

PERIODIC ORBITS FOR THE GENERALIZED YANG-MILLS HAMILTONIAN SYSTEM IN DIMENSION 6

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ABSTRACT. We apply the averaging theory to study a generalized Yang-Mills Hamiltonian system in dimension 6 with six parameters. We provide sufficient conditions on the six parameters of the system which guarantee the existence of continuous families of period orbits parameterized by the energy.

1. INTRODUCTION

We study the generalized classical Yang–Mills Hamiltonian system in dimension 6. It consists of a harmonic oscillator plus a homogenous potential of fourth degree with six real parameters a, b, c, d, e and f .

$$(1) \quad H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2 + x^2 + y^2 + z^2) + \frac{1}{4}(ax^4 + 2bx^2y^2 + 2cx^2z^2 + dy^4 + 2ey^2z^2 + fz^4).$$

When $z = p_z = 0$ the previous Hamiltonian contains the planar classical Yang-Mills Hamiltonian system. The periodic solutions of this system when $z = p_z = 0$ were studied in [1]. Our aim is to study the periodic solutions in the different energy levels $H = h$ of the Hamiltonian system associated to the Hamiltonian (1).

The mentioned planar Hamiltonian system ($z = p_z = 0$) for $a = 0$ was studied by Contopoulos and co-workers during many years, such a Hamiltonian is now known as the Contopoulos Hamiltonian which describes the perturbed central part of an elliptical or barred galaxy without escapes. For more details see the references [2], [3], and [4]. When the quadratic part $(x^2 + y^2)/2 = 0$ we obtain the mechanical Yang-Mills Hamiltonian $H = (p_x^2 + p_y^2)/2 + bx^2y^2/2$; where the term x^2y^2 characterizes the Yang-Mills potential, which arises in connection with the classical Yang-Mills field with gauge group $SU(2)$ for a homogeneous two-component field, see [7]. Several authors studied quartic homogeneous potentials (without quadratic terms), see for instance the references [8], [9] and [10]. Moreover, when $b \neq 0$ it is well known that the Hamiltonian of Yang-Mills is non integrable and strongly chaotic. Others studies and investigations related with generalizations of the mechanical YangMills Hamiltonian have treated quartic terms with three up to five terms in [11], [12], [13], and [14]. Maciejewski et. al. [14] studied generalized Yang Mills Hamiltonian systems, which have a quadratic potential plus a homogeneous of fourth degree potential with five parameters, and they proved the existence of connected branches of non stationary periodic trajectories emanating from the origin. Caranicolas and Varvoglis [11] studied a Hamiltonian with a quartic potential of three parameters plus a quadratic harmonic potential with frequencies ω_1 and

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ω_2 being two extra parameters of the form

$$(2) \quad H = \frac{1}{2}(p_x^2 + p_y^2 + \omega_1^2 x^2 + \omega_2^2 y^2) + \varepsilon(ax^4 + 2bx^2y^2 + cy^4),$$

and they calculated numerically families of periodic orbits and its characteristic curves. In [15] the generalization of the Yang-Mills potentials $H = (p_x^2 + p_y^2 + x^2 + y^2)/2 + ax^4/4 + bx^2y^2/2$ was studied with two real parameters a and b , in order that the problem be tractable in a two-dimensional parameter space, although these calculations can be generalized to higher dimensional parameter spaces. Here we study a harmonic oscillator plus a homogenous potential of fourth degree with six real parameters.

The Hamiltonian differential system associated to the Hamiltonian (1) is

$$(3) \quad \begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} = p_x, \\ \dot{y} &= \frac{\partial H}{\partial p_y} = p_y, \\ \dot{z} &= \frac{\partial H}{\partial p_z} = p_z, \\ \dot{p}_x &= -\frac{\partial H}{\partial x} = -x - x(ax^2 + by^2 + cz^2), \\ \dot{p}_y &= -\frac{\partial H}{\partial y} = -y - y(bx^2 + dy^2 + ez^2), \\ \dot{p}_z &= -\frac{\partial H}{\partial z} = -z - z(cx^2 + ey^2 + fz^2). \end{aligned}$$

The dot denotes derivative with respect to the independent variable t , the time.

We study the periodic orbits of the Yang-Mills Hamiltonian system by using the *averaging theory*, see section 2 for more details about this tool. More precisely through the averaging method we will provide sufficient conditions on the six parameters a, b, c, d, e and f for the existence of periodic orbits of our Yang-Mills system.

The periodic orbits studied in this paper are isolated in every energy level and they are of special interest because after the equilibrium points the periodic orbits are the most simple non-trivial solutions of the system, and their stability determines the kind of motion in their neighborhood.

The averaging method provides periodic orbits of a perturbed periodic non-autonomous differential system depending on a small parameter ε . Roughly speaking, the problem of finding periodic solutions of a differential system is reduced to find zeros of some convenient finite dimensional function. We check the conditions under which the averaging theory guarantees the existence of periodic orbits, and we find them as a function of the energy. In this way we can find analytically periodic orbits in any energy level as function of the six parameters of the Yang-Mills systems (3). We summarize our main result on the periodic orbits as follows.

Theorem 1. *At every positive energy level $H = h$ with $h > 0$ the Yang-Mills Hamiltonian system (3) has at least*

(a) *one periodic orbit if* $b c (a - b) (3a - b) (a - c) (3a - c) \neq 0$, $\left| \frac{3a - 2c}{c} \right| \leq 1$,
 $\left| \frac{3a - 2b}{b} \right| \leq 1$;

(b) *two periodic orbits if one of the following conditions hold:*

(i) $\left| \frac{3d - 2b}{b} \right| < 1$, $\left| \frac{2e - 3d}{e} \right| \leq 1$, $(b - 3d)(b - d)(d - e)(3d - e)be \neq 0$;

(ii) $\left| \frac{2e - 3f}{e} \right| \leq 1$, $\left| \frac{2c - 3f}{c} \right| < 1$, $ce(c - 3f)(c - f)(e - 3f)(e - f) \neq 0$;

(iii) $(c - 3f)(-3a + 2c - 3f) > 0$, $(c - 3a)(-3a + 2c - 3f) > 0$,

$$c(-3ae + 9af + 3bc - 9bf - c^2 + ce) \cdot$$

$$(-9ae + 9af + bc - 3bf - c^2 + 3ce) \neq 0$$

$$\left| \frac{6ae - 9af - 2bc + 6bf + c^2 - 2ce}{3ae + bc - 3bf - ce} \right| \leq 1;$$

(iv) $c(-ae + af + bc - bf - c^2 + ce) \cdot (ae - bc + bf - ce) \cdot$

$$(-ae + 3af + bc - bf - 3c^2 + ce) \neq 0,$$

$$\left| \frac{2ae - 3af - 2bc + 2bf + 3c^2 - 2ce}{-ae + bc - bf + ce} \right| \leq 1, (f - c)(a - 2c + f) > 0,$$

$$(c - a)(-a + 2c - f) > 0;$$

(c) *eight periodic orbits if one of the following conditions hold:*

(i) $\left| \frac{-3e^2 + 2c(-d + e) + 2b(e - f) + 3df}{c(d - e) + b(-e + f)} \right| < 1$,

$$e(-be + bf + cd - ce - 3df + 3e^2)(-be + bf + cd - ce - df + e^2) \cdot$$

$$(c(d - e) + b(f - e)) \neq 0, (f - e)(d - 2e + f) > 0, (d - e)(d - 2e + f) > 0;$$

(ii) $\left| \frac{6cd - 2be - 2ce + e^2 + 6bf - 9df}{3cd + be - ce - 3bf} \right| < 1$,

$$(-be + 3bf + 9cd - 3ce - 9df + e^2) \cdot (3cd + be - ce - 3bf) \cdot$$

$$(-3be + 9bf + 3cd - ce - 9df + e^2) \neq 0,$$

$$(-3d + 2e - 3f)(e - 3f) \neq 0, (-3d + 2e - 3f)(e - 3d) \neq 0.$$

Theorem 1 is proved in section 3.

2. THE AVERAGING THEORY OF FIRST ORDER

Now we shall present the basic results from averaging theory that we need for proving the results of this paper.

The next theorem provides a first order approximation for the periodic solutions of a periodic differential system, for the proof see Theorems 11.5 and 11.6 of Verhulst [5].

Consider the differential equation

$$(4) \quad \dot{\mathbf{x}} = \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0$$

with $\mathbf{x} \in D$, where D is an open subset of \mathbb{R}^n , $t \geq 0$. Moreover we assume that both $F_1(t, \mathbf{x})$ and $F_2(t, \mathbf{x}, \varepsilon)$ are T -periodic in t . We also consider in D the averaged differential equation

$$(5) \quad \dot{\mathbf{y}} = \varepsilon f_1(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0,$$

where

$$(6) \quad f_1(\mathbf{y}) = \frac{1}{T} \int_0^T F_1(t, \mathbf{y}) dt.$$

Under certain conditions, equilibrium solutions of the averaged equation turn out to correspond with T -periodic solutions of equation (4).

Theorem 2. *Consider the two initial value problems (4) and (5). Suppose:*

- (i) F_1 , its Jacobian $\partial F_1/\partial x$, its Hessian $\partial^2 F_1/\partial x^2$, F_2 and its Jacobian $\partial F_2/\partial x$ are defined, continuous and bounded by a constant independent of ε in $[0, \infty) \times D$ and $\varepsilon \in (0, \varepsilon_0]$.
- (ii) F_1 and F_2 are T -periodic in t (T independent of ε).

Then the following statements hold.

- (a) If p is an equilibrium point of the averaged equation (5) and

$$(7) \quad \det \left(\frac{\partial f_1}{\partial \mathbf{y}} \right) \Big|_{\mathbf{y}=p} \neq 0,$$

then there exists a T -periodic solution $\varphi(t, \varepsilon)$ of equation (4) such that $\varphi(0, \varepsilon) \rightarrow p$ as $\varepsilon \rightarrow 0$.

- (b) The stability or instability of the limit cycle $\varphi(t, \varepsilon)$ is given by the stability or instability of the equilibrium point p of the averaged system (5). In fact the singular point p has the stability behavior of the Poincaré map associated to the limit cycle $\varphi(t, \varepsilon)$.

For a proof of Theorem 2, see sections 6.3 and 11.8 in [5].

To apply the averaging theory we will do some transformations in the Hamiltonian differential system (3) in order to write it in the normal form of equation (4). So first we do a rescaling transformation with a factor $\sqrt{\varepsilon}$ in order to have a small parameter $\varepsilon > 0$ in the Hamiltonian system. Second, using a kind of generalized polar coordinates in \mathbb{R}^6 and taking as new independent variable an angle coordinate instead of the time, we obtain a 2π -periodic differential system. Third, fixing the energy level and omitting a redundant variable in every energy level, we will get the differential system written in the normal form for applying the averaging theorem of first order (i.e Theorem 2), and finally we shall prove the existence of some isolated periodic orbits in every energy level.

3. PROOF OF THEOREM 1

We do a rescaling using a small parameter $\varepsilon > 0$. In the Hamiltonian system (3) we change the variables (x, y, z, p_x, p_y, p_z) by (X, Y, Z, p_X, p_Y, p_Z) where $x = \sqrt{\varepsilon}X$, $y = \sqrt{\varepsilon}Y$, $z = \sqrt{\varepsilon}Z$, $p_x = \sqrt{\varepsilon}p_X$, $p_y = \sqrt{\varepsilon}p_Y$ and $p_z = \sqrt{\varepsilon}p_Z$. In the new variables, system (3) becomes

$$(8) \quad \begin{aligned} \dot{X} &= p_X, \\ \dot{Y} &= p_Y, \\ \dot{Z} &= p_Z, \\ \dot{p}_X &= -X - \varepsilon X(aX^2 + bY^2 + cZ^2), \\ \dot{p}_Y &= -Y - \varepsilon Y(bX^2 + dY^2 + eZ^2), \\ \dot{p}_Z &= -Z - \varepsilon Z(cX^2 + eY^2 + fZ^2). \end{aligned}$$

This system again is Hamiltonian with Hamiltonian

$$(9) \quad \begin{aligned} H &= \frac{1}{2}(p_X^2 + p_Y^2 + p_Z^2 + X^2 + Y^2 + Z^2) + \\ &\varepsilon \frac{1}{4} (aX^4 + 2bX^2Y^2 + 2cX^2Z^2 + dY^4 + 2eY^2Z^2 + fZ^4). \end{aligned}$$

As the change of variables is only a scale transformation for all $\varepsilon > 0$, the original and the transformed systems (3) and (8) have the same topological phase portrait, and additionally system (8) for ε sufficiently small is close to an integrable one.

The averaging theory needs the periodicity in the independent variable of the differential system, so we change the Hamiltonian (9) and the equations of motion (8) to a kind of generalized polar coordinates $(r, \theta, \rho, \alpha, R, \beta)$ in \mathbb{R}^6 . Thus the explicit change of variables is

$$(10) \quad \begin{aligned} X &= r \cos \theta, & Y &= \rho \cos(\theta + \alpha), & Z &= R \cos(\theta + \beta), \\ p_X &= r \sin \theta, & p_Y &= \rho \sin(\theta + \alpha), & p_Z &= R \sin(\theta + \beta). \end{aligned}$$

This change of variables is done when $r \geq 0$, $\rho \geq 0$ and $R \geq 0$.

The first integral H in the new coordinates becomes

$$(11) \quad \begin{aligned} H &= \frac{1}{2}(r^2 + \rho^2 + R^2) + \varepsilon \frac{1}{4} [ar^4 \cos^4 \theta + 2br^2 \rho^2 \cos^2 \theta \cos^2(\theta + \alpha) \\ &+ d\rho^4 \cos^4(\theta + \alpha) + 2R^2(cr^2 \cos^2 \theta + e\rho^2 \cos^2(\theta + \alpha)) \cos^2(\theta + \beta) \\ &+ fR^4 \cos^4(\theta + \beta)], \end{aligned}$$

and the new equations of motion are

$$\begin{aligned}
\dot{r} &= -\varepsilon r \cos \theta \sin \theta [a r^2 \cos^2 \theta + b \rho^2 \cos^2(\theta + \alpha) + c R^2 \cos^2(\theta + \beta)], \\
\dot{\theta} &= -1 - \varepsilon \cos^2 \theta [a r^2 \cos^2 \theta + b \rho^2 \cos^2(\theta + \alpha) + c R^2 \cos^2(\theta + \beta)], \\
\dot{\rho} &= -\varepsilon \rho \cos(\theta + \alpha) \sin(\theta + \alpha) [b r^2 \cos^2 \theta + d \rho^2 \cos^2(\theta + \alpha) \\
&\quad + e R^2 \cos^2(\theta + \beta)], \\
\dot{\alpha} &= \varepsilon [a r^2 \cos^4 \theta + \cos^2 \theta (b(-r^2 + \rho^2) \cos^2(\theta + \alpha) + c R^2 \cos^2(\theta + \beta)) \\
(12) \quad &\quad - \cos^2(\theta + \alpha) (d \rho^2 \cos^2(\theta + \alpha) + e R^2 \cos^2(\theta + \beta))], \\
\dot{R} &= -R \varepsilon \cos(\theta + \beta) \sin(\theta + \beta) [c r^2 \cos^2 \theta + e \rho^2 \cos^2(\theta + \alpha) \\
&\quad + f R^2 \cos^2(\theta + \beta)], \\
\dot{\beta} &= \varepsilon [a r^2 \cos^4 \theta + b \rho^2 \cos^2 \theta \cos^2(\theta + \alpha) - (c(r^2 - R^2) \\
&\quad \cos^2 \theta + e \rho^2 \cos^2(\theta + \alpha)) \cos^2(\theta + \beta) - f R^2 \cos^4(\theta + \beta)].
\end{aligned}$$

We observe in this last system that if we take the variable θ as the new independent variable of the system instead of t , we will obtain the periodicity necessary for applying the averaging theory.

From now on the independent variable will be θ . This means that our system has now only five equations. We denote by a prime the derivative with respect to θ and we expand the previous system in Taylor series in powers of ε . Thus, the system (12) becomes

$$\begin{aligned}
r' &= \varepsilon r \sin \theta \cos \theta [a r^2 \cos^2 \theta + b \rho^2 \cos^2(\theta + \alpha) + c R^2 \cos^2(\theta + \beta)] \\
&\quad + O(\varepsilon^2), \\
\rho' &= \varepsilon \rho \cos(\theta + \alpha) \sin(\theta + \alpha) [b r^2 \cos^2 \theta + d \rho^2 \cos^2(\theta + \alpha) \\
&\quad + e R^2 \cos^2(\theta + \beta)] + O(\varepsilon^2), \\
\alpha' &= \varepsilon [-a r^2 \cos^4 \theta + d \rho^2 \cos^4(\theta + \alpha) + e R^2 \cos^2(\theta + \alpha) \cos^2(\theta + \beta) + \\
(13) \quad &\quad \cos^2 \theta (b(r^2 - \rho^2) \cos^2(\theta + \alpha) - c R^2 \cos^2(\theta + \beta))] + O(\varepsilon^2), \\
R' &= \varepsilon R \sin(\theta + \beta) \cos(\theta + \beta) [c r^2 \cos^2 \theta + e \rho^2 \cos^2(\theta + \alpha) \\
&\quad + f R^2 \cos^2(\theta + \beta)] + O(\varepsilon^2), \\
\beta' &= \varepsilon [-\cos^2 \theta (a r^2 \cos^2 \theta + b \rho^2 \cos^2(\theta + \alpha)) + \\
&\quad (c(r^2 - R^2) \cos^2 \theta + e \rho^2 \cos^2(\theta + \alpha)) \cos^2(\theta + \beta) + \\
&\quad f R^2 \cos^4(\theta + \beta)] + \\
&\quad + O(\varepsilon^2).
\end{aligned}$$

Now system (13) is 2π -periodic respect to the variable θ . In order to apply the averaging theory we must fix the value of the first integral $H = h$ with $h > 0$. Otherwise when we apply the averaging theory to system (13) the Jacobian (7) will be zero because the periodic orbits are non-isolated leaving on a cylindric parameterized by the energy, see for more details [6]. So we shall restrict system (13) to every positive energy level $H = h > 0$.

By solving equation (11) for ρ we obtain two solutions, we choose the positive one,

and expanding it in Taylor series in ε we have

$$(14) \quad \rho = \sqrt{2h - r^2 - R^2} + O(\varepsilon).$$

Substituting ρ in the system (13) we obtain the differential system

$$(15) \quad \begin{aligned} r' &= \varepsilon r \sin \theta \cos \theta [a r^2 \cos^2 \theta + b(2h - r^2 - R^2) \cos^2(\theta + \alpha) \\ &\quad + cR^2 \cos^2(\theta + \beta)] + O(\varepsilon^2), \\ \alpha' &= \varepsilon [-ar^2 \cos^4 \theta + d(2h - r^2 - R^2) \cos^4(\theta + \alpha) + eR^2 \cos^2(\theta + \alpha) \cdot \\ &\quad \cos^2(\theta + \beta) + \cos^2 \theta (b(-2h + 2r^2 + R^2) \cos^2(\theta + \alpha) \\ &\quad - cR^2 \cos^2(\theta + \beta))] + O(\varepsilon^2), \\ R' &= \varepsilon R \sin(\theta + \beta) \cos(\theta + \beta) [c r^2 \cos^2 \theta + e(2h - r^2 - R^2) \cos^2(\theta + \alpha) \\ &\quad + fR^2 \cos^2(\theta + \beta)] + O(\varepsilon^2), \\ \beta' &= \varepsilon [-ar^2 \cos^4 \theta + \cos^2(\theta + \beta)(e(2h - r^2 - R^2) \cos^2(\theta + \alpha) \\ &\quad + fR^2 \cos^2(\theta + \beta)) + \cos^2 \theta (b(-2h + r^2 + R^2) \cos^2(\theta + \alpha) \\ &\quad + c(-R^2 + r^2) \cos^2(\theta + \beta))] + O(\varepsilon^2). \end{aligned}$$

Using the notation of the averaging theory, the function F_1 of (4) is

$F_1 = (F_{11}, F_{12}, F_{13}, F_{14})$ where

$$\begin{aligned} F_{11} &= r \sin \theta \cos \theta [a r^2 \cos^2 \theta + b(2h - r^2 - R^2) \cos^2(\theta + \alpha) + cR^2 \cos^2(\theta + \beta)], \\ F_{12} &= -ar^2 \cos^4 \theta + d(2h - r^2 - R^2) \cos^4(\theta + \alpha) + eR^2 \cos^2(\theta + \alpha) \cos^2(\theta + \beta) \\ &\quad + \cos^2 \theta (b(-2h + 2r^2 + R^2) \cos^2(\theta + \alpha) - cR^2 \cos^2(\theta + \beta)), \\ F_{13} &= R \sin(\theta + \beta) \cos(\theta + \beta) [c r^2 \cos^2 \theta + e(2h - r^2 - R^2) \cos^2(\theta + \alpha) \\ &\quad + fR^2 \cos^2(\theta + \beta)], \\ F_{14} &= -ar^2 \cos^4 \theta + e(2h - r^2 - R^2) \cos^2(\theta + \alpha) \cos^2(\theta + \beta) + fR^2 \cos^4(\theta + \beta) \\ &\quad + \cos^2 \theta (b(-2h + r^2 + R^2) \cos^2(\theta + \alpha) + c(-R^2 + r^2) \cos^2(\theta + \beta)). \end{aligned}$$

Computing the function $f_1 = (f_{11}, f_{12}, f_{13}, f_{14})$ using the expression (6) we obtain

$$\begin{aligned} f_{11}(r, \alpha, R, \beta) &= \frac{1}{8} r [b(-2h + r^2 + R^2) \sin 2\alpha - cR^2 \sin 2\beta], \\ f_{12}(r, \alpha, R, \beta) &= \frac{1}{8} [6dh - 3ar^2 - 3dr^2 - 3dR^2 + (-2bh + 2br^2 + bR^2)(2 + \cos 2\alpha) \\ &\quad + eR^2(2 + \cos 2(\alpha - \beta)) - cR^2(2 + \cos 2\beta)], \\ f_{13}(r, \alpha, R, \beta) &= \frac{1}{8} R [e(-2h + r^2 + R^2) \sin 2(\alpha - \beta) + cr^2 \sin 2\beta], \\ f_{14}(r, \alpha, R, \beta) &= \frac{1}{8} [4(-b + e)h - 3ar^2 + 2(b + c - e)r^2 + (2b - 2(c + e) + 3f) \cdot R^2 \\ &\quad + (-2h + r^2 + R^2)(b \cos 2\alpha - e \cos 2(\alpha - \beta)) + c(r^2 - R^2) \cos 2\beta]. \end{aligned}$$

We have to find the zeros $(r^*, \alpha^*, R^*, \beta^*)$ of

$$(16) \quad f_{1i}(r, \alpha, R, \beta) = 0 \quad \text{for } i = 1, 2, 3, 4.$$

and to check that the Jacobian determinant (7) at these zeros are different from zero.

Then

$$f_{11}(r, \alpha, R, \beta) = 0 \Rightarrow \text{either } r = 0 \quad \text{or} \quad r = \sqrt{2h - R^2 + \frac{cR^2 \sin 2\beta}{b \sin 2\alpha}}.$$

Case 1: $r = 0$. We substitute r in f_{1i} , for $i = 2, 3, 4$, we obtain

$$\begin{aligned} f_{12}(0, \alpha, R, \beta) &= \frac{1}{8}[6dh - 3dR^2 + (-2bh + bR^2)(2 + \cos 2\alpha) + eR^2 \\ &\quad (2 + \cos 2(\alpha - \beta)) - cR^2(2 + \cos 2\beta)], \\ f_{13}(0, \alpha, R, \beta) &= \frac{1}{8}Re(-2h + R^2) \sin 2(\alpha - \beta), \\ f_{14}(0, \alpha, R, \beta) &= \frac{1}{8}[4(-b + e)h + (2b - 2(c + e) + 3f)R^2 + (-2h + R^2) \\ &\quad \cdot (b \cos 2\alpha - e \cos 2(\alpha - \beta)) - cR^2 \cos 2\beta]. \end{aligned}$$

So

$$f_{13}(r, \alpha, R, \beta) = 0 \Rightarrow \begin{cases} e = 0, \\ R = 0, \\ R = \sqrt{2h}, \\ \alpha = \beta + \frac{k\pi}{2} \quad \text{with } k \in \mathbb{Z}. \end{cases}$$

Subcase 1.1: $e = 0$. Then $f_{13} = 0$ so the Jacobian is zero. So the averaging theory does not provide information on the periodic orbits in this case. Hence, in what follows we assume that $e \neq 0$.

Subcase 1.2: $R = 0$. We substitute R in f_{1i} , for $i = 2, 4$, and we obtain

$$\begin{aligned} f_{12}(0, \alpha, R, \beta) &= \frac{1}{8}[6dh - 2bh(2 + \cos 2\alpha)], \\ f_{14}(0, \alpha, R, \beta) &= \frac{1}{4}h[-2b + 2e - b \cos 2\alpha + e \cos 2(\alpha - \beta)]. \end{aligned}$$

So

$$f_{12}(r, \alpha, R, \beta) = 0 \Rightarrow \begin{cases} \alpha_1 = \frac{1}{2} \arccos \frac{3d - 2b}{b}, \\ \alpha_2 = -\frac{1}{2} \arccos \frac{3d - 2b}{b}. \end{cases}$$

If we substitute α_1 in $f_{14}(0, \alpha, R, \beta)$ we get

$$f_{14}(0, \alpha, R, \beta) = -\frac{1}{4}h \left[e \cos \left(\arccos \frac{2b - 3d}{b} + 2\beta \right) + 3d - 2e \right].$$

So

$$f_{14}(r, \alpha, R, \beta) = 0 \Rightarrow \begin{cases} \beta_{11} = -\frac{1}{2} \left(\arccos \frac{2b - 3d}{b} + \arccos \frac{2e - 3d}{e} \right), \\ \beta_{12} = -\frac{1}{2} \left(\arccos \frac{2b - 3d}{b} - \arccos \frac{2e - 3d}{e} \right). \end{cases}$$

If we substitute α_2 in $f_{14}(0, \alpha, R, \beta)$ we get

$$f_{14}(0, \alpha, R, \beta) = \frac{1}{4}h \left[e \cos \left(\arccos \frac{3d-2b}{b} + 2\beta \right) - 3d + 2e \right].$$

So

$$f_{14}(r, \alpha, R, \beta) = 0 \Rightarrow \begin{cases} \beta_{21} = -\frac{1}{2} \left(\arccos \frac{3d-2b}{b} + \arccos \frac{3d-2e}{e} \right), \\ \beta_{22} = -\frac{1}{2} \left(\arccos \frac{3d-2b}{b} - \arccos \frac{3d-2e}{e} \right). \end{cases}$$

The four zeros of (15) in the subcase $R = 0$ are

$$(17) \quad \begin{aligned} r_1^* &= 0, \\ \rho_1^* &= \sqrt{2h}, \\ \alpha_1^* &= \frac{1}{2} \arccos \frac{3d-2b}{b}, \\ R_1^* &= 0, \\ \beta_1^* &= \frac{1}{2} \left[-\arccos \frac{2b-3d}{b} \pm \arccos \frac{2e-3d}{e} \right]. \end{aligned}$$

$$(18) \quad \begin{aligned} r_2^* &= 0, \\ \rho_2^* &= \sqrt{2h}, \\ \alpha_2^* &= -\frac{1}{2} \arccos \frac{3d-2b}{b}, \\ R_2^* &= 0, \\ \beta_2^* &= -\frac{1}{2} \left[\arccos \frac{3d-2b}{b} \pm \arccos \frac{3d-2e}{e} \right]. \end{aligned}$$

Referencing to (10) we see clearly that the unique angle which plays a role in the initial conditions of the periodic solution is α because of the nullity of r and R so (15) admits two solutions if $\left| \frac{3d-2b}{b} \right| < 1$ and one solution if $\frac{3d-2b}{b} = 1$.

Now we calculate the Jacobian of f_1 applied in this possible solutions. By definition the Jacobian is

$$J_{f_1(s^*=(r^*, \alpha^*, R^*, \beta^*))} = |D_{r\alpha R\beta} f_1(s^*)| = \begin{vmatrix} \frac{\partial f_{11}}{\partial r} & \frac{\partial f_{11}}{\partial \alpha} & \frac{\partial f_{11}}{\partial R} & \frac{\partial f_{11}}{\partial \beta} \\ \frac{\partial f_{12}}{\partial r} & \frac{\partial f_{12}}{\partial \alpha} & \frac{\partial f_{12}}{\partial R} & \frac{\partial f_{12}}{\partial \beta} \\ \frac{\partial f_{13}}{\partial r} & \frac{\partial f_{13}}{\partial \alpha} & \frac{\partial f_{13}}{\partial R} & \frac{\partial f_{13}}{\partial \beta} \\ \frac{\partial f_{14}}{\partial r} & \frac{\partial f_{14}}{\partial \alpha} & \frac{\partial f_{14}}{\partial R} & \frac{\partial f_{14}}{\partial \beta} \end{vmatrix}.$$

So

$$(19) \quad \begin{cases} \left| \frac{3d-2b}{b} \right| < 1 \Rightarrow J_{f_1(s^*)} = -\frac{9}{64}h^4(b-3d)(b-d)(d-e)(3d-e). \\ \frac{3d-2b}{b} = 1 \Rightarrow J_{f_1(s^*)} = 0. \end{cases}$$

Summarizing the results of this subcase, the system (15) admits two solutions with the following existence's conditions $\left| \frac{3d-2b}{b} \right| < 1$, $\left| \frac{2e-3d}{e} \right| \leq 1$,

$$be(b-3d)(b-d)(d-e)(3d-e) \neq 0.$$

Subcase 1.3: $R = \sqrt{2h}$. We substitute $R = \sqrt{2h}$ in $f_{1i}(r, \alpha, R, \beta)$, for $i = 2, 4$ and we obtain

$$\begin{aligned} f_{12}(0, \alpha, R, \beta) &= \frac{1}{4}h[2e - 2c - c \cos 2\beta + e \cos 2(\alpha - \beta)], \\ f_{14}(0, \alpha, R, \beta) &= -\frac{1}{4}h[2c - 3f + c \cos 2\beta]. \end{aligned}$$

So

$$f_{14}(0, \alpha, R, \beta) = 0 \Rightarrow \begin{cases} \beta_3 = \frac{1}{2} \arccos \frac{3f-2c}{c}, \\ \beta_4 = -\frac{1}{2} \arccos \frac{3f-2c}{c}. \end{cases}$$

We substitute β_3 in f_{12} we have

$$f_{12}(0, \alpha, R, \beta) = \frac{h}{4} \left[2e - 3f - e \cos \left(2\alpha + \arccos \frac{2c-3f}{c} \right) \right].$$

$$f_{12}(0, \alpha, R, \beta) = 0 \Rightarrow \begin{cases} \alpha_{31} = \frac{1}{2} \left[\arccos \frac{2e-3f}{e} - \arccos \frac{2c-3f}{c} \right], \\ \alpha_{32} = \frac{1}{2} \left[-\arccos \frac{2e-3f}{e} - \arccos \frac{2c-3f}{c} \right] \end{cases}$$

We substitute β_4 in f_{12} we have

$$f_{12}(0, \alpha, R, \beta) = \frac{h}{4} \left[2e - 3f + e \cos \left(2\alpha + \arccos \frac{3f-2c}{c} \right) \right].$$

$$f_{12}(0, \alpha, R, \beta) = 0 \Rightarrow \begin{cases} \alpha_{41} = \frac{1}{2} \left[\arccos \frac{3f-2e}{e} - \arccos \frac{3f-2c}{c} \right], \\ \alpha_{42} = \frac{1}{2} \left[-\arccos \frac{3f-2e}{e} - \arccos \frac{3f-2c}{c} \right] \end{cases}$$

The four zeros of (15) are

$$\begin{aligned}
(20) \quad r_1^* &= 0, \\
\rho_1^* &= 0, \\
\alpha_1^* &= \frac{1}{2} \left[\pm \arccos \frac{2e - 3f}{e} - \arccos \frac{2c - 3f}{c} \right], \\
R_1^* &= \sqrt{2h}, \\
\beta_1^* &= \frac{1}{2} \arccos \frac{3f - 2c}{c}. \\
\\
(21) \quad r_2^* &= 0, \\
\rho_2^* &= 0, \\
\alpha_2^* &= \frac{1}{2} \left[\pm \arccos \frac{3f - 2e}{e} - \arccos \frac{3f - 2c}{c} \right], \\
R_2^* &= \sqrt{2h}, \\
\beta_2^* &= -\frac{1}{2} \arccos \frac{3f - 2c}{c}.
\end{aligned}$$

Referencing to (10) we see clearly that the unique angle which plays a role in the initial conditions of the periodic solution is β because of the nullity of r and ρ so (15) admits one of the above zeros if $\frac{-2c + 3f}{c} = 1$ and two solutions if $\left| \frac{-2c + 3f}{c} \right| < 1$.

So

$$(22) \quad \begin{aligned}
\left| \frac{-2c + 3f}{c} \right| < 1 &\Rightarrow J_{f_1(s^*)} = -\frac{9}{32} h^4 (c - 3f)(c - f)(e - 3f)(e - f). \\
\frac{-2c + 3f}{c} = 1 &\Rightarrow J_{f_1(s^*)} = 0.
\end{aligned}$$

Summarizing the conditions of the existence of the two solutions chosen we have

$$\left| \frac{2e - 3f}{e} \right| \leq 1, \quad \left| \frac{2c - 3f}{c} \right| < 1, \quad (c - 3f)(c - f)(e - 3f)(e - f) \neq 0 \text{ and } ce \neq 0.$$

Subcase 1.4: $\alpha = \beta + \frac{k\pi}{2}$. We substitute α in f_{1i} , for $i = 2, 4$ and we obtain

$$\begin{aligned}
f_{12}(0, \alpha, R, \beta) &= \frac{1}{8} \left(-2bh \left((-1)^k \cos 2\beta + 2 \right) + bR^2 \left((-1)^k \cos 2\beta + 2 \right) \right. \\
&\quad \left. - cR^2 (\cos 2\beta + 2) + 6dh - 3dR^2 + e \left((-1)^k + 2 \right) R^2 \right), \\
f_{14}(0, \alpha, R, \beta) &= \frac{1}{8} \left(R^2 (2b - 2(c + e) + 3f) + e(-1)^k (2h - R^2) \right. \\
&\quad \left. - b(-1)^k (2h - R^2) \cos 2\beta + 4h(e - b) - cR^2 \cos 2\beta \right).
\end{aligned}$$

So solving $f_{12}(0, \alpha, R, \beta) = 0$ with respect to the variable R we get

$$R = \sqrt{\frac{2h \left(b(-1)^k \cos 2\beta + 2b - 3d \right)}{b(-1)^k \cos 2\beta + 2b - c \cos 2\beta - 2c - 3d + e(-1)^k + 2e}}$$

We substitute R in $f_{14}(0, \alpha, R, \beta)$ and we have

$$f_{14}(0, \alpha, R, \beta) = \frac{\Sigma_1}{\Sigma_2} \quad \text{with}$$

$$\begin{aligned} \Sigma_1 &= h(\cos 2\beta (c(3d - e((-1)^k + 2)) - b(-1)^k e((-1)^k + 2))) \\ &\quad + h(3bf(-1)^k \cos 2\beta - 2b(e((-1)^k + 2))) \\ &\quad + h(c(6d - 2e((-1)^k + 2)) - 9df + e^2((-1)^k + 2)^2), \\ \Sigma_2 &= 4(\cos 2\beta (b(-1)^k - c) + 2b - 2c - 3d + e((-1)^k + 2)). \end{aligned}$$

So solving $f_{14}(0, \alpha, R, \beta) = 0$ with respect to the variable β we obtain

$$\beta = \pm \frac{1}{2} \arccos \left(\frac{-2e(-1)^k(b + c - 2e) - 4be + 6bf + 6cd - 4ce - 9df + 5e^2}{(-1)^k(2be - 3bf + ce) + be - 3cd + 2ce} \right).$$

If we substitute β in the expression of R we find

$$\begin{aligned} R &= \sqrt{\frac{\Delta_1}{\Delta_2}} \quad \text{with} \\ \Delta_1 &= 2h(b(e(e(-1)^k(4(-1)^k + 5) - 3d(2(-1)^k + 1)) - 2c((-1)^k - 1) \\ &\quad (e(-1)^k + 2))) + 2h(6bcd((-1)^k - 1) + 3cd(3d - e((-1)^k + 2))), \\ \Delta_2 &= b(e(-2(-1)^k(3d - 5e + 3f) - 3d + 8e - 3f)) - 2bc((-1)^k - 1) \\ &\quad (e(-1)^k - 3(d - e + f)) + b(3cd(3d - 2e((-1)^k + 2) + 3f)). \end{aligned}$$

- The zeros of (15) for $k = 0$

$$(23) \quad \begin{aligned} r_{*1} &= 0, \\ \rho_1^* &= \sqrt{\frac{2h(f - e)}{d - 2e + f}}, \\ \alpha_{*1} &= \pm \frac{1}{2} \arccos \left(\frac{-3e^2 + 2c(-d + e) + 2b(e - f) + 3df}{c(d - e) + b(-e + f)} \right), \\ R_{*1} &= \sqrt{\frac{2h(d - e)}{d - 2e + f}}, \\ \beta_{*1} &= \pm \frac{1}{2} \arccos \left(\frac{-3e^2 + 2c(-d + e) + 2b(e - f) + 3df}{c(d - e) + b(-e + f)} \right). \end{aligned}$$

- The zeros of (15) for $k = 2$

$$\begin{aligned}
r^*_{*2} &= 0, \\
\rho^*_{*2} &= \sqrt{\frac{2h(f-e)}{d-2e+f}}, \\
\alpha^*_{*2} &= \pm \frac{1}{2} \arccos \left(\frac{-3e^2 + 2c(-d+e) + 2b(e-f) + 3df}{c(d-e) + b(-e+f)} \right) + \pi, \\
R^*_{*2} &= \sqrt{\frac{2h(d-e)}{d-2e+f}}, \\
\beta^*_{*2} &= \pm \frac{1}{2} \arccos \left(\frac{-3e^2 + 2c(-d+e) + 2b(e-f) + 3df}{c(d-e) + b(-e+f)} \right).
\end{aligned}
\tag{24}$$

Referencing to (10) we see clearly that the angles which play a role in the initial conditions of the periodic solution are β and α because the nullity of r so if

$$\frac{-3e^2 + 2c(-d+e) + 2b(e-f) + 3df}{c(d-e) + b(-e+f)} = 1 \text{ our system (15) admits two solutions and}$$

eight solutions if $\left| \frac{-3e^2 + 2c(-d+e) + 2b(e-f) + 3df}{c(d-e) + b(-e+f)} \right| < 1$.

If $\left| \frac{-3e^2 + 2c(-d+e) + 2b(e-f) + 3df}{c(d-e) + b(-e+f)} \right| < 1$ So

$$\begin{aligned}
J_{f_1(s^*)} &= -\frac{9eh^4}{\Lambda} [(e-f)(-be+bf+cd-ce-3df+3e^2) \cdot \\
&\quad (d-e)(-be+bf+cd-ce-df+e^2)].
\end{aligned}
\tag{25}$$

Where $\Lambda = 32(d-2e+f)^3$.

If $\left| \frac{-3e^2 + 2c(-d+e) + 2b(e-f) + 3df}{c(d-e) + b(-e+f)} \right| = 1$ we have $J_{f_1(s^*)} = 0$.

Summarizing the conditions of the existence of this subcase

$$\left| \frac{-3e^2 + 2c(-d+e) + 2b(e-f) + 3df}{c(d-e) + b(-e+f)} \right| < 1,$$

$e(-be+bf+cd-ce-3df+3e^2)(-be+bf+cd-ce-df+e^2)(c(d-e)+b(f-e)) \neq 0$, $(f-e)(d-2e+f) > 0$, $(d-e)(d-2e+f) > 0$.

- The zeros of (15) for $k = 1$

$$\begin{aligned}
r^*_{*3} &= 0, \\
\rho^*_3 &= \sqrt{\frac{2h(e-3f)}{-3d+2e-3f}}, \\
\alpha^*_{*3} &= \pm \frac{1}{2} \arccos \left(-\frac{6cd-2be-2ce+e^2+6bf-9df}{3cd+be-ce-3bf} \right) + \frac{\pi}{2}, \\
R^*_{*3} &= \sqrt{\frac{2h(3d-e)}{3d-2e+3f}}, \\
\beta^*_{*3} &= \pm \frac{1}{2} \arccos \left(-\frac{6cd-2be-2ce+e^2+6bf-9df}{3cd+be-ce-3bf} \right).
\end{aligned}
\tag{26}$$

- The zeros of (15) for $k = 3$

$$\begin{aligned}
r^*_{*4} &= 0, \\
\rho^*_4 &= \sqrt{\frac{2h(e-3f)}{-3d+2e-3f}}, \\
\alpha^*_{*4} &= \pm \frac{1}{2} \arccos \left(-\frac{6cd-2be-2ce+e^2+6bf-9df}{3cd+be-ce-3bf} \right) + \frac{3\pi}{2}, \\
R^*_{*4} &= \sqrt{\frac{2h(3d-e)}{3d-2e+3f}}, \\
\beta^*_{*4} &= \pm \frac{1}{2} \arccos \left(-\frac{6cd-2be-2ce+e^2+6bf-9df}{3cd+be-ce-3bf} \right).
\end{aligned}
\tag{27}$$

Referencing to (10) we see clearly that the angles which play a role in the initial conditions of the periodic solution are β and α because the nullity of r so (15) admits two of the above zeros if $-\frac{6cd-2be-2ce+e^2+6bf-9df}{3cd+be-ce-3bf} = 1$ and eight solutions if

$$\left| -\frac{6cd-2be-2ce+e^2+6bf-9df}{3cd+be-ce-3bf} \right| < 1.$$

The Jacobian if $\left| -\frac{6cd-2be-2ce+e^2+6bf-9df}{3cd+be-ce-3bf} \right| < 1$ is

$$\begin{aligned}
J_{f_1(s^*)} &= \frac{eh^4(3d-e)}{\Omega} [(-be+3bf+9cd-3ce-9df+e^2) \cdot \\
&\quad (-3be+9bf+3cd-ce-9df+e^2)(e-3f)].
\end{aligned}
\tag{28}$$

Where $\Omega = 32(3d-2e+3f)^3$.

If $-\frac{6cd-2be-2ce+e^2+6bf-9df}{3cd+be-ce-3bf} = 1$ we have $J_{f_1(s^*)} = 0$.

Summarizing the conditions of the existence of this subcase

$$\begin{aligned}
\left| -\frac{6cd-2be-2ce+e^2+6bf-9df}{3cd+be-ce-3bf} \right| < 1, \\
(-be+3bf+9cd-3ce-9df+e^2) (-3be+9bf+3cd-ce-9df+e^2).
\end{aligned}$$

$(3cd + be - ce - 3bf) \neq 0$, $(-3d + 2e - 3f)(e - 3f) \neq 0$, $(-3d + 2e - 3f)(e - 3d) \neq 0$,

Case 2: $r = \sqrt{2h - R^2 + \frac{cR^2 \sin 2\beta}{b \sin 2\alpha}}$. Then substituting r in f_{1i} , for $i = 2, 3, 4$,

$$\begin{aligned} f_{12}(r, \alpha, R, \beta) &= \frac{1}{8b} \left(\frac{1}{4} R^2 \frac{\sin 2\beta}{\sin 2\alpha} (-6ac + 4bc \cos 2\alpha + 8bc - be \cos 4\alpha + be - 6cd) \right. \\ &\quad \left. + b(R^2(3a - 2(b + c - e)) - 6ah + b \cos 2\alpha(2h - R^2) + 4bh) + \right. \\ &\quad \left. bR^2 \cos 2\beta(e \cos 2\alpha - c) \right), \\ f_{13}(r, \alpha, R, \beta) &= \frac{cR \sin(2\beta) (b(2h - R^2) + R^2 \csc(2\alpha)(c \sin(2\beta) + e \sin(2(\alpha - \beta))))}{8b}, \\ f_{14}(r, \alpha, R, \beta) &= R^2(6ab - c(8b + e) + 6bf) + cR^2 \frac{\sin 2\beta}{\sin 2\alpha} (-6a + 2b \cos(2\alpha)) \\ &\quad + 4(b + c - e) + \sin 4\beta(c - e \cos 2\alpha) + e \cos 4\beta + \frac{1}{4}h(2c - 3a) \\ &\quad + 4bc \cos 2\beta (h - R^2). \end{aligned}$$

So

$$f_{13}(r, \alpha, R, \beta) = 0 \Rightarrow \begin{cases} c = 0, \\ R = 0, \\ R = \sqrt{\frac{2bh \sin 2\alpha}{b \sin 2\alpha - c \sin 2\beta - e \sin 2(\alpha - \beta)}}, \\ \beta = \frac{k\pi}{2} \quad \text{with } k \in \mathbb{Z}. \end{cases}$$

Subcase 2.1: $c = 0$. Then $f_{13} = 0$ and the Jacobian is equal to zero so we cannot apply averaging theory.

Subcase 2.2: $R = 0$. Then $r = \sqrt{2h}$. So substituting r and R in $f_{1i}(r, \alpha, R, \beta)$, for $i = 2, 4$, we obtain

$$\begin{aligned} f_{12}(r, \alpha, R, \beta) &= \frac{1}{4}h(-3a + b \cos(2\alpha) + 2b), \\ f_{14}(r, \alpha, R, \beta) &= \frac{1}{4}h(-3a + c \cos(2\beta) + 2c). \end{aligned}$$

Therefore solving $f_{12}(r, \alpha, R, \beta) = 0$ with respect to α we obtain $\alpha = \pm \frac{1}{2} \arccos \frac{3a - 2b}{b}$,

and solving $f_{14}(r, \alpha, R, \beta) = 0$ with respect to β we obtain $\beta = \pm \frac{1}{2} \arccos \frac{3a - 2c}{c}$.

The zeros of (15) in this subcase are

$$\begin{aligned}
 r^* &= \sqrt{2h}, \\
 \rho^* &= 0, \\
 \alpha^* &= \pm \frac{1}{2} \arccos \frac{3a-2b}{b}, \\
 R^* &= 0, \\
 \beta^* &= \pm \frac{1}{2} \arccos \frac{3a-2c}{c}.
 \end{aligned}
 \tag{29}$$

Referencing to (10) we see clearly that (15) admits only one solution because the nullity of R and ρ .

The Jacobian is

$$J_{f_1(s^*)} = -\frac{9}{32} h^4 (a-b)(3a-b)(a-c)(3a-c).
 \tag{30}$$

Summarize the conditions of existence of the solution of this subcase,

$$bc(a-b)(3a-b)(a-c)(3a-c) \neq 0, \quad \left| \frac{3a-2c}{c} \right| \leq 1, \quad \left| \frac{3a-2b}{b} \right| \leq 1.$$

$$\text{Subcase 2.3: } R = \sqrt{\frac{2bh \sin 2\alpha}{b \sin 2\alpha - e \sin 2(\alpha - \beta) - c \sin 2\beta}}.$$

$$\text{Then } r = \sqrt{\frac{2eh \sin 2(\alpha - \beta)}{c \sin 2\beta - b \sin 2\alpha + e \sin 2(\alpha - \beta)}} \quad \text{and}$$

$$\begin{aligned}
 f_{12}(r, \alpha, R, \beta) &= \frac{h}{D} \left[-\sin 2(\alpha - \beta)(-3ae + bc + 2be) + \sin 2\beta(2bc + be - 3cd) \right. \\
 &\quad \left. + 2b(e - c) \sin 2\alpha \right],
 \end{aligned}$$

$$\begin{aligned}
 f_{14}(r, \alpha, R, \beta) &= \frac{h}{D} \left[(2bc + ce - 3bf) \sin 2\alpha + (bc - 3ae + 2ce) \sin 2(\alpha - \beta) \right. \\
 &\quad \left. + 2c(-b + e) \sin 2\beta \right].
 \end{aligned}$$

Where $D = 4(b \sin 2\alpha - e \sin 2(\alpha - \beta) - c \sin 2\beta)$.

To calculate the zeros of this two last functions we need their numerators:

$$h(\sin 2(\alpha - \beta))(3ae - bc - 2be) + \sin 2\beta(2bc + be - 3cd) + 2b \sin 2\alpha(e - c),$$

$$h(\sin 2(\alpha - \beta)(3ae - bc - 2ce) + 2c(b - e) \sin 2\beta + \sin 2\alpha(-c(2b + e) + 3bf)).$$

Expanding the trigonometrical terms of these numerators and using one notation of $\sin \alpha = s$; $\cos \alpha = \pm \sqrt{1 - s^2}$; $\sin \beta = S$; $\cos \beta = \pm \sqrt{1 - S^2}$ we obtain

$$\begin{aligned}
 P_{12}(s, S) &= 2hs\sqrt{1-s^2}(-6aeS^2 + 3ae + 2bcS^2 - 3bc + 4beS^2) \\
 &\quad - 2hS\sqrt{1-S^2}(-6aes^2 + 3ae + 2bcs^2 - 3bc + 4bes^2 - 3be + 3cd),
 \end{aligned}$$

$$\begin{aligned}
 P_{14}(s, S) &= 2hs\sqrt{1-s^2}(-6aeS^2 + 3ae + 2bcS^2 - 3bc + 3bf + 4ceS^2 - 3ce) \\
 &\quad - 2hS\sqrt{1-S^2}(-6aes^2 + 3ae + 2bcs^2 - 3bc + 4ces^2).
 \end{aligned}$$

The other three notations provide the same previous expressions. So $P_{12}(s, S) = 0$ and $P_{14}(s, S) = 0$ implies that $Q_{12}(s, S) = 0$ and $Q_{14}(s, S) = 0$.

Where

$$(31) \quad \begin{aligned} Q_{12}(s, S) &= 4h^2S^2(1-S^2)(-6aes^2 + 3ae + 2bcs^2 - 3bc + 4bes^2 - 3be + 3cd)^2 \\ &\quad - 4h^2s^2(1-s^2)(-6aeS^2 + 3ae + 2bcS^2 - 3bc + 4beS^2)^2, \\ Q_{14}(s, S) &= 4h^2S^2(1-S^2)(-6aes^2 + 3ae + 2bcs^2 - 3bc + 4ces^2)^2 \\ &\quad - 4h^2s^2(1-s^2)(-6aeS^2 + 3ae + 2bcS^2 - 3bc + 3bf + 4ceS^2 - 3ce)^2. \end{aligned}$$

We calculate the resultant of Q_{12} and Q_{14} with respect to s and S , we obtain

$$(32) \quad \begin{aligned} R_{12}(S) &= 47775744h^{16}(-1+S)^4S^8(1+S)^4K^2(S)L^2(S), \\ R_{14}(s) &= 47775744h^{16}(-1+s)^4s^8(1+s)^4M^2(s)N^2(s). \end{aligned}$$

with $K(S)$ and $M(s)$ two polynomials of the form $AS^2 + B$ and $Cs^2 + D$ respectively with A, B, C, D constants, and $L(S)$ and $N(s)$ two polynomials of the form $ES^4 + FS^2 + G$ and $HS^4 + Is^2 + J$ respectively with E, F, G, H, I, J constants. So if we calculate s_0 and S_0 the zeros of R_{12} and R_{14} then (s_0, S_0) is a zero of (31).

Solving (32) we obtain 81 pairs of (s, S) . Only 9 of this pairs are solutions of (31). When we calculate (α, β) corresponding to (s, S) solution we find the zeros

$$\begin{aligned} z^* = (r^*, \rho^*, \alpha^*, R^*, \beta^*) \text{ of (15) which are } & s_{1,\pm}^* \left(\sqrt{\frac{2eh}{e+c}}, \sqrt{\frac{2ch}{e+c}}, \pm \frac{\pi}{2} + 2k\pi, 0, \pm \frac{\pi}{2} + 2l\pi \right); \\ & s_{2,\pm}^* \left(\sqrt{\frac{2eh}{e+c}}, \sqrt{\frac{2ch}{e+c}}, \pm \frac{\pi}{2} + 2m\pi, 0, 0 \right); \\ & s_{3,\pm}^* \left(\sqrt{\frac{2eh}{e-c}}, \sqrt{\frac{2ch}{e-c}}, 0, 0, \pm \frac{\pi}{2} + 2n\pi \right); \\ & s_4^* \left(\sqrt{\frac{2eh}{e-c}}, \sqrt{\frac{2ch}{e-c}}, 0, 0, 0 \right), \text{ with } k, l, m, n \in \mathbb{Z}. \end{aligned}$$

The Jacobian applied in all this zeros is equal to zero so we cannot apply averaging theory in this case.

Subcase 2.4: $\beta = \frac{k\pi}{2}$. So substituting β in f_{12} and f_{14} we obtain

$$\begin{aligned} f_{12}(r, \alpha, R, \beta) &= \frac{1}{8} \left(R^2(3a - 2(b+c-e)) - 6ah - R^2(\cos 2\alpha(b - e(-1)^k) \right. \\ &\quad \left. + c(-1)^k) + 2bh \cos 2\alpha + 4bh \right), \\ f_{14}(r, \alpha, R, \beta) &= \frac{1}{8} \left(3a(R^2 - 2h) + 2c((-1)^k + 2)(h - R^2) + 3fR^2 \right). \end{aligned}$$

Solving $f_{14} = 0$ with respect to R we obtain

$$R = \frac{\sqrt{2(-3ah + ch(-1)^k + 2ch)}}{\sqrt{-3a + 2c(-1)^k + 4c - 3f}}.$$

Subcase 2.4.1 $k = 1$ and $k = 3$. Then

$$R = \sqrt{\frac{2h(c-3a)}{-3a+2c-3f}}.$$

and f_{12} becomes

$$f_{12}(r, \alpha, R, \beta) = \frac{h(\cos(2\alpha)(-3ae - bc + 3bf + ce) + 6ae - 9af - 2bc + 6bf + c^2 - 2ce)}{4(3a - 2c + 3f)}.$$

Solving $f_{12} = 0$ with respect to α we obtain

$$\alpha = \pm \frac{1}{2} \arccos\left(\frac{6ae - 9af - 2bc + 6bf + c^2 - 2ce}{3ae + bc - 3bf - ce}\right).$$

Substituting α in r and R we obtain

- the zeros of (15) for $k = 1$

$$(33) \quad \begin{aligned} r^*_{*1} &= \sqrt{\frac{2h(c-3f)}{-3a+2c-3f}}, \\ \rho^*_{*1} &= 0, \\ \alpha^*_{*1} &= \pm \frac{1}{2} \arccos\left(\frac{6ae - 9af - 2bc + 6bf + c^2 - 2ce}{3ae + bc - 3bf - ce}\right), \\ R^*_{*1} &= \sqrt{\frac{2h(c-3a)}{-3a+2c-3f}}, \\ \beta^*_{*1} &= \frac{\pi}{2}. \end{aligned}$$

- the zeros of (15) for $k = 3$

$$(34) \quad \begin{aligned} r^*_{*2} &= \sqrt{\frac{2h(c-3f)}{-3a+2c-3f}}, \\ \rho^*_{*2} &= 0, \\ \alpha^*_{*2} &= \pm \frac{1}{2} \arccos\left(\frac{6ae - 9af - 2bc + 6bf + c^2 - 2ce}{3ae + bc - 3bf - ce}\right), \\ R^*_{*2} &= \sqrt{\frac{2h(c-3a)}{-3a+2c-3f}}, \\ \beta^*_{*2} &= \frac{3\pi}{2}. \end{aligned}$$

Referencing to (10) we see clearly that the unique angle which plays a role in the initial conditions of the periodic solution is β because the nullity of ρ so (15) admit two of the above zeros.

The Jacobian is

$$(35) \quad J_{f_1(s^*)} = \frac{ch^4}{DC} (3a - c)(c - 3f) (-3ae + 9af + 3bc - 9bf - c^2 + ce) \\ (-9ae + 9af + bc - 3bf - c^2 + 3ce).$$

Where $DC = 16(3a - 2c + 3f)^3$.

Summarizing the conditions of existence of the solution of this subcase,

$$(c - 3f)(-3a + 2c - 3f) > 0, (c - 3a)(-3a + 2c - 3f) > 0,$$

$$\left| \frac{6ae - 9af - 2bc + 6bf + c^2 - 2ce}{3ae + bc - 3bf - ce} \right| \leq 1,$$

$$c(-3ae + 9af + 3bc - 9bf - c^2 + ce)(-9ae + 9af + bc - 3bf - c^2 + 3ce) \neq 0.$$

Subcase 2.4.2 $k = 0$ and $k = 2$. Then

$$R = \sqrt{\frac{2h(c - a)}{-a + 2c - f}}.$$

and f_{12} becomes

$$f_{12}(r, \alpha, R, \beta) = \frac{h(\cos(2\alpha)(ae - c(b + e) + bf) + 2ae - 3af + 2b(f - c) + 3c^2 - 2ce)}{4(a - 2c + f)}.$$

Solving $f_{12} = 0$ with respect to α we obtain

$$\alpha = \pm \frac{1}{2} \arccos \left(\frac{2ae - 3af - 2bc + 2bf + 3c^2 - 2ce}{-ae + bc - bf + ce} \right).$$

Substituting α in r and R we obtain

- the zeros of (15) for $k = 0$

$$(36) \quad \begin{aligned} r^*_{*3} &= \sqrt{\frac{2h(f - c)}{a - 2c + f}}, \\ \rho^*_{*3} &= 0, \\ \alpha^*_{*3} &= \pm \frac{1}{2} \arccos \left(\frac{2ae - 3af - 2bc + 2bf + 3c^2 - 2ce}{-ae + bc - bf + ce} \right), \\ R^*_{*3} &= \sqrt{\frac{2h(a - c)}{a - 2c + f}}, \\ \beta^*_{*3} &= 0. \end{aligned}$$

- the zeros of (15) for $k = 2$

$$(37) \quad \begin{aligned} r^*_{*4} &= \sqrt{\frac{2h(f - c)}{a - 2c + f}}, \\ \rho^*_{*4} &= 0, \\ \alpha^*_{*4} &= \pm \frac{1}{2} \arccos \left(\frac{2ae - 3af - 2bc + 2bf + 3c^2 - 2ce}{-ae + bc - bf + ce} \right), \\ R^*_{*4} &= \sqrt{\frac{2h(a - c)}{a - 2c + f}}, \\ \beta^*_{*4} &= \pi. \end{aligned}$$

Referencing to (10) we see clearly that the unique angle which plays a role in the initial conditions of the periodic solution is β because the nullity of ρ so (15) admits two of the above zeros .

The Jacobian is

$$(38) \quad J_{f_1(s^*)} = -\frac{9ch^4}{DD}(a-c)(c-f)(-ae+af+bc-bf-c^2+ce) \cdot \\ (-ae+3af+bc-bf-3c^2+ce)(ae-bc+bf-ce)^2.$$

Where $DD = 16(a-2c+f)^3(-ae+bc-bf+ce)^2$.

Summarizing the existence's conditions of the solution,

$$c(-ae+af+bc-bf-c^2+ce)(-ae+3af+bc-bf-3c^2+ce)$$

$$(ae-bc+bf-ce) \neq 0, \left| \frac{2ae-3af-2bc+2bf+3c^2-2ce}{-ae+bc-bf+ce} \right| \leq 1,$$

$$(f-c)(a-2c+f) > 0, (c-a)(-a+2c-f) > 0.$$

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