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ON THE NON-INTEGRABILTY OF A HALPHEN SYSTEM

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Abstract. In this paper we study the Halphen system given by

$$\dot{x}_1 = x_2x_3 - x_1(x_2 + x_3) - \alpha_1^2(x_1 - x_2)(x_3 - x_1),$$
$$\dot{x}_2 = x_3x_1 - x_2(x_3 + x_1) - \alpha_2^2(x_2 - x_3)(x_1 - x_2),$$
$$\dot{x}_3 = x_1x_2 - x_3(x_1 + x_2) - \alpha_3^2(x_3 - x_1)(x_2 - x_3),$$

where x_1, x_2, x_3 are real variables and $\alpha_1, \alpha_2, \alpha_3$ are real constants. We show that when $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ then it does not admit any first integral that can be described by formal power series. As a corollary we also obtain the nonexistence of analytic first integrals. Furthermore, we also prove that it does not admit any first integral of Darboux type.

1. Introduction to the problem

In this paper we consider the system

$$\begin{aligned} \dot{x}_1 &= F_1(x_1, x_2, x_3) = x_2x_3 - x_1(x_2 + x_3) - \alpha_1^2(x_1 - x_2)(x_3 - x_1), \\ \dot{x}_2 &= F_2(x_1, x_2, x_3) = x_3x_1 - x_2(x_3 + x_1) - \alpha_2^2(x_2 - x_3)(x_1 - x_2), \\ \dot{x}_3 &= F_3(x_1, x_2, x_3) = x_1x_2 - x_3(x_1 + x_2) - \alpha_3^2(x_3 - x_1)(x_2 - x_3), \end{aligned} \tag{1}$$

where x_1, x_2, x_3 are real variables and $\alpha_1, \alpha_2, \alpha_3$ are real constants. We will call system (1) as the second Halphen system (already called the second system by Halphen himlself in [2]) since when $\alpha_1 = \alpha_2 = \alpha_3 = 0$ system (1) becomes the so-called classical Halphen system. The classical Halphen system is a famous model (see for instance [2, 5, 6]) that first appeared in Darboux's work (see [2]) and was later solved by Halphen in [6]. One of the facts that made it famous is that it was shown that this system is equivalent to the Einstein field equations for a diagonal self-dual Bianchi-IX metric with Euclidean signature (see [1, 5]). Also the classical Halphen system arises in the similarity reductions of associativity equations on a three-dimensional Frobenius manifold (see [3]).

From the point of view of the integrability, the classical Halphen system has been intensively studied using different theories. One of the main results in this direction is that system (1) with $\alpha_1 = \alpha_2 = \alpha_3 = 0$ can be explicitly integrated, since we can express its general solution in terms of elliptic integrals

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(see [6, 12, 13]) but the first integrals are not global and are multi-valued non-algebraic functions (see [14]). Another result that we want to mention is [11] where the authors proved by means of the so-called Darboux polynomials that system (1) with $\alpha_1 = \alpha_2 = \alpha_3 = 0$ does not admit a non-constant algebraic first integral and finally [15] where the author provides a complete characterization of the formal and analytic first integrals.

The question to be settled is the generic behavior of system system (1): it is chaotic or not? It is well known that the existence of complicated behavior of a system forbids its integrability and thus, a first step toward understanding the underlying mechanism of system (1) we will prove that when $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ then system (1) does not admit any non-constant global first integral that is formal or a global analytic first integral. We observe that this is a first step towards proving that this system is chaotic. However, this is not enough to show that the system is chaotic and to do that, we must also prove that the system has a positive Lyapunov exponent, or has homoclinic/heteroclinic transversal connections,...

The first aim of this paper is to show the non-existence of first integrals of system (1) that can be described by formal series. We will restrict to the case in which $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ the case in which $\alpha_1 = \alpha_2 = \alpha_3 = 0$ is completely studied in [15]. In [16] the author working with another generalization of the classical Halphen system was able to prove under some restrictive conditions on the parameters $\alpha_1, \alpha_2, \alpha_3$ the non-existence of Darboux first integrals. In the present paper we will be able to characterize for all values of $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ the non-existence of Darboux first integrals for the generalized Halphen system given in (1).

One of the ideas that we will use for characterizing the existence of formal first integrals is the following. The three planes

$$H_1 := x_1 - x_2 = 0, \quad H_2 := x_1 - x_3 = 0, \quad H_3 := x_2 - x_3 = 0 \tag{2}$$

are invariant by the flow of system (1) and, if $f := f(x_1, x_2, x_3)$ is a first integral of system (1), then for each $i = 1, 2, 3$, the restriction of f to $H_i = 0$ is also a first integral of system (1). Thus the method for proving our results will consist of studying completely the integrability of the reduced systems on each $H_i = 0$ to get the exact information on the integrals of the whole system (1).

A *formal first integral* $f = f(x_1, x_2, x_3)$ of system (1) is a non-constant formal power series in the variables x_1, x_2, x_3 such that

$$\sum_{k=1}^{3} \frac{\partial f}{\partial x_i} F_i(x_1, x_2, x_3) = 0.$$

The main results of this paper are the following ones.

Theorem 1. *For any* $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ *system* (1) *does not admit any formal first integral.*

Here an *analytic first integral* of system (1) is a non-constant analytic function which is constant over the trajectories of system (1). As a corollary of Theorem 1 we obtain the following result.

Theorem 2. *For any* $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ *system* (1) *does not admit any analytic first integral.*

A *rational first integral* $f = f(x_1, x_2, x_3)$ of system (1) is a non-constant rational function that is constant over the solutions of system (1).

Theorem 3. *For any* $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ *system* (1) *does not admit any rational first integral.*

Finally, we study the Darboux first integrals of system (1) (see below for the definition).

Theorem 4. *For any* $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ *system* (1) *does not admit any Darboux first integral.*

The paper is organized as follows. In section 2 we present some auxiliary results needed for proving Theorems 1 and 2. In section 3 we provide the proofs of Theorems 1 and 2. In section 4 we prove Theorem 3 and finally, in section 5 we will show Theorem 4.

2. Auxiliary results

Note that system (1) is invariant under the changes

$$(x_1, x_2, x_3, \alpha_1, \alpha_2, \alpha_3) \to (x_2, x_3, x_1, \alpha_2, \alpha_3, \alpha_1),$$
$$(x_1, x_2, x_3, \alpha_1, \alpha_2, \alpha_3) \to (x_3, x_1, x_2, \alpha_3, \alpha_1, \alpha_2). \tag{3}$$

The first auxiliary result can be proved easily using Newton's binomial formula and its proof has been omitted. For a proof we refer the reader to [15].

Lemma 5. *Let* $f = f(x_1, x_2, x_3)$ *be a formal power series such that in* $x_l = x_j$, $l, j \in \{1, 2, 3\}$, $l \neq j$, *we have* $f(x_1, x_2, x_3)|_{x_l = x_j} = \overline{f}$, *where* $\overline{f}$ *is a formal power series in the variables* x_j, x_k *with* $k \in \{1, 2, 3\}$, $k \neq j$ *and* $k \neq l$. *Then, there exists a formal series* $g = g(x_1, x_2, x_3)$ *such that* $f = \overline{f} + (x_l - x_j)g$.

We recall the definition of *Darboux polynomial* for system (1) with cofactor K. We say that $f = f(x_1, x_2, x_3)$ is a Darboux polynomial of sysem (1) if it satisfies

$$\frac{\partial f}{\partial x_1}\dot{x}_1 + \frac{\partial f}{\partial x_2}\dot{x}_2 + \frac{\partial f}{\partial x_3}\dot{x}_3 = Kf. \tag{4}$$

Furthermore, since the polynomials in the right-hand side of (1) have degree two, the cofactor K has degree at most one. We write it as $K = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3$. Note that $f = 0$ is an invariant algebraic surface for the flow of system (1), and a polynomial first integral of system (1) is a Darboux polynomial with zero cofactor.

We recall that if there exist invariant planes under the flow of system (1) and a Darboux polynomial of system (1) with cofactor K, then the restriction of f to each of the invariant planes is a Darboux polynomial of system (1) restricted to each one of the planes and with cofactors being the restriction of the cofactor K to each one of the planes. We note that, for real polynomial differential systems, such as system (1), when we look for their Darboux first integrals, we use in general complex Darboux polynomials and complex exponential factors. This is due to the fact that these objects appear in pairs (them and their conjugates), and this forces the Darboux first integral to become real if it exists.

The following two lemmas are well-known.

Lemma 6. *For $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ any Darboux polynomial f of system* (1) *has cofactor of the form $K = c_1x_1 + c_2x_2 + c_3x_3$.*

Lemma 7. *For $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ if we write f as sum of its homogeneous parts, $f = f_1 + \cdots + f_n$, then f is a Darboux polynomial of system* (1) *with cofactor K if and only if, for all $i = 1, \ldots, n$, f_i is a Darboux polynomial of system* (1) *with cofactor K.*

Lemma 8. *Let F be an analytic function and let $F = \sum_i F_i$ be its decomposition into homogeneous polynomials of degree i. Then F is an analytic first integral of the homogeneous polynomial differential system* (1) *if and only if F_i is a homogeneous polynomial first integral of system* (1) *for all i.*

The proof of Lemmas 6, 7 and 8 uses the homogeneity of system (1). Their proofs are well-known (see for instance [4] for the first two and [8] for the third). As usual $\mathbb{N}$ denotes the set of positive integers.

Lemma 9. *If we decompose the polynomial f into its irreducible factors in $\mathbb{C}[x_1, x_2, x_3]$ as $\prod_{j=1}^{s} f_j^{n_j}$ with $n_j \in \mathbb{N} \cup \{0\}$, then f is a Darboux polynomial if and only if every f_j is a Darboux polynomial. Moreover, if K and K_j are the cofactors of f and f_j, then $K = \sum_{j=1}^{s} n_j K_j$.*

Lemma 9 is proved, for instance in [4].

The following statement is important to investigate the rational integrability of polynomial systems. It was proved in [7].

Proposition 10. *The existence of a rational first integral for the polynomial differential system* (1) *implies either: the existence of a polynomial first integral (and thus a Darboux polynomial with zero cofactor), or the existence of two coprime Darboux polynomials with the same non–zero cofactor.*

An exponential factor F of the polynomial differential system (1) is a function $F = \exp(f/g) \notin \mathbb{C}$ with $f, g \in \mathbb{C}[x_1, x_2, x_3]$ coprime and satisfying

$$\frac{\partial F}{\partial x_1}\dot{x}_1 + \frac{\partial F}{\partial x_2}\dot{x}_2 + \frac{\partial f}{\partial x_3}\dot{x}_3 = LF, \tag{5}$$

for some polynomial L of degree one. The following result is well-known. Its proof and geometrical meaning is given in [9] and [10].

Proposition 11. *The following statements hold.*

(a) *If* $E = \exp(g_0/g_1)$ *is an exponential factor for the polynomial system* (1) *and* g_1 *is not a constant polynomial, then* $g_1 = 0$ *is an invariant algebraic curve.*

(b) *Eventually* e^{g_0} *can be exponential factors, coming from the multiplicity of the infinite invariant straight line.*

The following result given in [9] characterizes the algebraic multiplicity of an invariant algebraic surface using the number of exponential factors of system (1) associated with this invariant algebraic surface.

Theorem 12. *Given an irreducible invariant algebraic surface* $g_1 = 0$ *of degree* m *of system* (1)*, it has algebraic multiplicity* k *if and only if the vector field associated to system* (1) *has* $k-1$ *exponential factors of the form* $\exp(g_{0,i}/g_1^i)$*, where* $g_{0,i}$ *is a polynomial of degree at most* im*, and* $g_{0,i}$ *and* g_1 *are coprime for* $i = 1, \ldots, k-1$.

In view of Theorem 12 if we prove that e^{g_0/g_1} is not an exponential factor with degree $g_0 \leq$ degree g_1, there are no exponential factors associated to the invariant algebraic surface $g_1 = 0$.

A first integral G of system (1) is called *Darboux* if it is of the form

$$G = f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q}, \tag{6}$$

where $f_1, \ldots, f_p$ are Darboux polynomials and $F_1, \ldots, F_q$ are exponential factors, and $\lambda_j, \mu_k \in \mathbb{C}$ for $j = 1, \ldots, p$ and $k = 1, \ldots, q$.

We will need the following result whose proof is given in [4].

Theorem 13. *Suppose that system* (1) *admits* p *Darboux polynomials and with cofactors* K_i *and* q *exponential factors* F_j *with cofactors* L_j*. Then there exists* $\lambda_j, \mu_j \in \mathbb{C}$ *not all zero such that*

$$\sum_{i=1}^{q} \lambda_k K_i + \sum_{i=1}^{q} \mu_i L_i = 0$$

if and only if the function G *given in* (6) *(called of Darboux type) is a first integral of system* (1).

As we already pointed out in the introduction the planes $\{x_1 = x_2\}$, $\{x_1 = x_3\}$ and $\{x_2 = x_3\}$ are invariant under the flow of (1). Therefore, if f is a formal first integral of system (1) then

$$f_1(x_2, x_3) = f(x_2, x_2, x_3), \;\; f_2(x_2, x_3) = f(x_3, x_2, x_3), \;\; f_3(x_1, x_3) = f(x_1, x_3, x_3) \tag{7}$$

are formal first integrals of system (1) restricted to the planes $\{x_1 = x_2\}$, $\{x_1 = x_3\}$ and $\{x_2 = x_3\}$, respectively.

Proposition 14. *For* $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ *if* f *is a formal first integral of system* (1)*, then*

$$f = c_0 + (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) g,$$

where c_0 is some constant and $g := g(x_1, x_2, x_3)$ is a formal power series.

Proof. Let f be a formal first integral of system (1). We will first prove that $f_1 = c_0$ where f_1 is given in (7). Indeed, f_1 satisfies

$$-x_2^2\frac{\partial f_1}{\partial x_2} + (x_2^2 - 2x_2x_3 + \alpha_3^2(x_2 - x_3)^2)\frac{\partial f_1}{\partial x_3} = 0.$$

We now introduce the linear change of variables

$$y_2 = x_2, \quad z_2 = x_2 - x_3. \tag{8}$$

In these new variables, we have that $f_1(x_2, x_3) = h(y_2, z_2)$ and satisfies

$$-y_2^2\frac{\partial h}{\partial y_2} - (2y_2 + \alpha_3^2 z_2)z_2\frac{\partial h}{\partial z_2} = 0. \tag{9}$$

We will show that $h = c_0$. For this, we write h in power series in the variables y_2 and z_2 as

$$h = \sum_{k,l\geq 0} h_{k,l} y_2^k z_2^l.$$

Thus imposing that h satisfies (9) we get that

$$\begin{aligned} 0 &= \sum_{k,l\geq 0}(k + 2l)h_{k,l}y_2^{k+1}z_2^l + \alpha_3^2\sum_{k,l\geq 0} l h_{k,l}y_2^k z_2^{l+1} \\ &= \sum_{k,l\geq 0}\big((k + 2l - 1)h_{k-1,l} + \alpha_3^2(l - 1)h_{k,l-1}\big)y_2^k z_2^l, \end{aligned} \tag{10}$$

where $h_{m,n} = 0$ for $m < 0$ or $n < 0$. Now computing the different degrees in the variables y_2 and z_2 in (10), for $k, l \geq 0$ we get that

$$(k + 2l - 1)h_{k-1,l} + \alpha_3^2(l - 1)h_{k,l-1} = 0. \tag{11}$$

We claim that

$$h_{k,l} = 0 \quad \text{for } k, l \geq 0 \text{ if } (k, l) \neq (0, 0). \tag{12}$$

We will prove (12) by induction over l. For $l = 0$, equation (11) implies that $(k - 1)h_{k-1,0} = 0$ for all $k \geq 0$, which yields that $h_{k,0} = 0$ for all $k > 0$ and concludes the proof of (12) for $l = 0$. Now assume that (12) is true for $l = 0, \ldots, m - 1$ (with $m \geq 1$) and we will prove it for $l = m$. By inductive hypothesis, $h_{k,m}$ satisfies $(k+2m-1)h_{k-1,m} = 0$ for $k \geq 0$, which yields $h_{k,m} = 0$ for $k \geq 0$. Then (12) is proved for $l = m$, and by induction, (12) holds. Then from (12) we get that $h = h_{0,0} = c_0$ and consequently $f_1(x_2, x_3) = c_0$. Then using Lemma 5 with $x_l = x_1$ and $x_j = x_2$, we obtain

$$f = c_0 + (x_1 - x_2)g_0, \tag{13}$$

for some formal power series $g_0 := g_0(x_1, x_2, x_3)$.

Now, repeating for f_2 the arguments that we applied for f_1 we get that there exists a constant c_1 and a formal power series $g_1 = g_1(x_1, x_2, x_3)$ such that

$$f = c_1 + (x_1 - x_3)g_1, \tag{14}$$

Moreover, repeating for f_3 the arguments that we applied for f_1 we get that there exists a constant c_2 and a formal power series $g_2 = g_2(x_1, x_2, x_3)$ such that

$$f = c_2 + (x_2 - x_3)g_2, \tag{15}$$

Now evaluating equations (13)–(15) on $x_1 = x_2 = x_3 = 0$ we get $c_0 = c_1 = c_2$. Furthermore, equating (13)–(15), we get

$$(x_1 - x_2)g_0 = (x_1 - x_3)g_1 = (x_2 - x_3)g_2,$$

which clearly implies that there exists a formal power series $g := g(x_1, x_2, x_3)$ such that

(16)

$$g_0 = (x_1 - x_3)(x_2 - x_3)g, \quad g_1 = (x_1 - x_2)(x_2 - x_3)g, \quad g_2 = (x_1 - x_2)(x_1 - x_3)g.$$

Now the proposition follows from (13) and the first relation in (16). □

3. Proof of Theorems 1 and 2

Proof of Theorem 1. Let f be any formal first integral of system (1). By Proposition 14 we know that f can be written as

$$f = c_0 + (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)g, \tag{17}$$

for some constant c_0 and some formal power series $g := g(x_1, x_2, x_3)$. Imposing that f is a first integral of system (1), we get that, after simplifying by $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$, g must satisfy

$$\frac{dg}{dt} = \big((2-2\alpha_1^2+\alpha_2^2+\alpha_3^2)x_1+(2+\alpha_1^2-2\alpha_2^2+\alpha_3^2)x_2+(2+\alpha_1^2+\alpha_2^2-2\alpha_3^2)x_3\big)g, \tag{18}$$

where the derivative is evaluated along a solution of system (1). We will prove that $g = 0$. For this, we will proceed by reduction to the absurd. Assume $g \neq 0$ and we will reach a contradiction. We consider two different cases.

Case 1: g is not divisible by $x_1 - x_2$. In this case, using Lemma 5 with $x_l = x_1$ and $x_j = x_2$, we can write g as $g = g_0 + (x_1 - x_2)g_1$, where $g_0 := g_0(x_2, x_3) \neq 0$ and $g_1 := g_1(x_1, x_2, x_3)$ are formal power series. Then, g_0 satisfies (18) restricted to $x_1 = x_2$. Thus introducing again the change of variables (8) we have that, in these new variables, if $g_0(x_2, x_3) = h_0(y_2, z_2)$ then h_0 satisfies

$$-y_2^2\frac{\partial h_0}{\partial y_2} - (2y_2 + \alpha_3^2 z_2)z_2\frac{\partial h}{\partial z_2} = (6y_2 - (2 + \alpha_1^2 + \alpha_2^2 - 2\alpha_3^2)z_2)h_0. \tag{19}$$

Now, we write

$$h_0 = \sum_{j\geq 0} h_{0,j} z_2^j, \quad h_{0,j} = h_{0,j}(y_2) \quad \text{with } h_{0,j} \text{ being formal power series for each } j.$$

We claim that

$$h_{0,j} = 0 \quad \text{for } j \geq 0. \tag{20}$$

Clearly $h_{0,0}$ satisfies (19) restricted to $z_2 = 0$ that is,

$$-y_2^2 \frac{dh_{0,0}}{dy_2} = 6y_2 h_{0,0},$$

and solving it we get $h_{0,0} = c/y_2^6$, where $c \in \mathbb{C}$. Since $h_{0,0}$ is a formal series in the variable y_2, we have that $c = 0$, and thus, $h_{0,0} = 0$, which proves (20) for $j = 0$. Now we assume that (20) is true for $j = 0, \ldots, m-1$ with $m \geq 1$ and we will prove it for $k = m$. Clearly, by hypothesis of induction, we have that $h_0 = \sum_{j\geq 0} h_{0,j+m} z_2^{j+m}$ and then, from (19), after dividing by z_2^m we obtain

$$\begin{aligned} &- y_2^2 \sum_{j\geq 0} \frac{dh_{0,j+m}}{dy_2} z_2^j - (2y_2 + \alpha_3^2 z_2) \sum_{j\geq 0} (j+m) h_{0,j+m} z_2^j \\ &= (6y_2 - (2 + \alpha_1^2 + \alpha_2^2 - 2\alpha_3^2) z_2) \sum_{j\geq 0} h_{0,j+m} z_2^j. \end{aligned} \tag{21}$$

Then evaluating (21) on $z_2 = 0$ we get

$$-y_2^2 \frac{dh_{0,m}}{dy_2} = y_2(6 + 2m) h_{0,m},$$

whose solution is $h_{0,m} = c_m / y_2^{6+2m}$, where $c_m \in \mathbb{C}$. Since $h_{0,m}$ is a formal series in the variable y_2, we have that $c_m = 0$ and thus $h_{0,m} = 0$, which proves (20) for $j = m$. Then by the induction hypothesis, (20) holds, and from (20) we obtain that $h_0 = 0$. Hence $g_0 = 0$, in contradiction with the fact that g is not divisible by $x_1 - x_2$.

Case 2: g is divisible by $x_1 - x_2$. In this case, $g = (x_1 - x_2)^j h$ with $j \geq 1$, $h \neq 0$ and $h := h(x_1, x_2, x_3)$ is a formal power series such that is not divisible by $x_1 - x_2$ and satisfies, after dividing by $(x_1 - x_2)^j$, the equation

$$\begin{aligned} \frac{dh}{dt} &= \big((2 + \alpha_2^2 + \alpha_3^2 - \alpha_1^2(2+j))x_1 + (2 + \alpha_1^2 + \alpha_3^2 - \alpha_2^2(2+j))x_2 \\ &\quad + ((2 + \alpha_1^2 + \alpha_2^2)(1+j) - 2\alpha_3^2)x_3\big) h, \end{aligned}$$

where the derivative of h is evaluated along a solution of system (1). Then, applying to h the same arguments used for g in Case 1, we conclude that $h = 0$, a contradiction.

Hence $g = 0$ and the proof of the theorem follows from (17) and the definition of formal first integral. □

Proof of Theorem 2. By Theorem 1, system (1) has no polynomial first integrals. Now the proof of Theorem 2 is immediate from Lemma 8. □

4. Proof of Theorem 3

We recall that the equation defining a Darboux polynomial is given in (4) and that in view of Lemma 6 we can take $K = c_1x_1 + c_2x_2 + c_3x_3$ with $(c_1, c_2, c_3) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$. We shall prove Theorem 3 with the help of Proposition 10, Theorem 1 and the following result.

Theorem 15. *For* $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ *every Darboux polynomial of system* (1) *is of the form*

$$f = cH_1^{n_1} H_2^{n_2} H_3^{n_3} = c(x_1 - x_2)^{n_1}(x_1 - x_3)^{n_2}(x_2 - x_3)^{n_3},$$

where c *is some constant and* n_1, n_2, n_3 *are nonnegative integers. Furthermore the cofactor of* f *is*

$$\begin{aligned} K = {} & n_1(-2x_3 - \alpha_1^2(x_3 - x_1) + \alpha_2^2(x_2 - x_3)) + n_2(-2x_2 + \alpha_1^2(x_1 - x_2) \\ & + \alpha_3^2(x_3 - x_2)) + n_3(-2x_1 - \alpha_2^2(x_1 - x_2) + \alpha_3^2(x_3 - x_1)). \end{aligned} \tag{22}$$

The main objective of this section is to prove Theorem 15 since as it will be clear later will readily imply the proof of Theorem 3. For this we will study the Darboux polynomials of system (1) of degree one and of degree greater than one. This will be done in two separate propositions.

Proposition 16. *For* $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ *the unique homogeneous Darboux polynomials of system* (1) *of degree one are* $H_1 := x_1 - x_2$, $H_2 := x_1 - x_3$ *and* $H_3 := x_2 - x_3$, *respectively, with cofactors,*

$$\begin{aligned} K_1 &= -2x_3 - \alpha_1^2(x_3 - x_1) + \alpha_2^2(x_2 - x_3), \\ K_2 &= -2x_2 - \alpha_3^2(x_2 - x_3) + \alpha_1^2(x_1 - x_2), \\ K_3 &= -2x_1 - \alpha_2^2(x_1 - x_2) + \alpha_3^2(x_3 - x_1). \end{aligned}$$

Proof. Let $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ and let f be a homogeneous Darboux polynomial of system (1) of degree one, i.e., $f = b_1x_1 + b_2x_2 + b_3x_3$. Then the result follows easily using that f must satisfy (4). □

Proposition 17. *For* $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ *let* f *be an irreducible homogeneous Darboux polynomial of system* (1) *with degree at least two and cofactor* K, *as in Lemma 6. Then* $K = 0$.

To prove Proposition 17 we will show that each of the coefficients c_1, c_2 and c_3 in the definition of K given in Lemma 6 is zero for any Darboux polynomial of system (1) of degree greater than or equal to two. For this we shall need the following preliminary result, which describes the Darboux polynomials and their cofactors of system (1) restricted to each plane H_j for $j = 1, 2, 3$, defined in (2).

Proposition 18. *Let* $\bar{f} = \bar{f}(x_2, x_3)$ *be a homogeneous Darboux polynomial of degree* n *of system* (1) *restricted to* $x_1 = x_2$ *with cofactor* $K = (c_1+c_2)x_2+c_3x_3$, *where* $(c_1, c_2, c_3) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$. *Then if* $\bar{f} \neq 0$ *it is of the form*

$$\bar{f} = c_0 x_2^{n-l_2-l_3}(x_2 - x_3)^{l_2}(x_2 + \alpha_3^2(x_2 - x_3))^{l_3}, \quad c_0 \in \mathbb{C} \setminus \{0\}$$

and $l_2, l_3 \in \mathbb{N} \cup \{0\}$, $l_2 + l_3 \leq n$,

$$c_3 = \alpha_3^2(l_2 + l_3) \quad \textit{and} \quad c_1 + c_2 + c_3 = -n - l_2.$$

Proof. Let $\bar{f}$ be a homogeneous Darboux polynomial of system (1) restricted to $x_1 = x_2$ with cofactor $K = (c_1+c_2)x_2+c_3x_3$, where $(c_1, c_2, c_3) \in \mathbb{C}^3 \setminus \{(0, 0, 0)\}$. We note that system (1) restricted to $x_1 = x_2$ we get

$$\dot{x}_2 = -x_2^2, \quad \dot{x}_3 = x_2^2 - 2x_2x_3 + \alpha_3^2(x_2 - x_3)^2. \tag{23}$$

An easy computation shows that x_2, $x_2 - x_3$ and $x_2 + \alpha_3^2(x_2 - x_3)$ (when $\alpha_3 \neq 0$) are the unique Darboux polynomials of degree one.

We introduce the change of variables

$$y_2 = x_2, \quad z_2 = x_2 - x_3,$$

Then system (23) becomes

$$\dot{y}_2 = -y_2^2, \quad \dot{z}_2 = -z_2(2y_2 + \alpha_3^2 z_2). \tag{24}$$

We will look for Darboux polynomials of system (24). Let $f = f(y_2, z_2)$ a Darboux polynomial of system (24) with cofactor $K = d_2y_2 + d_3z_2$, i.e., it satisfies

$$-y_2^2\frac{\partial f}{\partial y_2} - z_2(2y_2 + \alpha_3^2 z_2)\frac{\partial f}{\partial y_2} = (d_2y_2 + d_3z_2)f. \tag{25}$$

If we set

$$u_2 = \frac{y_2^2}{z_2} \quad w_2 = y_2,$$

we get

$$u_2' = \alpha_3^2 w_2^2, \quad w_2' = -w_2^2.$$

Let $H_2 = u_2 + \alpha_3^2 w_2$. Then we obtain

$$H_2' = 0, \quad w_2' = -w_2^2.$$

So we rewrite $g = g(H_2, w_2) = f\Big(w_2, \frac{w_2^2}{H_2-\alpha_3^2 w_2}\Big)$ and it follows from (25) that

$$-w_2^2\frac{dg}{dw_2} = \Big(d_2w_2 + \frac{d_3w_2^2}{H_2 - \alpha_3^2 w_2}\Big)g.$$

Solving it we get

$$g = K(H_2)w_2^{-d_2}(H_2 - \alpha_3^2 w_2)^{d_3/\alpha_3^2},$$

where K is a function of H_2. Hence,

$$f = K\Big(\frac{y_2(y_2 + \alpha_3^2 z_2)}{z_2}\Big)z_2^{-d_3/\alpha_3^2}y_2^{2d_3/\alpha_3^2-d_2}.$$

Since f must be a homogeneous polynomial we get that

$$f = c_0 y_2^{l_1} z_2^{l_2}(y_2 + \alpha_3^2 z_2)^{l_3}, \quad c_0 \in \mathbb{C} \setminus \{0\}, \tag{26}$$

with $l_1 + l_2 + l_3 = n \geq 2$. So

$$d_3 = -\alpha_3^2(l_2 + l_3), \quad d_2 = -n - l_2, \quad l_1 = n - l_2 - l_3.$$

Going back to $\bar{f}$ and its cofactor we obtain

$$c_3 = \alpha_3^2(l_2 + l_3) \quad \text{and} \quad c_1 + c_2 + c_3 = -n - l_2.$$

This completes the proof. □

The following two propositions can be proved in a similar manner.

Proposition 19. *Let* $\bar{f} = \bar{f}(x_2, x_3)$ *be a homogeneous Darboux polynomial of degree* n *of system* (1) *restricted to* $x_1 = x_3$ *with cofactor* $K = c_2x_2 + (c_1 + c_3)x_3$, *where* $(c_1, c_2, c_3) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$. *Then if* $\bar{f} \neq 0$ *we have*

$$\bar{f} = d_0 x_3^{n-j_2-j_3}(x_3 - x_2)^{j_2}(x_3 + \alpha_2^2(x_3 - x_2))^{j_3}, \quad d_0 \in \mathbb{C} \setminus \{0\}$$

and

$$c_2 = \alpha_2^2(j_2 + j_3) \quad \textit{and} \quad c_1 + c_2 + c_3 = -n - j_2.$$

Proposition 20. *Let* $\bar{f} = \bar{f}(x_1, x_3)$ *be a homogeneous Darboux polynomial of degree* n *of system* (1) *restricted to* $x_2 = x_3$ *with cofactor* $K = c_1x_1 + (c_2 + c_3)x_3$, *where* $(c_1, c_2, c_3) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$. *Then*

$$\bar{f} = e_0 x_3^{n-m_2-m_3}(x_3 - x_1)^{m_2}(x_3 + \alpha_1^2(x_3 - x_1))^{m_3}, \quad e_0 \in \mathbb{C} \setminus \{0\},$$

and

$$c_1 = \alpha_1^2(m_2 + m_3) \quad \textit{and} \quad c_1 + c_2 + c_3 = -n - m_2.$$

Proof of Proposition 17. Let $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ and f an irreducible Darboux polynomial of system (1) of degree $n \geq 2$ and with cofactor $K = c_1x_1 + c_2x_2 + c_3x_3$ with $(c_1, c_2, c_3) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$. From the fact that f is irreducible it is clear that $f_1 \neq 0$, $f_2 \neq 0$ and $f_3 \neq 0$ (we recall (7) for the definition of f_i for $i = 1, 2, 3$), otherwise f would be divisible by $x_1 - x_2$, or by $x_1 - x_3$, or $x_2 - x_3$, a contradiction.

Furthermore since f_1, f_2 and f_3 are respectively homogeneous Darboux polynomials of system (1) restricted to $x_1 = x_2$, $x_1 = x_3$ and $x_2 = x_3$ respectively, we can apply Proposition 18 with $\bar{f} = f_1$, Proposition 19 with $\bar{f} = f_2$ and Proposition 20 with $\bar{f} = f_3$.

It follows from Propositions 18, 19 and 20 that

$$c_1 + c_2 + c_3 = -n - l_2 = -n - j_2 = -n - m_2$$

that is $l_2 = j_2 = m_2$. Furthermore, we have that

$$c_1 + c_2 + c_3 = -n - l_2 = \alpha_1^2(l_2 + m_3) + \alpha_2^2(l_2 + j_3) + \alpha_3^2(l_2 + l_3). \tag{27}$$

Note that the right-hand side of (27) is always greater than or equal to zero while the quantity $-n - l_2$ is always less than or equal to zero. This implies that they must be zero, that is, $n = l_2 = 0$, in contradiction with the fact that $n \geq 2$. This concludes the proof of the proposition. □

Proof of Theorem 15. If f has degree one, the proof follows directly from Proposition 16. Now assume that system (1) has an irreducible Darboux polynomial of degree at least two with cofactor $K \neq 0$ given as in Lemma 6. Then from Lemma 7 we can assume that f is a homogeneous irreducible Darboux polynomial of degree at least two and cofactor $K \neq 0$. Then from Proposition 17 we reach a contradiction. Thus all Darboux polynomials with cofactor $K \neq 0$ given as in Lemma 6, must come from Darboux polynomials of degree one, i.e., H_1, H_2 or H_3. Furthermore, from Theorem 1 we know that all Darboux polynomials with zero cofactor, that is, polynomial first integrals are constants. This fact, together with Lemma 9 imply the proof of Theorem 15. □

Proof of Theorem 3. By Theorem 15, it follows that every Darboux polynomial of system (1) is of the form

$$f = cH_1^{n_1}H_2^{n_2}H_3^{n_3} = c(x_1-x_2)^{n_1}(x_1-x_3)^{n_2}(x_2-x_3)^{n_3},$$

where c is some constant and n_1, n_2, n_3 are nonnegative integers. Furthermore the cofactor of f is given in (22).

From Proposition 10 and Theorem 1, the existence of a nonconstant rational first integral implies the existence of two coprime Darboux polynomials with the same nonzero cofactor. So, the first integral must be of the form $R/S = c_0H_1^{n_1}H_2^{n_2}H_3^{n_3}/c_1(cH_1^{m_1}H_2^{m_2}H_3^{m_3})$ with at least one m_i and n_i nonzero, and the cofactor of R and S must be equal. Then, according to (22), the equality of the cofactors of R and S implies that

$$\begin{aligned}(m_1-n_1)(-2x_3-\alpha_1^2(x_3-x_1)+\alpha_2^2(x_2-x_3))+(m_2-n_2)(-2x_2+\alpha_1^2(x_1-x_2)\\ +\alpha_3^2(x_3-x_2))+(m_3-n_3)(-2x_1-\alpha_2^2(x_1-x_2)+\alpha_3^2(x_3-x_1))=0.\end{aligned}$$

Hence using that α_i are real we get $m_i = n_i$ for $i = 1, 2, 3$ in contradiction with the fact that R and S are coprime. Thus the theorem is proved. □

5. Proof of Theorem 4

We recall that the equation defining the exponential factor $F = \exp(h/g)$ with cofactor L for system (1) is

$$\dot{x}_1\frac{\partial}{\partial x_1}\Big(\frac{h}{g}\Big)+\dot{x}_2\frac{\partial}{\partial x_2}\Big(\frac{h}{g}\Big)+\dot{x}_1\frac{\partial}{\partial x_3}\Big(\frac{h}{g}\Big)=L, \tag{28}$$

where we have simplified the common factor F and

$$L = b_0 + b_1x_1 + b_2x_2 + b_3x_3. \tag{29}$$

According to Propositions 11 and 12 and Theorems 1 and 15, if system (1) has exponential factors, they must be of the form $\exp(h/(H_1^{n_1}H_2^{n_2}H_3^{n_3}))$, where $h \in \mathbb{C}[x_1, x_2, x_3]$, $n_1, n_2, n_3 \in \mathbb{N} \cup \{0\}$ and the degree of h is less than or equal to $n_1 + n_2 + n_3$.

Proposition 21. *For $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ system (1) does not admit exponential factors.*

Proof. We start by showing that there are no exponential factors of the form $\exp(h)$. Applying (28) with $g = 1$ we get

$$\dot{x}_1\frac{\partial h}{\partial x_1}+\dot{x}_2\frac{\partial h}{\partial x_2}+\dot{x}_3\frac{\partial h}{\partial x_3}=L \tag{30}$$

with L given in (29). Taking $x_1 = x_2 = x_3 = 0$ in (30) we obtain $b_0 = 0$. Now letting $x_1 = x_2 = 0$ in (30) we get

$$\alpha_3^2 x_3^2 \frac{\partial h}{\partial x_3}\bigg|_{x_1=x_2=0} = b_3 x_3. \tag{31}$$

Equation (31) implies that $b_3 = 0$. Analogously, setting $x_1 = x_3 = 0$ in (30) we get $b_2 = 0$, and setting $x_2 = x_3 = 0$ in (30) we get $b_1 = 0$. Thus $L = 0$ and from (30) we get that h is a polynomial first integral of system (1), in contradiction with Theorem 1.

Assume now that $\exp(h/(H_1^{n_1} H_2^{n_2} H_3^{n_3}))$ is an exponential factor of system (1) where n_1, n_2, n_3 are nonnegative integers with at least one of them positive, h is coprime with H_1, H_2 and H_3 and the degree of h is at most $n_1 + n_2 + n_3$. Then h satisfies

$$\dot{x}_1 \frac{\partial h}{\partial x_1} + \dot{x}_2 \frac{\partial h}{\partial x_2} + \dot{x}_3 \frac{\partial h}{\partial x_3} - \left(\frac{\dot{H}_1}{H_1} n_1 + \frac{\dot{H}_2}{H_2} n_2 + \frac{\dot{H}_3}{H_3} n_3\right) h = L H_1^{n_1} H_2^{n_2} H_3^{n_3}. \tag{32}$$

Without loss of generality we can assume that $n_1 > 0$. Taking $H_1 = 0$ in (32) and denoting by h_1 the restriction of h to $H_1 = 0$ we conclude that h_1 satisfies

$$\begin{aligned} &- x_2^2 \frac{\partial h_1}{\partial x_2} + (x_2^2 - 2x_2x_3 + \alpha_3^2 (x_2 - x_3)^2) \frac{\partial h_1}{\partial x_3} \\ &= \big(n_1((\alpha_1^2 + \alpha_2^2)(x_2 - x_3) - 2x_3) + (n_2 + n_3)(\alpha_3^2 x_3 - (\alpha_3^2 + 2)x_2\big) h_1. \end{aligned} \tag{33}$$

Since by hypothesis, h is coprime with H_1, we have that $h_1 \neq 0$. Furthermore, from (33) h_1 is a Darboux polynomial of system (1) restricted to $x_1 = x_2$ and with cofactor $K = n_1((\alpha_1^2 + \alpha_2^2)(x_2 - x_3) - 2x_3) + (n_2 + n_3)(\alpha_3^2 x_3 - (\alpha_3^2 + 2)x_2)$. In view of Lemma 7, we can assume that h_1 is homogeneous and we know that $n \leq n_1 + n_2 + n_3$. Then in view of Proposition 18, we must have

$$\alpha_3^2(l_3 + l_2) = -(2 + \alpha_1^2 + \alpha_2^2) n_1 + \alpha_3^2 (n_2 + n_3) \quad \text{and} \quad c_1 + c_2 + c_3 = -n - l_2. \tag{34}$$

Computing $c_1 + c_2 + c_3$ from (33) we get that $c_1 + c_2 + c_3 = -2(n_1 + n_2 + n_3)$. Hence

$$n + l_2 = 2(n_1 + n_2 + n_3).$$

Since $n \leq n_1 + n_2 + n_3$ and $l_2 \leq n$ we get that $l_2 = n = n_1 + n_2 + n_3$, $l_1 = l_3 = 0$. Then the first identity in (34) becomes

$$\alpha_3^2 n_1 = -(2 + \alpha_1^2 + \alpha_2^2) n_1,$$

which is not possible due to the fact that $n_1 > 0$ and thus the right-hand side is negative while the left-hand side is not. This completes the proof of the proposition. □

Proof of Theorem 4. From Propositions 11 and 21 and Theorems 1 and 15, if system (1) has a Darboux first integral G then $G = c H_1^{\lambda_1} H_2^{\lambda_2} H_3^{\lambda_3}$ where $c, \lambda_i, \lambda_2, \lambda_3 \in \mathbb{C}$. Since G is a first integral it must hold that

$$\begin{aligned} &\lambda_1(-2x_3 - \alpha_1^2(x_3 - x_1) + \alpha_2^2(x_2 - x_3)) + \lambda_2(-2x_2 + \alpha_1^2(x_1 - x_2) \\ &\quad + \alpha_3^2(x_3 - x_2)) + \lambda_3(-2x_1 - \alpha_2^2(x_1 - x_2) + \alpha_3^2(x_3 - x_1)) = 0. \end{aligned}$$

Hence, using that α_i are real for $i = 1, 2, 3$, we get that $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and completes the proof of the theorem. □

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