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Spatial bi–stacked central configurations formed by two dual regular polyhedra

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Abstract

In this paper we prove the existence of two new families of spatial stacked central configurations, one consisting of eight equal masses on the vertices of a cube and six equal masses on the vertices of a regular octahedron, and the other one consisting of twenty masses at the vertices of a regular dodecahedron and twelve masses at the vertices of a regular icosahedron. The masses on the two different polyhedra are in general different. We note that the cube and the octahedron, the dodecahedron and the icosahedron are dual regular polyhedra. The tetrahedron is itself dual. There are also spatial stacked central configurations formed by two tetrahedra, one and its dual.

Keywords: n–body problem, spatial central configurations, dual regular polyhedra
2000 MSC: 70F10, 70F15

1. Introduction

We consider the spatial N–body problem

$$m_k \ddot{\mathbf{q}}_k = - \sum_{\substack{j=1 \\ j \neq k}}^{N} G\, m_k\, m_j \frac{\mathbf{q}_k - \mathbf{q}_j}{|\mathbf{q}_k - \mathbf{q}_j|^3} \ ,$$

$k = 1, \ldots, N$, where $\mathbf{q}_k \in \mathbb{R}^3$ is the position vector of the punctual mass m_k in an inertial coordinate system, and G is the gravitational constant which can be taken equal to one by choosing conveniently the unit of time. We fix the center of mass $\sum_{i=1}^{N} m_i\, \mathbf{q}_i / \sum_{i=1}^{N} m_i$ of the system at the origin of $\mathbb{R}^{3N}$. The *configuration space* of the N–body problem in $\mathbb{R}^3$ is

$$\mathcal{E} = \{(\mathbf{q}_1, \ldots, \mathbf{q}_N) \in \mathbb{R}^{3N} \ : \ \sum_{i=1}^{N} m_i\, \mathbf{q}_i = 0, \ \mathbf{q}_i \neq \mathbf{q}_j \text{ for } i \neq j\} \ .$$

Given $m_1, \ldots, m_N$ a configuration $(\mathbf{q}_1, \ldots, \mathbf{q}_N) \in \mathcal{E}$ is *central* if there exists a positive constant λ such that

$$\ddot{\mathbf{q}}_k = -\lambda\, \mathbf{q}_k \ ,$$

$k = 1, \ldots, N$. Thus a central configuration $(\mathbf{q}_1, \ldots, \mathbf{q}_N) \in \mathcal{E}$ of the N–body problem with positive masses $m_1, \ldots, m_N$ is a solution of the system of the N vectorial equations with $N+1$ unknowns ($\mathbf{q}_k$ for $k = 1, \ldots, N$ plus $\lambda > 0$)

$$\sum_{\substack{j=1 \\ j \neq k}}^{N} m_j \frac{\mathbf{q}_k - \mathbf{q}_j}{|\mathbf{q}_k - \mathbf{q}_j|^3} = \lambda\, \mathbf{q}_k. \tag{1}$$

Central configurations play a main role in celestial mechanics, the transition of behaviour or bifurcations of a N–body system happens at central configurations, total collapse and some kind of escapes are asymptotic to central configurations. We also must mention that they generate the homographic solutions, i.e. solutions where the configuration of the bodies is central for all time, in other words the configuration is similar with itself for all time, these are the unique explicitly solutions of the N–body problem known until now (see [14] for more details).

The simplest spatial central configuration for the 4–body problem for 4 arbitrary masses is the tetrahedron, actually this is the unique spatial central configuration for $N = 4$ (modulo homotheties and rotations). In general for $N \geq 5$, N equal masses at the vertices of a regular polyhedron of N vertices always form a central configuration (see [1]). For $N \geq 5$ many particular central configurations have been discovered, but still is interesting to find new classes of central configurations.

In the literature the reader can find several papers studying spatial central configurations consisting of nested regular polyhedra, see for instance [2, 3, 4] and the references therein.

Another important class of central configurations are the so called *stacked central configurations*, where a proper subgroup of particles form by themselves a central configuration. They were introduced by Hampton in 2005 in a planar five problem [6], since then they have been studied by many authors, see for instance [5, 13, 10] for some planar stacked central configurations and [17, 15, 11, 12, 7] for some spatial stacked central configurations. In this paper we are interested in a particular class of spatial stacked central configurations formed by two *dual* regular polyhedra; that is, the vertices of one polyhedron correspond to the faces of the other one and viceversa. More precisely, we will study central configurations formed by a cube and its dual, the octahedron. We will also analyze the central configurations formed by the dodecahedron and its dual, the icosahedron. Since both regular polyhedra are by itself a central configuration, the above classes form a *bi–stacked central configurations*. As far as we know, this is the first time that this kind of central configurations are detected with two different dual polyhedra, because for the tetrahedron such kind of central configurations where studied in [4, 16]. Note that the dual of a regular tetrahedron is another regular tetrahedron facing in the opposite directions.

Let P_{n_1} denote a regular polyhedron with n_1 vertices. The simplest way to create the dual polyhedron of P_{n_1} is by finding the barycenter of each of its faces, and then connecting these barycenters so that they become the vertices of the new dual polyhedron, which we denote by $P^d_{n_1}$.

Definition 1. *Given a central configuration formed by two dual regular polyhedra P_{n_1} and $P^d_{n_1}$, we say that it is of (see Figure 1):*

- *Type A if $P^d_{n_1}$ is inside P_{n_1}.*
- *Type B if the faces of $P^d_{n_1}$ intersect the faces of P_{n_1}.*
- *Tipe C if $P^d_{n_1}$ is outside P_{n_1}.*

The main result of this paper is the following.

Theorem 2. *We consider n_1 masses equal to M at the vertices of a regular polyhedron P_{n_1} of n_1 vertices. Without loss of generality we can assume that $M = 1$. We consider n_2 additional masses equal to m at the vertices of the dual polyhedron $P^d_{n_1}$ scaled by a factor a, then the following statements hold.*

(a) *For each regular polyhedron P_{n_1} we can find a function $m(a)$ and a non–empty set $\mathcal{D} \in \mathbb{R}^+$ such that for all $a \in \mathcal{D}$ the configuration formed by the vertices of P_{n_1} and $P^d_{n_1}$ with $m = m(a)$ is central. Moreover the set $\mathcal{D}$ contains central configurations of the three Types A, B and C.*

(b) *For each regular polyhedron P_{n_1} different from the dodecahedron and the icosahedron we can find two values $\mu_1 < \mu_2$ such that*

- (b.1) *If $m \in (0, \mu_1)$, then there are three central configurations, two of Type A and one of Type B.*
- (b.2) *If $m = \mu_1$, then there are two central configurations, one of Type A and one of Type B.*
- (b.3) *If $m = (\mu_1, \mu_2)$, then there is a unique central configurations of Type B.*
- (b.4) *If $m = \mu_2$, then there are two central configurations one of Type B and one of Type C.*
- (b.5) *If $m = (\mu_2, +\infty)$, then there are three central configurations, one of Type B and two of Type C.*

(c) *For the dodecahedron and the icosahedron we can find three values $\mu_1 < \mu_2 < \mu_3$ such that*

- (c.1) *If $m \in (0, \mu_1)$, then there are three central configurations, one of Type A and two of Type B.*
- (c.2) *If $m = \mu_1$, then there are two central configurations, one of Type A and one of Type B.*
- (c.3) *If $m = (\mu_1, \mu_2)$, then there is a unique central configurations of Type A.*
- (c.4) *If $m = \mu_2$, then there are two central configurations one of Type A and one of Type C.*
- (c.5) *If $m = (\mu_2, \mu_3]$, then there are three central configurations, one of Type A and two of Type C.*
- (c.6) *If $m = (\mu_3, +\infty)$, then there are three central configurations, one of Type A, one of Type B and one of Type C.*

More information on the central configurations described in Theorem 2 is provided in Theorems 4 and 6 of Section 2 for the cube and the octahedron, in Theorems 8 and 10 of Section 3 for the dodecahedron and the icosahedron, and in Theorems 11 and 12 of Section 4 for the tetrahedron.

2. Central configurations formed by a cube and an octahedron

We consider 8 equal masses $m_1 = \cdots = m_8 = 1$ at the vertices of a cube with positions

$$\begin{array}{lll} \mathbf{q}_1 = (1,1,1), & \mathbf{q}_2 = (-1,1,1), & \mathbf{q}_3 = (1,-1,1), \\ \mathbf{q}_4 = (1,1,-1), & \mathbf{q}_5 = (-1,-1,1), & \mathbf{q}_6 = (-1,1,-1), \\ \mathbf{q}_7 = (1,-1,-1), & \mathbf{q}_8 = (-1,-1,-1). & \end{array}$$

We consider 6 additional equal masses $m_9 = \cdots = m_{14} = m$ at the vertices of a regular octahedron with positions

$$\begin{array}{lll} \mathbf{q}_9 = (a,0,0), & \mathbf{q}_{10} = (-a,0,0), & \mathbf{q}_{11} = (0,a,0) \\ \mathbf{q}_{12} = (0,-a,0), & \mathbf{q}_{13} = (0,0,a), & \mathbf{q}_{14} = (0,0,-a), \end{array}$$

for some $a > 0$.

It is easy to check that the center of mass of these two polyhedra is located at the origin of coordinates. After some computations we can see that for this kind of configuration equations (1) become

$$\begin{aligned} & m\left(\frac{a+3}{R_2^{3/2}} - \frac{a-3}{R_1^{3/2}}\right) + \frac{2\sqrt{3}+9\sqrt{2}+18}{72} - \lambda = 0, \\ & m\left(\frac{\sqrt{2}}{a^2} + \frac{1}{4a^2}\right) + \frac{4(a-1)}{R_1^{3/2}} + \frac{4(a+1)}{R_2^{3/2}} - a\,\lambda = 0, \end{aligned} \tag{2}$$

where $R_1 = a^2 - 2a + 3$ and $R_2 = a^2 + 2a + 3$.

We compute λ from the first equation of (2) and we substitute it into the second equation. The resulting equation can be written as $m\,f(a) + g(a) = 0$ where

$$\begin{aligned} f(a) &= \frac{\sqrt{2}}{a^2} + \frac{1}{4a^2} - \frac{a^2+3\,a}{R_2^{3/2}} + \frac{a^2-3\,a}{R_1^{3/2}}, \\ g(a) &= -a\left(\frac{2\sqrt{3}+9\sqrt{2}+18}{72}\right) + \frac{4(a-1)}{R_1^{3/2}} + \frac{4(a+1)}{R_2^{3/2}}. \end{aligned}$$

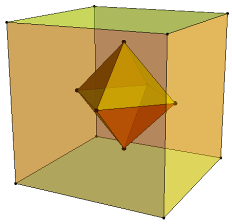
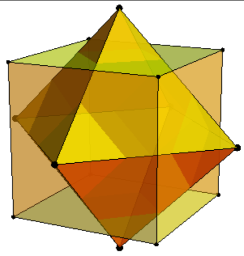
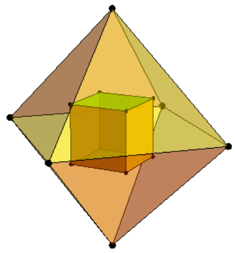
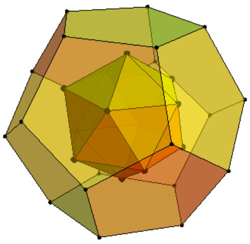
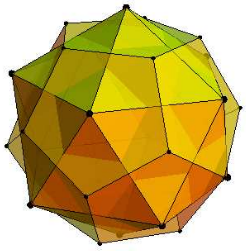
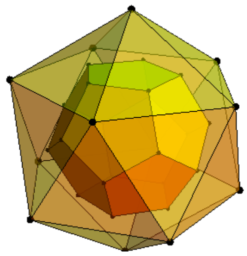
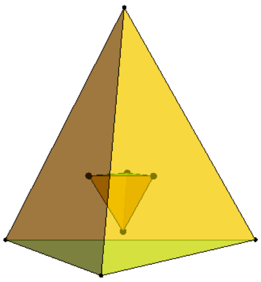
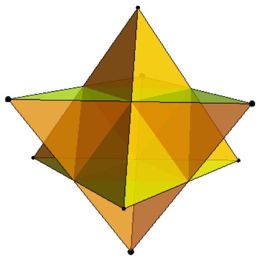
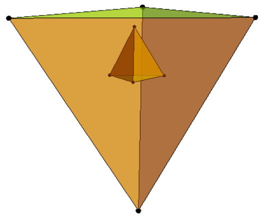
n_1	Type A $P^d_{n_1}$ inside P_{n_1}	Type B the faces of $P^d_{n_1}$ and P_{n_1} intersect	Type C $P^d_{n_1}$ outside P_{n_1}
8	 $a \in (0,1)$	 $a \in [1,3]$	 $a \in (3,+\infty)$
20	 $a \in (0,1)$	 $a \in [1,15-6\sqrt{5}]$	 $a \in (15-6\sqrt{5},+\infty)$
4	 $a \in (0,1)$	 $a \in [1,9]$	 $a \in (9,+\infty)$

Figure 1: Plots of central configurations formed by two dual regular polyhedra.

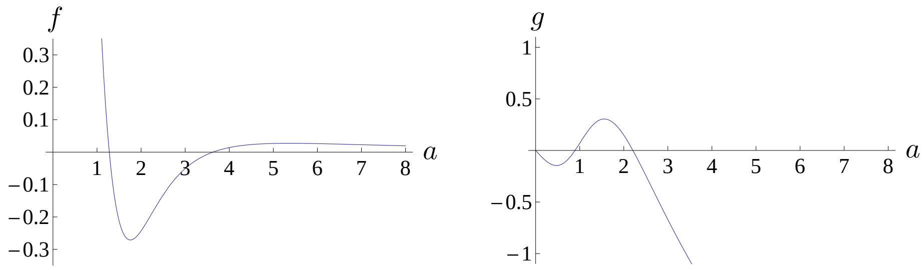


Figure 2: The graphs of f and g

Therefore

$$m = m(a) = -\frac{g(a)}{f(a)}. \tag{3}$$

The solution of (2) given by (3) provides a central configuration if and only if $m > 0$.

Next we analyze the sign of the functions f and g for $a > 0$.

Lemma 3. *The functions f and g satisfy the following properties for $a > 0$ (see the graphs of f and g in Figure 2).*

(a) *The functions f and g are defined for all $a > 0$.*

(b) *Let $\alpha_1 = 1.278175\ldots$ and $\alpha_2 = 3.628586\ldots$. Then $f(\alpha_1) = 0$, $f(\alpha_2) = 0$, $f(a) > 0$ when $a \in (0, \alpha_1) \cup (\alpha_2, +\infty)$ and $f(a) < 0$ when $a \in (\alpha_1, \alpha_2)$.*

(c) *Let $\beta_1 = 0.8932884\ldots$ and $\beta_2 = 2.2083166\ldots$. Then $g(\beta_1) = 0$, $g(\beta_2) = 0$, $g(a) > 0$ when $a \in (\beta_1, \beta_2)$ and $g(a) < 0$ when $a \in (0, \beta_1) \cup (\beta_2, +\infty)$.*

Proof. It is easy to check that equations $R_1 = 0$ and $R_2 = 0$ have no real solutions, so $R_1 > 0$ and $R_2 > 0$ for all $a \in \mathbb{R}$. Therefore f and g are defined for all $a > 0$. This proves statement (a).

By solving equation $f(a) = 0$ (see Appendix A) we get two real solutions with $a > 0$, $a = \alpha_1$ and $a = \alpha_2$. Then by analyzing the sign of f for $a > 0$ we get statement (b).

Finally we solve equation $g(a) = 0$ and we get also two real roots with $a > 0$, $a = \beta_1$ and $a = \beta_2$ (see Appendix B). By analyzing the sign of g for $a > 0$ we get statement (c).

¿From Lemma 3 we see that $m(a) > 0$ when $a \in (0,\beta_1) \cup (\alpha_1,\beta_2) \cup (\alpha_2,+\infty)$. In short we have proved the following result.

Theorem 4. *For each $a \in (0,\beta_1)\cup(\alpha_1,\beta_2)\cup(\alpha_2,+\infty)$ we can find a unique value of $m = m(a) = -g(a)/f(a)$ such that the configuration formed by the cube nested with an octahedron is central. The central configuration is of Type A when $a \in (0,\beta_1)$, it is of Type B when $a \in (\alpha_1,\beta_2)$, and it is of Type C when $a \in (\alpha_2,+\infty)$ (see Figure 1).*

We note that Theorem 4 corresponds to statement (a) of Theorem 2 for the cube and its dual polyhedron, the octahedron. If in Theorem 4 we replace m by $1/m$ and a by $1/a$, then we obtain statement (a) of Theorem 2 for the octahedron and its dual polyhedron, the cube.

Now we are interested in the number of central configurations for each value of $m > 0$.

Lemma 5. *The function $m(a) = -g(a)/f(a)$ when $a > 0$ satisfies the following properties (see Figure 3).*

(a) $m(\beta_1) = m(\beta_2) = 0$,

(b) $\lim_{a\to 0^+} m(a) = 0$,

(c) $\lim_{a\to\alpha_1^+} m(a) = +\infty$ *and* $\lim_{a\to\alpha_2^+} m(a) = +\infty$,

(d) Let $\gamma_1 = 0.7049775\ldots$ and $\gamma_2 = 4.645077\ldots$. Then $a = \gamma_1$ is a local maximum of m with $m(\gamma_1) = \mu_1 = 0.03840360\ldots$, $a = \gamma_2$ is a local minimum with $m(\gamma_2) = \mu_2 = 74.48633\ldots$, m is increasing in $(0,\gamma_1)\cup(\gamma_2,+\infty)$, and it is decreasing in $(\gamma_1,\alpha_1)\cup(\alpha_1,\alpha_2)\cup(\alpha_2,\gamma_2)$.

Proof. Statement (a) follows from Lemma 3(c), and statements (b) and (c) follow immediately from the computation of the corresponding limits.

The derivative of m becomes $m'(a) = -\widetilde{g}(a)/f(a)^2$ where

$$\begin{aligned}\widetilde{g}(a) \quad = \quad & \frac{32a\left(a^2+3\right)}{R_1^2R_2^2} + \frac{90+81\sqrt{2}+2\sqrt{3}+8\sqrt{6}}{96a^2} - \\ & \frac{32a\left(a^4-10a^2+9\right)}{R_1^{5/2}R_2^{5/2}} + \frac{\left(18+9\sqrt{2}+2\sqrt{3}\right)a^5+72\left(1+4\sqrt{2}\right)}{36a^3} \\ & \left(\frac{\left(a^2-5a+3\right)}{R_1^{5/2}} - \frac{\left(a^2+5a+3\right)}{R_2^{5/2}}\right).\end{aligned}$$

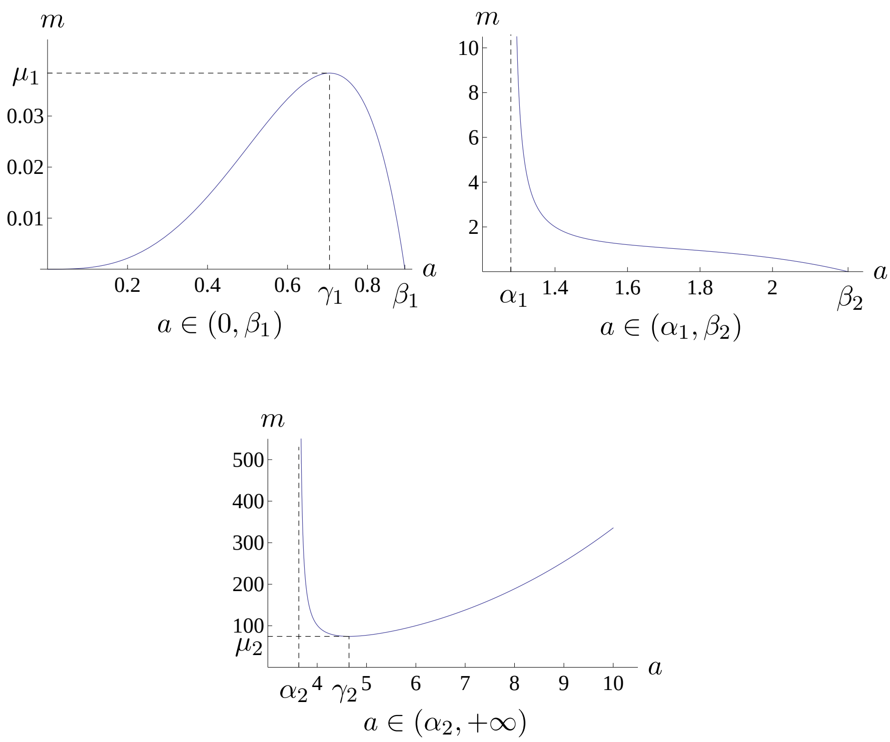


Figure 3: The graphic of the function $m = m(a) = -g(a)/f(a)$.

By proceeding in a similar way than in Appendix A and Appendix B we can transform equation $\widetilde{g}(a)$ into a polynomial equation of degree 44. By solving numerically this polynomial equation we get the following real solutions satisfying equation $\widetilde{g}(a) = 0$

$$a = -1.704481\ldots, \qquad a = -0.6412853\ldots,$$
$$a = \gamma_1 = 0.7049775\ldots, \qquad a = \gamma_2 = 4.645077\ldots.$$

Then by analyzing the sign of $\widetilde{g}(a)$ for $a > 0$ (remember that we only are interested in positive values of a) we get that $\widetilde{g}(a) > 0$ when $a \in (0, \gamma_1) \cup (\gamma_2, +\infty)$ and $\widetilde{g}(a) < 0$ when $a \in (\gamma_1, \alpha_1) \cup (\alpha_1, \alpha_2) \cup (\alpha_2, \gamma_2)$. This proves statement (d).

¿From Lemma 5 and Figure 3 we have that m increases from 0 to μ_1 when $a \in (0, \gamma_1)$, it decreases from μ_1 to 0 when $a \in (\gamma_1, \beta_1)$, it decreases form $+\infty$ to 0 when $a \in (\alpha_1, \beta_2)$, it decreases from $+\infty$ to μ_2 when $a \in (\alpha_2, \gamma_2)$. Finally m increases from μ_2 to $+\infty$ when $a \in (\gamma_2, +\infty)$. In short we have proved the following result.

Theorem 6. *Let α_1, α_2, β_1 and β_2 be defined as in Lemma 3, let μ_1 and μ_2 be defined as in Lemma 5(d), and let $D_1 = (0, \gamma_1)$, $D_2 = (\gamma_1, \beta_1)$, $D_3 = (\alpha_1, \beta_2)$, $D_4 = (\alpha_2, \gamma_2)$ and $D_5 = (\gamma_2, +\infty)$.*

(a) For each value of $m \in (0, \mu_1)$ there are three different central configurations, one with $a \in D_1$ (Type A), one with $a \in D_2$ (Type A) and one with $a \in D_3$ (Type B).

(b) For $m = \mu_1$ there are two different central configurations, one with $a = \gamma_1$ (Type A), and other with $a \in D_3$ (Type B).

(c) For $m \in (\mu_1, \mu_2)$ there is a unique central configuration with $a \in D_3 (Type B)$.

(c) For $m = \mu_2$ there are two central configurations, one with $a \in D_3$ (Type B) and other with $a = \gamma_2$ (Type C).

(d) For $m \in (\mu_2, +\infty)$ there are three different central configurations, one with $a \in D_3$ (Type B), one with $a \in D_4$ (Type C), and one with $a \in D_5$ (Type C).

We note that Theorem 6 corresponds to statement (b) of Theorem 2 for the cube and its dual polyhedron, the octahedron. As above if in Theorem 6 we replace m by $1/m$ and a by $1/a$, then we obtain statement (b) of Theorem 2 for the octahedron and its dual polyhedron, the cube.

3. Central configurations formed by a dodecahedron and an icosahedron

We consider the dodecahedron with vertices

$\mathbf{q}_1 = (1,1,1)$, $\mathbf{q}_2 = (-1,1,1)$, $\mathbf{q}_3 = (1,-1,1)$,
$\mathbf{q}_4 = (1,1,-1)$, $\mathbf{q}_5 = (-1,-1,1)$, $\mathbf{q}_6 = (-1,1,-1)$,
$\mathbf{q}_7 = (1,-1,-1)$, $\mathbf{q}_8 = (-1,-1,-1)$, $\mathbf{q}_9 = (0,1/\phi,\phi)$,
$\mathbf{q}_{10} = (0,-1/\phi,\phi)$ $\mathbf{q}_{11} = (0,1/\phi,-\phi)$, $\mathbf{q}_{12} = (0,-1/\phi,-\phi)$,
$\mathbf{q}_{13} = (1/\phi,\phi,0)$, $\mathbf{q}_{14} = (-1/\phi,\phi,0)$, $\mathbf{q}_{15} = (1/\phi,-\phi,0)$,
$\mathbf{q}_{16} = (-1/\phi,-\phi,0)$, $\mathbf{q}_{17} = (\phi,0,\ 1/\phi)$, $\mathbf{q}_{18} = (-\phi,0,1/\phi)$,
$\mathbf{q}_{19} = (\phi,0,-1/\phi)$, $\mathbf{q}_{20} = (-\phi,0,-1/\phi)$,

where $\phi = (1+\sqrt{5})/2$ is the golden ratio. Now we construct its dual icosahedron. The dodecahedron is composed by 12 regular pentagonal faces. Let $F\{i,j,k,\ell,m\}$ denote the face of the dodecahedron corresponding two the pentagon with vertices i,j,k,ℓ,m and let K_i be the barycenter of this pentagon. The 12 faces of the dodecahedron and its corresponding barycenters are given in Table 1.

We consider 20 equal masses $m_1 = \cdots = m_{20} = 1$ at the vertices of the dodecahedron with vertices $\mathbf{q}_i$ for $i = 1,\ldots,20$, and we consider 12 additional equal masses $m_{21} = \cdots = m_{32} = m$ at the positions $\mathbf{q}_i = a\,K_{i-20}$ for $i = 21,\ldots,32$ and for some $a > 0$.

It is easy to check that the center of mass of these two polyhedra is located at the origin of coordinates. After some computations we can see that the equations of the central configurations (1) for the configuration formed by the $\mathbf{q}_i$ with $i = 1,\ldots,32$, become

$$
\begin{aligned}
& m\left[\sqrt{5-2\sqrt{5}}\left(\frac{3\left(5-2\sqrt{5}\right)-a}{R_1^{3/2}}+\frac{a+3\left(5-2\sqrt{5}\right)}{R_2^{3/2}}\right)+\right.\\
&\qquad\left.\sqrt{85-38\sqrt{5}}\left(\frac{3\sqrt{5}-a}{R_3^{3/2}}+\frac{a+3\sqrt{5}}{R_4^{3/2}}\right)\right]+\\
&\qquad\frac{1}{36}\left(18+9\sqrt{2}+\sqrt{3}+9\sqrt{5}\right)-\lambda=0,\\
& m\left[\frac{\sqrt{2\left(65-29\sqrt{5}\right)}+5\left(3\sqrt{5}-5\right)}{8a^2}\right]+5\sqrt{\frac{1}{2}\left(65-29\sqrt{5}\right)}\\
&\qquad\left(\frac{a-1}{R_1^{3/2}}+\frac{a+1}{R_2^{3/2}}+\frac{a-\sqrt{5}+2}{R_3^{3/2}}+\frac{a+\sqrt{5}-2}{R_4^{3/2}}\right)-\\
&\qquad\frac{1}{10}\left(5+\sqrt{5}\right)a\lambda=0,
\end{aligned}
\tag{4}
$$

Face	Barycenter
$F\{1,2,9,13,14\}$	$K_1 = \left(0, \frac{2\phi^2+2\phi+1}{5\phi}, \frac{\phi+2}{5}\right)$
$F\{1,3,9,10,17\}$	$K_2 = \left(\frac{\phi+2}{5}, 0, \frac{2\phi^2+2\phi+1}{5\phi}\right)$
$F\{1,4,13,17,19\}$	$K_3 = \left(\frac{2\phi^2+2\phi+1}{5\phi}, \frac{\phi+2}{5}, 0\right)$
$F\{2,5,9,10,18\}$	$K_4 = \left(\frac{1}{5}(-\phi-2), 0, \frac{2\phi^2+2\phi+1}{5\phi}\right)$
$F\{2,6,14,18,20\}$	$K_5 = \left(-\frac{2\phi^2+2\phi+1}{5\phi}, \frac{\phi+2}{5}, 0\right)$
$F\{3,5,10,15,16\}$	$K_6 = \left(0, -\frac{2\phi^2+2\phi+1}{5\phi}, \frac{\phi+2}{5}\right)$
$F\{3,7,15,17,19\}$	$K_7 = \left(\frac{2\phi^2+2\phi+1}{5\phi}, \frac{1}{5}(-\phi-2), 0\right)$
$F\{4,6,11,13,14\}$	$K_8 = \left(0, \frac{2\phi^2+2\phi+1}{5\phi}, \frac{1}{5}(-\phi-2)\right)$
$F\{4,7,11,12,19\}$	$K_9 = \left(\frac{\phi+2}{5}, 0, -\frac{2\phi^2+2\phi+1}{5\phi}\right)$
$F\{5,8,16,18,20\}$	$K_{10} = \left(-\frac{2\phi^2+2\phi+1}{5\phi}, \frac{1}{5}(-\phi-2), 0\right)$
$F\{6,8,11,12,20\}$	$K_{11} = \left(\frac{1}{5}(-\phi-2), 0, -\frac{2\phi^2+2\phi+1}{5\phi}\right)$
$F\{7,8,12,15,16\}$	$K_{12} = \left(0, -\frac{2\phi^2+2\phi+1}{5\phi}, \frac{1}{5}(-\phi-2)\right)$

Table 1: The 12 faces of the dodecahedron and its corresponding barycenters.

where $R_1 = a^2 - 2a - 6\sqrt{5} + 15$, $R_2 = a^2 + 2a - 6\sqrt{5} + 15$, $R_3 = a^2 + 2\left(2-\sqrt{5}\right)a - 6\sqrt{5} + 15$, and $R_4 = a^2 - 2\left(2-\sqrt{5}\right)a - 6\sqrt{5} + 15$.

We compute λ from the first equation of (4) and we substitute it into the second equation. The resulting equation can be written as $m\,f(a)+g(a)=0$ where

$$
\begin{aligned}
f(a) \;=\;& \frac{\sqrt{2\left(65-29\sqrt{5}\right)}+5\left(3\sqrt{5}-5\right)}{8a^2} - \sqrt{\frac{5-\sqrt{5}}{10}} \\
& \left(\frac{\left(3\left(5-2\sqrt{5}\right)-a\right)a}{R_1^{3/2}} + \frac{\left(a+3\left(5-2\sqrt{5}\right)\right)a}{R_2^{3/2}}\right) - \\
& \sqrt{\frac{65-29\sqrt{5}}{10}}\left(\frac{\left(3\sqrt{5}-a\right)a}{R_3^{3/2}} + \frac{\left(a+3\sqrt{5}\right)a}{R_4^{3/2}}\right), \\
g(a) \;=\;& -\frac{1}{360}\left(5+\sqrt{5}\right)\left(18+9\sqrt{2}+\sqrt{3}+9\sqrt{5}\right)a + \\
& 5\sqrt{\frac{65-29\sqrt{5}}{2}}\left(\frac{a-1}{R_1^{3/2}} + \frac{a+1}{R_2^{3/2}} + \frac{a-\sqrt{5}+2}{R_3^{3/2}} + \frac{a+\sqrt{5}-2}{R_4^{3/2}}\right).
\end{aligned}
$$

Therefore

$$
m = m(a) = -\frac{g(a)}{f(a)}. \tag{5}
$$

The solution of (4) given by (5) provides a central configuration if and only if $m > 0$.

Next we analyze the sign of the functions f and g for $a > 0$.

Lemma 7. *The functions f and g satisfy the following properties for $a > 0$ (see the graphs of f and g in Figure 4).*

(a) The functions f and g are defined for all $a > 0$.

(b) Let $\alpha_1 = 0.9515686\ldots$ and $\alpha_2 = 1.582290\ldots$. Then $f(\alpha_1) = 0$, $f(\alpha_2) = 0$, $f(a) > 0$ when $a \in (0,\alpha_1) \cup (\alpha_2,+\infty)$ and $f(a) < 0$ when $a \in (\alpha_1,\alpha_2)$.

(c) Let $\beta_1 = 1.062451\ldots$ and $\beta_2 = 1.560908\ldots$. Then $g(\beta_1) = 0$, $g(\beta_2) = 0$, $g(a) > 0$ when $a \in (\beta_1,\beta_2)$ and $g(a) < 0$ when $a \in (0,\beta_1)\cup(\beta_2,+\infty)$.

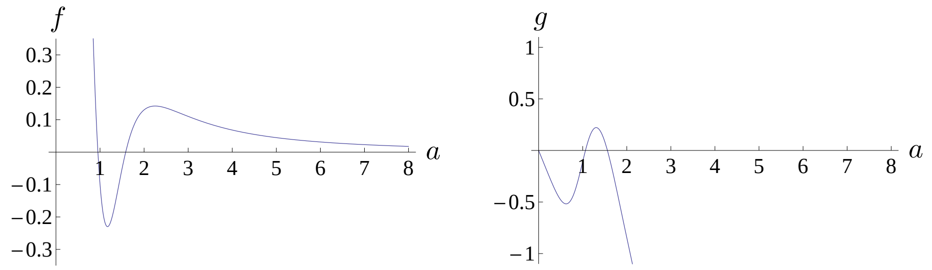


Figure 4: The graphs of f and g

Proof. Equations $R_1 = 0$, $R_2 = 0$, $R_3 = 0$ and $R_4 = 0$ have no real solutions, so $R_1 > 0$, $R_2 > 0$, $R_3 > 0$ and $R_4 > 0$ for all $a > 0$. Therefore f and g are defined for all $a > 0$. This proves statement (a).

Using the notion of resultants we find all the real solutions of equation $f(a) = 0$ with $a > 0$ with the help of an algebraic manipulator as Mathematica. These reals solutions are $a = \alpha_1$ and $a = \alpha_2$ (see Appendix C for details). To complete the prove of statement (b) we analyze the sign of f on the intervals $(0, \alpha_1)$, (α_1, α_2), and $(\alpha_2, +\infty)$.

By doing the same for the function g we get statement (c) (see also Appendix C).

¿From Lemma 7 we see that $m(a) > 0$ when $a \in (0, \alpha_1) \cup (\beta_1, \beta_2) \cup (\alpha_2, +\infty)$. In short we have proved the following result.

Theorem 8. *For each $a \in (0, \alpha_1) \cup (\beta_1, \beta_2) \cup (\alpha_2, +\infty)$ we can find a unique value of $m = m(a) = -g(a)/f(a)$ such that the configuration formed by the dodecahedron nested with an icosahedron is central. The central configuration is of Type A when $a \in (0, \alpha_1)$, it is of Type B when $a \in (\beta_1, \beta_2)$ and when $a \in (\alpha_2, 15 - 6\sqrt{5})$, and it is of Type C when $a \in (15 - 6\sqrt{5}, +\infty)$ (see Figure 1).*

We note that Theorem 8 corresponds to statement (a) of Theorem 2 for the dodecahedron and its dual polyhedron, the icosahedron. If in Theorem 8 we replace m by $1/m$ and a by $1/a$, then we obtain statement (a) of Theorem 2 for the icosahedron and its dual polyhedron, the dodecahedron.

Now we are interested in the number of central configurations for each value of $m > 0$.

Lemma 9. *The function $m(a) = -g(a)/f(a)$ when $a > 0$ satisfies the following properties (see Figure 5).*

(a) $m(\beta_1) = m(\beta_2) = 0$,

(b) $\lim_{a\to 0^+} m(a) = 0$,

(c) $\lim_{a\to\alpha_1^-} m(a) = +\infty$ *and* $\lim_{a\to\alpha_2^+} m(a) = +\infty$,

(d) Let $\gamma_1 = 1.478858\ldots$ *and* $\gamma_2 = 1.674216\ldots$. *Then* $a = \gamma_1$ *is a local maximum of* m *with* $m(\gamma_1) = \mu_1 = 1.735523\ldots$, $a = \gamma_2$ *is a local minimum with* $m(\gamma_2) = \mu_2 = 4.193060\ldots$, m *increases in* $a \in (0, \alpha_1) \cup (\beta_1, \gamma_1) \cup (\gamma_2, +\infty)$ *and decreases when* $a \in (\gamma_1, \beta_2) \cup (\alpha_2, \gamma_2)$.

Proof. Statement (a) of Lemma 9 follows from Lemma 7(c), and statements (b) and (c) follow immediately from the computation of the corresponding limits.

The derivative of m is $m'(a) = -\widetilde{g}(a)/f(a)^2$ where $\widetilde{g}(a) = g'(a)\,f(a) - g(a)\,f'(a)$. Working in a similar way as in Appendix C we solve equation $\widetilde{g}(a) = 0$, and we get two real solutions with $a > 0$, $a = \gamma_1$ and $a = \gamma_2$. Then by analyzing the sign of $\widetilde{g}(a)$ for $a > 0$ we get that $\widetilde{g}(a) > 0$ when $a \in (\gamma_1, \beta_2)\cup(\alpha_2, \gamma_2)$ and that $\widetilde{g}(a) < 0$ when $a \in (0, \alpha_1)\cup(\beta_1, \gamma_1)\cup(\gamma_2, +\infty)$. This proves statement (d).

¿From Lemma 9 and Figure 5 we have that m increases from 0 to $+\infty$ when $a \in (0, \alpha_1)$, it increases from 0 to μ_1 when $a \in (\beta_1, \gamma_1)$, it decreases form μ_1 to 0 when $a \in (\gamma_1, \beta_2)$, it decreases from $+\infty$ to μ_2 when $a \in (\alpha_2, \gamma_2)$. Finally m increases from μ_2 to $+\infty$ when $a \in (\gamma_2, +\infty)$. In short we have proved the following result.

Theorem 10. *Let* α_1, α_2, β_1 *and* β_2 *be defined as in Lemma 7, let* $\xi = 15 - 6\sqrt{5}$, *let* μ_1 *and* μ_2 *be defined as in Lemma 9(d), let* $\mu_3 = 48.84223\ldots$ *and let* $D_1 = (0, \alpha_1)$, $D_2 = (\beta_1, \gamma_1)$, $D_3 = (\gamma_1, \beta_2)$, $D_4 = (\alpha_2, \xi]$, $D_5 = (\xi, \gamma_2)$ *and* $D_6 = (\gamma_2, +\infty)$.

(a) For each value of $m \in (0, \mu_1)$ *there are three different central configurations, one with* $a \in D_1$ *(Type A), one with* $a \in D_2$ *(Type B) and one with* $a \in D_3$ *(Type B).*

(b) For $m = \mu_1$ *there are two different central configurations, one with* $a = \gamma_1$ *(Type B), and other with* $a \in D_1$ *(Type A).*

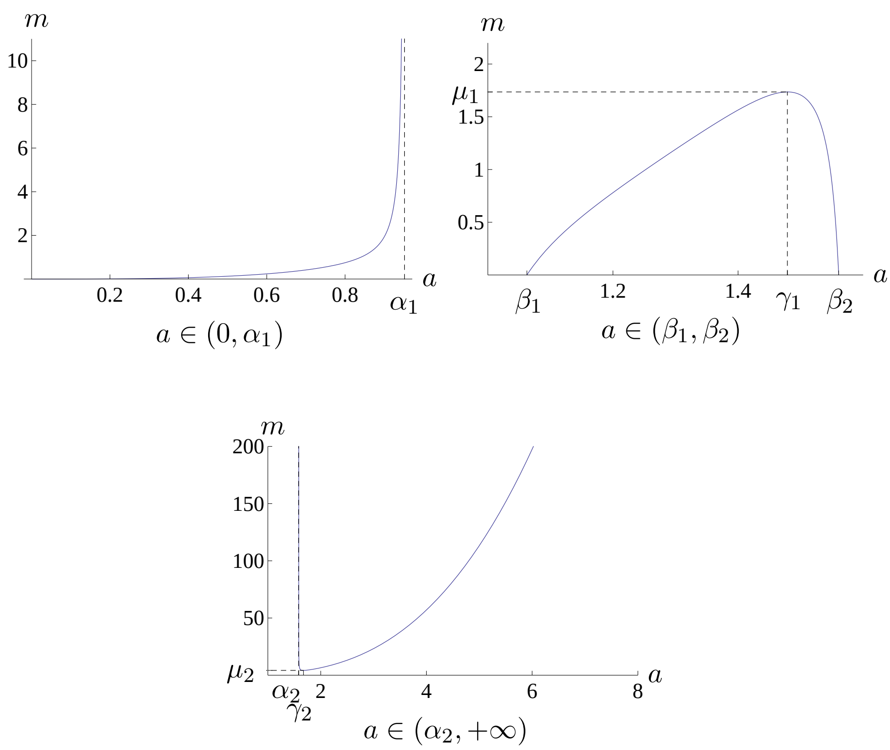


Figure 5: The graphic of the function $m = m(a) = -g(a)/f(a)$.

(c) For $m \in (\mu_1, \mu_2)$ there is a unique central configuration with $a \in D_1$ (Type A).

(c) For $m = \mu_2$ there are two central configurations, one with $a \in D_1$ (Type A) and other with $a = \gamma_2$ (Type C).

(d) For $m \in (\mu_2, \mu_3]$ there are three different central configurations, one with $a \in D_1$ (Type A), one with $a \in D_5$ (Type C), and one with $a \in D_6$ (Type C).

(e) For $m \in (\mu_3, +\infty)$ there are three different central configurations, one with $a \in D_1$ (Type A), one with $a \in D_4$ (Type B), and one with $a \in D_6$ (Type C).

We note that Theorem 10 corresponds to statement (c) of Theorem 2 for the dodecahedron and its dual polyhedron, the icosahedron. If in Theorem 10 we replace m by $1/m$ and a by $1/a$, then we obtain statement (c) of Theorem 2 for the icosahedron and its dual polyhedron, the dodecahedron.

4. Central configurations formed by two dual tetrahedra

We consider the regular tetrahedron with vertices

$$\mathbf{q}_1 = (0, 0, \sqrt{3/2}), \qquad \mathbf{q}_2 = (0, 2/\sqrt{3}, -1/\sqrt{6}),$$
$$\mathbf{q}_3 = (1, -1/\sqrt{3}, -1/\sqrt{6}), \qquad \mathbf{q}_4 = (-1, -1/\sqrt{3}, -1/\sqrt{6}).$$

In order to construct its dual tetrahedron we compute the barycenter of each one of its triangular faces. As in the previous section, we denote by $F\{i, j, k\}$ the face of the tetrahedron corresponding to the triangle with vertices i, j, k and we denote by K_i its barycenter. The faces of the tetrahedron and its corresponding barycenters are given in Table 2.

We consider 4 equal masses $m_1 = \cdots = m_4 = 1$ at the vertices of the tetrahedron with vertices $\mathbf{q}_i$ for $i = 1, \ldots, 4$ and we consider 4 additional equal masses $m_5 = \cdots = m_8 = m$ at the positions $\mathbf{q}_i = a\, K_{i-4}$ for $i = 5, \ldots, 8$ and for some $a > 0$. We note that the barycenters K_1, K_2, K_3 and K_4 are the vertices of a new regular tetrahedron which corresponds to the initial one scaled by a factor 1/3 and rotated with the rotation matrix

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Face	Barycenter
$F\{1,2,3\}$	$K_1 = \left(\frac{1}{3}, \frac{1}{3\sqrt{3}}, \frac{1}{3\sqrt{6}}\right)$
$F\{1,2,4\}$	$K_2 = \left(-\frac{1}{3}, \frac{1}{3\sqrt{3}}, \frac{1}{3\sqrt{6}}\right)$
$F\{1,3,4\}$	$K_3 = \left(0, -\frac{2}{3\sqrt{3}}, \frac{1}{3\sqrt{6}}\right)$
$F\{2,3,4\}$	$K_4 = \left(0, 0, -\frac{1}{\sqrt{6}}\right)$

Table 2: The faces of the tetrahedron and its corresponding barycenters.

Therefore this configuration corresponds to the configuration of Type II studied in Theorem 2 of [4], and studied also in [16].

By writing the results of [4] and [16] but using the notation of this paper we get the following two theorems.

Theorem 11. *Let be*

$$\begin{aligned} f(a) &= \frac{2\sqrt{\frac{2}{3}}(a-9)a}{(a^2-2a+9)^{3/2}} + \frac{9}{2a^2} - \frac{2\sqrt{\frac{2}{3}}a}{(a+3)^2}, \\ g(a) &= \frac{6\sqrt{6}(a-1)}{(a^2-2a+9)^{3/2}} - \frac{a}{6} + \frac{2\sqrt{6}}{(a+3)^2}, \end{aligned}$$

and let $\alpha_1 = 2.145669\ldots$ *and* $\alpha_2 = 19.60823\ldots$ *be the zeros of* f *with* $a > 0$, *and* $\beta_1 = 0.4589907\ldots$ *and* $\beta_2 = 4.194495\ldots$ *be the zeros of* g *with* $a > 0$.

For each $a \in (0, \beta_1) \cup (\alpha_1, \beta_2) \cup (\alpha_2, +\infty)$ *we can find a unique value of* $m = m(a) = -g(a)/f(a)$ *such that the configuration formed by the vertices of the two dual tetrahedra is central. The central configuration is of Type A when* $a \in (0, \beta_1)$, *it is of Type B when* $a \in (\alpha_1, \beta_2)$, *and it is of Type C when* $a \in (\alpha_2, +\infty)$ *(see Figure 1).*

We note that Theorem 11 corresponds to statement (a) of Theorem 2 for two dual tetrahedra.

Theorem 12. *Let* $\gamma_1 = 0.3418567\ldots$ *and* $\gamma_2 = 26.32681\ldots$ *be the zeros of* $-(g(a)/f(a))'$ *with* $a > 0$, *let be* $m(\gamma_1) = \mu_1 = 0.0003471823\ldots$ *and* $m(\gamma_2) = \mu_2 = 2880.330\ldots$, *and let be* $D_1 = (0, \gamma_1)$, $D_2 = (\gamma_1, \beta_1)$, $D_3 = (\alpha_1, \beta_2)$, $D_4 = (\alpha_2, \gamma_2)$ *and* $D_5 = (\gamma_2, +\infty)$.

(a) *For each value of* $m \in (0, \mu_1)$ *there are three different central configurations, one with* $a \in D_1$ *(Type A), one with* $a \in D_2$ *(Type A) and one with* $a \in D_3$ *(Type B).*

(b) *For* $m = \mu_1$ *there are two different central configurations, one with* $a = \gamma_1$ *(Type A), and other with* $a \in D_3$ *(Type B).*

(c) *For* $m \in (\mu_1, \mu_2)$ *there is a unique central configuration with* $a \in D_3$ *(Type B).*

(c) *For* $m = \mu_2$ *there are two central configurations, one with* $a \in D_3$ *(Type B) and other with* $a = \gamma_2$ *(Type C).*

(d) *For* $m \in (\mu_2, +\infty)$ *there are three different central configurations, one with* $a \in D_3$ *(Type B), one with* $a \in D_4$ *(Type C), and one with* $a \in D_5$ *(Type C).*

We note that Theorem 12 corresponds to statement (b) of Theorem 2 for two dual tetrahedra.

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Appendix A.

In this appendix we solve the equation $f(a) = 0$ defined in Section 2. This equation can be written as

$$\frac{\sqrt{2}}{a^2} + \frac{1}{4a^2} + \frac{a^2 - 3\,a}{R_1^{3/2}} = \frac{a^2 + 3\,a}{R_2^{3/2}}.$$

By squaring booth sides of this equation and isolating the terms containing $R_1^{3/2}$ we get

$$\frac{1}{\sqrt{2}\,a^4} + \frac{33}{16\,a^4} + \frac{a^4 - 6a^3 + 9a^2}{R_1^3} - \frac{a^4 + 6a^3 + 9a^2}{R_2^3} = -\frac{\left(1 + 4\sqrt{2}\right)(a - 3)}{2\,a\,R_1^{3/2}}.$$

We square again both sides of this equation and we drop the denominators by multiplying both sides of the resulting equation by the least common denominator. Then we get the polynomial equation

$$
\begin{aligned}
&-64\left(33+8\sqrt{2}\right)a^{26}+\left(1217+528\sqrt{2}\right)a^{24}+\\
&\left(836428+10944\sqrt{2}\right)a^{22}+\left(19232\sqrt{2}-4158990\right)a^{20}-\\
&20\left(731837+8400\sqrt{2}\right)a^{18}+\left(33217887-2686992\sqrt{2}\right)a^{16}+\\
&\left(52167192-10097280\sqrt{2}\right)a^{14}-4\left(44891329+8079888\sqrt{2}\right)a^{12}\\
&-216\left(1309799+179376\sqrt{2}\right)a^{10}-243\left(1270667+27312\sqrt{2}\right)a^{8}+\\
&2916\left(84671+61296\sqrt{2}\right)a^{6}+747954\left(1217+528\sqrt{2}\right)a^{4}+\\
&708588\left(1217+528\sqrt{2}\right)a^{2}+531441\left(1217+528\sqrt{2}\right)=0.
\end{aligned}
$$

We find numerically (for instance with the help of Mathematica) all real solutions of this polynomial equation and we get

$$a=\pm1.278175\ldots,\qquad a=\pm2.841837\ldots,\qquad a=\pm3.628586\ldots.$$

Finally we check which of these solutions are also solutions of the equation $f(a)=0$, and we get that the real solutions of $f(a)=0$ are exactly

$$a=1.278175\ldots,\qquad \text{and}\qquad a=3.628586\ldots.$$

Appendix B.

In this appendix we solve the equation $g(a)=0$ defined in Section 2. By proceeding as in Appendix 1 we transform equation $g(a)=0$ into the polynomial equation

$$\left(13369 + 9396\sqrt{2} + 3288\sqrt{3} + 2292\sqrt{6}\right) a^{28} +$$
$$12\left(13369 + 9396\sqrt{2} + 3288\sqrt{3} + 2292\sqrt{6}\right) a^{26} +$$
$$114\left(13369 + 9396\sqrt{2} + 3288\sqrt{3} + 2292\sqrt{6}\right) a^{24} +$$
$$4\left(1192183 + 897804\sqrt{2} + 409512\sqrt{3} + 318156\sqrt{6}\right) a^{22} +$$
$$3\left(-8368051 - 4603068\sqrt{2} + 423096\sqrt{3} + 992388\sqrt{6}\right) a^{20} +$$
$$24\left(-10239223 - 6247692\sqrt{2} - 676776\sqrt{3} + 45876\sqrt{6}\right) a^{18} -$$
$$4\left(371547449 + 233330868\sqrt{2} + 37414104\sqrt{3} + 10910964\sqrt{6}\right) a^{16} -$$
$$216\left(-37480969 + 12039948\sqrt{2} + 1963944\sqrt{3} + 597708\sqrt{6}\right) a^{14} -$$
$$27\left(-1448334005 + 228991644\sqrt{2} + 37873032\sqrt{3} + 11908956\sqrt{6}\right) a^{12} +$$
$$4860\left(-26736799 - 948780\sqrt{2} - 84840\sqrt{3} + 25620\sqrt{6}\right) a^{10} +$$
$$4374\left(-78274029 + 132156\sqrt{2} + 234568\sqrt{3} + 228092\sqrt{6}\right) a^{8} +$$
$$2916\left(212211739 + 4522716\sqrt{2} + 1296648\sqrt{3} + 805788\sqrt{6}\right) a^{6} +$$
$$531441\left(13369 + 9396\sqrt{2} + 3288\sqrt{3} + 2292\sqrt{6}\right) a^{4} -$$
$$1088391168\left(83 + 54\sqrt{2} + 12\sqrt{3} + 6\sqrt{6}\right) a^{2} = 0$$

We find numerically all real solutions of this polynomial equation obtaining

$$a = 0, \qquad a = \pm 0.8932884\ldots,$$
$$a = \pm 1.032168\ldots, \qquad a = \pm 2.208316\ldots.$$

Finally we check which of these solutions are also solutions of the equation $g(a) = 0$, and we get that the real solutions of $g(a) = 0$ are

$$a = 0, \qquad a = \pm 0.8932884\ldots, \qquad a = \pm 2.208316\ldots.$$

Appendix C.

In this appendix we solve equations $f(a) = 0$ and $g(a) = 0$ defined in Section 3. Let $r_1 = R_1^{1/2}$, $r_2 = R_2^{1/2}$, $r_3 = R_3^{1/2}$ and $r_4 = R_4^{1/2}$. Then f can be written as

$$
\begin{aligned}
f(a) \quad = \quad & \frac{\sqrt{2\left(65-29\sqrt{5}\right)+5\left(3\sqrt{5}-5\right)}}{8a^2} - \sqrt{\frac{5-\sqrt{5}}{10}} \\
& \left(\frac{\left(3\left(5-2\sqrt{5}\right)-a\right)a}{r_1^3} + \frac{\left(a+3\left(5-2\sqrt{5}\right)\right)a}{r_2^3}\right) - \\
& \sqrt{\frac{65-29\sqrt{5}}{10}}\left(\frac{\left(3\sqrt{5}-a\right)a}{r_3^3} + \frac{\left(a+3\sqrt{5}\right)a}{r_4^3}\right).
\end{aligned}
$$

First we eliminate the denominators of the fractions which appear in $f(a)$ by multiplying equation $f(a)=0$ by the least common multiple of all the denominators. We obtain a new equation $F_0=0$, which is a polynomial in the variables a, r_1, r_2, r_3 and r_4. We consider four additional polynomials

$$
F_1 = r_1^2 - R_1, \qquad F_2 = r_2^2 - R_2, \qquad F_3 = r_3^2 - R_3, \qquad F_4 = r_4^2 - R_4.
$$

Clearly, the solutions of equation $f(a)=0$ are also solutions of system

$$
F_0 = 0, \qquad F_1 = 0, \qquad F_2 = 0, \qquad F_3 = 0, \qquad F_4 = 0. \tag{C.1}
$$

We solve the system of polynomial equations (C.1) by means of resultants.

The *resultant* of two polynomials $P(x)$ and $Q(x)$ with leading coefficient one, $\mathrm{Res}[P,Q]$, is the expression formed by the product of all the differences $a_i - b_j$ for $i=1,2,\ldots,n$, $j=1,2,\ldots,m$ where a_i with $i=1,2,\ldots,n$ are the roots of P and b_j with $j=1,2,\ldots,m$ are the roots of Q. In order to see how to compute $\mathrm{Res}[P,Q]$, see for instance [8] and [9]. The main property of the resultant is that if P and Q have a common root then necessarily $\mathrm{Res}[P,Q]=0$.

Consider now two multivariable polynomials, say $P(x,y)$ and $Q(x,y)$. These polynomials can be considered as polynomials in x with polynomial coefficients in y. The resultant with respect to x, which is denoted by $\mathrm{Res}[P,Q,x]$, is a polynomial in the variable y satisfying the following property: if (x_0,y_0) is a solution of system $P(x,y)=0$, $Q(x,y)=0$ then $\mathrm{Res}[P,Q,x](y_0)=0$.

In order to solve system (C.1), we eliminate the variable r_1 by doing the resultant $\mathrm{Res}[F_0,F_1,r_1]$. We obtain a new polynomial equation $\widetilde{F}_0=0$ with the variables a, r_2, r_3 and r_4. Next we eliminate the variable r_2 by doing the resultant $\widetilde{F}_1=\mathrm{Res}[\widetilde{F}_0,F_2,r_2]$, we eliminate the variable r_3 by doing the resultant $\widetilde{F}_2=\mathrm{Res}[\widetilde{F}_1,F_3,r_3]$, and finally we eliminate the variable r_4 by doing the resultant $\widetilde{F}_3 = \text{Res}[\widetilde{F}_2, F_4, r_4]$. We obtain in this way a polynomial equation $\widetilde{F}_3 = 0$ of degree 204 in the variable a.

We find numerically all real solutions of this polynomial equation obtaining 18 different solutions of which only two are real solutions of the initial equation $f(a) = 0$ with $a > 0$. These two solutions are

$$a = 0.9515686\ldots, \qquad a = 1.582290\ldots.$$

We repeat the same steps for the equation $g(a) = 0$ obtaining in this case a polynomial equation $\widetilde{F}_3 = C\, a^4\, \widetilde{F} = 0$, where $\widetilde{F}$ is a polynomial of degree 204 in the variable a, and C is a constant.

As above we find numerically all real solutions of this polynomial equation obtaining 19 different solutions of which only two are real solutions of the initial equation $g(a) = 0$ with $a > 0$. These two solutions are

$$a = 1.062451\ldots, \qquad a = 1.560908\ldots.$$

References

[1] F. Cedó and J. Llibre, Symmetric central configurations of the spatial n–body problem, J. of Geometry and Physics 6 (1989) 367–394.

[2] M. Corbera and J. Llibre, Central configurations for the spatial $2n$–body problem of nested regular polyhedra, J. of Geometry and Physics 58 (2008) 1241–1252.

[3] M. Corbera and J. Llibre, Central configurations of three nested regular polyhedra for the spatial $3n$–body problem, J. of Geometry and Physics 59 (2009) 321–339.

[4] M. Corbera and J. Llibre, Central configurations of nested rotated regular tetrahedra, J. of Geometry and Physics 59 (2009) 1379-1394.

[5] A. C. Fernandes and L.F. Mello, On stacked central configurations with n–bodies when one body is removed, J. Math. Anal. Appl. 405 (2013) 320–325.

[6] M. Hampton, Stacked central configurations: new examples in the planar five–body problem, Nonlinearity **18** (2005) 2299–2304.

[7] M. Hampton and M. Santoprete, Seven-body central configurations: a family of central configurations in the spatial seven-body problem, Celestial Mech. Dynam. Astronom. 99 (2007) 293-305.

[8] S. Lang, *Algebra*, 3rd. Edition, Addison–Wesley, 1993.

[9] P. Olver, *Classical invariant theory*, London Math. Soc. Student Texts, Vol. 44, Cambbridge Univ. Press, New York, 1999.

[10] J. Llibre, L.F. Mello, and E. Perez-Chavela, New stacked central configurations for the planar 5–body problem, Celestial Mech. Dynam. Astronom. 110 (2011) 43-52

[11] L.F. Mello and A.C. Fernandes, Stacked Central Configurations for the Spatial Seven-Body Problem, Qual. Theory Dyn. Syst. 12 (2013) 101-114.

[12] L.F. Mello, F.E. Chaves, A.C. Fernandes, B.A. Garcia, Stacked central configurations for the spatial six-body problem, J. Geom. Phys. 59 (2009) 1216-1226.

[13] A. Oliveira and H. Cabral, On stacked central configurations of the planar coorbital satellites problem, Discrete Contin. Dyn. Syst. 32 (2012) 3715-3732.

[14] D. Saari, On the role and properties of central configurations, Cel. Mech. **21** (1980) 9–20.

[15] X. Su and T. An, Twisted stacked central configurations for the spatial seven-body problem, J. Geom. Phys. 70 (2013) 164-171.

[16] X. Liu, T. Feng, and X. Chen, Eight-body central configuration for conjugate-nested regular tetrahedrons, J. of Chongqing University (English Edition) 6 (2007) 294–298.

[17] F. Zhao and J. Chen, Stacked central configurations for Newtonian N+2p–body problems, J. Math. Anal. Appl. 407 (2013) 541-544.