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Algebraic and analytical tools for the study of the period function

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Abstract. In this paper we consider analytic planar differential systems having a first integral of the form $H(x,y) = A(x) + B(x)y + C(x)y^2$ and an integrating factor $\kappa(x)$ not depending on y. Our aim is to provide tools to study the period function of the centers of this type of differential system and to this end we prove three results. Theorem A gives a characterization of isochronicity, a criterion to bound the number of critical periods and a necessary condition for the period function to be monotone. Theorem B is intended for being applied in combination with Theorem A in an algebraic setting that we shall specify. Finally, Theorem C is devoted to study the number of critical periods bifurcating from the period annulus of an isochrone perturbed linearly inside a family of centers. Four different applications are given to illustrate these results.

1 Introduction

The present paper is concerned with the period function of centers. A critical point p of a planar differential system is a *center* if it has a punctured neighbourhood that consists entirely of periodic orbits surrounding p. The largest punctured neighbourhood with this property is called the *period annulus* of the center and, in what follows, it will be denoted by $\mathscr{P}$. The *period function* of the center assigns to each periodic orbit in $\mathscr{P}$ its period. If the period function is constant, then the center is said to be *isochronous*. Since the period function is defined on the set of periodic orbits in $\mathscr{P}$, in order to study its qualitative properties usually the first step is to parameterize this set. This can be done, for instance, by taking a transversal section on $\mathscr{P}$ or, in case that the differential system has global a first integral, by using the energy level of the periodic orbits. If $\{\gamma_s\}_{s\in(0,1)}$ is such a parameterization, then $s \longmapsto T(s) := \{\text{period of } \gamma_s\}$ is a smooth map that provides the qualitative properties of the period function that we are concerned about. In particular the existence of *critical periods*, which are isolated critical points of this function, i.e., $\hat{s} \in (0,1)$ such that $T'(s) = \alpha(s-\hat{s})^k + \mathrm{o}\big((s-\hat{s})^k\big)$ with $\alpha \neq 0$ and $k \geqslant 1$. In this case we shall say that $\gamma_{\hat{s}}$ is a *critical periodic orbit* of multiplicity k of the center. One can readily see that the number, character (maximum or minimum) and distribution of the critical periods do not depend on the particular parameterization of the set of periodic orbits used.

Questions related to the behaviour of the period function have been extensively studied. Let us quote, for instance, the problems of isochronicity (see [9, 11, 25]), monotonicity (see [5, 33, 34]) or bifurcation of critical periods (see [6, 10, 20]). These are in fact subproblems of a more general question that asks for a bound (if any) on the number of critical periods that a given a family of centers can have. In this regard the most studied family is the quadratic one. According to Chicone's conjecture, see [6, 7], if a quadratic

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system has a center with a non monotonic period function then, by an affine transformation and a constant rescaling of time, it can be transformed to the Loud normal form and, in this case, it has at most two critical periods. This conjecture is the analogue for the period function of the second part of *Hilbert's 16th problem* for quadratic systems (see [31]), which asks for a bound on the number of limit cycles that these systems can have. This similarity is not only conceptual but also in the techniques used and in the degree of difficulty. Chicone's conjecture has attracted interest of many authors (see [8, 12, 14, 16, 23, 24, 27, 34, 35] and references therein) and there is much analytic evidence that it is true. The majority of the results are concerned about monotonicity because there is a lack of tools to investigate centers with non monotonic period function. To our knowledge the only exception to this are [23, 24, 36, 37], which succeed in studying quadratic centers with non monotonic period function by showing that it verifies a Picard-Fuchs equation. The problem with this approach is that we must have a rational first integral to begin with.

The goal of the present paper is to provide tools to bound the number of critical periods of a center. Thus, Theorem A can be viewed as the continuation of the results in [29], where similar tools are developed for potential systems. We generalize them to consider a wider class of differential systems, which includes the ones in Loud normal form. The second result that we obtain, Theorem B, is addressed to an algebraic setting where Theorem A is particularly useful. Finally, Theorem C is devoted to bound the number of critical periods bifurcating from the period annulus of an isochronous center. Let us note in this regard that in our opinion it is lacking an accurate definition of this notion in the literature. In this paper we propose one, see Definition 2.3, which is in fact more general because it can also be applied to the critical periods bifurcating from a center or a polycycle.

The organization of the paper is as follows. Section 2 is devoted to introduce the definitions of the notions used henceforth and to state the main results. Then, after proving some preliminary lemmas, we show Theorems A, B and C in Section 3. In Section 4 we give four different applications of these results. More precisely, Proposition 4.1 constitutes a new result, whereas in Propositions 4.2, 4.3 and 4.4 we revisit already known results with the aim of showing the simplicity of our tools in relation to the previous approaches.

2 Definitions and statement of the results

Throughout the present paper we understand an analytic planar differential system

$$\begin{cases} \dot{x} = p(x,y), \\ \dot{y} = q(x,y, \end{cases} \tag{1}$$

to be given and that it satisfies the following *standing hypothesis*:

(H) The differential system (1) has a center at the origin and an analytic first integral of the form $H(x,y) = A(x) + B(x)y + C(x)y^2$ with $A(0) = 0$. In addition its integrating factor, say κ, depends only on x.

Let us point out that the center at the origin can be degenerate or not. Moreover, for the sake of simplicity in the exposition, we suppose that the differential system (1) is defined on the whole plane, although with the obvious modifications the results also hold in case that it is defined on a neighbourhood of the origin.

Our first result, Theorem A, extends the results obtained in [29] for potential systems to systems verifying hypothesis **(H)**. Note in this regard that the potential systems correspond to $B \equiv 0$ and constant C and κ. In order to state it some additional definitions are needed. Henceforth (x_ℓ, x_r) will be the interval obtained by projecting on the x-axis the period annulus $\mathscr{P}$ of the center at the origin of the differential system (1). Note then that $0 \in (x_\ell, x_r)$. Moreover we define

$$M := \frac{4AC - B^2}{4|C|}. \tag{2}$$

The hypothesis **(H)** implies, see Lemma 3.1, that M is a well defined analytic function on (x_ℓ, x_r) with $M(0)=0$ and $xM'(x)>0$ for all $x\in(x_\ell,x_r)\setminus\{0\}$. Accordingly there exist a unique analytic function σ on (x_ℓ,x_r) with $\sigma(x)=-x+\mathrm{o}(x)$ such that $M=M\circ\sigma$. Note that σ is an *involution* with $\sigma(0)=0$. (Recall that a mapping σ is said to be an involution if $\sigma\circ\sigma=Id$ and $\sigma\neq Id$.) Given an analytic function f on $(x_\ell,x_r)\setminus\{0\}$ we define its σ-*balance* to be

$$\mathscr{B}_\sigma\big(f\big)(x):=\frac{f(x)-f\big(\sigma(x)\big)}{2}.$$

Taking these definitions into account, the first result of the present paper is the following:

Theorem A. *Suppose that the analytic differential system (1) satisfies the hypothesis **(H)**. Setting $\mu_0=-1$, define recursively*

$$\mu_i:=\left(\frac{1}{2}+\frac{1}{2i-3}\right)\mu_{i-1}+\frac{\sqrt{|C|}M}{(2i-3)\kappa}\left(\frac{\kappa\,\mu_{i-1}}{\sqrt{|C|}M'}\right)' \quad \text{and} \quad \ell_i:=\frac{\kappa\,\mu_i}{\sqrt{|C|}M'} \quad \text{for } i\geqslant 1.$$

Then the center at the origin verifies the following:

(*a*) *It is isochronous if, and only if, $\mathscr{B}_\sigma(\ell_1)\equiv 0$, which in turn is equivalent to $\mathscr{B}_\sigma(\ell_i)\equiv 0$ for all $i\geqslant 1$.*

(*b*) *If the number of zeros of $\mathscr{B}_\sigma(\ell_i)$ on $(0,x_r)$, counted with multiplicities, is $n\geqslant 0$ and it holds that $i>n$, then the number of critical periods of the center, counted with multiplicities, is at most n.*

(*c*) *If the period function is monotone, then $\mathscr{B}_\sigma(g)$ is monotone on $(0,x_r)$, where*

$$g(x):=\frac{1}{\sqrt{M(x)}}\int_0^x\frac{\kappa(s)\,ds}{\sqrt{|C(s)|}}.$$

We stress that the statement in (*b*) can be used to show monotonicity, which corresponds to take $n=0$. Let us advance that the proof of Theorem A has two main ingredients. The first one is Lemma 3.2, which provides a coordinate transformation that conjugates any differential system verifying hypothesis **(H)** with a potential system. The second ingredient is of course the results obtained in [29].

Remark 2.1. Since $\mathscr{B}_\sigma(f\circ\sigma)=-\mathscr{B}_\sigma(f)$ for any function f and σ maps the interval $(0,x_r)$ to $(x_\ell,0)$, one can replace $(0,x_r)$ by $(x_\ell,0)$ in the statement of Theorem A. □

The main difficulty to apply Theorem A is that in general it is not possible to compute the involution σ explicitly. If σ and ℓ_i are algebraic functions, then one can bypass this inconvenience by taking advantage of the *multipolynomial resultant* (see [3,13]). This was already used in the applications that appear in [29] without taking the multiplicity into account. Our second result goes futher in this approach. More concretely we prove the following:

Theorem B. *Let σ be an analytic involution on (x_ℓ,x_r) with $\sigma(0)=0$ and let ℓ be an analytic function on $(x_\ell,x_r)\setminus\{0\}$. Assume that ℓ and σ are algebraic, i.e., that there exists $L,S\in\mathbb{C}[x,y]$ such that $L\big(x,\ell(x)\big)\equiv 0$ and $S\big(x,\sigma(x)\big)\equiv 0$. Let us define $T(x,y):=Res_z\big(L(x,z),L(y,z)\big)$ and $\mathscr{R}(x):=Res_y\big(S(x,y),T(x,y)\big)$. Finally let $s(x)$ and $t(x)$ be, respectively, the leading coefficients of $S(x,y)$ and $T(x,y)$ with respect to y. Then the following hold:*

(*a*) *If $\mathscr{B}_\sigma\big(\ell\big)(x_0)=0$ for some $x_0\in(x_\ell,x_r)\setminus\{0\}$, then $\mathscr{R}(x_0)=0$.*

(*b*) *If $s(x)$ and $t(x)$ do not vanish simultaneously at x_0, then the multiplicity of $\mathscr{B}_\sigma\big(\ell\big)$ at x_0 is not greater than the multiplicity of $\mathscr{R}$ at x_0.*

Although Theorem B is conceived to be used in combination with Theorem A, it can also be useful in other situations where studying the zeros of some σ-balance is needed. This is the case for instance of the criterions proved in [19, 30], that give sufficient conditions for a collection of Abelian integrals to have some kind of Chebyshev property. One can also take the following remark into account when applying Theorem B.

Remark 2.2. Since $\mathrm{Res}(fg,h)=\mathrm{Res}(f,h)\,\mathrm{Res}(g,h)$, we can factorize $T=T_1T_2\ldots T_k$, cross out any T_i such that $T_i\big(x,\sigma(x)\big)\neq 0$ for all $x\in(x_\ell,x_r)$ and apply Theorem B with the resulting polynomial. Moreover, a sufficient condition for σ to be algebraic is that the function M such that $M=M\circ\sigma$ is algebraic. Indeed, if $P\big(x,M(x)\big)\equiv 0$ with $P\in\mathbb{C}[x,y]$, then it turns out that $S(x,y):=\mathrm{Res}_z\big(P(x,z),P(y,z)\big)$ is a polynomial verifying $S\big(x,\sigma(x)\big)\equiv 0$. □

Let us turn now to the statement of our last result, which is concerned with the bifurcation of critical periods from the period annulus of an isochrone. Since in our opinion it is lacking an accurate definition of this notion in the literature, we begin by proposing one. As a matter of fact, with the aim of applying it in other problems, this definition is addressed to a more general setting. So consider a family X_ν of analytic planar vector fields that depend analytically on a parameter $\nu\in\mathbb{R}^n$. Suppose in addition that the origin is a center of X_ν for all ν.

Definition 2.3. Let L be an invariant set of $X_{\bar\nu}$. We define the *criticality* of the pair $(L,X_{\bar\nu})$ with respect to the deformation X_ν to be

$$\mathrm{Crit}\big((L,X_{\bar\nu}),X_\nu\big):=\sup\big\{\mathcal{N}_K: K\subset L, K \text{ is a compact invariant set of } X_{\bar\nu}\big\}$$

where, for such a K, $\mathcal{N}_K$ is the smallest integer having the property that there exists a neighbourhood V of K and $\delta>0$ such that, for every ν with $\|\nu-\bar\nu\|<\delta$, the vector field X_ν has no more than $\mathcal{N}_K$ critical periodic orbits contained in V. □

In other words, what we call the criticality $\mathrm{Crit}\big((L,X_{\bar\nu}),X_\nu\big)$ is the maximal number of critical periodic orbits that tend to L as $\nu\to\bar\nu$. The easiest situation is when L is the center itself because then the criticality can be computed by studying the ideal generated by the *period constants* (see [6] for instance). We point out that Definition 2.3 is a verbatim adaptation to our setting of the notion of cyclicity that L. Gavrilov introduces in [17] to study the number of *limit cycles* that bifurcate from an open period annulus. In what follows we shall assume that the center of $X_{\bar\nu}$ is isochronous and we will be concerned about the criticality of $L=\mathscr{P}$, its period annulus. To state our result we need an additional definition.

Definition 2.4. Let $f_1,f_2,\ldots,f_n$ be analytic functions on an interval I of $\mathbb{R}$. We say that $\{f_1,f_2,\ldots,f_n\}$ is an *extended Chebyshev system* on I if any nontrivial linear combination $\alpha_1f_1+\alpha_2f_2+\ldots+\alpha_nf_n$ has at most $n-1$ zeros on I counted with multiplicities. □

It is well-known that a sufficient condition for $\{f_1,f_2,\ldots,f_n\}$ to be an extended Chebyshev system on I is that none of the leading principal minors of its Wronskian vanishes on I, see for instance [22].

In our last result we will suppose that the family $\{X_\nu,\nu\in\mathbb{R}^n\}$ verifies the hypothesis **(H)**, meaning that the functions A, B, C and κ depend analytically on ν. Therefore, for each ν, the function $M(\,.\,;\nu)$ defines an involution $\sigma(\,.\,;\nu)$ and, by applying Theorem A, we obtain $\ell_i(\,.\,;\nu)$ for $i\geqslant 1$. Then, using also the notation $\nu=(\nu_1,\nu_2,\ldots,\nu_n)$, we prove the following:

Theorem C. *Let $\{X_\nu,\nu\in\mathbb{R}^n\}$ be an analytic family of vector fields verifying hypothesis **(H)**. Assume that the center at the origin is isochronous for $\nu=\bar\nu$ and let (x_ℓ,x_r) be the projection on the x-axis of its period annulus $\mathscr{P}$. Let us define $\xi:=\lim_{x\to x_r}M(x;\bar\nu)$, $g(x):=sgn(x)\sqrt{M(x;\bar\nu)}$ and, for $k=1,2,\ldots,n$,*

$$L_k(x):=\frac{\hat L_k\big(\sqrt{x}\,\big)-\hat L_k\big(-\sqrt{x}\,\big)}{\sqrt{x}},\quad \textit{where } \hat L_k:=\left(\frac{\partial\ell_n(\,.\,;\bar\nu)}{\partial\nu_k}-\frac{\ell_n'(\,.\,;\bar\nu)}{M'(\,.\,;\bar\nu)}\frac{\partial M(\,.\,;\bar\nu)}{\partial\nu_k}\right)\circ g^{-1}.$$

If $\{L_1, L_2, \ldots, L_n\}$ is an extended Chebyshev system on $[0, \xi)$, then the criticality of the pair $(\mathscr{P}, X_{\bar{\nu}})$ with respect to any one-parameter regular deformation $X_{\nu(\varepsilon)}$ with $\nu(0) = \bar{\nu}$ is at most $n-1$.

By a one-parameter *regular* deformation we mean to take a germ of analytic curve $\varepsilon \longmapsto \nu(\varepsilon)$ which is regular at $\nu(0) = \bar{\nu}$, i.e., such that $\nu'(0) \neq 0$. We stress that it is not known whether the maximal number of critical periodic orbits bifurcating from $\mathscr{P}$ under a given multi-parameter deformation X_ν is achieved by some one-parameter deformation $X_{\nu(\varepsilon)}$. However it constitutes an easier problem that gives information about the criticality $\operatorname{Crit}\big((L, X_{\bar{\nu}}), X_\nu\big)$. For reader's convenience, we recall the standard approach to study one-parameter deformations. To this end let $\zeta\colon (0,1) \longrightarrow \mathbb{R}^2$ be an analytic transversal section to $X_{\bar{\nu}}$ on $\mathscr{P}$ and, for each $s \in (0,1)$, let $T(s;\nu)$ be the period of the periodic orbit of X_ν passing through $\zeta(s)$. Then one considers the Taylor development of $T(s;\nu(\varepsilon))$ at $\varepsilon = 0$, say $T(s;\nu(\varepsilon)) = \sum_{i\geqslant 0} T_i(s)\varepsilon^i$. Note that T_0 is constant because by assumption the center is isochronous for $\varepsilon = 0$. Hence, if the center of $X_{\nu(\varepsilon)}$ is non-isochronous for $\varepsilon \neq 0$, there exists some $\ell \geqslant 1$ such that

$$T'(s;\nu(\varepsilon)) = T'_\ell(s)\varepsilon^\ell + \mathrm{o}(\varepsilon^\ell),$$

where T'_ℓ is not identically zero and the remainder is uniform in s on each compact subinterval of $(0,1)$. In this case one can readily show, by applying the Weierstrass Preparation Theorem, that the number of zeros of $T'_\ell(s)$ for $s \in (0,1)$, counted with multiplicities, is an upper bound for the number of critical periods that bifurcate from $\mathscr{P}$. A lower bound is given by the number of simple zeros of $T'_\ell(s)$ in $(0,1)$ by using the Implicit Function Theorem. To our knowledge all the previous papers addressed to this issue (see [4, 10, 15, 16, 20]) follow this approach after computing a formula for T'_1 and T'_2. Note in this regard that $T'_1 \not\equiv 0$ only in case that the deformation is regular, so that as a matter of fact this assumption is implicitly required in those papers as well. This approach is similar to the use of the so called *Melnikov functions* for studying the bifurcation of limit cycles arising from the perturbation of an integrable center. In the present paper we tackle the problem in a different way, instead of expanding the period function in ϵ, we expand the "test function" $\mathscr{B}_\sigma(\ell_i)$ given by Theorem A. This lead us to introduce the notion of criticality, see Definition 2.3, which is more general than the one used in the papers quoted above, even for the case $L = \mathscr{P}$. Indeed, the latter is addressed to one-parameter deformations only and, in this case, it is defined to be the number of simple zeros of T'_ℓ. Besides the novelty in our approach, the interest of Theorem C lies in the fact that it is extremely easy to apply in comparison with the results appearing in the previous papers. We illustrate this with Proposition 4.3, that reproves a result appearing in [15].

Let us conclude this section with an open problem concerning Definition 2.3. Is it true that the criticality of $(\mathscr{P}, X_{\bar{\nu}})$ under a given multi-parameter analytic deformation X_ν can be achieved by an appropriate one-parameter analytic deformation $X_{\nu(\varepsilon)}$, provided that this criticality is finite? L. Gavrilov proves in [18] the analogue property for the cyclicity of an open period annulus by using techniques that seem feasible to be adapted to the problem for critical periods.

3 Proof of the main results

Our first result shows that the function M, as introduced in (2), defines an involution σ on (x_ℓ, x_r).

Lemma 3.1. *If the analytic differential system (1) satisfies the hypothesis **(H)**, then*

(a) κ and C are nonvanishing on (x_ℓ, x_r).

(b) M is an analytic function on (x_ℓ, x_r) with $M(0) = 0$ and $xM'(x) > 0$ for all $x \neq 0$.

Proof. Let us fix some $\bar{x} \in (x_\ell, x_r)$. Since there are periodic orbits in $\mathscr{P}$ intersecting twice the straight-line $x = \bar{x}$ and H is constant on the integral curves of (1), there exist $y_1, y_2 \in \mathbb{R}$ with $y_1 < y_2$ such that $H(\bar{x}, y_1) = H(\bar{x}, y_2)$. Due to $H(\bar{x}, y) = A(\bar{x}) + B(\bar{x})y + C(\bar{x})y^2$, it follows that $(\bar{x}, y) \in \mathscr{P}$ for all $y \in [y_1, y_2]$

and $C(\bar{x}) \neq 0$. This, on account of the analyticity of $p(x,y) = -\frac{B(x)+2C(x)y}{\kappa(x)}$ implies that $\kappa(\bar{x}) \neq 0$. This proves (a). We claim that $H(p) \neq 0$ for all $p \in \mathscr{P}$. To show this let us note that $H(0,0) = 0$ and

$$\nabla H(x,y) = \kappa(x)\big(q(x,y), -p(x,y)\big). \tag{3}$$

By contradiction, suppose that there exists $\hat{p} \in \mathscr{P}$ such that $H(\hat{p}) = 0$. Let $\hat{\gamma}$ be the periodic orbit of (1) passing through $\hat{p}$ and note that $H(p) = 0$ for all $p \in \hat{\gamma}$. Let $\mathscr{U}$ be the bounded component of $\mathbb{R}^2 \setminus \{\hat{\gamma}\}$. By compactness, H takes the maximum and minimum on $\bar{\mathscr{U}}$. Since H vanishes on $\hat{\gamma}$ and the origin, this implies that there exists at least one point $(x_0, y_0) \in \mathscr{U} \setminus \{(0,0)\}$ such that $\nabla H(x_0, y_0) = (0,0)$. This is not possible because, on account of (3) and $\kappa(x_0) \neq 0$, then (x_0, y_0) would be a critical point of (1) inside $\mathscr{P}$. So the claim is true.

In what follows we suppose, without lost of generality, that H is positive on $\mathscr{P}$. In order to show (b) note first that $M(0) = 0$ because $B(0) = 0$, due to $q(0,0) = 0$, and $A(0) = 0$. In addition, since $H(0,y) = C(0)y^2$, we have that C is positive on (x_ℓ, x_r). Consider now any $\bar{x} \in (x_\ell, x_r) \setminus \{0\}$ and take $y_1, y_2 \in \mathbb{R}$ with $y_1 < y_2$ verifying that $H(\bar{x}, y_1) = H(\bar{x}, y_2) > 0$ and $(\bar{x}, y) \in \mathscr{P}$ for all $y \in [y_1, y_2]$. If there exits $\bar{y} \in \mathbb{R}$ such that $H(\bar{x}, \bar{y}) = 0$ then, due to $C(\bar{x}) > 0$, it would follow that $\bar{y} \in (y_1, y_2)$. However this is not possible because then $(\bar{x}, \bar{y}) \in \mathscr{P}$ with $H(\bar{x}, \bar{y}) = 0$, contradicting the previous claim. Thus $A(\bar{x}) + B(\bar{x})y + C(\bar{x})y^2 = 0$ has no real solutions, which yields $(B^2 - 4AC)(\bar{x}) < 0$. Consequently $M(\bar{x}) > 0$. Since $\bar{x}$ is arbitrary, (b) will follow once we show that $M'(\bar{x}) \neq 0$. To this end note that $H(\bar{x}, y_1) = H(\bar{x}, y_2)$ implies that there exists $\tilde{y} \in (y_1, y_2)$ such that $H_y(\bar{x}, \tilde{y}) = 0$. Then $H_x(\bar{x}, \tilde{y}) \neq 0$, otherwise $(\bar{x}, \tilde{y}) \in \mathscr{P}$ would be a critical point due to (3). Since $\tilde{y} = -\left(\frac{B}{2C}\right)(\bar{x})$ and an easy computation shows that

$$H_x\left(\bar{x}, -\left(\frac{B}{2C}\right)(\bar{x})\right) = -\frac{M'(\bar{x})}{2},$$

we get $M'(\bar{x}) \neq 0$. This proves the validity of the result. ∎

The proof of the next result is a straightforward computation that for the sake of shortness is omitted. The interested reader is referred to [14, Lemma 5] for the idea behind the coordinate transformation used.

Lemma 3.2. *Suppose that the analytic differential system* (1) *satisfies the hypothesis* ***(H)****. Then*

$$u = f(x) := \int_0^x \frac{\kappa(s)\, ds}{\sqrt{2|C(s)|}} \quad \text{and } v = \sqrt{2|C(x)|}\, y + \frac{B(x)}{\sqrt{2|C(x)|}}$$

is an analytic change of variables on $\mathscr{P}$ *that brings system* (1) *to the potential system* $\{\dot{u} = -v,\ \dot{v} = V'(u)\}$, *where* $V(u) = \big(M \circ f^{-1}\big)(u)$.

The proof of Theorem A strongly relies on the results in [29], which show its validity in the particular case that (1) is a potential system (i.e., for $B \equiv 0$, $C \equiv 1/2$ and $\kappa \equiv 1$). For reader's convenience we restate these results according to the notation used in the present paper. So let us suppose that the origin is a center for the potential system

$$\begin{cases} \dot{u} = -v, \\ \dot{v} = V'(u), \end{cases} \tag{4}$$

where V is an analytic function with $V(0) = 0$. Let (u_ℓ, u_r) be the projection on the u-axis of its period annulus and consider moreover the involution $\hat{\sigma}$ defined by $V = V \circ \hat{\sigma}$, which is analytic on (u_ℓ, u_r). The following proposition collects the results in [29] that we shall need.

Proposition 3.3. *Setting* $\hat{\mu}_0 = -1$, *define recursively*

$$\hat{\mu}_i := \left(\frac{1}{2} + \frac{1}{2i-3}\right)\hat{\mu}_{i-1} + \frac{V}{2i-3}\left(\frac{\hat{\mu}_{i-1}}{V'}\right)' \quad \text{and } \hat{\ell}_i := \frac{\hat{\mu}_i}{V'} \text{ for } i \geqslant 1.$$

Then the center at the origin of the potential system (4) *verifies the following:*

(*a*) *It is isochronous if, and only if,* $\mathscr{B}_{\hat\sigma}\big(\hat\ell_1\big)\equiv 0$*, which in turn is equivalent to* $\mathscr{B}_{\hat\sigma}\big(\hat\ell_i\big)\equiv 0$ *for all* $i\geqslant 1$.

(*b*) *If the number of zeros of* $\mathscr{B}_{\hat\sigma}\big(\hat\ell_i\big)$ *on* $(0,u_r)$*, counted with multiplicities, is* $n\geqslant 0$ *and it holds that* $i>n$*, then the number of critical periods of the center of* (4)*, counted with multiplicities, is at most* n.

(*c*) *If the period function is monotone, then* $\mathscr{B}_{\hat\sigma}\left(\frac{Id}{\sqrt{V}}\right)$ *is monotone on* $(0,u_r)$.

Proof. Let γ_h be the periodic orbit of (4) inside the energy level $\big\{\frac{1}{2}v^2+V(u)=h\big\}$ and let $T(h)$ be its period. Then [29, Theorem A] shows that

$$T'(h)=\frac{1}{h^i}\int_{\gamma_h}\hat\mu_i(u)\,v^{2i-3}du\ \text{ for all } i\geqslant 1.$$

Let the projection of γ_h on the u-axis be the interval (u_h^-,u_h^+). Then $V(u_h^\pm)=h$, so that $\hat\sigma(u_h^+)=u_h^-$, and $0\in(u_h^-,u_h^+)$ for all h. Taking this into account we get

$$h^iT'(h)=4\int_0^{u_h^+}\mathscr{P}_{\hat\sigma}\big(\hat\mu_i\big)(u)\big(2h-2V(u)\big)^{\frac{2i-3}{2}}du,\ \text{ where } \mathscr{P}_{\hat\sigma}\big(\hat\mu_i\big)(u):=\frac{\hat\mu_i(u)-\hat\mu_i\big(\hat\sigma(u)\big)\hat\sigma'(u)}{2}.$$

In order to prove (*a*) let us suppose first that the center is isochronous. Then, since only non-degenerate centers can be isochronous, $V(0)=V'(0)=0$ and $V''(0)\neq 0$. One can easily prove by induction that this implies $\hat\mu_i(0)=0$ for all $i\geqslant 1$ and, accordingly, the functions $\hat\ell_i$ are analytic at $u=0$. Therefore $\mathscr{P}_{\hat\sigma}\big(\hat\mu_i\big)$ is analytic at $u=0$ for all $i\geqslant 1$. Taking this into account and the fact that $u_h^\pm\longrightarrow 0$ as h tends to 0, it is clear from the above expression that $T'\equiv 0$ implies $\mathscr{P}_{\hat\sigma}\big(\hat\mu_i\big)\equiv 0$. This shows that the isochronicity of the center implies that $\mathscr{B}_{\hat\sigma}\big(\hat\ell_i\big)\equiv 0$ for all $i\geqslant 1$ because, due to $V'(u)=V'\big(\hat\sigma(u)\big)\hat\sigma'(u)$, we have that

$$\mathscr{B}_{\hat\sigma}\big(\hat\ell_i\big)=\mathscr{B}_{\hat\sigma}\left(\frac{\hat\mu_i}{V'}\right)=V'\mathscr{P}_{\hat\sigma}\big(\hat\mu_i\big).$$

Since the reverse implication is obvious, (*a*) is proved. The assertion in (*b*) follows from the above equality as well because [29, Theorem A] shows that if for some i the number of zeros of $\mathscr{P}_{\hat\sigma}\big(\hat\ell_i\big)$ on $(0,u_r)$, counted with multiplicities, is $n<i$ then the number of critical periods of the center of (4), counted with multiplicities, is at most n. Finally (*c*) follows straightforward from [29, Theorem 2.6]. ■

Proof of Theorem A. By applying Lemma 3.2 we know that the coordinate transformation

$$u=f(x):=\int_0^x\frac{\kappa(s)\,ds}{\sqrt{2|C(s)|}}\ \text{ and } v=\sqrt{2|C(x)|}\,y+\frac{B(x)}{\sqrt{2|C(x)|}}$$

brings system (1) to the potential system (4) with $V:=M\circ f^{-1}$. The period annulus of the center of the latter system is the image of $\mathscr{P}$ by the above coordinate transformation and $(u_\ell,u_r)=f\big((x_\ell,x_r)\big)$. Moreover the involution defined by V is $\hat\sigma:=f\circ\sigma\circ f^{-1}$, where recall that σ is the one defined by M. Indeed, the equality $V(u)=V\big(\hat\sigma(u)\big)$ writes as $\big(M\circ f^{-1}\big)(u)=\big(M\circ f^{-1}\big)\big(\hat\sigma(u)\big)$, which is equivalent to $M\big(f^{-1}(u)\big)=\big(M\circ\sigma\big)\big(f^{-1}(u)\big)$, and this is true because $M=M\circ\sigma$ by definition.

We apply Proposition 3.3 to this potential system and we thus obtain the functions $\hat\mu_i$ and $\hat\ell_i$ defined in its statement in terms of $V=M\circ f^{-1}$ and $\hat\sigma=f\circ\sigma\circ f^{-1}$. We claim that $\hat\mu_i=\mu_i\circ f^{-1}$ for all $i\geqslant 0$. We prove this by induction. The base case $i=0$ is obvious because $\mu_0=\hat\mu_0=-1$. To show the inductive step we note that

$$f'=\frac{\kappa}{\sqrt{2|C|}}\ \text{ and } V'=\frac{M'}{f'}\circ f^{-1}=\frac{\sqrt{2|C|}M'}{\kappa}\circ f^{-1}. \tag{5}$$

Thus, since $\hat{\mu}_{i-1}=\mu_{i-1}\circ f^{-1}$ by the inductive hypothesis, we obtain

$$\begin{aligned}\hat{\mu}_i &= \left(\frac{1}{2}+\frac{1}{2i-3}\right)\hat{\mu}_{i-1}+\frac{V}{2i-3}\left(\frac{\hat{\mu}_{i-1}}{V'}\right)' \\ &= \left(\frac{1}{2}+\frac{1}{2i-3}\right)\mu_{i-1}\circ f^{-1}+\frac{M\circ f^{-1}}{2i-3}\left(\frac{\kappa\,\mu_{i-1}}{\sqrt{2|C|}M'}\circ f^{-1}\right)' \\ &= \left(\frac{1}{2}+\frac{1}{2i-3}\right)\mu_{i-1}\circ f^{-1}+\left(\frac{\sqrt{|C|}M}{(2i-3)\kappa}\left(\frac{\kappa\,\mu_{i-1}}{\sqrt{|C|}M'}\right)'\right)\circ f^{-1}.\end{aligned}$$

Hence $\hat{\mu}_i=\mu_i\circ f^{-1}$, and so the claim is true. Consequently we get that

$$\begin{aligned}\mathscr{B}_{\hat{\sigma}}\big(\hat{\ell}_i\big)(u) = \mathscr{B}_{\hat{\sigma}}\left(\frac{\hat{\mu}_i}{V'}\right)(u) &= \mathscr{B}_{\hat{\sigma}}\left(\frac{\kappa\,\mu_i}{\sqrt{2|C|}M'}\circ f^{-1}\right)(u) \\ &= \left(\frac{\kappa\,\mu_i}{\sqrt{2|C|}M'}\right)\big(f^{-1}(u)\big)-\left(\frac{\kappa\,\mu_i}{\sqrt{2|C|}M'}\right)\big(f^{-1}(\hat{\sigma}(u))\big) \\ &= \left(\frac{\kappa\,\mu_i}{\sqrt{2|C|}M'}\right)\big(f^{-1}(u)\big)-\left(\frac{\kappa\,\mu_i}{\sqrt{2|C|}M'}\right)\big(\sigma(f^{-1}(u))\big) \\ &= \mathscr{B}_{\sigma}\left(\frac{\kappa\,\mu_i}{\sqrt{2|C|}M'}\right)\big(f^{-1}(u)\big)=\frac{\mathscr{B}_{\sigma}\big(\ell_i\big)\big(f^{-1}(u)\big)}{\sqrt{2}}.\end{aligned}$$

The first equality above follows from the definition of $\hat{\ell}_i$, the second one taking (5) and $\hat{\mu}_i=\mu_i\circ f^{-1}$ into account, while in the third one we use the definition of $\hat{\sigma}$-balance. We obtain the fourth equality using that $\hat{\sigma}=f\circ\sigma\circ f^{-1}$ and, finally, we get the fifth and sixth equalities from the definitions of σ-balance and ℓ_i, respectively. Accordingly $\mathscr{B}_{\sigma}\big(\ell_i\big)=\sqrt{2}\,\mathscr{B}_{\hat{\sigma}}\big(\hat{\ell}_i\big)\circ f$ and, since f maps diffeomorphically (x_ℓ,x_r) to (u_ℓ,u_r) with $f(0)=0$, it is clear that (a) and (b) follow, respectively, by applying (a) and (b) in Proposition 3.3.

Finally, in order to prove (c) we note that

$$\begin{aligned}\mathscr{B}_{\hat{\sigma}}\left(\frac{Id}{\sqrt{V}}\right)(u) &= \frac{u}{\sqrt{V(u)}}-\frac{\hat{\sigma}(u)}{\sqrt{V\big(\hat{\sigma}(u)\big)}}=\frac{\big(f\circ f^{-1}\big)(u)}{\sqrt{\big(M\circ f^{-1}\big)(u)}}-\frac{\hat{\sigma}(u)}{\sqrt{\big(M\circ f^{-1}\big)\big(\hat{\sigma}(u)\big)}} \\ &= \frac{\big(f\circ f^{-1}\big)(u)}{\sqrt{\big(M\circ f^{-1}\big)(u)}}-\frac{\big(f\circ\sigma\circ f^{-1}\big)(u)}{\sqrt{\big(M\circ\sigma\circ f^{-1}\big)(u)}}=\mathscr{B}_{\sigma}\left(\frac{f}{\sqrt{M}}\right)\big(f^{-1}(u)\big).\end{aligned}$$

Therefore $\mathscr{B}_{\hat{\sigma}}\Big(\frac{Id}{\sqrt{V}}\Big)\circ f=\mathscr{B}_{\sigma}\Big(\frac{f}{\sqrt{M}}\Big)$ and so the assertion in (c) follows by applying (c) in Proposition 3.3 because f is a diffeomorphism. This proves the result. ∎

Proof of Theorem B. By a well-known property of the resultant, see [3, Chapter 3], $T(x,y)$ is inside the ideal generated by $L(x,z)$ and $L(y,z)$ in $\mathbb{C}[x,y,z]$. Therefore

$$T(x,y)=a(x,y,z)L(x,z)+b(x,y,z)L(y,z)\ \text{ for some } a,b\in\mathbb{C}[x,y,z]. \tag{6}$$

This implies that $T(\bar{x},\bar{y})=0$ at any point such that $\ell(\bar{x})=\ell(\bar{y})$. Indeed, this follows by evaluating the above equality at $(\bar{x},\bar{y},\bar{z})$, where $\bar{z}=\ell(\bar{x})=\ell(\bar{y})$, and using that, by hypothesis, $L\big(x,\ell(x)\big)\equiv 0$. Thus

$$S(x,y)=0 \text{ if } y=\sigma(x) \text{ and } T(x,y)=0 \text{ if } \ell(x)=\ell(y). \tag{7}$$

Suppose that $\mathscr{B}_\sigma\big(\ell\big)(x) = \ell(x) - \ell\big(\sigma(x)\big)$ has a zero at $x_0 \in (x_\ell, x_r) \setminus \{0\}$. Let us define $y_0 := \sigma(x_0)$ and $p_0 := (x_0, y_0)$. Then, on account of (7), the algebraic curves $S = 0$ and $T = 0$ intersect at $p_0 \in \mathbb{R}^2$. Hence, due to $\mathscr{R}(x) = \mathrm{Res}_y\big(S(x,y), T(x,y)\big)$, this shows that $\mathscr{R}(x_0) = 0$ and so (a) follows.

Let us turn now to the proof of (b). In what follows $\mathbb{C}\{x,y\}$ will denote the ring of convergent power series. Recall, see [21, §3.2] for instance, that if $g \in \mathbb{C}\{x,y\}$ is irreducible then the intersection multiplicity of any $f \in \mathbb{C}\{x,y\}$ with g is given by

$$i_0(f,g) := \mathrm{ord}_t\, f\big(x(t), y(t)\big) = \sup\big\{m \in \mathbb{N} : t^m \text{ divides } f\big(x(t), y(t)\big)\big\},$$

where $t \longmapsto \big(x(t), y(t)\big)$ is a parametrization for the plane curve germ defined by g. In case that $g = g_1 g_2 \ldots g_s$ with g_i irreducible, then $i_0(f,g) := i_0(f,g_1) + \ldots + i_0(f,g_s)$. For convenience we translate $p_0 = (x_0, y_0)$ to the origin by setting $\hat{S}(x,y) := S(x+x_0, y+y_0)$ and $\hat{T}(x,y) := T(x+x_0, y+y_0)$. The polynomial $\hat{S}$ may be irreducible or not in $\mathbb{C}\{x,y\}$ but, since the intersection multiplicity is additive with respect to the branches and $\hat{S}\big(t, \sigma(t+x_0) - y_0\big) = 0$ for all $t \approx 0$, we have that

$$i_{p_0}(S,T) = i_0(\hat{S}, \hat{T}) \geqslant \mathrm{ord}_t\, \hat{T}\big(t, \sigma(t+x_0) - y_0\big) = \mathrm{ord}_t\, T\big(t+x_0, \sigma(t+x_0)\big). \tag{8}$$

Let us set $\hat{L}(x,z) := L\big(x+x_0, z+\ell(x_0)\big)$. Then, on account of $\hat{L}(x,z) = 0$ for $z = \ell(x+x_0) - \ell(x_0)$ and the fact that $x \longmapsto \ell(x+x_0) - \ell(x_0)$ is analytic at $x = 0$, the Weierstrass Division Theorem, see [1, pp. 238–339] or [21, Theorem 1.8], easily implies that $z - \ell(x+x_0) + \ell(x_0)$ divides $\hat{L}(x,z)$ as elements of $\mathbb{C}\{x,z\}$. Hence $\hat{L}(x,z) = \mathscr{U}(x,z)\big(z - \ell(x+x_0) + \ell(x_0)\big)$ with $\mathscr{U} \in \mathbb{C}\{x,z\}$. Accordingly

$$L(x,z) = \mathscr{U}\big(x - x_0, z - \ell(x_0)\big)\big(z - \ell(x)\big) \text{ for } x \approx x_0 \text{ and } z \approx \ell(x_0).$$

Since $\ell(x_0) = \ell(y_0)$, we can evaluate the above expression at $z = \ell(y)$ with $y \approx y_0$. In doing so we obtain

$$L\big(x, \ell(y)\big) = \mathscr{U}\big(x - x_0, \ell(y) - \ell(x_0)\big)\big(\ell(y) - \ell(x)\big) \text{ for } (x,y) \approx (x_0, y_0).$$

Therefore, on account of $L\big(y, \ell(y)\big) \equiv 0$, from (6) we get that if $(x,y) \approx (x_0, y_0)$ then

$$T(x,y) = a\big(x, y, \ell(y)\big) L\big(x, \ell(y)\big) = a\big(x, y, \ell(y)\big) \mathscr{U}\big(x - x_0, \ell(y) - \ell(x_0)\big)\big(\ell(y) - \ell(x)\big).$$

Due to $y_0 = \sigma(x_0)$, we can evaluate this expression at $x = t + x_0$ and $y = \sigma(t+x_0)$ with $t \approx 0$ and conclude that $\mathrm{ord}_t\, T\big(t+x_0, \sigma(t+x_0)\big) \geqslant \mathrm{ord}_t\big\{\ell(t+x_0) - \ell\big(\sigma(t+x_0)\big)\big\} = \mathrm{mult}\big(\mathscr{B}_\sigma(\ell), x_0\big)$. Thus, from (8),

$$\mathrm{mult}\big(\mathscr{B}_\sigma(\ell), x_0\big) \leqslant i_{p_0}(S,T). \tag{9}$$

Let us suppose finally that $\{(x_0, y) \in \mathbb{C}^2 : S(x_0,y) = T(x_0,y) = 0\} = \{p_0, p_1, \ldots, p_d\}$. Then, taking into account that $\mathscr{R}(x) = \mathrm{Res}_y\big(S(x,y), T(x,y)\big)$, we get

$$\mathrm{mult}(\mathscr{R}, x_0) = \sum_{k=0}^{d} i_{p_k}(S,T) \geqslant i_{p_0}(S,T). \tag{10}$$

The first equality above is a well-known result on intersection theory (see [2, Proposition 5] or [13, §1.6]) and it is here where the assumption that $s(x)$ and $t(x)$ do not vanish simultaneously at $x = x_0$ is needed. Thus (b) follows because (9) and (10) imply that $\mathrm{mult}\big(\mathscr{B}_\sigma(\ell), x_0\big) \leqslant \mathrm{mult}(\mathscr{R}, x_0)$, as desired. This finishes the proof of the result. ∎

The next result is a "local version" of the assertion (b) in Theorem A. In its statement, and in what follows, for a given periodic orbit $\bar{\gamma}$, $\mathrm{Int}\,(\bar{\gamma})$ stands for the bounded connected component of $\mathbb{R}^2 \setminus \{\bar{\gamma}\}$.

Lemma 3.4. *Following the notation in Theorem A, suppose that the number of zeros, counted with multiplicities, of $\mathscr{B}_\sigma(\ell_i)$ on $(0, \bar{x})$, where $\bar{x} < x_r$, is $n \geqslant 0$ and that it holds $i > n$. Then system (1) has at most $n-1$ critical periodic orbits in $\mathrm{Int}\,(\bar{\gamma})$, where $\bar{\gamma}$ is the periodic orbit inside the level curve $\{H = M(\bar{x})\}$.*

Proof. To show this it suffices to note that the discriminant of $A(x)+B(x)y+C(x)y^2 = M(\bar{x})$ with respect to y is $M(x) - M(\bar{x})$, so that the projection of $\bar{\gamma}$ on the x-axis is $\big(\sigma(\bar{x}), \bar{x}\big)$, and then the result follows by applying (b) in Theorem A to the "period annulus" $\mathrm{Int}\,(\bar{\gamma}) \setminus \{(0,0)\}$. ∎

Proof of Theorem C. Let $\varepsilon \longmapsto \nu(\varepsilon)$ be a germ of analytic curve regular at $\nu = \bar{\nu}$, i.e., such that $\nu(\varepsilon) = \bar{\nu} + \hat{\nu}\varepsilon + \mathrm{o}(\varepsilon)$ with $\hat{\nu} \in \mathbb{R}^n \setminus \{0\}$, and consider the one-parameter family of centers $X_{\nu(\varepsilon)}$, which by assumption verifies the hypothesis **(H)**. Then, by (b) in Lemma 3.1, $M\big(x; \nu(\varepsilon)\big) = a(\varepsilon)x^{2k} + \mathrm{o}(x^{2k})$ with $a(\varepsilon) > 0$ and $k \in \mathbb{N}$. Since the center at the origin is isochronous for $\varepsilon = 0$, and only non-degenerated centers can be isochronous, we have that $k = 1$. Hence

$$g_\varepsilon(x) := \mathrm{sgn}(x)\sqrt{M(x;\nu(\varepsilon))} = x\sqrt{\frac{M(x;\nu(\varepsilon))}{x^2}}$$

is an analytic diffeomorphism on the projection of the period annulus of $X_{\nu(\varepsilon)}$ on the x-axis. Note moreover that $g_\varepsilon(0) = 0$ and $\sigma(x;\nu(\varepsilon)) = g_\varepsilon^{-1}\big(-g_\varepsilon(x)\big)$, where recall that $\sigma(\,.\,;\nu)$ is the involution defined by $M(\,.\,;\nu)$. Therefore, if we define

$$F(x;\nu) := \ell_n(x;\nu) - \ell_n\big(\sigma(x;\nu);\nu\big),$$

then $x \longmapsto F\big(g_\varepsilon^{-1}(x);\nu(\varepsilon)\big)$ is an odd function, so that

$$\frac{F\big(g_\varepsilon^{-1}(\sqrt{x}\,);\nu(\varepsilon)\big)}{\sqrt{x}} = \frac{\ell_n\big(g_\varepsilon^{-1}(\sqrt{x}\,);\nu(\varepsilon)\big) - \ell_n\big(g_\varepsilon^{-1}(-\sqrt{x}\,);\nu(\varepsilon)\big)}{\sqrt{x}}$$

is analytic at $x = 0$. Since, by (a) in Theorem A, $F(x;\nu(0)) = 0$ for all x, it follows that

$$\frac{\partial}{\partial\varepsilon}\frac{F\big(g_\varepsilon^{-1}(\sqrt{x}\,);\nu(\varepsilon)\big)}{\sqrt{x}}\bigg|_{\varepsilon=0} = \frac{1}{\sqrt{x}}\sum_{k=1}^{n}\hat{\nu}_k\frac{\partial F\big(g_0^{-1}(\sqrt{x}\,);\nu(0)\big)}{\partial\nu_k},$$

where we write the perturbative direction as $\hat{\nu} = (\hat{\nu}_1, \hat{\nu}_2, \ldots, \hat{\nu}_n)$. In order to compute the partial derivative $F_{\nu_k}(\,.\,;\bar{\nu})$ we first note that $F(x;\bar{\nu}) \equiv 0$ implies $\ell_n'\big(x;\bar{\nu}\big) - \ell_n'\big(\sigma(x;\bar{\nu});\bar{\nu}\big)\sigma'(x;\bar{\nu}) = 0$ for all x. Consequently

$$\ell_n'\big(\sigma(x;\bar{\nu});\bar{\nu}\big) = \frac{\ell_n'(x;\bar{\nu})}{\sigma'(x;\bar{\nu})}. \tag{11}$$

Similarly, $M(x;\nu) = M\big(\sigma(x;\nu);\nu\big)$ for all x and ν, implies

$$M'\big(\sigma(x;\bar{\nu});\bar{\nu}\big) = \frac{M'(x;\bar{\nu})}{\sigma'(x;\bar{\nu})} \tag{12}$$

and $M_{\nu_k}(x;\bar{\nu}) = M'\big(\sigma(x;\bar{\nu});\bar{\nu}\big)\sigma_{\nu_k}(x;\bar{\nu}) + M_{\nu_k}\big(\sigma(x;\bar{\nu});\bar{\nu}\big)$. The combination of these two equalities yields

$$\frac{\sigma_{\nu_k}(x;\bar{\nu})}{\sigma'(x;\bar{\nu})} = \frac{M_{\nu_k}(x;\bar{\nu}) - M_{\nu_k}\big(\sigma(x;\bar{\nu});\bar{\nu}\big)}{M'(x;\bar{\nu})} = \frac{\mathscr{B}_{\bar{\sigma}}\big(M_{\nu_k}(\,.\,;\bar{\nu})\big)(x)}{M'(x;\bar{\nu})}, \tag{13}$$

where recall that $\bar{\sigma} = \sigma(\,.\,;\bar{\nu})$. Note in addition that, from (11) and (12), we obtain

$$\frac{\ell_n'\big(\bar{\sigma}(x);\bar{\nu}\big)}{M'\big(\bar{\sigma}(x);\bar{\nu}\big)} = \frac{\ell_n'\big(x;\bar{\nu}\big)}{M'\big(x;\bar{\nu}\big)}. \tag{14}$$

By using these identities we get that

$$
\begin{aligned}
\frac{\partial F(x;\bar\nu)}{\partial\nu_k} &= \frac{\partial \ell_n(x;\bar\nu)}{\partial\nu_k} - \frac{\partial \ell_n\big(\sigma(x;\bar\nu);\bar\nu\big)}{\partial\nu_k} - \ell_n'\big(\sigma(x;\bar\nu);\bar\nu\big)\frac{\partial\sigma(x;\bar\nu)}{\partial\nu_k}\\
&= \mathscr{B}_{\bar\sigma}\left(\frac{\partial \ell_n(\,.\,;\bar\nu)}{\partial\nu_k}\right)(x) - \frac{\ell_n'(x;\bar\nu)}{\sigma'(x;\bar\nu)}\frac{\partial\sigma(x;\bar\nu)}{\partial\nu_k}\\
&= \mathscr{B}_{\bar\sigma}\left(\frac{\partial \ell_n(\,.\,;\bar\nu)}{\partial\nu_k}\right)(x) - \frac{\ell_n'(x;\bar\nu)}{M'(x;\bar\nu)}\,\mathscr{B}_{\bar\sigma}\left(\frac{\partial M(\,.\,;\bar\nu)}{\partial\nu_k}\right)(x)\\
&= \mathscr{B}_{\bar\sigma}\left(\frac{\partial \ell_n(\,.\,;\bar\nu)}{\partial\nu_k} - \frac{\ell_n'(\,.\,;\bar\nu)}{M'(\,.\,;\bar\nu)}\,\frac{\partial M(\,.\,;\bar\nu)}{\partial\nu_k}\right)(x).
\end{aligned}
$$

To be more precise, in the first equality above we take the definition of F into account, whereas the second and third equalities follow by using (11) and (13), respectively. Finally the fourth equality follows from (14). Hence, on account of the definition of $\hat L_k$, we have proved that

$$\frac{\partial F(x;\bar\nu)}{\partial\nu_k} = \mathscr{B}_{\bar\sigma}\big(\hat L_k\circ g_0\big)(x) = \hat L_k\big(g_0(x)\big) - \hat L_k\big(-g_0(x)\big),$$

where in the last equality we take $\bar\sigma(x) = g_0^{-1}\big(-g_0(x)\big)$ into account. Consequently

$$\frac{1}{\sqrt{x}}\,\frac{\partial F\big(g_0^{-1}(\sqrt{x}\,);\nu(0)\big)}{\partial\nu_k} = \frac{\hat L_k(\sqrt{x}\,) - \hat L_k(-\sqrt{x}\,)}{\sqrt{x}} = L_k(x).$$

Accordingly, since $F\big(\,.\,;\nu(0)\big) = 0$, we can assert that

$$\frac{F\big(g_\varepsilon^{-1}(\sqrt{x}\,);\nu(\varepsilon)\big)}{\sqrt{x}} = \varepsilon\sum_{k=1}^{n}\hat\nu_k L_k\big(x\big) + \mathrm{o}(\varepsilon).$$

Recall at this point that, by hypothesis, $\{L_1, L_2, \ldots, L_n\}$ is an extended Chebyshev system on $[0,\xi)$. Define $h_\alpha := \xi - \alpha^2$ if ξ is finite and $h_\alpha := 1/\alpha^2$ otherwise. Then $\hat\nu_1 L_1 + \hat\nu_2 L_2 + \ldots + \hat\nu_n L_n$ has at most $n-1$ zeros on $[0,h_\alpha]$, counted with multiplicities, for any $\alpha\approx 0$. By the Weierstrass Preparation Theorem and an easy compactness argument, there exits $\varepsilon_0 > 0$ such that if $\varepsilon\in(-\varepsilon_0,\varepsilon_0)\setminus\{0\}$, then $x\longmapsto F\big(g_\varepsilon^{-1}(\sqrt{x}\,);\nu(\varepsilon)\big)$ has at most $n-1$ zeros for $x\in[0,h_\alpha]$, counted with multiplicities. Next we perform the change of variable $u = g_\varepsilon^{-1}(\sqrt{x}\,)$, which satisfies $M\big(u;\nu(\varepsilon)\big) = x$ and, for $\varepsilon = 0$, maps the interval $(0,\xi)$ to $(0,x_r)$. In doing so, due to $h_\alpha \longrightarrow \xi$ as α tends to zero, we can assert that for all $\delta > 0$ there exists $\varepsilon_0 > 0$ such that if $\varepsilon\in(-\varepsilon_0,\varepsilon_0)\setminus\{0\}$, then

$$F\big(u;\nu(\varepsilon)\big) = \ell_n\big(u;\nu(\varepsilon)\big) - \ell_n\big(\sigma(u;\nu(\varepsilon));\nu(\varepsilon)\big)$$

has at most $n-1$ zeros for $u\in(0,x_r-\delta)$, counted with multiplicities. (Here, and in what follows, we assume $x_r < +\infty$. The case $x_r = +\infty$ follows exactly the same way replacing $x_r - \delta$ by $1/\delta$ with the obvious modifications.) Note at this point that $F\big(\,.\,;\nu(\varepsilon)\big)$ is precisely the $\sigma\big(\,.\,;\nu(\varepsilon)\big)$-balance of $\ell_n\big(\,.\,;\nu(\varepsilon)\big)$.

We claim that the criticality of $(\mathscr{P}, X_{\bar\nu})$ with respect to $X_{\nu(\varepsilon)}$ is at most $n-1$. By contradiction, suppose that it is greater. Then, see Definition 2.3, there exists a compact invariant set K_c of $X_{\nu(0)}$ inside $\mathscr{P}$ with the property that, for any neighbourhood V of K_c and any $\eta > 0$, there exists some $\rho\in(-\eta,\eta)$ such that $X_{\nu(\rho)}$ has at least n critical periodic orbits contained in V. Let V_c be one of these neighbourhoods verifying that $\bar V_c$ is compact and inside $\mathscr{P}$. For the sake of clarity we split the proof of the claim:

1. Let $\bar\delta > 0$ be small enough such that the periodic orbit of $X_{\nu(0)}$ inside the level curve

$$\big\{(x,y)\in\mathscr{P} : H\big(x,y;\nu(0)\big) = M\big(x_r - \bar\delta;\nu(0)\big)\big\}\,,$$

say γ_0, satisfies that $\bar V_c \subset \mathrm{Int}\,(\gamma_0)$.

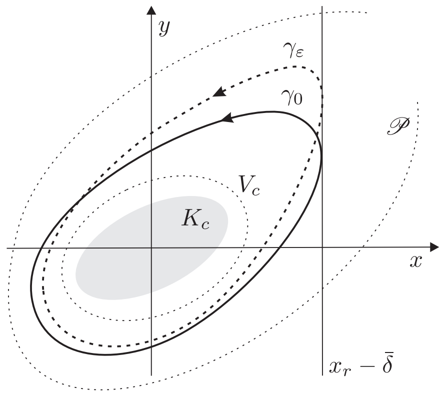

Figure 1: Sketch of the argument in the proof of Theorem C to show by contradiction that the criticality of the pair $(\mathscr{P}, X_{\bar{\nu}})$ with respect to $X_{\nu(\varepsilon)}$ is at most $n-1$.

2. By continuity, we can take $\bar{\varepsilon}_1 > 0$ small enough such that if $\varepsilon \in (-\bar{\varepsilon}_1, \bar{\varepsilon}_1)$, then the periodic orbit of $X_{\nu(\varepsilon)}$ inside the level curve

$$\big\{(x,y) \in \mathscr{P} : H\big(x,y;\nu(\varepsilon)\big) = M\big(x_r - \bar{\delta};\nu(\varepsilon)\big)\big\},$$

say γ_ε, still verifies that $\bar{V}_c \subset \operatorname{Int}(\gamma_\varepsilon)$.

3. Let $\bar{\varepsilon}_2 > 0$ small enough such that, taking any ε with $0 < |\varepsilon| < \bar{\varepsilon}_2$, then $F\big(x;\nu(\varepsilon)\big)$ has at most $n-1$ zeros for $x \in (0, x_r - \bar{\delta})$, counted with multiplicities.

4. Define $\bar{\varepsilon}_3 := \min(\bar{\varepsilon}_1, \bar{\varepsilon}_2)$. Then, if $\varepsilon \in (-\bar{\varepsilon}_3, \bar{\varepsilon}_3) \setminus \{0\}$, by applying Lemma 3.4 it follows that $X_{\nu(\varepsilon)}$ has at most $n-1$ critical periodic orbits inside $\operatorname{Int}(\gamma_\varepsilon)$ and, on the other hand, $\bar{V}_c \subset \operatorname{Int}(\gamma_\varepsilon)$. Clearly this contradicts the fact that, by construction, there exists $\rho \in (-\bar{\varepsilon}_3, \bar{\varepsilon}_3)$ such that $X_{\nu(\rho)}$ has at least n critical periodic orbits contained in V_c.

This shows the validity of the claim and concludes the proof of the result. ■

Remark 3.5. It is worth pointing out that in fact we have proved that the criticality of $\big(\mathscr{P} \cup \{(0,0)\}, X_{\bar{\nu}}\big)$ with respect to $X_{\nu(\varepsilon)}$ is at most $n-1$. In other words, the result also applies to the critical periods that bifurcate from the center. □

4 Applications

The family of the so-called *Loud's dehomogenized centers* $\{X_\nu, \nu \in \mathbb{R}^2\}$, where $\nu := (D, F)$ and

$$X_\nu \quad \begin{cases} \dot{x} = -y + xy, \\ \dot{y} = x + Dx^2 + Fy^2, \end{cases} \tag{15}$$

constitutes an appropriate testing ground for the tools developed in the present paper because it is perhaps the most studied one from the point of view of the period function (see [8, 10, 14, 16, 23, 24, 26–28, 34, 36, 37] and references therein). This family verifies the hypothesis **(H)** since $\kappa(x) = (1-x)^{-2F-1}$ is an integrating factor with first integral given by $H(x,y) = A(x) + C(x)y^2$, where

$$A(x) = \int_0^x (s + Ds^2)\kappa(s)ds \;\text{ and }\; C(x) = \frac{1}{2}(1-x)^{-2F}.$$

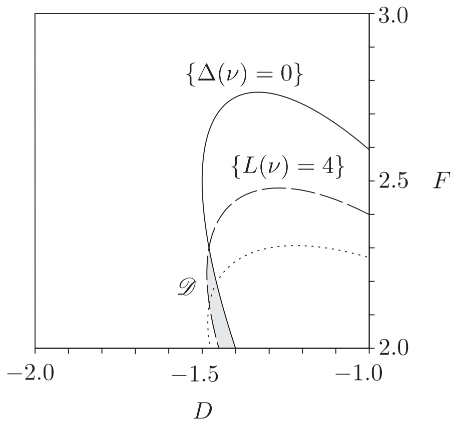

Figure 2: Numerical drawing of the region $\mathscr{D}$ in Proposition 4.1.

In fact a computation shows that if $F \notin \{0, \frac{1}{2}, 1\}$ then $A(x) = (1-x)^{-2F}q(x) - q(0)$, where

$$q(x) := \tfrac{D}{2(F-1)}\,x^2 - \tfrac{D-F+1}{(F-1)(2F-1)}\,x + \tfrac{D-F+1}{2F(F-1)(2F-1)}. \tag{16}$$

Our first application concerns with some lower bounds for the number of critical periods of the differential system (15) established in [27]. To be more precise, Theorem 5.2 in that paper determines a region $\mathscr{U}$ in the parameter plane where the corresponding center has at least two critical periods. We determine a region $\mathscr{D}$ with the same property that, although intersects $\mathscr{U}$, verifies that $\mathscr{D}\setminus\mathscr{U}$ is nonempty. Accordingly we provide a new region with at least two critical periods. Figure 2 displays a numerical drawing of $\mathscr{D}$ in light grey. (The dotted line in this figure, which is a component of $\partial\mathscr{U}$, splits $\mathscr{D}$ in two regions and $\mathscr{D}\setminus\mathscr{U}$ is the smallest one.) In order to state our result we note that the first period constant of the center at the origin of (15) is given by $\Delta(\nu) := 4F^2-5F+10DF-D+10D^2+1$, see [27]. Setting $U := \{\nu\in\mathbb{R}^2 : F>2, F+D>0, D<0\}$, let us define $\mathscr{D} := \{\nu \in U : \Delta(\nu) > 0 \text{ and } L(\nu) < 4\}$, where

$$L(\nu) := \frac{\sqrt{2}\,(1-\rho)^{-F}}{F\sqrt{-q(0)}} \text{ with } \rho := \frac{F(1+D-F)+\sqrt{F(F-1)(D+F)(F-1-D)}}{DF(2F-1)}.$$

Taking this notation into account we prove the following:

Proposition 4.1. *If $\nu_0 \in \mathscr{D}$ then the center at the origin of X_{ν_0} has at least two critical periods.*

Proof. First we will show that the period function is not monotone by applying (c) in Theorem A. With this end in view recall that the involution $z=\sigma(x)$ is defined by means of $M(x)=M(z)$, which in this case writes as $(1-x)^{-2F}q(x) = (1-z)^{-2F}q(z)$, where q is the second degree polynomial in (16). The implicit derivation of this equality yields

$$z = \sigma(x) = -x - \frac{2}{3}(1+D+2F)x^2 - \frac{4}{9}(1+D+2F)^2x^3 + \mathrm{o}(x^3). \tag{17}$$

On the other hand an easy computation shows that

$$g(x) = \frac{1}{\sqrt{M(x)}}\int_0^x \frac{\kappa(s)\,ds}{\sqrt{|C(s)|}} = \frac{\sqrt{2}}{F}\frac{(1-x)^{-F}-1}{\sqrt{(1-x)^{-2F}q(x)-q(0)}}$$

and then from (17) it follows that $\mathscr{B}_\sigma\big(g\big)(x) = g(x) - g\big(\sigma(x)\big) = 4 + \frac{\Delta}{9}x^2 + \mathrm{o}(x^2)$, where Δ is the first period constant of the center. Thus $\mathscr{B}_\sigma\big(g\big)$ is increasing at $x=0$ by assumption. If $\nu \in U$ then, see [27, §3.1], the projection on the x-axis of the period annulus of the center of X_ν is $(-\infty, x_r)$ with x_r being the smallest root of $q(x)=0$, which one can check it is $x=\rho$. Thus $\sigma(x)\longrightarrow -\infty$ as x tends to $x_r=\rho$ and

$$\lim_{x\longrightarrow x_r}\mathscr{B}_\sigma\big(g\big)(x)=\lim_{x\longrightarrow x_r}\big(g(x)-g\big(\sigma(x)\big)\big)=\frac{\sqrt{2}}{F}\left(\frac{(1-\rho)^{-F}-1}{\sqrt{-q(0)}}-\frac{-1}{\sqrt{-q(0)}}\right)=\frac{\sqrt{2}\,(1-\rho)^{-F}}{F\sqrt{-q(0)}}=L,$$

where in the second equality we used that $q(\rho)=0$ with $\rho\in(0,1)$ and that, on account of $F>2$, we have $(1-x)^{-2F}q(x)\longrightarrow 0$ as x tends to $-\infty$. Clearly $\mathscr{B}_\sigma\big(g\big)$ can not be monotone on $(0,x_r)$ in case that it is increasing at $x=0$ and

$$\lim_{x\longrightarrow x_r}\mathscr{B}_\sigma\big(g\big)(x)<\mathscr{B}_\sigma\big(g\big)(0).$$

These conditions are given by $\Delta>0$ and $L<4$, respectively, and so by (c) in Theorem A we can assert that the period function is not monotone for $\nu\in U$ verifying both inequalities. We take now advantage of [27, Theorem A], which shows that if $F>2$ then the period function is increasing near the outer boundary of $\mathscr{P}$. On the other hand the period function is increasing near the center as well because $\Delta>0$. Therefore if $\nu_0\in\mathscr{D}$ then X_{ν_0} must have at least two critical periods, and this shows the validity of the result. ■

Part (b) in Theorem A provides a criterion to bound the number of critical periods of a center. Our aim with the next result is to illustrate its applicability in combination with Theorem B. To this end we reprove partially the main result in [36], which is devoted to study the period function of system (15) with $F=2$. The author proves that the center is isochronous for $D=-\frac{1}{2}$, it has one critical period for $D\in(-2,-\frac{7}{5})$ and a monotone period function for all the other values. The proof relies upon obtaining a Picard-Fuchs equation for the period function and it is technically non-trivial. In the next result we use $\mathscr{B}_\sigma\big(\ell_2\big)$ to study $D\in(-2,0)$. We restrict to this interval for the sake of shortness but one could use $\mathscr{B}_\sigma\big(\ell_1\big)$ similarly to show the monotonicity for $D\notin(-2,0)$.

Proposition 4.2. *Consider the center at the origin of system (15) with $F=2$ and $D\in(-2,0)$. Then it is isochronous for $D=-\frac{1}{2}$ and it has at most one critical period for $D\neq-\frac{1}{2}$.*

Proof. For $F=2$ and $D\in(-2,0)$, the projection of $\mathscr{P}$ on the x-axis is $(-\infty,\rho)$ where $x=\rho$ is the smallest zero of the second degree polynomial q in (16), see [27, §3.1]. One can also verify that

$$M(x)=\frac{6x^2+4(D-1)x^3-(D-1)x^4}{12(1-x)^4}\ \text{ and }\ \ell_2(x)=\frac{\sqrt{2}\,(1+2D)\,(x-1)^3\,p(x)}{144\,(1+Dx)^5},$$

with $p(x)=D(D-1)^2x^5-2(D-2)(D-1)^2x^4+(D-1)\big((10D^2-9D+23)x^3+60x\big)+(52-56D+40D^2)x^2+36$. By applying (a) in Theorem A this shows that the center is isochronous for $D=-\frac{1}{2}$.

Consider next the case $D\neq-\frac{1}{2}$. Following (b) in Theorem A, we shall study the zeros of $\mathscr{B}_\sigma(\ell_2)$. To this end we note that M and ℓ_2 are algebraic because $P\big(x,M(x)\big)\equiv 0$ and $L\big(x,\ell_2(x)\big)\equiv 0$ with

$$P(x,y)=6x^2+4(D-1)x^3-(D-1)x^4-12(1-x)^4y$$

and

$$L(x,y)=\sqrt{2}\,(1+2D)\,(x-1)^3\,p(x)-144(1+Dx)^5y.$$

For convenience, see Remark 2.1, we shall prove that $\mathscr{B}_\sigma(\ell_2)$ has at most one root, counted with multiplicity, on $(-\infty,0)$. One can check that $\mathrm{Res}_z\big(P(x,z),P(y,z)\big)=12(x-y)\,S(x,y)$, where

$$\begin{aligned}S(x,y)=&\big(6-17(D-1)xy-6Dx^2y^2\big)(x+y)\\&+(D-1)\big(4\,(1+xy)(x^2+y^2)-x^3-y^3\big)+4\,(7D-1)(xy-1)xy,\end{aligned}$$

so that $S\big(x,\sigma(x)\big)\equiv 0$. Taking S and L we can thus apply Theorem B to bound the multiplicity of the zeros of $\mathscr{B}_\sigma(\ell_2)$. Some computations show that $T(x,y):=\mathrm{Res}_z\big(L(x,z),L(y,z)\big)=(1+2D)(x-y)\,\hat{T}(x,y)$, where $\hat{T}$ is a polynomial of degree 12 with coefficients in $\mathbb{R}[D]$. By Remark 2.2 we can apply Theorem B taking $\hat{T}$ instead of T. In doing so we obtain

$$\mathscr{R}(x):=\mathrm{Res}_y\big(S(x,y),\hat{T}(x,y)\big)=(D+1)^5(x-1)^6(1+Dx)^4\,\hat{r}(x),$$

where $\hat{r}$ is a polynomial of degree 30 with coefficients in $\mathbb{R}[D]$. Let us also note that the leading coefficients of $S(x,y)$ and $\hat{T}(x,y)$ with respect to y, say $s(x)$ and $t(x)$, do not vanish simultaneously because $\mathrm{Res}_x\big(s(x),t(x)\big)=-D^7(D-1)^4(D+2)^5(D+1)^5\neq 0$ for all $D\in(-2,0)$.

We claim that $\mathscr{R}$ has at most one zero on $(-\infty,0)$ counted with multiplicity. To show this, for each $D\in(-2,0)$, denote by $\mathcal{Z}(D)$ the number of zeros of $\hat{r}$ on $(-\infty,0)$, counted with multiplicities. One can check that $\hat{r}(0)=-7776(5D+7)(16D^3+24D^2+21D+11)(2\,D+1)^2(D-1)^3$ and that $36D^{11}(D+2)^5(D-1)^6$ is the leading coefficient of $\hat{r}(x)$. The roots of these two polynomials on $(-2,0)$ are $D=-\frac{7}{5}$, $D=-\frac{1}{2}$ and $D=D_1$ where $D_1\approx-0.8925$ is the real zero of $16\,D^3+24\,D^2+21\,D+11$. In addition, the discriminant of $\hat{r}(x)$ with respect to x, say $\mathscr{D}(D)$, is a polynomial of degree 1102 in D. After factorizing it, a careful analysis by applying Sturm's theorem shows that the only roots of $\mathscr{D}$ on $(-2,0)$ are $D\in\{-\frac{7}{5},-1,-\frac{1}{2},D_2,D_3,D_4,D_5\}$, with $D_2\approx-1.031$, $D_3\approx-0.712$, $D_4\approx-0.281$ and $D_5\approx-0.037$. It is clear therefore that $\mathcal{Z}(D)$ for $D\in(-2,0)$ can only change at $D\in\{-\frac{7}{5},-1,-\frac{1}{2},D_1,D_2,D_3,D_4,D_5\}$. These values split the interval $(-2,0)$ into several subintervals, taking a parameter in each one and applying Sturm's theorem repeatedly we can assert that $\mathcal{Z}(D)=1$ for $D\in(-2,-\frac{7}{5})\cup(D_1,0)$ and $\mathcal{Z}(D)=0$ for $D\in(-\frac{7}{5},D_1)$. Accordingly $\mathscr{R}$ has at most one zero on $(-\infty,0)$ counted with multiplicity. Thus, by Theorem B, it follows that if $D\in(-2,0)\setminus\{-\frac{1}{2}\}$ then $\mathscr{B}_\sigma(\ell_2)$ has at most one zero, counted with multiplicity, on $(-\infty,0)$. On account of (b) in Theorem A, this proves that the center has at most one critical period for $D\in(-2,0)\setminus\{-\frac{1}{2}\}$, as desired. ■

The authors of [15] give a formula in Theorem 2 to study the bifurcation of critical periods from the period annulus of an isochrone perturbed linearly inside a family of centers. They apply it, see [15, Theorem 3], to the perturbation of the four quadratic isochrones inside the family of Loud's dehomogenized centers $\{X_\nu,\nu\in\mathbb{R}^2\}$. To use the mentioned formula, a normalizer and the explicit solution of the isochronous center are needed. Then the proof of Theorem 3 follows after long and very complicated computations. In case of centers verifying the hypothesis **(H)**, as the family of Loud's dehomogenized centers does, the application of Theorem C simplifies the problem a lot. To illustrate this we reprove the following result, which corresponds to the case (d) in [15, Theorem 3]. Let us mention that the proof in [15] is particularly difficult for this case because it involves the complete elliptic integrals of the first and second kinds.

Proposition 4.3. *Any one-parameter regular deformation of the isochrone at $\bar{\nu}=(-\frac{1}{2},2)$ inside the family $\{X_\nu,\nu\in\mathbb{R}^2\}$ in (15) gives rise to at most one critical periodic orbit bifurcating from its period annulus.*

Proof. For consistency we keep the notation of Theorem C, thus we set $\nu_1=D$ and $\nu_2=F$. Some easy computations show that $M(x;\bar{\nu})=\frac{x^2(x-2)^2}{8(x-1)^4}$ and that the projection of $\mathscr{P}$ on the x-axis is $(-\infty,1-1/\sqrt{2})$. Hence $x_r=1-1/\sqrt{2}$. Moreover, on account of $g(x)^2=M(x;\bar{\nu})$, we get that $g^{-1}(x)=1-\big(1+2\sqrt{2}\,x\big)^{-1/2}$. Since one can verify that $\ell_2(\,.\,;\bar{\nu})=0$,

$$\frac{\partial\ell_2(x;\bar{\nu})}{\partial\nu_1}=\frac{(x-1)^3}{\sqrt{2}}\ \text{ and }\ \frac{\partial\ell_2(x;\bar{\nu})}{\partial\nu_1}=\frac{(x-1)^4}{2\sqrt{2}},$$

some computations show that $\hat{L}_1(x)=\frac{-1}{\sqrt{2}}\big(1+2\sqrt{2}\,x\big)^{-3/2}$ and $\hat{L}_2(x)=\frac{-1}{2\sqrt{2}}\big(1+2\sqrt{2}\,x\big)^{-2}$. Accordingly

$$L_1(x)=\frac{\big(1-2\sqrt{2x}\big)^{-3/2}-\big(1+2\sqrt{2x}\big)^{-3/2}}{\sqrt{2x}}\ \text{ and }\ L_2(x)=\frac{4}{(1-8x)^2}.$$

A straightforward exercise shows that $L_1'L_2 - L_1L_2'$ is non-vanishing on $[0, x_r)$. Since this is obvious for L_2, we have that $\frac{L_1}{L_2}$ is a smooth monotone function on $[0, x_r)$, which trivially implies that $\{L_1, L_2\}$ is an extended Chebyshev system there. Now the result follows by applying Theorem C. ■

We point out that if $n = 2$, i.e., $\nu \in \mathbb{R}^2$, then the proof of Theorem C also gives the perturbative direction in order that a critical period bifurcates. For instance, in the previous result we have that $\{L_1, L_2\}$ is an extended Chebyshev system on $[0, x_r)$ because $\frac{L_1}{L_2}$ is monotone there. Since $\big(\frac{L_1}{L_2}\big)(0) = \frac{3}{2}$ and $\big(\frac{L_1}{L_2}\big)(x_r) = 0$, the image of $(0, x_r)$ by $\frac{L_1}{L_2}$ is $(0, \frac{3}{2})$. Then it readily follows that the critical period can only appear in case that the perturbation is given by $\nu(\varepsilon) = \bar{\nu} + \varepsilon\hat{\nu} + \mathrm{o}(\varepsilon)$ with $\hat{\nu} = (\hat{\nu}_1, \hat{\nu}_2)$ such that $-\frac{\hat{\nu}_1}{\hat{\nu}_2} \in (0, \frac{3}{2})$.

Our last application is to illustrate the isochronicity criterion given by (a) in Theorem A. To this end we show the following result, which was proved previously by Sáez and Szántó [32, Theorem B].

Proposition 4.4. *The Kolmogorov cubic system*

$$\begin{cases} \dot{x} = x\big((1-a)(1-2x) - a(x^2 - y^2)\big), \\ \dot{y} = y\big((a-1)(1-2y) - a(x^2 - y^2)\big), \end{cases}$$

has an isochronous center at $\big(\frac{1}{2}, \frac{1}{2}\big)$ *for all* $a > 1$.

Proof. We first translate the singular point at $\big(\frac{1}{2}, \frac{1}{2}\big)$ to the origin by performing the coordinate transformation $\big\{x = \frac{1}{2}(1 - u - \sqrt{a-1}v), y = \frac{1}{2}(1 - u + \sqrt{a-1}v)\big\}$. Then, after a constant rescaling of time, the differential system writes as

$$\begin{cases} \dot{u} = v(-1 + 2u - au^2), \\ \dot{v} = u - u^2 + v^2 - auv^2. \end{cases}$$

This differential system fulfils the hypothesis **(H)** because one can check that $H(u, v) = \frac{u^2+v^2}{1-2u+au^2}$ is a first integral with integrating factor given by $\kappa(u) = \frac{2}{(1-2u+au^2)^2}$. In this case $M(u) = \frac{u^2}{1-2u+au^2}$ and the involution associated to M is the Möbius transformation $\sigma(u) = \frac{u}{2u-1}$. Some computations show that $\ell_1(u) = \frac{(1-2u+au^2)^{3/2}}{2(1-u)^3}$ and one can verify that $\mathscr{B}_\sigma\big(\ell_1\big)(u) = 0$ for $u < 1/2$. Therefore by (a) in Theorem A we can assert that the center is isochronous. This proves the result. ■

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