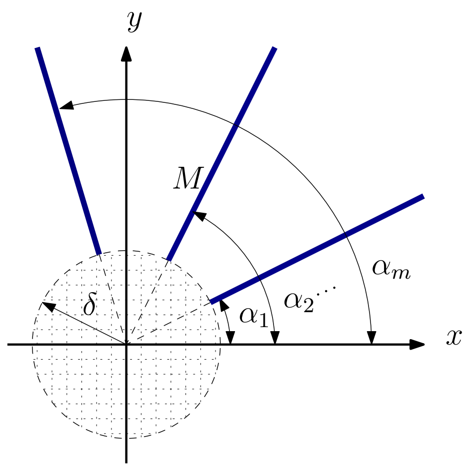

Figure 3. $M = {h_\alpha}^{-1}(0) \subset \mathbb{R}^2$.

We stress that only the behavior of the set M outside the ball $\overline{B_\delta(0,0)}$ is considered. Formally,

$$M = \bigcup_{i=1}^{m} L_i,$$

where $L_i \cap (B_\delta(0,0))^c$ is the segment starting at the point $(\delta\cos(\alpha_i), \delta\sin(\alpha_i))$, which is supported by the line starting at the point $(0,0)$ and passing through $(\delta\cos(\alpha_i), \delta\sin(\alpha_i))$, for $i = 1, 2, \ldots, m$.

We call by an *α–piecewise discontinuous Liénard polynomial differential equation of degree n* the following system

$$\begin{aligned} \dot{x} &= y + \operatorname{sgn}(h_\alpha(x,y))F_n(x), \\ \dot{y} &= -x. \end{aligned} \tag{18}$$

Now, consider the condition

(C) L_i is a straight line starting at the point $(0,0)$ and passing through $(\delta\cos(\alpha_i), \delta\sin(\alpha_i))$, for $i = 1, 2, \ldots, m$.

Assuming condition C, we define $H(m,n)$ as the lower upper bound for the maximum number of limit cycles of the system (18) for any $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m) \in \mathbb{T}^m$ (m–Torus), with $m = 2, 4, 6, \ldots$, such that $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_m \leq 2\pi$.

Such kind of problem is fairly known in the literature and its origin is based in the 16th Hilbert's problem [13], see for instance: Écalle [10]; Ilyashenko [14]; Ilyashenko and Panov [15]; Lins, de Melo and Pugh [17]; Dumortier, Panazzolo and Roussarie [9]; and De Maesschalck and Dumortier [8].

We present, in the following theorem, a lower bound for $H(m,n)$.

Theorem 3. *The inequality $H(m,n) \geq n$ hold for $m = 2, 4, 6, \cdots$ and $n \in \mathbb{N}$.*

Here, we shall give in detail a proof of Theorem 3 for $m = 2$ and $n \in \mathbb{N}$. The proof of Theorem 3 for all $m = 2, 4, 6, \ldots$ will follow similary.

Clearly, taking $\alpha_2 = \alpha_1 + \pi$ the system (18) becomes a 2–piecewise discontinuous Liénard polynomial differential equation, for what, Llibre and Teixeira [19] has proved that $\lfloor n/2 \rfloor$ is a lower bound for the maximum number of limit cycles of this system when $\alpha_1 = \pi/2$. In the following proposition, we assure that this result holds for every $\alpha_1 \in (0, \pi)$.

Proposition 4. *Assume that $\alpha_2 = \alpha_1 + \pi$ and $\alpha_1 \in (0,\pi)$. Then $\lfloor n/2 \rfloor$ is a lower bound for the maximum number of limit cycles of the differential system* (18).

When the symmetry $\alpha_2 = \alpha_1 + \pi$ is broken, many others limit cycles can appear, as we can see in the following proposition.

Proposition 5. *We take $\alpha = (\alpha_1, \alpha_2)$ and assume that one of the following hypotheses holds:*

(a) *(i)* $\sin(\alpha_1)\cos(\alpha_1) \geq 0$, *(ii)* $\sin(\alpha_2)\cos(\alpha_2) \leq 0$, *and* $\alpha_2 - \alpha_1 < pi$. *Moreover, one of inequalities (i) or (ii) is strictly;*

(b) *(j)* $\sin(\alpha_1)\cos(\alpha_1) \leq 0$, *(jj)* $\sin(\alpha_2)\cos(\alpha_2) \geq 0$, *and* $\alpha_2 - \alpha_1 > pi$. *Moreover, one of inequalities (j) or (jj) is strictly.*

Then n is a lower bound for the maximum number of limit cycles of the differential system (18).

Note that: all points $(\alpha_1, \alpha_2) \in \mathbb{T}^2$ such that $(\alpha_1, \alpha_2) \in (0, \pi/2) \times (\pi/2, \pi)$ or $(\alpha_1, \alpha_2) \in (\pi, 3\pi/2) \times (3\pi/2, 2\pi)$ satisfy hypothesis (a), moreover both inequalities (i) and (ii) are strictly; all points $(\alpha_1, \alpha_2) \in \mathbb{T}^2$ such that $(\alpha_1, \alpha_2) \in (\pi/2, \pi) \times (\alpha_1 + \pi, 2\pi)$ satisfy hypothesis (b), moreover both inequalities (j) and (jj) are strictly.

Clearly, Proposition 5 implies the inequality $H(2, n) \geq n$. Thus, once proved Proposition 5, the Theorem 3 is valid for $m = 2$.

To prove Propositions 4 and 5, and Theorem 3 we recall the *Descartes Theorem* about the number of zeros of a real polynomial (see [5]).

Descartes Theorem *Consider the real polynomial* $p(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \cdots + a_{i_r}x^{i_r}$ *with* $0 \leq i_1 < i_2 < \cdots < i_r$ *and* $a_{i_j} \neq 0$ *real constants for* $j \in \{1, 2, \cdots, r\}$. *When* $a_{i_j}a_{i_{j+1}} < 0$*, we say that* a_{i_j} *and* $a_{i_{j+1}}$ *have a variation of sign. If the number of variations of signs is* m*, then* $p(x)$ *has at most* m *positive real roots. Moreover, it is always possible to choose the coefficients of* $p(x)$ *in such a way that* $p(x)$ *has exactly* $r - 1$ *positive real roots.*

Firstly, we prove the Proposition 5, since its proof will be used to prove Proposition 4 and Theorem 3.

• *Proof of Proposition 5.* To prove that N is a lower bound for the maximum number of limit cycles of the system (18) we shall find a polynomial function $F_n(x)$ of degree n such that the differential system (18) has N limit cycles. Thus, taking $F_n(x) = \varepsilon P_n(x)$, with $P_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, the system (18) becomes

$$\begin{aligned} \dot{x} &= y + \varepsilon \operatorname{sgn}(h_\alpha(x, y))P_n(x), \\ \dot{y} &= -x. \end{aligned} \tag{19}$$

In order to prove the proposition we have to identify in the system (18) the elements of Corollary B, thus

$$F_{1,1}(x, y) = F_{2,1}(x, y) = F_{2,2}(x, y) = 0,$$

and

$$F_{1,2}(x, y) = P_n(x),$$

Computing the averaged function (6), for the system (19), we have that

$$
\begin{aligned}
f(r) = & \int_0^{2\pi} \delta^* F_{1,1}(\theta, r)\cos(\theta) + \delta^* F_{2,1}(\theta, r)\sin(\theta) d\theta \\
& + \int_0^{2\pi} \operatorname{sign}\left(\delta^* h(\theta, r)\right)\big(\delta^* F_{1,2}(\theta, r)\cos(\theta) + \delta^* F_{2,2}(\theta, r)\sin(\theta)\big) d\theta, \\
= & \int_0^{2\pi} \cos(\theta) P_n\left((r+\delta)\cos(\theta)\right)\operatorname{sign}((\theta-\alpha_1)(\theta-\alpha_2)) d\theta, \\
= & \sum_{l=0}^{n} a_l (r+\delta)^l \int_0^{2\pi} \cos^{l+1}(\theta)\operatorname{sign}((\theta-\alpha_1)(\theta-\alpha_2)) d\theta, \\
= & \sum_{l=0}^{n} a_l (r+\delta)^l \left(\int_0^{\alpha_1} \cos^{l+1}(\theta) d\theta - \int_{\alpha_1}^{\alpha_2} \cos^{l+1}(\theta) d\theta + \int_{\alpha_2}^{2\pi} \cos^{l+1}(\theta) d\theta\right), \\
= & \sum_{l=0}^{n} a_l \, b_l \, (r+\delta)^l,
\end{aligned}
$$

with

$$
b_l = \int_0^{\alpha_1} \cos^{l+1}(\theta) d\theta - \int_{\alpha_1}^{\alpha_2} \cos^{l+1}(\theta) d\theta + \int_{\alpha_2}^{2\pi} \cos^{l+1}(\theta) d\theta.
$$

So, for $l = 0, 1$, it is easy to see that

$$
\begin{aligned}
b_0 &= 2\sin(\alpha_1) - 2\sin(\alpha_2), \\
b_1 &= \alpha_1 - \alpha_2 + \pi + \cos(\alpha_1)\sin(\alpha_1) - \cos(\alpha_2)\sin(\alpha_2).
\end{aligned}
$$

Now, using the identity, for $l > 0$,

$$
\int \cos^{l+1}(\theta) d\theta = \frac{\cos^l(\theta)\sin(\theta)}{l+1} + \frac{l}{l+1} \int \cos^{l-1}(\theta) d\theta,
$$

we conclude that, for $l > 1$,

$$\begin{aligned}
b_l = & \int_0^{\alpha_1} \cos^{l+1}(\theta)d\theta - \int_{\alpha_1}^{\alpha_2} \cos^{l+1}(\theta)d\theta + \int_{\alpha_2}^{2\pi} \cos^{l+1}(\theta)d\theta, \\
= & \frac{\cos^l(\alpha_1)\sin(\alpha_1)}{l+1} + \frac{l}{l+1}\int_0^{\alpha_1} \cos^{l-1}(\theta)d\theta \\
& -\frac{\cos^l(\alpha_2)\sin(\alpha_2)}{l+1} + \frac{\cos^l(\alpha_1)\sin(\alpha_1)}{l} - \frac{l}{l+1}\int_{\alpha_1}^{\alpha_2} \cos^{l-1}(\theta)d\theta \\
& -\frac{\cos^l(\alpha_2)\sin(\alpha_2)}{l+1} + \frac{l}{l+1}\int_{\alpha_2}^{2\pi} \cos^{l-1}(\theta)d\theta, \\
= & \frac{2}{l+1}(\cos^l(\alpha_1)\sin(\alpha_1) - \cos^l(\alpha_2)\sin(\alpha_2)) \\
& +\frac{l}{l+1}\left(\int_0^{\alpha_1} \cos^{l-1}(\theta)d\theta - \int_{\alpha_1}^{\alpha_2} \cos^{l-1}(\theta)d\theta + \int_{\alpha_2}^{2\pi} \cos^{l-1}(\theta)d\theta\right), \\
= & \frac{2}{l+1}(\cos^l(\alpha_1)\sin(\alpha_1) - \cos^l(\alpha_2)\sin(\alpha_2)) + \frac{l}{l+1}b_{l-2}.
\end{aligned}$$

Proceeding by induction under l, we have that, for $l \geq 0$,

$$b_{2l} = \frac{2\sin(\alpha_1)}{2l+1}\sum_{j=0}^{l} D_1(l,j)\cos^{2j}(\alpha_1) - \frac{2\sin(\alpha_2)}{2l+1}\sum_{j=0}^{l} D_1(l,j)\cos^{2j}(\alpha_2),$$

and

$$\begin{aligned}
b_{2l+1} = & \frac{\sin(\alpha_1)}{l+1}\sum_{j=0}^{l} D_2(l,j)\cos^{2j+1}(\alpha_1) - \frac{\sin(\alpha_2)}{l+1}\sum_{j=0}^{l} D_2(l,j)\cos^{2j+1}(\alpha_2) \\
& +2\frac{(2l+1)!!}{(2l+2)!!}\left(\alpha_1 - \alpha_2 + \pi\right),
\end{aligned}$$

where

$$D_1(p,q) = \frac{(2p)!!(2q-1)!!}{(2q)!!(2p-1)!!} \quad \text{and} \quad D_2(p,q) = \frac{(2p+1)!!(2q)!!}{(2q+1)!!(2p)!!}, \tag{20}$$

for $p, q \in \mathbb{Z}$. Here, $n!!$, for $n \in \mathbb{N}$, denotes as usual the *Double Factorial*:

$$\begin{aligned}
(2n+1)!! = & \ 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n+1), \\
(2n)!! = & \ 2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n).
\end{aligned}$$

Following Arfken [2], these are related to the regular factorial function by

$$(2n)!! = 2^n n! \quad \text{and} \quad (2n+1)!! = \frac{(2n+1)!}{2^n n!}. \tag{21}$$

It is also defined $(-1)!! = 1$, a special case that does not follow from equation (21).

Each hypothesis, (a) and (b), implies that $b_l \neq 0$ for $l = 0, 1, 2, \ldots, n$. By Descartes Theorem and choosing the coefficients a_l conveniently the polynomial $g(r) = f(r - \delta)$ has n positive roots r_k for $k = 1, 2, \ldots, n$. Therefore $f'(r_k) \neq 0$ for $k = 1, 2, \ldots, n$, since the polynomial has degree n. Now, we choose $\delta > 0$ such that the circles of radius r_k, for $k = 1, 2, \ldots, n$, are contained in Σ_0. Hence, by Corollary B, for $|\varepsilon| > 0$ sufficiently small the differential equation (19) will have n limit cycles near the circles of radius r_k for $k = 1, 2, \ldots, n$. Hence, the proposition is proved. □

• *Proof of Proposition 4.* Since $\alpha_2 = \alpha_1 + \pi$ and $\alpha_1 \in (0, \pi)$, we have that $\cos(\alpha_2) = -\cos(\alpha_1)$ and $\sin(\alpha_2) = -\sin(\alpha_1)$ with $\sin(\alpha) \neq 0$. Therefore

$$b_{2l} = \frac{4\sin(\alpha_1)}{2l+1} \sum_{j=0}^{l} D_1(l, j) \cos^{2j}(\alpha_1) \neq 0,$$

and $b_{2l+1} = 0$ for all $l = 0, 1, \cdots, \lceil n/2 \rceil - 1$.

By Descartes Theorem and choosing the coefficients a_l conveniently the polynomial $g(r) = f(r-\delta)$ has $\lfloor n/2 \rfloor$ positive roots r_k for $k = 1, 2, \ldots, \lfloor n/2 \rfloor$. Clearly the other roots of that polynomial of degree $2\lfloor n/2 \rfloor$ are $-r_k$ for $l = 0, 1, 2, \ldots, \lfloor n/2 \rfloor$. Therefore $f'(r_k) \neq 0$ for $k = 1, 2, \ldots, \lfloor n/2 \rfloor$. Now, choose $\delta > 0$ such that the circles of radius r_k, for $k = 1, 2, \ldots, \lfloor n/2 \rfloor$, are contained in Σ_0. Hence, by Corollary B, for $|\varepsilon| > 0$ sufficiently small the differential equation (19) will have $\lfloor n/2 \rfloor$ limit cycles near the circles of radius r_k for $k = 1, 2, \ldots, \lfloor n/2 \rfloor$. Hence, the proposition is proved. □

• *Proof of Theorem 3.* We take $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m) \in \mathbb{T}^m$ (m–Torus), with $m = 2, 4, 6, \ldots$, such that $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < 2\pi$, and denote $\alpha_0 = 0$ and $\alpha_{m+1} = 2\pi$. For $F_n(x) = \varepsilon P_n(x)$, with $P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, the system (18) becomes

$$\begin{aligned} \dot{x} &= y + \varepsilon \mathrm{sgn}(h_\alpha(x, y)) P_n(x), \\ \dot{y} &= -x. \end{aligned} \tag{22}$$

Computing the averaged function (6), for the system (22), we have that

$$f(r) = \sum_{l=0}^{n} a_l \, b_l \, (r + \delta)^l,$$

with

$$b_l = \sum_{i=0}^{m} (-1)^i \int_{\alpha_i}^{\alpha_{i+1}} \cos^{l+1}(\theta) d\theta.$$

So, for $l=0,1$, it is easy to see that

$$b_0 = 2\sum_{i=1}^{m/2}\left(\sin(\alpha_{2i-1})-\sin(\alpha_{2i})\right),$$

and

$$b_1 = \sum_{i=1}^{m/2}\left(\sin(\alpha_{2i-1})\cos(\alpha_{2i-1})-\sin(\alpha_{2i})\cos(\alpha_{2i})\right)+\pi+\sum_{i=1}^{m/2}\left(\alpha_{2i-1}-\alpha_{2i}\right).$$

Proceeding analogously to the proof of Proposition 4, we obtain

$$b_l = \frac{2}{l+1}\sum_{i=1}^{m/2}\left(\sin(\alpha_{2i-1})\cos^l(\alpha_{2i-1})-\sin(\alpha_{2i})\cos^l(\alpha_{2i})\right)+\frac{l}{l+1}b_{l-2}.$$

By induction under l, we have that, for $l=0,1,\ldots,\lfloor n/2\rfloor$,

$$\begin{aligned} b_{2l} = & \frac{2}{2l+1}\sum_{i=1}^{m/2}\sum_{j=0}^{l}D_1(l,j)\sin(\alpha_{2i-1})\cos^{2j}(\alpha_{2i-1}) \\ & -\frac{2}{2l+1}\sum_{i=1}^{m/2}\sum_{j=0}^{l}D_1(l,j)\sin(\alpha_{2i})\cos^{2j}(\alpha_{2i}), \end{aligned}$$

and, for $l=0,1,\ldots,\lceil n/2\rceil-1$,

$$\begin{aligned} b_{2l+1} = & \frac{2}{2l+1}\sum_{i=1}^{m/2}\sum_{j=0}^{l}D_2(l,j)\sin(\alpha_{2i-1})\cos^{2j+1}(\alpha_{2i-1}) \\ & -\frac{2}{2l+1}\sum_{i=1}^{m/2}\sum_{j=0}^{l}D_2(l,j)\sin(\alpha_{2i})\cos^{2j+1}(\alpha_{2i}) \\ & 2\frac{(2l+1)!!}{(2l+2)!!}\left(\pi+\sum_{i=1}^{m/2}\left(\alpha_{2i-1}-\alpha_{2i}\right)\right), \end{aligned}$$

where D_1 and D_2 are defined in (20).

Now, we take the sequence $(\beta_i)_{i\in\mathbb{N}}\subset[\pi/4,\pi/2)$ such that

$$\beta_i = \frac{\pi}{2}-\frac{\pi}{4i}.$$

Thus, for every $s>1$, we have that

$$1 > \frac{\cos(\beta_{i+1})}{\cos(\beta_i)} > \left(\frac{\cos(\beta_{i+1})}{\cos(\beta_i)}\right)^s > 0.$$

Moreover, for every $i\in\mathbb{N}$, it follows that

$$\sin\left(\frac{\pi}{2i}\right) > \sin\left(\frac{\pi}{2(i+1)}\right) > 0.$$

Therefore

$$\frac{\cos(\beta_i)}{\cos(\alpha_{i+1})}\frac{\sin(\beta_i)}{\sin(\beta_{i+1})} = \frac{\sin\left(\dfrac{\pi}{2i}\right)}{\sin\left(\dfrac{\pi}{2(i+1)}\right)} > 1.$$

Hence, for $i \in \mathbb{N}$

$$\frac{\sin(\beta_i)}{\sin(\beta_{i+1})} > \frac{\cos(\beta_{i+1})}{\cos(\beta_i)}.$$

So, for $s > 1$,

$$\frac{\sin(\beta_{2i-1})}{\sin(\beta_{2i})} > \frac{\cos(\beta_{2i})}{\cos(\alpha_{2i-1})} > \left(\frac{\cos(\beta_{2i})}{\cos(\beta_{2i-1})}\right)^s,$$

which implies that

$$\sin(\beta_{2i-1})\cos^s(\beta_{2i-1}) > \sin(\beta_{2i})\cos^s(\alpha_{2i}).$$

Choosing $\alpha_i = \beta_i$, for $i = 1, 2, \ldots, m$, and $s = 2j$, for $j = 0, 1, \ldots, l$, we have that $b_{2l} > 0$, for $l = 0, 1, \ldots, \lfloor n/2 \rfloor$. Now, choosing $s = 2j + 1$, for $j = 0, 1, \ldots, l$, we have that

$$\sum_{i=1}^{m/2}(\alpha_{2i-1} - \alpha_{2i}) < \sum_{i=1}^{\infty}(\beta_{2i-1} - \beta_{2i}) = -\frac{\ln(2)}{4}\pi.$$

So,

$$2\frac{(2l+1)!!}{(2l+2)!!}\left(\pi + \sum_{i=1}^{m/2}(\alpha_{2i-1} - \alpha_{2i})\right) > 0.$$

Therefor $b_{2l+1} > 0$, for $l = 0, 1, \cdots, \lceil n/2 \rceil - 1$.

Since $b_l \neq 0$ for $l = 0, 1, \ldots, n$, by Descartes Theorem and choosing the coefficients a_l conveniently the polynomial $g(r) = f(r - \delta)$ has n positive roots r_k for $k = 1, 2, \ldots, n$. Therefore $f'(r_k) \neq 0$ for $k = 1, 2, \ldots, n$, since the polynomial has degree n. Now, we choose $\delta > 0$ such that the circles of radius r_k, for $k = 1, 2, \ldots, n$, are contained in Σ_0. Hence, by Corollary B, for $|\varepsilon| > 0$ sufficiently small the differential equation (22) will have n limit cycles near the circles of radius r_k for $k = 1, 2, \ldots, n$. Hence, the Theorem 3 is proved.

□

5.2. **Application 2.** Consider the function $h(x, y) = (x^2 - 1)(y^2 - 1)$. Thus the set of discontinuity $M = h^{-1}(0)$ is represented by the bold lines in the Figure 1.

Now, consider the equation

$$x''(t) = -x' + \varepsilon\, y\, \mathrm{sign}(h(x, x')). \tag{23}$$

Proposition 6. *For $|\varepsilon| > 0$ sufficiently small there exist a periodic solution $x(t, \varepsilon)$ of the system* (23) *such that $|(x(0, \varepsilon), x'(0, \varepsilon)| \to \sqrt{4 + 2\sqrt{2}}$ as $\varepsilon \to 0$.*

• *Proof.* Firstly, we have to identify the elements of Corollary B in the system (23). Thus

$$F_1^1(x,y) = F_2^1(x,y) = F_1^2(x,y) = 0 \quad \text{and} \quad F_2^2(x,y) = y.$$

The averaged function of the system (23) is given by

$$f(|a| + \sqrt{2}) = \left(2\sqrt{2} + |a|\right)\left(\pi + 8\,\text{arccsc}\left(2\sqrt{2} + |a|\right)\right),$$

which has $|a| = \sqrt{4 + 2\sqrt{2}}$ as a solution. Moreover

$$\frac{df}{dr}(r)\Big|_{r=\sqrt{4+2\sqrt{2}}-\sqrt{2}} = 8(\sqrt{2} - 1) \neq 0.$$

Thus hypothesis H5 of Corollary B holds. Clearly, hypothesis H1 – H4 also hold. Hence, by Corollary B, the proof has been concluded. □

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