This is a preprint of: "Global configurations of singularities for quadratic differential systems with total finite multiplicity three and at most two real singularities", Joan Carles Artés, Jaume Llibre, Dana Schlomiuk, Nicolae Vulpe, *Qual. Theory Dyn. Syst.*, vol. 13, 305–351, 2014.
DOI: [10.1007/s12346-014-0119-7]

# GLOBAL CONFIGURATIONS OF SINGULARITIES FOR QUADRATIC DIFFERENTIAL SYSTEMS WITH TOTAL FINITE MULTIPLICITY THREE AND AT MOST TWO REAL SINGULARITIES

Abstract. In this work we consider the problem of classifying all configurations of singularities, finite and infinite, of quadratic differential systems, with respect to the *geometric equivalence relation* defined in [2]. This relation is deeper than the *topological equivalence relation* which does not distinguish between a focus and a node or between a strong and a weak focus or between foci (or saddles) of different orders. Such distinctions are however important in the production of limit cycles close to the foci (or loops) in perturbations of the systems. The notion of *geometric equivalence relation* of configurations of singularities allows us to incorporate all these important geometric features which can be expressed in purely algebraic terms. This equivalence relation is also deeper than the *qualitative equivalence relation* introduced in [19]. The *geometric classification* of all configurations of singularities, finite and infinite, of quadratic systems was initiated in [3] where the classification was done for systems with total multiplicity $m_f$ of finite singularities less than or equal to one. That work was continued in [4] where the geometric classification was done for the case $m_f = 2$. The case $m_f = 3$ has been split in two separate papers because of its length. The subclass of three real distinct singular points was done in [5] and we complete this case here.

In this article we obtain *geometric classification* of singularities, finite and infinite, for the remaining three subclasses of quadratic differential systems with $m_f = 3$ namely: (i) systems with a triple singularity (19 configurations); (ii) systems with one double and one simple real singularities (62 configurations) and (iii) systems with one real and two complex singularities (74 configurations). We also give here the global bifurcation diagrams of configurations of singularities, both finite and infinite, with respect to the *geometric equivalence relation*, for these subclasses of systems. The bifurcation set of this diagram is algebraic. The bifurcation diagram is done in the 12-dimensional space of parameters and it is expressed in terms of invariant polynomials. This provides an algorithm for computing the geometric configuration of singularities for any quadratic system in this class.

## 1. Introduction and statement of main results

We consider here differential systems of the form

$$\frac{dx}{dt} = p(x,y), \qquad \frac{dy}{dt} = q(x,y), \tag{1}$$

where $p, q \in \mathbb{R}[x,y]$, i.e. $p$, $q$ are polynomials in $x$, $y$ over $\mathbb{R}$. We call *degree* of a system (1) the integer $m = \max(\deg p, \deg q)$. In particular we call *quadratic* a differential system (1) with $m = 2$. We denote here by **QS** the whole class of real quadratic differential systems.

The study of the class **QS** has proved to be quite a challenge since hard problems formulated more than a century ago, are still open for this class. It is expected that we have a finite number of phase portraits in **QS**. We have phase portraits for several subclasses of **QS** but to obtain the complete topological classification of these systems, which occur rather often in applications, is a daunting task. This is partly due to the elusive nature of limit cycles and partly to the rather large number of parameters involved. This family of systems depends on twelve parameters but due to the group action of real affine transformations and time

---

1991 *Mathematics Subject Classification.* Primary 58K45, 34C05, 34A34.

*Key words and phrases.* quadratic vector fields, infinite and finite singularities, affine invariant polynomials, Poincaré compactification, configuration of singularities, geometric equivalence relation.

The first two authors are partially supported by a MINECO/FEDER grant number MTM2008-03437 and an AGAUR grant number 2014SGR 568. The first author is also partially supported by NSERC. The second author is also supported by ICREA Academia and FP7-PEOPLE2012-IRSES number 318999. The second and fourth authors are partially supported by the grant FP7-PEOPLE-2012-IRSES-316338. The third author is supported by NSERC. The fourth author is also supported by the grant 12.839.08.05F from SCSTD of ASM and partially by NSERC.

homotheties, the class ultimately depends on five parameters which is still a rather large number of parameters. For the moment only subclasses depending on at most three parameters were studied globally, including global bifurcation diagrams (for example [1]). On the other hand we can restrict the study of the whole quadratic class by focusing on specific global features of the systems in this family. We may thus focus on the global study of singularities and their bifurcation diagram. The singularities are of two kinds: finite and infinite. The infinite singularities are obtained by compactifying the differential systems on the sphere, on the Poincaré disk or on the projective plane as defined in Subection 2 (see [16], [20]).

The global study of quadratic vector fields began with the study of these systems in the neighborhood of infinity ([15], [22], [27], [28]). In [6] the authors classified topological (adding also the distinction between nodes and foci) the whole quadratic class, according to configurations of their finite singularities.

To reduce the number of phase portraits in half in topological classifications problems of quadratic systems, the *topological equivalence relation* was taken to mean the existence of a homeomorphism of the phase plane carrying orbits to orbits and *preserving or reversing* the orientation.

We use the concepts and notations introduced in [2] and [3] which we describe in Section 2. To distinguish among the foci (or saddles) we use the notion of order of the focus (or of the saddle) defined using the the algebraic concept of Poincaré-Lyapunov constants. We call strong focus (or strong saddle) a focus (or a saddle) whose linearization matrix has non-zero trace. Such a focus (or saddle) will be denoted by $f$ (respectively $s$). A focus (or saddle) with trace zero is called a weak focus (weak saddle). We denote by $f^{(i)}$ ($s^{(i)}$) the weak foci (weak saddles) of order $i$ and by $c$ and $\$$ the centers and integrable saddles. For more notations see Subsection 2.5.

In the topological classification no distinction was made among the various types of foci or saddles, strong or weak of various orders. However these distinctions of an algebraic nature are very important in the study of perturbations of systems possessing such singularities. Indeed, the maximum number of limit cycles which can be produced close to the weak foci in perturbations depends on the orders of the foci.

There are also three kinds of simple nodes: nodes with two characteristic directions (the generic nodes), nodes with one characteristic direction and nodes with an infinite number of characteristic directions (the star nodes). The three kinds of nodes are distinguished algebraically. Indeed, the linearization matrices of the two direction nodes have distinct eigenvalues, they have identical eigenvalues and they are not diagonal for the one direction nodes and they have identical eigenvalues and they are diagonal for the star nodes (see [2], [3], [4]). We recall that the star nodes and the one direction nodes could produce foci in perturbations.

Furthermore, a generic node at infinity may or may not have the two exceptional curves lying on the line at infinity. This leads to two different situations for the phase portraits. For this reason we split the generic nodes at infinity in two types as indicated in Subsection 2.5.

The *geometric equivalence relation* (see further below) for finite or infinite singularities, introduced in [2] and used in [3], [4] and [5], takes into account such distinctions. This equivalence relation is also deeper than the *qualitative equivalence relation* introduced by Jiang and Llibre in [19] because it distinguishes among the foci (or saddles) of different orders and among the various types of nodes. This equivalence relation induces also a deeper distinction among the more complicated degenerate singularities.

In quadratic systems these could be of orders 1, 2 or 3 [11]. For details on Poincaré-Lyapunov constants and weak foci of various orders we refer to [26], [20]. As indicated before, algebraic information plays a fundamental role in the study of perturbations of systems possessing such singularities. In [31] necessary and sufficient conditions for a quadratic system to have weak foci (saddles) of orders $i$, i=1,2,3 are given in invariant form.

For the purpose of classifying **QS** according to their singularities, finite or infinite, we use the *geometric equivalence relation* which involves only algebraic methods. It is conjectured that there are over 1000 distinct *geometric configurations* of singularities. The first step in this direction was done in [2] where the global classification of singularities at infinity of *the whole class* **QS**, was done according to the *geometric equivalence relation* of configurations of infinite singularities. This work was then partially extended to also incorporate finite singularities. We initiated this work in [3] where this classification was done for the case of singularities with a total finite multiplicity $m_f \leq 1$, continued it in [4] where the classification was done for $m_f = 2$ and in [5] where the classification was done for 3 distinct real finite singularities.

In the present article our goal is to go one step further in the *geometric classification* of global configurations of singularities by studying here the case of finite singularities with total finite multiplicity three and at most two real finite singularities.

We recall below the notion of *geometric configuration of singularities* defined in [4] for both finite and infinite singularities. We distinguish two cases:

1) If we have a finite number of infinite singular points and a finite number of finite singularities we call *geometric configuration of singularities*, finite and infinite, the set of all these singularities each endowed with its own multiplicity together with the local phase portraits around real singularities endowed with additional geometric structure involving the concepts of tangent, order and blow–up equivalences defined in Section 4 of [2] and using the notations described here in Subsection 2.5.

2) If the line at infinity $Z = 0$ is filled up with singularities, in each one of the charts at infinity $X \neq 0$ and $Y \neq 0$, the corresponding system in the Poincaré compactification (see Section 2) is degenerate and we need to do a rescaling of an appropriate degree of the system, so that the degeneracy be removed. The resulting systems have only a finite number of singularities on the line $Z = 0$. In this case we call *geometric configuration of singularities*, finite and infinite, the set of all points at infinity (they are all singularities) in which we single out the singularities at infinity of the "reduced" system, taken together with their local phase portraits and we also take the local phase portraits of finite singularities each endowed with additional geometric structure to be described in Section 2.

**Remark 1.** *We note that the geometric equivalence relation for configurations is much deeper than the topological equivalence. Indeed, for example the topological equivalence does not distinguish between the following three configurations which are geometrically non-equivalent: 1)* $n$, $f$; $\overline{\binom{1}{1}}SN$, ©, ©, *2)* $n$, $f^{(1)}$; $\overline{\binom{1}{1}}SN$, ©, ©, *and 3)* $n^d$, $f^{(1)}$; $SN$, ©, © *where* $n$ *and* $n^d$ *mean singularities which are nodes, respectively two directions and one direction nodes, capital letters indicate points at infinity,* © *in case of a complex point and* $SN$ *a saddle–node at infinity and* $\binom{1}{1}$ *encodes the multiplicities of the saddle-node* $SN$*. For more details see the subsection on notation of Subsection 2.5 on notation.*

The invariants and comitants of differential equations used for proving our main result are obtained following the theory of algebraic invariants of polynomial differential systems, developed by Sibirsky and his disciples (see for instance [30, 33, 24, 8, 14]).

Our results are stated in the following theorem.

**Main Theorem.** $(A)$ *We consider here all configurations of singularities, finite and infinite, of quadratic vector fields with finite singularities of total multiplicity* $m_f = 3$ *possessing at most two real finite singularities. These configurations are classified in* DIAGRAMS 1–3 *according to the geometric equivalence relation. We have 155 geometrically distinct configurations of singularities, finite and infinite. More precisely 74 geometrically distinct configurations with one real and two complex finite singularities; 62 geometrically distinct configurations with one double and one simple real finite singularities and 19 with one real finite singularity which is triple.*

$(B)$ *Necessary and sufficient conditions for each one of the 155 different geometric equivalence classes can be assembled from these diagrams in terms of 22 invariant polynomials, with respect to the action of the affine group and time rescaling, given in Section 2.6 and appearing in the DIAGRAMS 1-3, polynomials defined in the webpage: http://mat.uab.es/∼artes/articles/qvfinvariants/qvfinvariants.html.*

($C$) *The* DIAGRAMS 1–3 *actually contain the global bifurcation diagrams in the 12-dimensional space of parameters, of the global geometric configurations of singularities, finite and infinite, of these subclasses of quadratic differential systems and provide an algorithm for finding for any given system in any of the three families considered, its respective geometric configuration of singularities.*

## 2. Concepts and results in the literature useful for this paper

2.1. **Compactification on the sphere and on the Poincaré disk.** Planar polynomial differential systems (1) can be compactified on the 2–dimensional sphere as follows. We first include the affine plane $(x, y)$ in $\mathbb{R}^3$, with its origin at $(0, 0, 1)$, and we consider it as the plane $Z = 1$. We then use a central projection to send the vector field to the upper and to the lower hemisphere. The vector fields thus obtained on the two hemispheres are analytic and diffeomorphic to our vector field on the $(x, y)$ plane. By a theorem stated by Poincaré and proved in [17] there exists an analytic vector field on the whole sphere which simultaneously extends the vector fields on the two hemispheres to the whole sphere. We call *Poincaré compactification on the sphere* of the planar polynomial system, the restriction of the vector field thus obtained on the sphere, to the upper hemisphere completed with the equator. For more details we refer to [16]. The vertical projection of this vector field defined on the upper hemisphere and completed with the equator, yields a diffeomorphic vector field on the unit disk, called the *Poincaré compactification on the disk* of the polynomial differential system. By a *singular point at infinity* of a planar polynomial vector field we mean a singular point of the vector field which is located on the equator of the sphere, also located on the boundary circle of the Poincaré disk.

2.2. **Compactification on the projective plane.** For a polynomial differential system (1) of degree $m$ with real coefficients we associate the differential equation $\omega_1 = q(x, y)dx - p(x, y)dy = 0$. This equation defines two foliations with singularities, one on the real and one on the complex affine planes. We can compactify these foliations with singularities on the real respectively complex projective plane with homogeneous coordinates $X, Y, Z$. This is done by introducing the homogeneous coordinates via the equations $x = X/Z, y = Y/Z$ and taking the pull-back $i_*(\omega_1)$ of the form $\omega_1$ of the inclusion $(x, y) \mapsto [x : y : 1]$ into the projective plane. We obtain a foliation with singularities on $P_2(\mathbb{K})$ ($\mathbb{K}$ equal to $\mathbb{R}$ or $\mathbb{C}$) defined by the equation $\omega = A(X, Y, Z)dX + B(X, Y, Z)dY + C(X, Y, Z)dZ = 0$ on the projective plane over $\mathbb{K}$, which is called the *compactification on the projective plane* of the foliation with singularities defined by $\omega_1 = 0$ on the affine plane $\mathbb{K}^2$. This is true because $A$, $B$, $C$ are homogeneous polynomials over $K$, defined by $A(X, Y, Z) = ZQ(X, Y, Z)$, $Q(X, Y, Z) = Z^m q(X/Z, Y/Z)$, $B(X, Y, Z) = ZP(X, Y, Z)$, $P(X, Y, Z) = Z^m p(X/Z, Y/Z)$ and $C(X, Y, Z) = YP(X, Y, Z) - XQ(X, Y, Z)$. The *points at infinity* of the foliation defined by $\omega_1 = 0$ on the affine plane are the singular points of the type $[X : Y : 0] \in P_2(\mathbb{K})$ and the line $Z = 0$ is called the *line at infinity* of this foliation. The singular points of the foliation $F$ are the solutions of the three equations $A = 0$, $B = 0$, $C = 0$. In view of the definitions of $A, B, C$ it is clear that the singular points at infinity are the points of intersection of $Z = 0$ with $C = 0$. For more details see [20], or [2] or [3].

2.3. **Assembling multiplicities of singularities in divisors of the line at infinity and in zero-cycles of the plane.** An isolated singular point $p$ at infinity of a polynomial vector field of degree $n$ has two types of multiplicities: the maximum number $m$ of finite singularities which can split from $p$, in small perturbations of the system within polynomial systems of degree $n$, and the maximum number $m'$ of infinite singularities which can split from $p$, in small such perturbations of the system. We encode the two in the column $(m, m')^t$. We then encode the global information about all isolated singularities at infinity using formal sums called *cycles* and *divisors* as defined in [23] or in [20] and used in [20], [28], [3], [2].

We have two formal sums (*divisors on the line at infinity* $Z = 0$ of the complex affine plane) $D_S(P, Q; Z) = \sum_w I_w(P, Q)w$ and $D_S(C, Z) = \sum_w I_w(C, Z)w$ where $w \in \{Z = 0\}$ and where by $I_w(F, G)$ we mean the intersection multiplicity at $w$ of the curves $F(X, Y, Z) = 0$ and $G(X, Y, Z) = 0$ on the complex projective

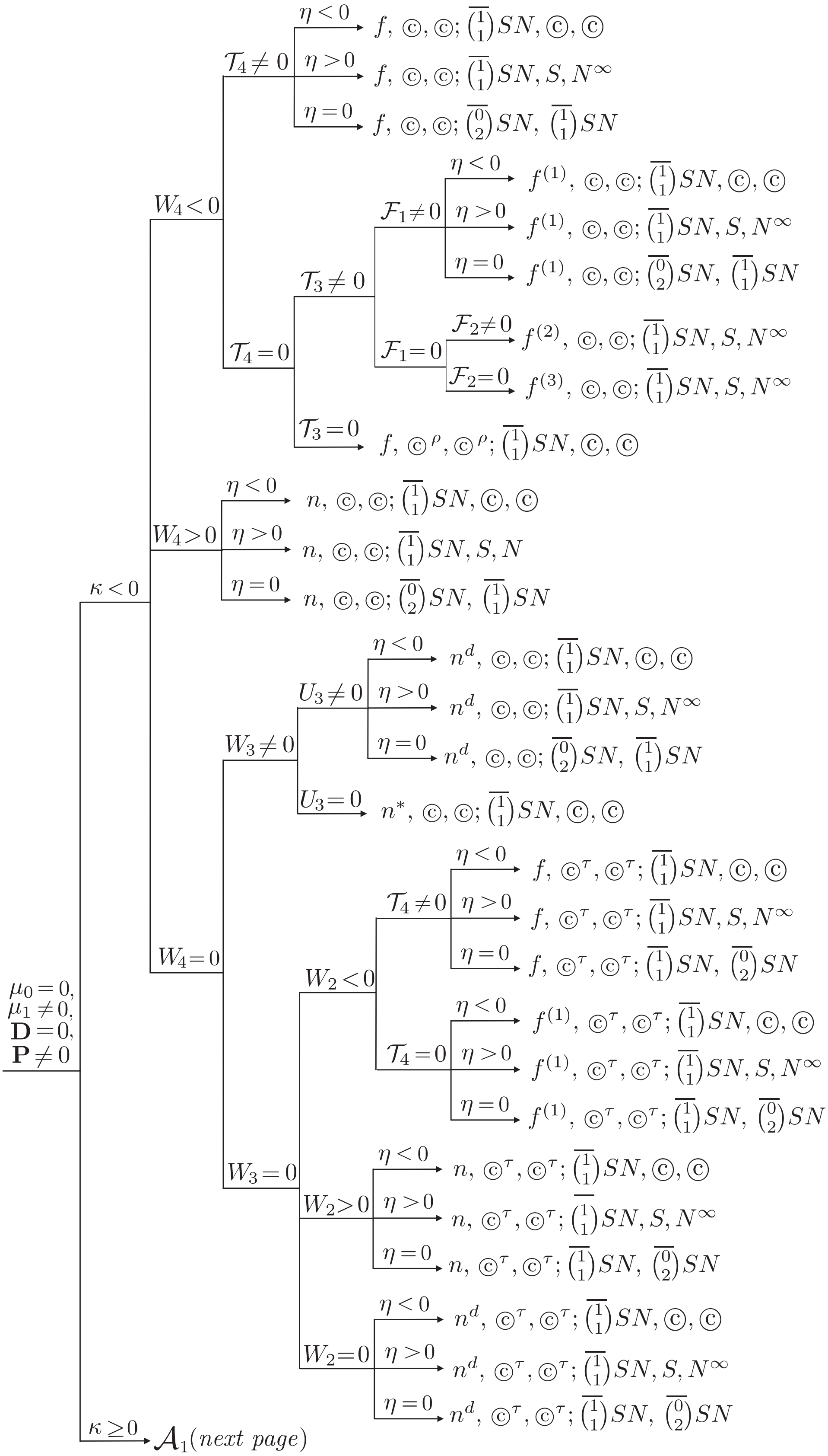


DIAGRAM 1. **Global configurations: the case $\mu_0 = 0$, $\mu_1 \neq 0$, $\mathbf{D} > 0$.**

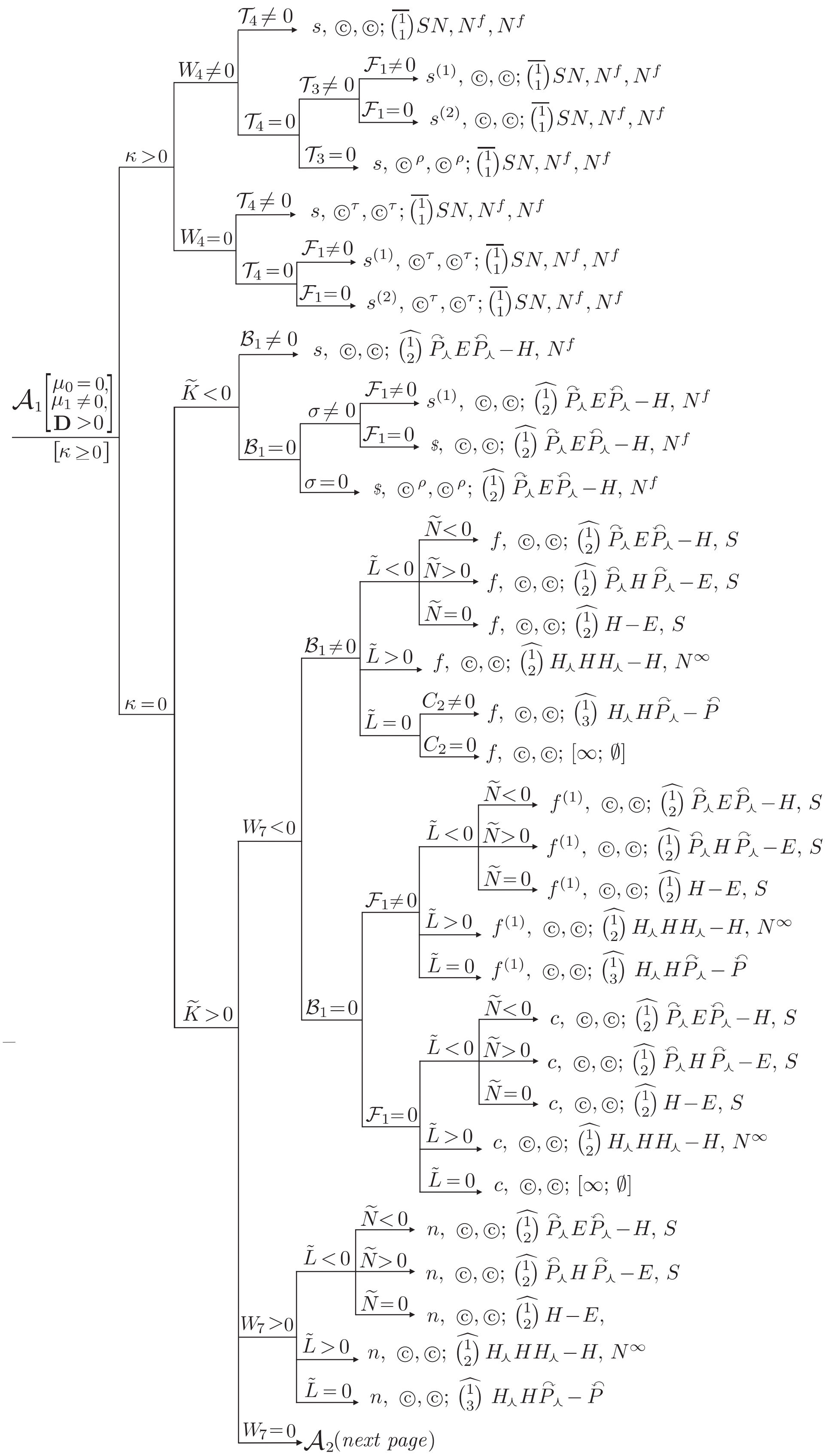

DIAGRAM 1 (*continued*). **Global configurations: the case $\mu_0 = 0$, $\mu_1 \neq 0$, $D > 0$.**

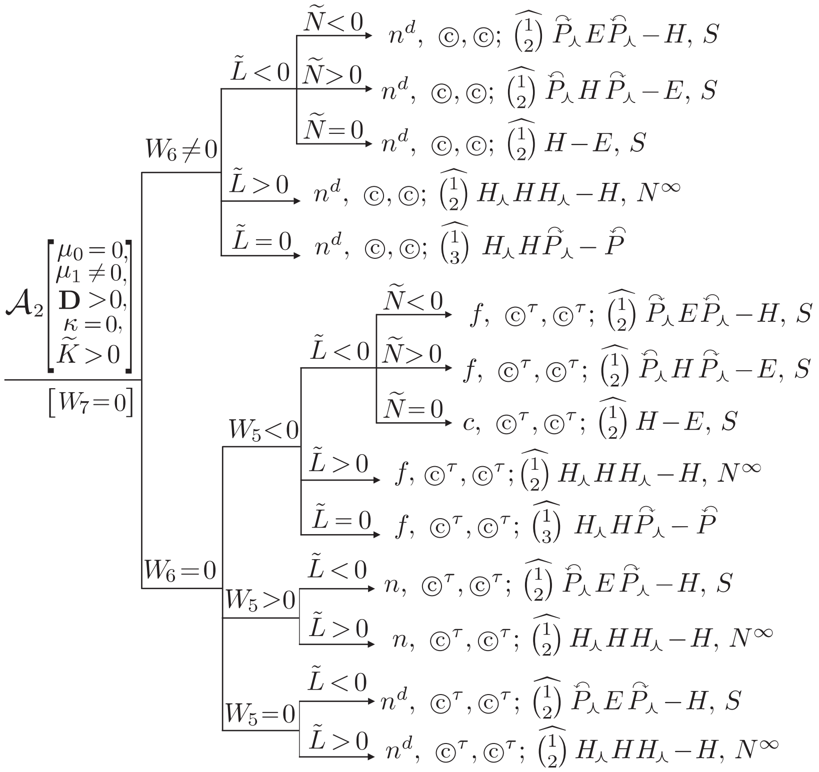


DIAGRAM 1 (*continued*). **Global configurations: the case $\mu_0 = 0$, $\mu_1 \neq 0$, $D > 0$.**

plane. For more details see [20]. Following [28] we encode the above two divisors on the line at infinity into just one but with values in the ring $\mathbb{Z}^2$:

$$D_S = \sum_{\omega\in\{Z=0\}} \begin{pmatrix} I_w(P,Q) \\ I_w(C,Z) \end{pmatrix} w.$$

For a system (1) with isolated finite singularities we consider the formal sum (*zero-cycle on the plane*) $D_S(p,q) = \sum_{\omega\in\mathbb{R}^2} I_w(p,q)w$ encoding the multiplicities of all finite singularities. For more details see [20], [1].

## 2.4. **Some geometrical concepts.** Firstly we recall some terminology.

We call *elemental* a singular point with its both eigenvalues not zero.

We call *semi–elemental* a singular point with exactly one of its eigenvalues equal to zero.

We call *nilpotent* a singular point with both its eigenvalues zero but with its Jacobian matrix at this point not identically zero.

We call *intricate* a singular point with its Jacobian matrix identically zero.

The *intricate* singularities are usually called in the literature *linearly zero*. We use here the term *intricate* to indicate the rather complicated behavior of phase curves around such a singularity.

In this section we use the same concepts we considered in [2], [3], [4], [5], such as *orbit $\gamma$ tangent to a semi–line $L$ at $p$*, *well defined angle at $p$*, *characteristic orbit at a singular point $p$*, *characteristic angle at a singular point*, *characteristic direction at $p$*. If a singular point has an infinite number of characteristic directions, we will it a *star–like* point.

It is known that the neighborhood of any isolated singular point of a polynomial vector field, which is not a focus or a center, is formed by a finite number of sectors which could only be of three types: parabolic, hyperbolic and elliptic (see [16]). It is also known that any degenerate singular point can be desingularized by

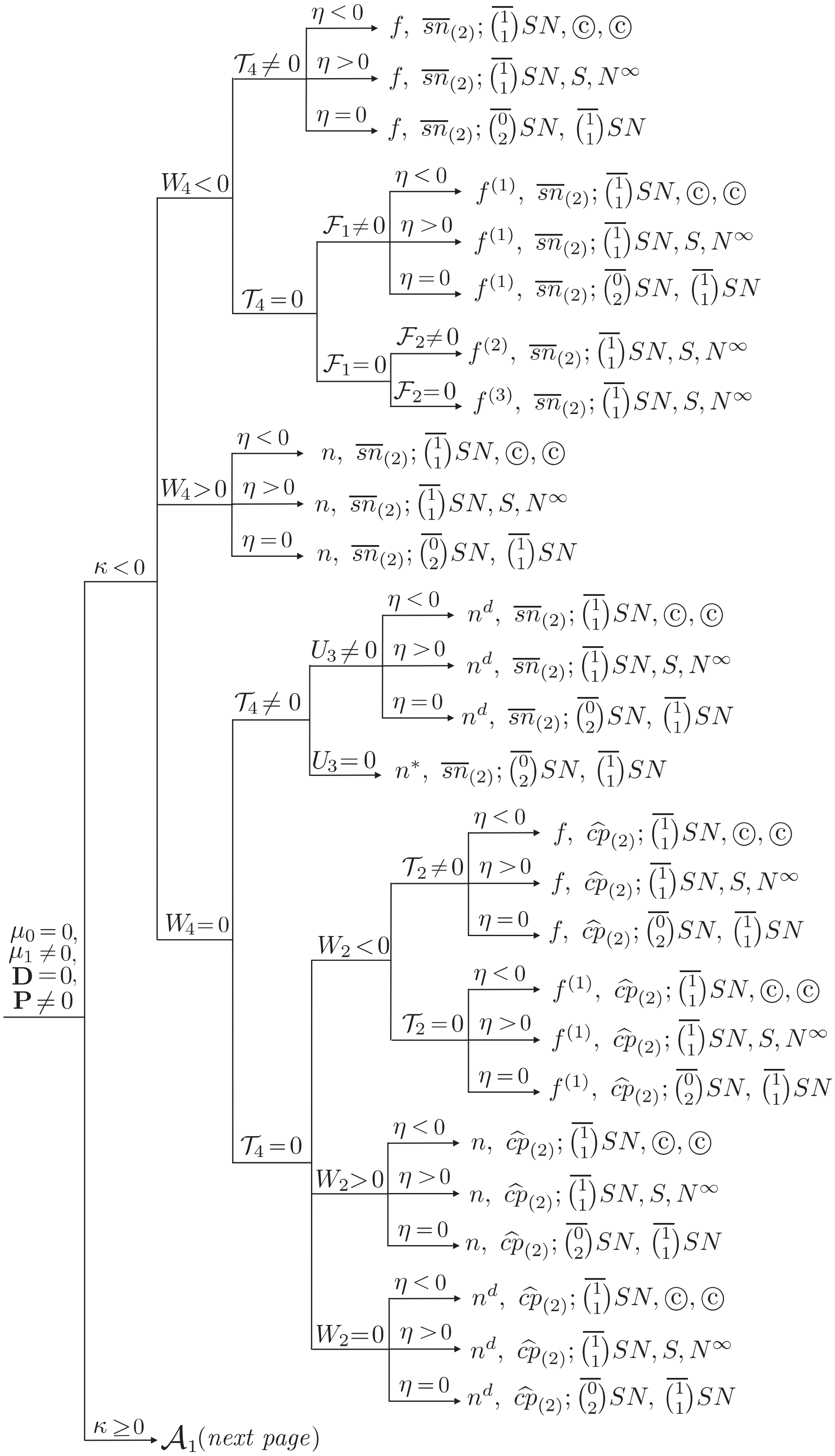


DIAGRAM 2. **Global configurations: the case $\boldsymbol{\mu_0=0}$, $\boldsymbol{\mu_1\neq 0}$, $\mathbf{D}=0$, $\mathbf{P}\neq 0$.**

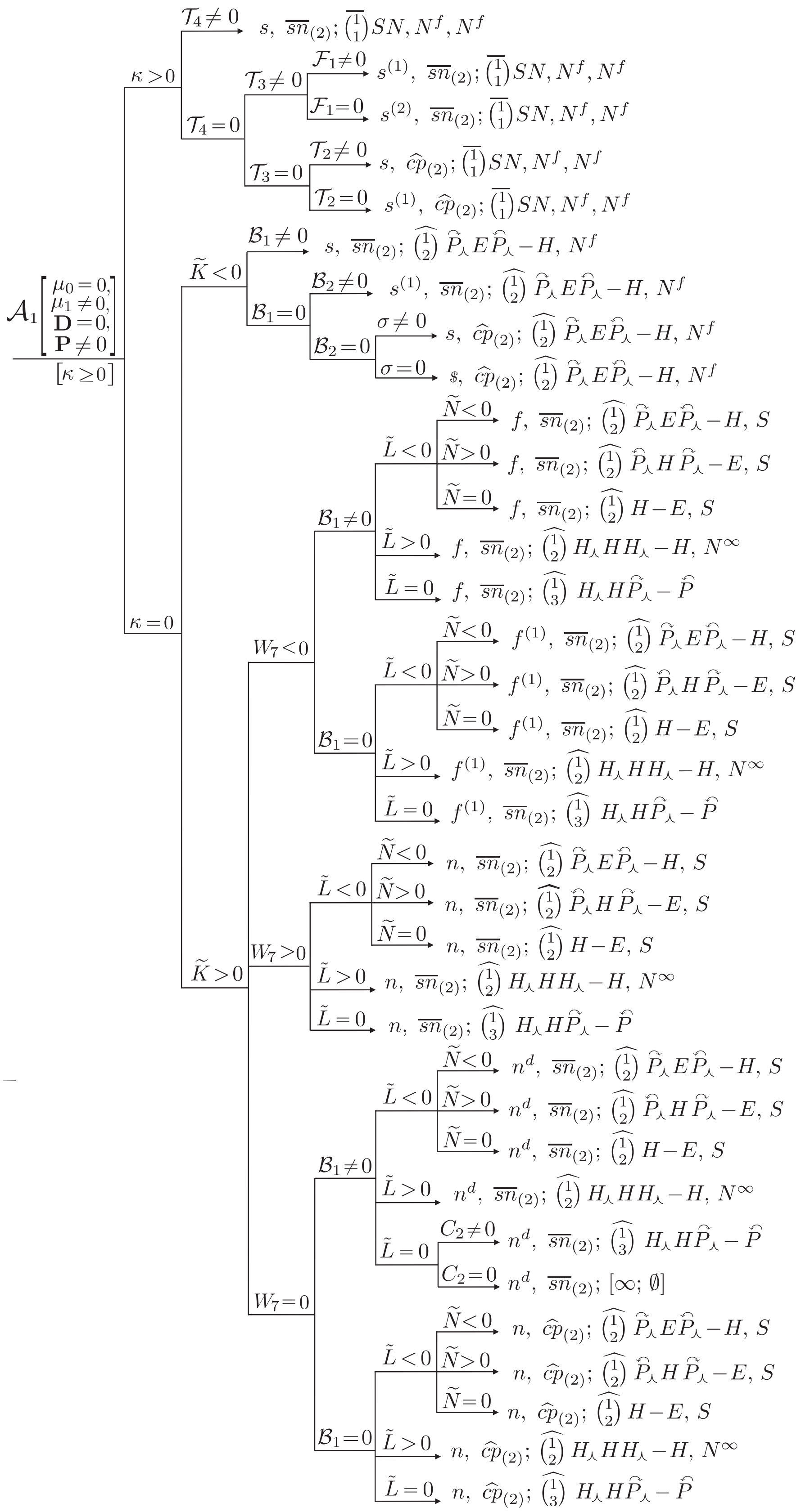


Diagram 2 (*continued*). **Global configurations: the case $\mu_0=0$, $\mu_1\neq 0$, $\mathbf{D}=0$, $\mathbf{P}\neq 0$.**

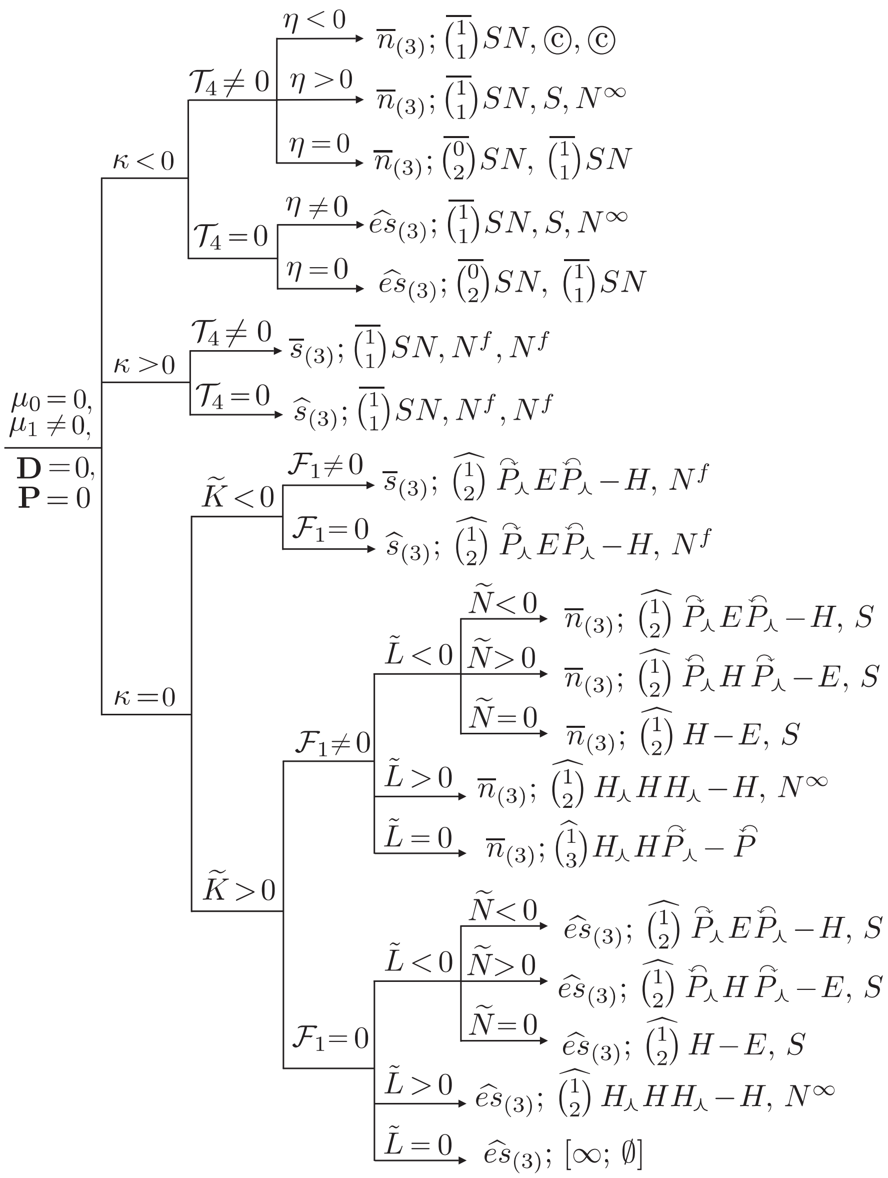


Diagram 3. **Global configurations: the case $\boldsymbol{\mu_0 = 0}$, $\boldsymbol{\mu_1 \neq 0}$, $\mathbf{D} = \mathbf{P} = 0$.**

means of a finite number of changes of variables, called blow–up's, into elemental ans semi-elemental singular points (for more details see the Section on blow–up in [2] or [16]).

Topologically equivalent local phase portraits can be distinguished according to the algebraic properties of their phase curves. For example they can be distinguished algebraically in the case when the singularities possess distinct numbers of characteristic directions.

The usual definition of a sector is of topological nature and it is local, defined with respect to a neighborhood around the singular point. We work with a new notion, namely of *geometric local sector*, introduced in [2], based on the notion of *borsec*, term meaning "border of a sector" (a new kind of sector, i.e. *geometric sector*) which takes into account orbits tangent to the half-lines of the characteristic directions at a singular point. For example a generic or semi–elemental node $p$ has two characteristic directions generating four half lines at $p$. For each one of these half lines at $p$ there exists at least one orbit tangent to that half line at $p$ and we pick an orbit tangent to that half line at $p$. Removing these four orbits together with the singular point, we are left with four sectors which we call *geometric local sectors* and we call *borsecs* these four orbits. The notion of *geometric local sector* and of *borsec* was extended for nilpotent and intricate singular points using the process of desingularization as indicated in [2]. We end up with the following definition: We call *geometric local sector* of a singular point $p$ with respect to a sufficiently small neighborhood $V$, a region in $V$ delimited by two consecutive borsecs. As mentioned these are defined using the desingularization process.

A nilpotent or intricate singular point can be desingularized by passing to polar coordinates or by using rational changes of coordinates. The first method has the inconvenience of using trigonometrical functions, and this becomes a serious problem when a chain of blow–ups are needed in order to complete the desingularization of the degenerate point. The second uses rational changes of coordinates, convenient for our polynomial systems. In such a case two blow–ups in different directions are needed and information from both must be glued together to obtain the desired portrait. Here for desingularization we use this second possibility, i.e. with rational changes of coordinates $(x, y) \to (x, zx)$ for the blow-up in the $y$-direction. This is a diffeomorphism for $x \neq 0$. The line $x = 0 = y$, the $z$-axis in $\mathbb{R}^3$, or $x = 0$ in the plane $(x, z)$ is called the blow-up line. Analogously we use the change $(x, y) \to (zy, y)$ for the blow-up in the $x$-direction. This is a diffeomorphism for $y \neq 0$. The blow-up line is again the $z$-axis in $\mathbb{R}^3$ or on the plane $(y, z)$.

The two directional blow–ups can be reduced to only one 1–direction blow–up but making sure that the direction in which we do a blow–up is not a characteristic direction, not to lose information by blowing–up in the chosen direction. This can be easily solved by a simple linear change of coordinates of the type $(x, y) \to (x + ky, y)$ where $k$ is a constant (usually 1). It seems natural to call this linear change a *$k$–twist* as the $y$–axis gets turned with some angle depending on $k$. It is obvious that the phase portrait of the degenerate point which is studied cannot depend on the set of $k$'s used in the desingularization process.

We recall that after a complete desingularization all singular points are elemental or semi–elemental. For more details and a complete example of the desingularization of an intricate singular point see [4].

Generically a *geometric local sector* is defined by two borsecs arriving at the singular point with two different well defined angles and which are consecutive. If this sector is parabolic, then the solutions can arrive at the singular point with one of the two characteristic angles, and this is a geometrical information that can be revealed with the blow–up.

There is also the possibility that two borsecs defining a geometric local sector at a point $p$ are tangent to the same halph–line at $p$. Such a sector will be called a *cusp–like sector* which can either be hyperbolic, elliptic or parabolic denoted by $H_{\curlywedge}$, $E_{\curlywedge}$ and $P_{\curlywedge}$ respectively. In the case of parabolic sectors we want to include the information about how the orbits arrive at the singular points namely tangent to one or to the other borsec. We distinguish the two cases by writing $\overset{\curvearrowleft}{P}$ if they arrive tangent to the borsec limiting the previous sector in clockwise sense or $\overset{\curvearrowright}{P}$ if they arrive tangent to the borsec limiting the next sector. In the case of a cusp–like parabolic sector, all orbits must arrive with only one well determined angle, but the distinction between $\overset{\curvearrowleft}{P}$ and $\overset{\curvearrowright}{P}$ is still valid because it occurs at some stage of the desingularization and this can be algebraically determined. Example of descriptions of complicated intricate singular points are $\overset{\curvearrowright}{P} E \overset{\curvearrowleft}{P} HHH$ and $E \overset{\curvearrowleft}{P}_{\curlywedge} HH \overset{\curvearrowright}{P}_{\curlywedge} E$.

A star–like point can either be a node or something much more complicated with elliptic and hyperbolic sectors included. In case there are hyperbolic sectors, they must be cusp–like. Elliptic sectors can either be cusp–like or star–like.

2.5. **Notations for singularities of polynomial differential systems.** In this work we limit ourselves to the class of quadratic systems with finite singularities of total multiplicity three and at most two real singularities. In [2] we introduced convenient notations which we also used in [3] and [4] some of which we also need here. Because these notations are essential for understanding the bifurcation diagram, we indicate below the notations needed for this article.

The finite singularities will be denoted by small letters and the infinite ones by capital letters. In a sequence of singular points we always place the finite ones first and then infinite ones, separating them by a semicolon';'.

**Elemental points:** We use the letters '$s$','$S$' for "saddles"; $\$$ for "integrable saddles"; '$n$', '$N$' for "nodes"; '$f$' for "foci"; '$c$' for "centers" and ⓒ (respectively Ⓒ) for complex finite (respectively infinite) singularities. We distinguish the finite nodes as follows:

- '$n$' for a node with two distinct eigenvalues (generic node);
- '$n^d$' (a one–direction node) for a node with two identical eigenvalues whose Jacobian matrix is not diagonal;
- '$n^*$' (a star–node) for a node with two identical eigenvalues whose Jacobian matrix is diagonal.

The case $n^d$ (and also $n*$) corresponds to a real finite singular point with zero discriminant. The case of a finite complex singular point with zero discriminant will be denoted by ⓒ$^\tau$.

In the case of an elemental infinite generic node, we want to distinguish whether the eigenvalue associated to the eigenvector directed towards the affine plane is, in absolute value, greater or lower than the eigenvalue associated to the eigenvector tangent to the line at infinity. This is relevant because this determines if all the orbits except one on the Poincaré disk arrive at infinity tangent to the line at infinity or transversal to this line. We will denote them as '$N^\infty$' and '$N^f$' respectively.

Finite elemental foci and saddles are classified as strong or weak foci, respectively strong or weak saddles. The strong foci or saddles are those with non-zero trace of the Jacobian matrix evaluated at them. In this case we denote them by '$s$' and '$f$'. When the trace is zero, except for centers, and saddles of infinite order (i.e. with all their Poincaré-Lyapounov constants equal to zero), it is known that the foci and saddles, in the quadratic case, may have up to 3 orders. We denote them by '$s^{(i)}$' and '$f^{(i)}$' where $i = 1, 2, 3$ is the order. In addition we have the centers which we denote by '$c$' and saddles of infinite order (integrable saddles) which we denote by '$\$$'.

A finite real singular point with zero trace will be denoted by ⓒ$^\rho$.

Foci and centers cannot appear as singular points at infinity and hence there is no need to introduce their order in this case. In case of saddles, we can have weak saddles at infinity but the maximum order of weak singularities in cubic systems is not yet known. For this reason, a complete study of weak saddles at infinity cannot be done at this stage. Due to this, in [2], [3], [4], [5] and here we chose not even to distinguish between a saddle and a weak saddle at infinity.

All non–elemental singular points are multiple points, in the sense that there are perturbations which have at least two elemental singular points as close as we wish to the multiple point. For finite singular points we denote with a subindex their multiplicity as in '$\overline{s}_{(5)}$' or in '$\widehat{es}_{(3)}$' (the notation '$^-$' indicates that the saddle is semi–elemental and '$\widehat{es}_{(3)}$' indicates that the singular point is nilpotent, in this case a triple *elliptic saddle* (i.e. it has two sectors, one elliptic and one hyperbolic)). In order to describe the two kinds of multiplicity for infinite singular points we use the concepts and notations introduced in [28]. Thus we denote by '$\binom{a}{b}$...' the maximum number $a$ (respectively $b$) of finite (respectively infinite) singularities which can be obtained by perturbation of the multiple point. For example '$\overline{\binom{1}{1}}SN$' means a saddle–node at infinity produced by the collision of one finite singularity with an infinite one; '$\overline{\binom{0}{3}}S$' means a saddle produced by the collision of 3 infinite singularities. The meaning of the notation '$^-$' in the general case will be described in the next paragraph.

**Semi–elemental points:** They can either be nodes, saddles or saddle–nodes, finite or infinite (see [16]). We denote the semi–elemental ones always with an overline, for example '$\overline{sn}$', '$\overline{s}$' and '$\overline{n}$' with the corresponding multiplicity. In the case of infinite points we put '$^-$' on top of the parenthesis with multiplicities.

Moreover, in cases which will be later explained, an infinite saddle–node may be denoted by '$\overline{\binom{1}{1}}NS$' instead of '$\overline{\binom{1}{1}}SN$'. Semi–elemental nodes could never be '$n^d$' or '$n^*$' since their eigenvalues are always different. In case of an infinite semi–elemental node, the type of collision determines whether the point is denoted by '$N^f$' or by '$N^\infty$'. The point '$\overline{\binom{2}{1}}N$' is an '$N^f$' and '$\overline{\binom{0}{3}}N$' is an '$N^\infty$'.

**Nilpotent points:** They can either be saddles, nodes, saddle–nodes, elliptic saddles, cusps, foci or centers (see [16]). The first four of these could be at infinity. We denote the nilpotent singular points with a hat '$\widehat{\ }$' as in $\widehat{es}_{(3)}$ for a finite nilpotent elliptic saddle of multiplicity 3 and $\widehat{cp}_{(2)}$ for a finite nilpotent cusp point of multiplicity 2. In the case of nilpotent infinite points, we will put the '$\widehat{\ }$' on top of the parenthesis with multiplicity, for example $\widehat{\binom{1}{2}}PEP-H$ (the meaning of $PEP-H$ will be explained below). The relative position of the sectors of an infinite nilpotent point, with respect to the line at infinity, can produce topologically different phase portraits. This forces us to use a notation for these points similar to the notation which we will use for the intricate points.

**Intricate points:** It is known that the neighborhood of any singular point of a polynomial vector field (except for foci and centers) is formed by a finite number of sectors which could only be of three types: parabolic, hyperbolic and elliptic (see [16]). Then, a reasonable way to describe intricate and nilpotent points is to use a sequence formed by the types of their sectors. The description we give is the one which appears in the clockwise direction (starting anywhere) once the blow–down of the desingularization is done. Thus in *non-degenerate* quadratic systems (that is, the components of the system are coprime), we have just seven possibilities for finite intricate singular points of multiplicity four (see [6]) which are the following ones: $phpphp_{(4)}$; $phph_{(4)}$; $hh_{(4)}$; $hhhhhh_{(4)}$; $peppep_{(4)}$; $pepe_{(4)}$; $ee_{(4)}$.

For infinite intricate and nilpotent singular points, we insert a dash (hyphen) between the sectors to split those which appear on one side or the other of the equator of the sphere. In this way we will distinguish between $\binom{2}{2}PHP-PHP$ and $\binom{2}{2}PPH-PPH$. Whenever we have an infinite nilpotent or intricate singular point, we will always start with a sector bordering the infinity (to avoid using two dashes).

For the description of the topological phase portraits around the isolated singular points the information described above is sufficient. However we are interested in additional geometrical features such as the number of characteristic directions which figure in the final global picture of the desingularization. In order to add this information we need to introduce more notation. If two borsecs, limiting orbits of a sector, arrive at the singular point with the same direction, then the sector will be denoted by $H_{\curlywedge}$, $E_{\curlywedge}$ or $P_{\curlywedge}$. The index in this notation refers to the cusp–like form of limiting trajectories of the sectors. Moreover, in the case of parabolic sectors we need to distinguish whether the orbits arrive tangent to the half-line at the singular point of one borsec or to the other. We distinguish the two cases by $\overset{\curvearrowleft}{P}$ if they arrive tangent to the borsec limiting the previous sector in clockwise sense or $\overset{\curvearrowright}{P}$ if they arrive tangent to the borsec limiting the next sector. A parabolic sector will be $P^*$ when all orbits arrive with all possible slopes between the two consecutive borsecs. In the case of a cusp–like parabolic sector, all orbits must be tangent to the same half-line at the singularity, but the distinction between $\overset{\curvearrowleft}{P}$ and $\overset{\curvearrowright}{P}$ is still valid if we consider the different desingularizations we obtain from them.

Finally there is also the possibility that we have an infinite number of infinite singular points.

**Line at infinity filled up with singularities:** It is known that any such system has in a sufficiently small neighborhood of infinity one of 6 topological distinct phase portraits (see [29]). The way to determine these phase portraits is by studying the reduced systems on the infinite local charts after removing the degeneracy of the systems within these charts. In case a singular point still remains on the line at infinity we study such a point. In [29] the tangential behavior of the solution curves was not considered in the case of a node. If after the removal of the degeneracy in the local charts at infinity a node remains, this could either be of the type $N^d$, $N$ and $N^\star$ (this last case does not occur in quadratic systems as it was shown in [2]). Since no eigenvector of such a node $N$ (for quadratic systems) will have the direction of the line at infinity we do not need to distinguish $N^f$ and $N^\infty$ (see [2]). After removal of the degeneracy, other types of singular points at infinity of quadratic systems can be saddles, foci, centers, semi–elemental saddle–nodes or nilpotent elliptic saddles. We also have the possibility of no singularities after removal of the degeneracy. To convey the way these singularities were obtained as well as their nature, we use the notation $[\infty;\, \emptyset]$, $[\infty;\, N]$, $[\infty;\, N^d]$, $[\infty;\, S]$, $[\infty;\, C]$, $[\infty;\, \overline{\binom{2}{0}}SN]$ and $[\infty;\, \widehat{\binom{3}{0}}ES]$.

2.6. **Affine invariant polynomials and preliminary results.** Consider real quadratic systems of the form

$$(2)\qquad \begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(x,y),\\ \frac{dy}{dt} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(x,y), \end{aligned}$$

with homogeneous polynomials $p_i$ and $q_i$ $(i = 0, 1, 2)$ in $x, y$ which are defined as follows:

$$\begin{aligned} p_0 = a_{00},\quad p_1(x,y) = a_{10}x + a_{01}y,\quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2,\\ q_0 = b_{00},\quad q_1(x,y) = b_{10}x + b_{01}y,\quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

Let $\tilde{a} = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ be the 12-tuple of the coefficients of systems (2) and denote $\mathbb{R}[\tilde{a}, x, y] = \mathbb{R}[a_{00}, \dots, b_{02}, x, y]$.

It is known that on the set **QS** of all quadratic differential systems (2) acts the group $Aff(2,\mathbb{R})$ of affine transformations on the plane (cf. [28]). For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of $G$ on **QS**. We can identify the set **QS** of systems (2) with a subset of $\mathbb{R}^{12}$ via the map $\mathbf{QS} \longrightarrow \mathbb{R}^{12}$ which associates to each system (2) the 12–tuple $\tilde{a} = (a_{00}, \dots, b_{02})$ of its coefficients. We associate to this group action polynomials in $x, y$ and parameters which behave well with respect to this action, the $GL$–comitants, the $T$–comitants and the $CT$–comitants. For their constructions we refer the reader to the paper [28] (see also [30]). In the statement of our main theorem intervene the following 22 invariant polynomials constructed in these article: $\mu_0, \mu_1, C_2, D, \kappa, \sigma, \eta, \mathcal{F}_1, \mathcal{F}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4, \mathcal{W}_5, \mathcal{W}_6, \mathcal{W}_7, \tilde{K}, \tilde{L}, \tilde{N}, \mathcal{B}_1, U_3$. In the proof of our main theorem there appear 19 other invariant polynomials: $\mathcal{B}_2, E_1, \mathcal{F}, \mathcal{F}_3, \mathcal{F}_4, G_9, G_{10}, \mathcal{H}, \tilde{M}, P, \mathcal{T}_1, \mathcal{T}_2, U_1, U_2, U_4, U_5, U_6, Y_1, Y_2$. We refer the reader interested in the construction of these polynomials to the following associated web-page:

http://mat.uab.es/∼artes/articles/qvfinvariants/qvfinvariants.html.

## 3. The proof of the Main Theorem

According to [31] for a quadratic system to have finite singularities of total multiplicity three (i.e. $m_f = 3$) the conditions $\mu_0 = 0$ and $\mu_1 \neq 0$ must be satisfied. Since the subclass with three real finite simple singularities was considered in [5] (i.e. the additional condition $\mathbf{D} < 0$ holds) we consider here three subclasses:

- systems with one real and two complex finite singularities ($\mu_0 = 0,\ \mu_1 \neq 0,\ \mathbf{D} > 0$);
- systems with one double and one simple real finite singularities ($\mu_0 = 0,\ \mu_1 \neq 0,\ \mathbf{D} = 0,\ \mathbf{P} \neq 0$);
- systems with one triple real finite singularity ($\mu_0 = 0,\ \mu_1 \neq 0,\ \mathbf{D} = 0,\ \mathbf{P} = 0$).

We observe that the systems from each one of the above mentioned subclasses have finite singularities of total multiplicity 3 and therefore by [2] the following lemma is valid.

**Lemma 1.** *The geometric configurations of singularities at infinity of the family of quadratic systems possessing finite singularities of total multiplicity 3 (i.e. $\mu_0 = 0$, $\mu_1 \neq 0$) are classified in* Diagram 4 *according to the geometric equivalence relation.*

3.1. **Systems with one real and two complex finite singularities.** Consider quadratic systems (2) with real coefficients and variables $x$ and $y$. Assume that these systems possess one real and two complex finite singularities. Then according to [31] via an affine transformation and time rescaling these systems could be brought to the form:

$$(3)\qquad \begin{aligned} \dot{x} &= 2(h - gu)x + g(u^2 + 1)y + gx^2 - 2hxy,\\ \dot{y} &= 2(m - lu)x + l(u^2 + 1)y + lx^2 - 2mxy \end{aligned}$$

possessing the real elemental singular point $M_1(0,0)$ and the two complex singularities $M_{2,3}(u \pm i, 1)$. For these systems calculations yield

$$\mu_0 = 0,\quad \mu_1 = 4(gm - hl)^2(1 + u^2)x,\quad \kappa = 128h^2(gm - hl).$$

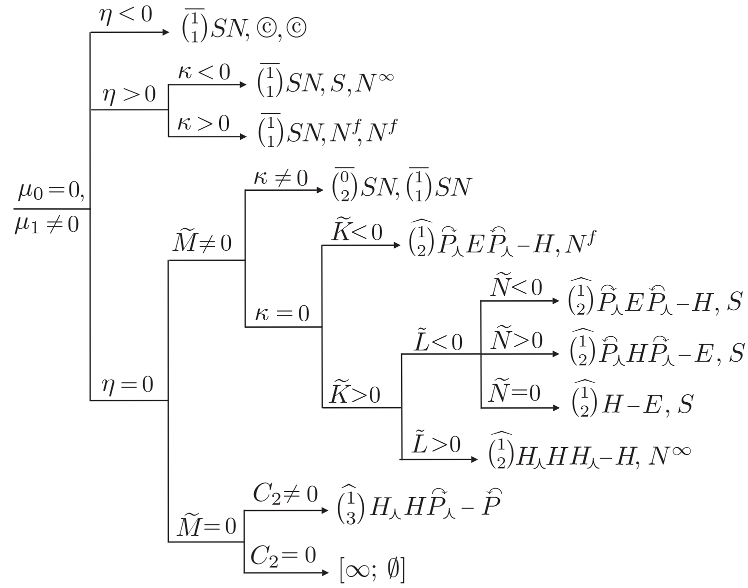


DIAGRAM 4. **Configurations of infinite singularities: the case $\mu_0 = 0$, $\mu_1 \neq 0$.**

Since for the above systems we have $\mu_1 \neq 0$ (i.e. $gm-hl \neq 0$) we observe that the condition $\kappa = 0$ is equivalent to $h = 0$.

3.1.1. **The case** $\kappa \neq 0$**.** Then $h \neq 0$ and we may assume $h = 1$ due to a time rescaling. So we get the systems

$$\begin{aligned}\dot{x} &= 2(1-gu)x + g(u^2+1)y + gx^2 - 2xy,\\ \dot{y} &= 2(m-lu)x + l(u^2+1)y + lx^2 - 2mxy\end{aligned} \tag{4}$$

and for the singular points $M_1(0,0)$ and $M_{2,3}(u \pm i, 1)$ we have the following values for the traces $\rho_i$, for the determinants $\Delta_i$, for the discriminants $\tau_i$ and for the linearization matrix $\mathcal{M}_1$ (in the case of a real singularity):

$$\begin{gathered}\mathcal{M}_1 = \begin{pmatrix} 2(1-gu) & g(1+u^2)\\ 2(m-lu) & l(1+u^2)\end{pmatrix}, \quad \rho_1 = 2 + l - 2gu + lu^2,\\ \Delta_1 = 2(l-gm)(1+u^2), \quad \tau_1 = \rho_1^2 - 4\Delta_1;\\ \rho_{2,3} = l - 2mu + lu^2 \pm 2i(g-m), \quad \Delta_{2,3} = 4(gm-l)(1 \mp iu), \quad \tau_{2,3} = \rho_{2,3}^2 - 4\Delta_{2,3}.\end{gathered} \tag{5}$$

Then for the above mentioned systems we calculate

$$\begin{gathered}\kappa = -64\Delta_1/(1+u^2), \quad \widetilde{K} = 2\Delta_1 x^2/(1+u^2), \quad G_9 = \Delta_1/(2(1+u^2)),\\ \mathcal{T}_4 = 2\rho_1\rho_2\rho_3\Delta_1^2/(1+u^2), \quad \mathcal{T}_3 = 2\big[\rho_1(\rho_2+\rho_3) + \rho_2\rho_3\big]\Delta_1^2/(1+u^2),\\ \mathcal{T}_2 = 2(\rho_1+\rho_2+\rho_3)\Delta_1^2/(1+u^2), \quad \eta = 4\big[(g+2m)^2 - 8l\big],\\ \widetilde{M} = -8\big[(g+2m)^2 - 6l\big]x^2 + 16(g+2m)xy - 32y^2,\\ W_4 = 4\tau_1\tau_2\tau_3\Delta_1^4/(1+u^2)^2, \quad W_2 = 4(\tau_1+\tau_2+\tau_3)\Delta_1^4/(1+u^2)^2,\\ W_3 = 4\big[\tau_1(\tau_2+\tau_3) + \tau_2\tau_3\big]\Delta_1^4/(1+u^2)^2.\end{gathered} \tag{6}$$

**Remark 2.** *We observe that* $\widetilde{M} \neq 0$ *and the condition* $\kappa \neq 0$ *implies* $\widetilde{K}G_9 \neq 0$*. Moreover we have* $\operatorname{sign}(\widetilde{K}) = -\operatorname{sign}(\kappa)$.

3.1.1.1. *The subcase* $\kappa < 0$. This implies $\widetilde{K} > 0$ and according to [6] (see Table 1, lines 124 - 135) the elemental real singular point is an anti-saddle and its type is governed by the invariant polynomials $W_i$, $i = 4, 3, 2$ (as in the case we consider we have $G_9 \neq 0$).

3.1.1.1.1. *The possibility* $W_4 < 0$. Clearly for the complex singularities $M_{2,3}$ we have $\tau_3 = \bar{\tau}_2$ (also $\rho_3 = \bar{\rho}_2$) and then $\tau_2\tau_3 > 0$. So considering (6) we have $\tau_1 < 0$, i.e. the real singular point is a focus.

**1)** *Assume first* $\mathcal{T}_4 \neq 0$. Then the focus is strong and considering Lemma 1 and the condition $\widetilde{M} \neq 0$ (see Remark 2) we arrive at the following three configurations of singularities:

- $f$, ©, ©; $\overline{\binom{1}{1}}SN$, ©, © : *Example* $\Rightarrow$ ($g = -2$, $l = 1$, $m = 1$, $u = 0$) (if $\eta < 0$);
- $f$, ©, ©; $\overline{\binom{1}{1}}SN, S, N^\infty$ : *Example* $\Rightarrow$ ($g = -2$, $l = -1$, $m = 1$, $u = 0$) (if $\eta > 0$);
- $f$, ©, ©; $\overline{\binom{0}{2}}SN$, $\overline{\binom{1}{1}}SN$ : *Example* $\Rightarrow$ ($g = -2$, $l = 0$, $m = 1$, $u = 0$) (if $\eta = 0$).

**2)** *Suppose now* $\mathcal{T}_4 = 0$.

***a)*** *The case* $\mathcal{T}_3 \neq 0$. Considering (6) we have $\rho_1 = 0$ (i.e. $l = 2(gu-1)/(1+u^2)$) and we calculate

$$(7)\qquad \begin{gathered} W_4 = -16\Delta_1^5\tau_2\tau_3/(1+u^2)^2, \quad \mathcal{T}_3 = 2\rho_2\rho_3\Delta_1^2/(1+u^2), \quad \mathcal{F} = -\rho_2\rho_3\Delta_1/(4(1+u^2)), \\ \mathcal{F}_1 = 2\Delta_1\big[(1+u^2)(2g^2u - 2gmu - 5g + 2m) + 4(g+u)\big]/(1+u^2). \end{gathered}$$

***a***$_1$**)** *The subcase* $\mathcal{F}_1 \neq 0$. As $\mathcal{T}_3\mathcal{F} < 0$ considering [31] (see Main Theorem, ($b_1$)) we have a first order weak focus. So by Lemma 1 and Remark 2 we get the three global configurations:

- $f^{(1)}$, ©, ©; $\overline{\binom{1}{1}}SN$, ©, © : *Example* $\Rightarrow$ ($g = 3$, $l = 2$, $m = -3$, $u = 1$) (if $\eta < 0$);
- $f^{(1)}$, ©, ©; $\overline{\binom{1}{1}}SN, S, N^\infty$ : *Example* $\Rightarrow$ ($g = 3$, $l = 2$, $m = -4$, $u = 1$) (if $\eta > 0$);
- $f^{(1)}$, ©, ©; $\overline{\binom{0}{2}}SN$, $\overline{\binom{1}{1}}SN$ : *Example* $\Rightarrow$ ($g = 3$, $l = 2$, $m = -7/2$, $u = 1$) (if $\eta = 0$).

***a***$_2$**)** *The subcase* $\mathcal{F}_1 = 0$. Then the order of the weak focus is at least two. We claim that in this case the condition $gu - 1 \neq 0$ holds. Indeed if $gu - 1 = 0$ (then $u \neq 0$) we set $g = 1/u$ (this implies $l = 2(gu-1)/(1+u^2) = 0$) and we calculate for systems (4)

$$\mathcal{F}_1 = -4m(1+u^2)/u^2, \quad \kappa = 128m/u.$$

Therefore due to $\kappa \neq 0$ we have $\mathcal{F}_1 \neq 0$ and we obtain a contradiction which proves our claim. So $gu - 1 \neq 0$ and considering (7) the condition $\mathcal{F}_1 = 0$ gives

$$m = \frac{4u - g + 2g^2u - 5gu^2 + 2g^2u^3}{2(gu-1)(1+u^2)}$$

and calculations yield

$$(8)\qquad \begin{gathered} \mathcal{F}_1 = 0, \quad \mathcal{F}_2 = -\frac{(2gu-1)^2(5gu-4)\big[g^2 + (gu-2)^2\big]^4}{(gu-1)^5(1+u^2)^2}, \\ \mathcal{T}_3 = \frac{2(2gu-1)^2\big[g^2+(gu-2)^2\big]^3}{(gu-1)^4(1+u^2)}, \quad \eta = \frac{4\Phi(g,u)\big[g^2+(gu-2)^2\big]^3}{(gu-1)^2(1+u^2)^2}, \end{gathered}$$

where $\Phi(g,u) = 9g^2u^2(1+u^2) - 4gu(3+4u^2) + 4(1+2u^2)$ is a quadratic polynomial with respect to $g$ whose discriminant equals $-16u^4(3+2u^2) \leq 0$. So for the expressions above we get $\eta > 0$ and we observe that the conditions $\mathcal{F}_2 = 0$ and $\mathcal{T}_3 \neq 0$ imply $5gu - 4 = 0$. In this case $u \neq 0$, otherwise we have $\mathcal{F}_2 \neq 0$. Therefore we get $g = 4/(5u)$ and we calculate:

$$\begin{gathered} \mathcal{F}_1 = \mathcal{F}_2 = 0, \quad \mathcal{F}_3 = 2^{10}3^2 5^{-6}(4+9u^2)^4u^{-8}(1+u^2)^{-2} \neq 0, \\ \mathcal{F}_4 = 2^{13}3^3 5^{-7}(4+9u^2)^4u^{-7}(1+u^2)^{-2} \neq 0. \end{gathered}$$

Hence by [31] we could not have a center. So considering Lemma 1 we arrive at the two global configurations:

- $f^{(2)}$, ©, ©; $\overline{\binom{1}{1}}SN, S, N^\infty$ : *Example* $\Rightarrow$ ($g = 2/3$, $l = -1/3$, $m = -4/3$, $u = 1$) (if $\mathcal{F}_2 \neq 0$);
- $f^{(3)}$, ©, ©; $\overline{\binom{1}{1}}SN, S, N^\infty$ : *Example* $\Rightarrow$ ($g = 4/5$, $l = -1/5$, $m = -11/5$, $u = 1$) (if $\mathcal{F}_2 = 0$).

***b)*** *The case* $\mathcal{T}_3 = 0$. Considering (6) it is clear that the condition $\mathcal{T}_4 = \mathcal{T}_3 = 0$ gives $\rho_2 = \rho_3 = 0$ and this implies $g = m$ and $l = 2mu/(1+u^2)$. Then calculations yield

$$W_4 = 256\Delta_1^6\tau_1/(1+u^2)^3, \quad \kappa = -64\Delta_1/(1+u^2),$$
$$\eta = 4m(9m-16u+9mu^2)/(1+u^2), \quad \rho_2 = \rho_3 = 0, \quad \rho_1 = 2. \tag{9}$$

We claim that in this case the condition $W_4 < 0$ implies $\eta < 0$. Indeed, we observe that the curves $\eta = 0$ and $\tau_1 = 0$ have only complex points of intersection. So it is sufficient to determine the sign of the polynomial $\eta$ at a point where $\tau_1 < 0$. At the point $(m,u) = (-1/2,-2)$ we have $\tau_1 = -2 < 0$ and $\eta = -19/5 < 0$ and this proves our claim.

Thus due to $\rho_1 \neq 0$ the focus is strong and we get the next configuration:

- $f$, ©$^\rho$, ©$^\rho$; $\overline{\binom{1}{1}}SN$, ©, © : *Example* ⇒ $(g = 1/2,\ l = 6/13,\ m = 1/2,\ u = 3/2)$.

3.1.1.1.2. *The possibility* $W_4 > 0$. In this case by [6] (see Table 1, line 124) the elemental singularity is a generic node and we arrive at the next three global configurations of singularities:

- $n$, ©, ©; $\overline{\binom{1}{1}}SN$, ©, © : *Example* ⇒ $(g = 1,\ l = 2,\ m = 1,\ u = 0)$ (if $\eta < 0$);
- $n$, ©, ©; $\overline{\binom{1}{1}}SN, S, N^\infty$ : *Example* ⇒ $(g = 1,\ l = 17/16,\ m = 1,\ u = 0)$ (if $\eta > 0$);
- $n$, ©, ©; $\overline{\binom{0}{2}}SN$, $\overline{\binom{1}{1}}SN$ : *Example* ⇒ $(g = 1,\ l = 9/8,\ m = 1,\ u = 0)$ (if $\eta = 0$).

3.1.1.1.3. *The possibility* $W_4 = 0$. We consider two cases: $W_3 \neq 0$ and $W_3 = 0$.

**1)** *Assume first* $W_3 \neq 0$. Considering (6) we obtain $\tau_1 = 0$ and we have a node with coinciding eigenvalues. We claim that this node will be a star node if and only if $U_3 = 0$.

Indeed taking into account the linearization matrix $\mathcal{M}_1$ from (5) it is clear that we have a star node if and only if $g = 0$, $l = 2/(1+u^2)$ and $m = 2u/(1+u^2)$. In this case a straightforward computation gives us $U_3 = 0$.

Conversely, assume that $U_3 = 0$. Then we have

$$\text{Coefficient}[U_3, y^5] = -12g(1+u^2)\left[g^2 + (gu-2)^2\right] = 0 \ \Leftrightarrow\ g = 0$$

and this implies

$$\text{Coefficient}[U_3, xy^4] = -24l(1+u^2)(l+lu^2-2) = 0, \quad \kappa = -128l.$$

So due to $\kappa \neq 0$ we obtain $l = 2/(1+u^2)$ and then we have

$$\text{Coefficient}[U_3, x^2y^3] = 288(2u-m-mu^2)/(1+u^2) = 0.$$

Therefore we get $m = 2u/(1+u^2)$ and this proves our claim.

We observe that in the case $U_3 = 0$ we obtain $\eta = -64/(1+u^2)^2 < 0$ and according to Lemma 1 we get the three global configurations of singularities

- $n^d$, ©, ©; $\overline{\binom{1}{1}}SN$, ©, © : *Example* ⇒ $(g = -1/4,\ l = 4,\ m = 2,\ u = 0)$ (if $\eta < 0$);
- $n^d$, ©, ©; $\overline{\binom{1}{1}}SN, S, N^\infty$ : *Example* ⇒ $(g = -1/16,\ l = 1,\ m = 2,\ u = 0)$ (if $\eta > 0$);
- $n^d$, ©, ©; $\overline{\binom{0}{2}}SN$, $\overline{\binom{1}{1}}SN$ : *Example* ⇒ $(g = 0,\ l = 2,\ m = 2,\ u = 0)$ (if $\eta = 0$) in the case $U_3 \neq 0$ and

the unique global configurations of singularities

- $n^*$, ©, ©; $\overline{\binom{1}{1}}SN$, ©, © : *Example* ⇒ $(g = 0,\ l = 2,\ m = 0,\ u = 0)$ in the case $U_3 = 0$.

**2)** *Suppose now* $W_3 = 0$. Considering (6) we get $\tau_2 = \tau_3 = 0$. However for systems (4) these conditions lead to a quite complicated system of polynomial equations. So we decide to construct a more convenient canonical form and this will be done in Lemma 2. According to this lemma in the case $\kappa < 0$ we obtain 12 geometrically distinct configurations and the corresponding invariant conditions which characterize them.

3.1.1.2. *The subcase* $\kappa > 0$. Then $\Delta_1 < 0$ and the real singular point is a saddle. On the other hand considering (6) we obtain $\eta > 0$ and according to Lemma 1 at infinity we get the unique configuration: $\overline{\binom{1}{1}}SN, N^f, N^f$.

3.1.1.2.1. *The possibility* $W_4 \neq 0$. In this case the discriminants of the complex points are different from zero.

**1)** *Assume first* $\mathcal{T}_4 \neq 0$. In this case the saddle is strong and we arrive at the global configurations of singularities

- $s$, ©, ©; $\overline{\binom{1}{1}}SN, N^f, N^f$ : *Example* $\Rightarrow (g = 1,\, l = -1,\, m = 1,\, u = 0)$.

**2)** *Suppose now* $\mathcal{T}_4 = 0$.

***a*)** *The case* $\mathcal{T}_3 \neq 0$. Then $\rho_1 = 0$, i.e. we have a weak saddle whose order depends on the polynomial $\mathcal{F}_1$.

***a*$_1$)** *The subcase* $\mathcal{F}_1 \neq 0$. The weak saddle has order one and we get the global configurations of singularities

- $s^{(1)}$, ©, ©; $\overline{\binom{1}{1}}SN, N^f, N^f$ : *Example* $\Rightarrow (g = 2,\, l = 1,\, m = 1,\, u = 1)$.

***a*$_2$)** *The subcase* $\mathcal{F}_1 = 0$. Then we have a weak saddle whose order is at least two. We claim that in this case the condition $\mathcal{F}_2 \neq 0$ holds, i.e. we could not have a third order weak saddle. Indeed, as it was shown earlier (see the compartment 3.1.1.1.1, p. ***a*$_2$**)) the conditions $\mathcal{T}_4 = \mathcal{F}_1 = 0$ yields

$$l = \frac{2(gu-1)}{1+u^2}, \quad m = \frac{4u - g + 2g^2u - 5gu^2 + 2g^2u^3}{2(gu-1)(1+u^2)}$$

and then we obtain the value of $\mathcal{F}_2$ given in (8). So due to $\mathcal{T}_3 \neq 0$ the condition $\mathcal{F}_2 = 0$ implies $5gu - 4 = 0$. In this case $u \neq 0$, otherwise we have $\mathcal{F}_2 \neq 0$. Therefore we get $g = 4/(5u)$ and we calculate:

$$\mathcal{F}_1 = \mathcal{F}_2 = 0, \quad \kappa = -768(4 + 9u^2)/(25u^2(1+u^2)) < 0$$

and we obtain a contradiction which proves our claim. Thus we obtain the unique global configurations of singularities which is:

- $s^{(2)}$, ©, ©; $\overline{\binom{1}{1}}SN, N^f, N^f$ : *Example* $\Rightarrow (g = 0,\, l = -2,\, m = 0,\, u = 0)$.

***b*)** *The case* $\mathcal{T}_3 = 0$. Considering (6) it is clear that the condition $\mathcal{T}_4 = \mathcal{T}_3 = 0$ gives $\rho_2 = \rho_3 = 0$ and this implies $g = m$ and $l = 2mu/(1+u^2)$. Then we have $\rho_1 = 2 \neq 0$, i.e. the saddle is strong:

- $s$, $©^{\rho}$, $©^{\rho}$; $\overline{\binom{1}{1}}SN, N^f, N^f$ : *Example* $\Rightarrow (g = 1,\, l = 0,\, m = 1,\, u = 0)$.

3.1.1.2.2. *The possibility* $W_4 = 0$. Since the real singularity is a saddle we conclude that in this case both complex points have zero discriminants (i.e. $W_4 = W_3 = 0$). In this case by Lemma 2 we get the three geometrically distinct configurations $(Cf_{13})$, $(Cf_{14})$ and $(Cf_{15})$ and the corresponding invariant conditions which characterize them.

3.1.1.3. *The subcase* $\kappa = 0$. Then $h = 0$ and we have $\mu_1 = 4g^2m^2(1+u^2)x \neq 0$. Therefore we may assume $g = 1$ due to a time rescaling and we get the 3-parameter family of systems

$$\begin{aligned} \dot{x} &= -2ux + (u^2+1)y + x^2, \\ \dot{y} &= 2(m - lu)x + l(u^2+1)y + lx^2 - 2mxy. \end{aligned} \tag{10}$$

For the singular points $M_1(0,0)$ and $M_{2,3}(u \pm i, 1)$ we have the following values for the traces $\rho_i$, for the determinants $\Delta_i$, for the discriminants $\tau_i$ and for the linearization matrix $\mathcal{M}_1$ (in the case of real singularity):

$$\begin{gathered} \mathcal{M}_1 = \begin{pmatrix} -2u & 1+u^2 \\ 2(m-lu) & l(1+u^2) \end{pmatrix}, \quad \rho_1 = l - 2u + lu^2, \quad \Delta_1 = -2m(1+u^2), \quad \tau_1 = \rho_1^2 - 4\Delta_1; \\ \rho_{2,3} = l - 2mu + lu^2 \pm 2i(1-m), \quad \Delta_{2,3} = 4m(1 \mp iu), \quad \tau_{2,3} = \rho_{2,3}^2 - 4\Delta_{2,3}. \end{gathered} \tag{11}$$

Then for systems above we calculate

$$
\begin{aligned}
&\mu_1 = 4m^2(1+u^2)x, \quad \widetilde{K} = -4mx^2, \quad G_9 = 0,\\
&W_7 = 3\tau_1\tau_2\tau_3\Delta_1^6/(1+u^2)^2/16, \quad W_5 = 9(\tau_1+\tau_2+\tau_3)\Delta_1^6/(1+u^2)^2/4,\\
&W_6 = \big[\tau_1(\tau_2+\tau_3)+\tau_2\tau_3\big]\Delta_1^6/(1+u^2)^2/16, \quad \mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_2 = \mathcal{T}_1 = 0,\\
&\sigma = \rho_1 - 2(m-1)x, \quad \mathcal{B}_1 = \rho_1\rho_2\rho_3\Delta_1,\\
&\mathcal{B}_2 = -(m-1)^2(1+u^2)\big[\rho_1(\rho_2+\rho_3)+\rho_2\rho_3\big]\Delta_1.
\end{aligned}
\tag{12}
$$

We observe that the condition $\mu_1 \neq 0$ implies $\widetilde{K} \neq 0$ and by (12) we have $\operatorname{sign}(\widetilde{K}) = \operatorname{sign}(\Delta_1)$. According to [6] and [31] the above polynomials are responsible for the types of the finite singularities of systems (10). In order to describe the infinite singularities, considering Lemma 1 we calculate the additional invariant polynomials:

$$
\begin{aligned}
&\kappa = \eta = 0, \quad \widetilde{M} = -8(1+2m)^2x^2, \quad \tilde{L} = 8(1+2m)x^2,\\
&C_2 = -lx^3 + (1+2m)x^2y, \quad \widetilde{N} = -4m(1+m)x^2.
\end{aligned}
\tag{13}
$$

So by Lemma 1 we have the next remark.

**Remark 3.** *Systems* (10) *could possess at infinity only one of the following seven configurations of singularities:*

$$
\begin{array}{lcl}
\widetilde{M} \neq 0,\ \widetilde{K} < 0 & \Rightarrow & \widehat{\binom{1}{2}}\, \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ N^f;\\
\widetilde{M} \neq 0,\ \widetilde{K} > 0,\ \tilde{L} < 0,\ \widetilde{N} < 0 & \Rightarrow & \widehat{\binom{1}{2}}\, \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ S;\\
\widetilde{M} \neq 0,\ \widetilde{K} > 0,\ \tilde{L} < 0,\ \widetilde{N} > 0 & \Rightarrow & \widehat{\binom{1}{2}}\, \widehat{P}_{\curlywedge} H \widehat{P}_{\curlywedge} - E,\ S;\\
\widetilde{M} \neq 0,\ \widetilde{K} > 0,\ \tilde{L} < 0,\ \widetilde{N} = 0 & \Rightarrow & \widehat{\binom{1}{2}}\, H - E,\ S;\\
\widetilde{M} \neq 0,\ \widetilde{K} > 0,\ \tilde{L} > 0 & \Rightarrow & \widehat{\binom{1}{2}}\, H_{\curlywedge} H H_{\curlywedge} - H,\ N^{\infty};\\
\widetilde{M} = 0,\ C_2 \neq 0 & \Rightarrow & \widehat{\binom{1}{3}}\ H_{\curlywedge} H \widehat{P}_{\curlywedge} - \widehat{P};\\
\widetilde{M} = C_2 = 0 & \Rightarrow & [\infty;\ \emptyset].
\end{array}
$$

3.1.1.3.1. *The possibility* $\widetilde{K} < 0$. Then $\Delta_1 < 0$ and the real singularity is a saddle. We observe that the condition $\widetilde{K} < 0$ implies $m > 0$ and then $\widetilde{M} \neq 0$.

**1)** *Assume first* $\mathcal{B}_1 \neq 0$. In this case $\rho_1 \neq 0$ and the saddle is strong. So considering Remark 3 we obtain the configuration

- $s,\ \copyright, \copyright;\ \widehat{\binom{1}{2}}\, \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ N^f$ : *Example* $\Rightarrow (l = 1,\ m = 1,\ u = 0)$.

**2)** *Suppose now* $\mathcal{B}_1 = 0$. Then $\rho_1\rho_2\rho_3 = 0$ and we consider two cases: $\mathcal{B}_2 \neq 0$ and $\mathcal{B}_2 = 0$.

***a*)** *The case* $\mathcal{B}_2 \neq 0$. As $\rho_3 = \bar{\rho}_2$ considering (12) we have $\rho_2\rho_3 \neq 0$ and then $\rho_1 = 0$, i.e. $l = 2u/(1+u^2)$ and the saddle is weak. We calculate

$$
\begin{aligned}
&\mathcal{B}_1 = \mathcal{H} = 0, \quad \mathcal{B}_2 = 8m(m-1)^4(1+u^2)^3,\\
&\sigma = 2(1-m)x, \quad \mathcal{F}_1 = 8m(m-1)u(1+u^2)
\end{aligned}
\tag{14}
$$

and clearly the conditions $\mathcal{B}_2 \neq 0$ and $\widetilde{K} < 0$ imply $\sigma \neq 0$ and $\mathcal{B}_2 > 0$. So according to [31] (see the statements $(e_1)$ and $(e_3), [\beta]$ of the Main Theorem) the weak saddle is of order one if $\mathcal{F}_1 \neq 0$ and it is integrable if $\mathcal{F}_1 = 0$. Considering Remark 3 we get the two configurations

- $s^{(1)},\ \copyright, \copyright;\ \widehat{\binom{1}{2}}\, \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ N^f$ : *Example* $\Rightarrow (l = 1,\ m = 2,\ u = 1)$ (if $\mathcal{F}_1 \neq 0$);
- $\$,\ \copyright, \copyright;\ \widehat{\binom{1}{2}}\, \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ N^f$ : *Example* $\Rightarrow (l = 0,\ m = 2,\ u = 0)$ (if $\mathcal{F}_1 = 0$).

***b*)** *The case* $\mathcal{B}_2 = 0$. We claim that the condition $\mathcal{B}_1 = \mathcal{B}_2 = 0$ implies $\sigma = 0$. Indeed, we observe that the condition $m = 1$ has to be satisfied, otherwise we must have $\rho_2 = \rho_3 = 0$ and this implies $m = 1$. Therefore as $m=1$ we get $\rho_1=\rho_2=\rho_3$ and then the condition $\mathcal{B}_1=\rho_1^3\Delta_1=0$ implies $\sigma=\rho_1=0$, i.e. our claim is proved.

Thus we have $m=1$ and $l=2u/(1+u^2)$ and we get Hamiltonian systems. So we arrive at the global configurations of singularities

- $\$,\ \copyright^{\rho},\copyright^{\rho};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge}E\widehat{P}_{\curlywedge}-H,\ N^f$ : *Example* $\Rightarrow (l=0,\ m=1,\ u=0)$.

**Remark 4.** *Considering* (13) *we observe that the condition* $\tilde{L}=0$ *is equivalent to* $\widetilde{M}=0$.

3.1.1.3.2. *The possibility* $\widetilde{K}>0$. In this case $\Delta_1>0$, i.e. the real singular point is an anti-saddle and as $G_9=0$, according to [6] (see Table 1, lines 127 - 129, 132-135) its type is governed by the invariant polynomials $W_7$, $W_6$ and $W_5$

**1)** *Assume first* $W_7<0$. By (12) due to $\tau_2\tau_3>0$ we have $\tau_1<0$ and we conclude, that the real singularity is a focus.

***a)*** *The case* $\mathcal{B}_1\neq 0$. Then $\rho_1\neq 0$ and the focus is strong. Considering Remarks 3 and 4 we arrive at the next six global configurations of singularities

- $f,\ \copyright,\copyright;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge}E\widehat{P}_{\curlywedge}-H,\ S$ : *Example* $\Rightarrow (l=1,\ m=-3/2,\ u=0)$ (if $\tilde{L}<0,\ \widetilde{N}<0$);
- $f,\ \copyright,\copyright;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge}H\widehat{P}_{\curlywedge}-E,\ S$ : *Example* $\Rightarrow (l=1,\ m=-2/3,\ u=0)$ (if $\tilde{L}<0,\ \widetilde{N}>0$);
- $f,\ \copyright,\copyright;\ \widehat{\binom{1}{2}}\ H-E,\ S$ : *Example* $\Rightarrow (l=1,\ m=-1,\ u=0)$ (if $\tilde{L}<0,\ \widetilde{N}=0$);
- $f,\ \copyright,\copyright;\ \widehat{\binom{1}{2}}\ H_{\curlywedge}HH_{\curlywedge}-H,\ N^{\infty}$ : *Example* $\Rightarrow (l=1,\ m=-1/3,\ u=0)$ (if $\tilde{L}>0$);
- $f,\ \copyright,\copyright;\ \widehat{\binom{1}{3}}\ H_{\curlywedge}H\widehat{P}_{\curlywedge}-\widehat{P}$ : *Example* $\Rightarrow (l=1,\ m=-1/2,\ u=0)$ (if $\tilde{L}=0,\ C_2\neq 0$).
- $f,\ \copyright,\copyright;\ [\infty;\ \emptyset]$ : *Example* $\Rightarrow (l=0,\ m=-1/2,\ u=1)$ (if $\tilde{L}=0,\ C_2=0$).

***b)*** *The case* $\mathcal{B}_1=0$. Then $\rho_1\rho_2\rho_3=0$ and as $\Im(\rho_2)=-\Im(\rho_3)=2(1-m)$ we obtain $\rho_2\rho_3\neq 0$ due to $\widetilde{K}>0$ (i.e. $m<0$). So we get $\rho_1=0$ and this gives $l=2u/(u^2+1)$. Then following [31] we calculate

$$\mathcal{B}_1=\mathcal{H}=0,\quad \sigma=2(1-m)x,\quad \mathcal{F}_1=8mu(m-1)(1+u^2),\quad \widetilde{K}=-4mx^2,$$
$$\mathcal{B}_2=8(m-1)^4m(1+u^2)^3,\quad C_2=x\big[-2ux+(1+u)^2(1+2m)y\big]/(1+u)^2$$

and clearly the condition $\widetilde{K}>0$ implies $\mathcal{B}_2<0$ and $\sigma\neq 0$. So according to [31] (see Main Theorem) we have a weak focus of the first order if $\mathcal{F}_1\neq 0$ (the statement $(e_2)$) and we have a center if $\mathcal{F}_1=0$ (the statement $(e_4),[\beta]$).

***b$_1$)*** *The subcase* $\mathcal{F}_1\neq 0$. Then $u\neq 0$ (this implies $C_2\neq 0$) and we arrive at the next five global configurations of singularities

- $f^{(1)},\ \copyright,\copyright;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge}E\widehat{P}_{\curlywedge}-H,\ S$ : *Example* $\Rightarrow (l=1,\ m=-3/2,\ u=1)$ (if $\tilde{L}<0,\ \widetilde{N}<0$);
- $f^{(1)},\ \copyright,\copyright;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge}H\widehat{P}_{\curlywedge}-E,\ S$ : *Example* $\Rightarrow (l=1,\ m=-2/3,\ u=1)$ (if $\tilde{L}<0,\ \widetilde{N}>0$);
- $f^{(1)},\ \copyright,\copyright;\ \widehat{\binom{1}{2}}\ H-E,\ S$ : *Example* $\Rightarrow (l=1,\ m=-1,\ u=1)$ (if $\tilde{L}<0,\ \widetilde{N}=0$);
- $f^{(1)},\ \copyright,\copyright;\ \widehat{\binom{1}{2}}\ H_{\curlywedge}HH_{\curlywedge}-H,\ N^{\infty}$ : *Example* $\Rightarrow (l=1,\ m=-1/3,\ u=1)$ (if $\tilde{L}>0$);
- $f^{(1)},\ \copyright,\copyright;\ \widehat{\binom{1}{3}}\ H_{\curlywedge}H\widehat{P}_{\curlywedge}-\widehat{P}$ : *Example* $\Rightarrow (l=1,\ m=-1/2,\ u=1)$ (if $\tilde{L}=0$).

***b$_2$)*** *The subcase* $\mathcal{F}_1=0$. Then $u=0$ and hence $l=0$. So considering (13) in this case the condition $\tilde{L}=0$ implies also $C_2=0$.

- $c,\ \copyright,\copyright;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge}E\widehat{P}_{\curlywedge}-H,\ S$ : *Example* $\Rightarrow (l=0,\ m=-3/2,\ u=0)$ (if $\tilde{L}<0,\ \widetilde{N}<0$);
- $c,\ \copyright,\copyright;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge}H\widehat{P}_{\curlywedge}-E,\ S$ : *Example* $\Rightarrow (l=0,\ m=-2/3,\ u=0)$ (if $\tilde{L}<0,\ \widetilde{N}>0$);
- $c,\ \copyright,\copyright;\ \widehat{\binom{1}{2}}\ H-E,\ S$ : *Example* $\Rightarrow (l=0,\ m=-1,\ u=0)$ (if $\tilde{L}<0,\ \widetilde{N}=0$);

- $c,\ ©,©;\ \widehat{\binom{1}{2}}\ H_\curlywedge HH_\curlywedge - H,\ N^\infty$: *Example* $\Rightarrow (l=0,\ m=-1/3,\ u=0)$ (if $\tilde{L}>0$);
- $c,\ ©,©;\ [\infty;\ \emptyset]$: *Example* $\Rightarrow (l=0,\ m=-1/2,\ u=0)$ (if $\tilde{L}=0$).

**2)** *Suppose now* $W_7>0$. In this case we have $\tau_1>0$ and the anti-saddle is a generic node.

**Remark 5.** *We observe that the condition* $C_2=0$ *(i.e.* $l=0$ *and* $m=-1/2$*) implies* $W_7=-3(1+u^2)^6/4<0$.

By this remark we conclude that the condition $W_7>0$ implies $C_2\neq 0$. So considering Remarks 3 and 4 we arrive at the next five global configurations of singularities

- $n,\ ©,©;\ \widehat{\binom{1}{2}}\ \widehat{P}_\curlywedge E\widehat{P}_\curlywedge - H,\ S$ : *Example* $\Rightarrow (l=4,\ m=-3/2,\ u=0)$ (if $\tilde{L}<0,\ \widetilde{N}<0$);
- $n,\ ©,©;\ \widehat{\binom{1}{2}}\ \widehat{P}_\curlywedge H\widehat{P}_\curlywedge - E,\ S$ : *Example* $\Rightarrow (l=4,\ m=-2/3,\ u=0)$ (if $\tilde{L}<0,\ \widetilde{N}>0$);
- $n,\ ©,©;\ \widehat{\binom{1}{2}}\ H-E,\ S$ : *Example* $\Rightarrow (l=4,\ m=-1,\ u=0)$ (if $\tilde{L}<0,\ \widetilde{N}=0$);
- $n,\ ©,©;\ \widehat{\binom{1}{2}}\ H_\curlywedge HH_\curlywedge - H,\ N^\infty$ : *Example* $\Rightarrow (l=4,\ m=-1/3,\ u=0)$ (if $\tilde{L}>0$);
- $n,\ ©,©;\ \widehat{\binom{1}{3}}\ H_\curlywedge H\widehat{P}_\curlywedge - \widehat{P}$ : *Example* $\Rightarrow (l=4,\ m=-1/2,\ u=0)$ (if $\tilde{L}=0$).

**3)** *Assume finally* $W_7=0$. Considering (12) we obtain $\tau_1\tau_2\tau_3=0$.

***a*)** *The case* $W_6\neq 0$. Then $\tau_2\tau_3\neq 0$ and we get $\tau_1=0$, i.e. the real singularity is a node with coinciding eigenvalues. Considering the corresponding linearization matrix $\mathcal{M}_1$ given by (11) we conclude that this node could not be a star node. Thus considering Remarks 3, 5 and 4 we arrive at the next five global configurations of singularities:

- $n^d,\ ©,©;\ \widehat{\binom{1}{2}}\ \widehat{P}_\curlywedge E\widehat{P}_\curlywedge - H,\ S$ : *Example* $\Rightarrow (l=4,\ m=-2,\ u=0)$ (if $\tilde{L}<0,\ \widetilde{N}<0$);
- $n^d,\ ©,©;\ \widehat{\binom{1}{2}}\ \widehat{P}_\curlywedge H\widehat{P}_\curlywedge - E,\ S$ : *Example* $\Rightarrow (l=5/2,\ m=-25/32,\ u=0)$ (if $\tilde{L}<0,\ \widetilde{N}>0$);
- $n^d,\ ©,©;\ \widehat{\binom{1}{2}}\ H-E,\ S$ : *Example* $\Rightarrow (l=2\sqrt{2},\ m=-1,\ u=0)$ (if $\tilde{L}<0,\ \widetilde{N}=0$);
- $n^d,\ ©,©;\ \widehat{\binom{1}{2}}\ H_\curlywedge HH_\curlywedge - H,\ N^\infty$ : *Example* $\Rightarrow (l=1,\ m=-1/8,\ u=0)$ (if $\tilde{L}>0$);
- $n^d,\ ©,©;\ \widehat{\binom{1}{3}}\ H_\curlywedge H\widehat{P}_\curlywedge - \widehat{P}$ : *Example* $\Rightarrow (l=2,\ m=-1/2,\ u=0)$ (if $\tilde{L}=0$).

***b*)** *The case* $W_6=0$. By (12) the conditions $W_7=W_6=0$ give $\tau_2=\tau_3=0$ and this implies for systems (10)

$$\begin{aligned}\Re(\tau_2)&=(l-2mu+lu^2)^2-4(1+m)^2=0,\\ \Im(\tau_2)&=4\big[l(m-1)(1+u^2)-2m(1+m)u\big]=0.\end{aligned}\tag{15}$$

As $\widetilde{K}>0$ (i.e. $m<0$) we have $m-1\neq 0$ and hence the equations above yield

$$l=\varepsilon\frac{4m^2(1+m)^2}{(m^2+1)^2},\quad u=\varepsilon\frac{m^2-1}{2m},\quad \varepsilon=\pm 1.$$

Then we calculate:

$$\begin{gathered}W_7=W_6=0,\quad W_5=\frac{9(1+m^2)^{10}(1+4m+m^2)}{16m^4}=9(m^2+1)^4\Delta_1^2\tau_1/4,\\ \mathcal{B}_1=-4\varepsilon(m+1)(1+m^2)^4/m^2=8(1+m^2)\Delta_1\rho_1,\\ \widetilde{N}=-4m(1+m)x^2,\quad \widetilde{K}=-4mx^2,\quad \tilde{L}=8(1+2m)x^2.\end{gathered}$$

***b*$_1$)** *The subcase* $W_5<0$. Then $\tau_1<0$ and the real singular point is a focus. Clearly the focus is strong if $\rho_1\neq 0$ (i.e. $\mathcal{B}_1\neq 0$) and it is weak if $\rho_1=0$. In the second case we have $m=-1$ and then calculations yield

$$\mathcal{B}_1=\mathcal{H}=\mathcal{F}_1=0,\quad \sigma=4x,\quad \mathcal{B}_2=-128<0.$$

According to [31] (see Main Theorem, the statement $(e_4),[\beta]$) we have a center. On the other hand, due to $\widetilde{K}\neq 0$, the condition $\mathcal{B}_1=0$ is equivalent to $\widetilde{N}=0$. So considering Remarks 3, 5 and 4 we arrive at the next five global configurations of singularities

- $f,\ ©^\tau,©^\tau;\ \widehat{\binom{1}{2}}\ \widehat{P}_\curlywedge E\widehat{P}_\curlywedge - H,\ S$ : *Example* $\Rightarrow (l=16/25,\ m=-2,\ u=-3/4)$ (if $\tilde{L}<0,\ \widetilde{N}<0$);

- $f,\ ©^{\tau}, ©^{\tau};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\lambda} H \widehat{P}_{\lambda} - E,\ S:$ *Example* $\Rightarrow (l = 16/169,\ m = -2/3,\ u = 5/12)$ (if $\tilde{L} < 0,\ \widetilde{N} > 0$);
- $c,\ ©^{\tau}, ©^{\tau};\ \widehat{\binom{1}{2}}\ H - E,\ S:$ *Example* $\Rightarrow (l = 0,\ m = -1,\ u = 0)$ (if $\tilde{L} < 0,\ \widetilde{N} = 0$);
- $f,\ ©, ©;\ \widehat{\binom{1}{2}}\ H_{\lambda} H H_{\lambda} - H,\ N^{\infty}:$ *Example* $\Rightarrow (l = 4/25,\ m = -1/3,\ u = 4/3)$ (if $\tilde{L} > 0$);
- $f,\ ©^{\tau}, ©^{\tau};\ \widehat{\binom{1}{3}}\ H_{\lambda} H \widehat{P}_{\lambda} - \widehat{P}:$ *Example* $\Rightarrow (l = 4/25,\ m = -1/2,\ u = 3/4)$ (if $\tilde{L} = 0$).

$\boldsymbol{b}_2$) *The subcase* $W_5 > 0$. Then $\tau_1 > 0$ and the real singular point is a generic node. On the other hand the conditions $\widetilde{K} > 0$ and $\tau_1 = 1 + 4m + m^2 > 0$ imply either $m < -2 - \sqrt{3}$ or $\sqrt{3} - 2 < m < 0$. So the condition $\tilde{L} < 0$ gives $m < -1/2$ and then we have $m < -2 - \sqrt{3}$ and this implies $\widetilde{N} < 0$. We also observe that in this case $\tilde{L} \neq 0$. Thus by Remarks 3 we get the next two global configurations of singularities

- $n,\ ©^{\tau}, ©^{\tau};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\lambda} E \widehat{P}_{\lambda} - H,\ S:$ *Example* $\Rightarrow (l = 576/289,\ m = -4,\ u = -15/8)$ (if $\tilde{L} < 0$);
- $n,\ ©^{\tau}, ©^{\tau};\ \widehat{\binom{1}{2}}\ H_{\lambda} H H_{\lambda} - H,\ N^{\infty}:$ *Example* $\Rightarrow (l = 36/289,\ m = -1/4,\ u = 15/8)$ (if $\tilde{L} > 0$).

$\boldsymbol{b}_3$) *The subcase* $W_5 = 0$. Then $\tau_1 = 0$ and considering the corresponding matrix we get a node with one direction. We obtain $m_{1,2} = -2 \pm \sqrt{3}$ ($m_2 < m_1 < 0$) and we get $L < 0$ for $m = m_2$ and $L > 0$ for $m = m_1$. So denoting

$$l_i = 4m_i^2(1 + m_i)^2/(1 + m_i^2)^2, \quad u_i = (m_i^2 - 1)/(2m_i), \quad i = 1, 2$$

we arrive at the two configurations

- $n^d,\ ©^{\tau}, ©^{\tau};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\lambda} E \widehat{P}_{\lambda} - H,\ S:$ *Example* $\Rightarrow (l = l_2,\ m = m_2,\ u = u_2)$ (if $\tilde{L} < 0$);
- $n^d,\ ©^{\tau}, ©^{\tau};\ \widehat{\binom{1}{2}}\ H_{\lambda} H H_{\lambda} - H,\ N^{\infty}:$ *Example* $\Rightarrow (l = l_1,\ m = m_1,\ u = u_1)$ (if $\tilde{L} > 0$).

Thus the case $\kappa = 0$ is completely examined.

In what follows we consider the family of systems (4) with the special property: the discriminants corresponding to the complex points vanish. However for these systems the discriminants are polynomials quite big which imply complicated computations. In order to avoid such computations we will use a new normal form of this family of systems.

Assume that a non–degenerate system (2) possesses singular points $M_{1,2}(\alpha \pm i\beta, \gamma \pm i\delta)$ ($\beta^2 + \delta^2 \neq 0$). Via the substitution $x \leftrightarrow y$ we may assume $\beta \neq 0$ and then we can consider $\beta = 1$ due to the change $x \to x/\beta$. Therefore the affine transformation $\tilde{x} = -\delta x + y + \alpha\delta - \gamma$, $\tilde{y} = x - \alpha$, moves the points $M_{1,2}(\alpha \pm i, \gamma \pm i\delta)$ to the points $\widetilde{M}_{1,2}(0, \pm i)$. Then from the identities $P(0, i) = Q(0, i) = 0$ we obtain: $d = f = 0$, $k = a$ and $n = b$ and introducing some new parameters we obtain the systems

$$\dot{x} = a + cx + gx^2 + 2hxy + ay^2, \quad \dot{y} = b + ex + lx^2 + 2mxy + by^2 \tag{16}$$

with $a^2 + b^2 \neq 0$, having the singular points $M_{1,2}(0, \pm i)$. For these systems the condition $\mu_0 = 0$ and $\mu_1 \neq 0$ must be fulfilled (i.e. only one finite real singular point must be at infinity).

On the other hand the invariant polynomial $\mu_0$ is the discriminant of the form $\tilde{K}(\tilde{a}, x, y)$. So in this case the homogeneous quadratic polynomial $\tilde{K}$ has the form: $\tilde{K} = (ux + vy)^2$. Moreover the linear form $ux + vy$ is a common factor of the quadratic parts of systems (2) and, hence, the fourth finite singularity has coalesced with the infinite singular point $N(-v, u, 0)$. We observe that the condition $v \neq 0$ has to be satisfied, otherwise the common factor of the quadratic parts of systems (16) will be $x$ and this implies $a = b = 0$ and we get degenerate systems. So $v \neq 0$ and via the transformation $x_1 = x$ and $y_1 = ux/v + y$, which preserves the singular points $M_{1,2}(0, \pm i)$, we get $\tilde{K}(\tilde{a}, x, y) = \delta y^2$. This means that the common factor of the homogeneous quadratic parts of systems (16) will be $y$ and therefore the transformation applied, implies the conditions $g = l = 0$.

Thus we arrive at the family of systems

$$\dot{x} = a + cx + 2hxy + ay^2, \quad \dot{y} = b + ex + 2mxy + by^2, \tag{17}$$

possessing three distinct finite singularities: $M_{1,2}(0, \pm i)$ and one real singularity $M_3$.

**Lemma 2.** *Assume that for a system* (17) *the conditions* $\kappa \neq 0$ *and* $W_4 = W_3 = 0$ *hold, i.e.* $\tau_1 = \tau_2 = 0$. *Then this system could possess only one of the following 15 geometrical distinct global configurations of singularities:*

- $f,\ \copyright^{\tau}, \copyright^{\tau}; \overline{\binom{1}{1}}SN, \copyright, \copyright$ *(if* $\kappa < 0,\ W_2 < 0,\ \mathcal{T}_4 \neq 0,\ \eta < 0$*);* $(Cf_1)$
- $f,\ \copyright^{\tau}, \copyright^{\tau}; \overline{\binom{1}{1}}SN, S, N^{\infty}$ *(if* $\kappa < 0,\ W_2 < 0,\ \mathcal{T}_4 \neq 0,\ \eta > 0$*);* $(Cf_2)$
- $f,\ \copyright^{\tau}, \copyright^{\tau}; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$ *(if* $\kappa < 0,\ W_2 < 0,\ \mathcal{T}_4 \neq 0,\ \eta = 0$*);* $(Cf_3)$
- $f^{(1)},\ \copyright^{\tau}, \copyright^{\tau}; \overline{\binom{1}{1}}SN, \copyright, \copyright$ *(if* $\kappa < 0,\ W_2 < 0,\ \mathcal{T}_4 = 0,\ \eta < 0$ *);* $(Cf_4)$
- $f^{(1)},\ \copyright^{\tau}, \copyright^{\tau}; \overline{\binom{1}{1}}SN, S, N^{\infty}$ *(if* $\kappa < 0,\ W_2 < 0,\ \mathcal{T}_4 = 0,\ \eta > 0$ *);* $(Cf_5)$
- $f^{(1)},\ \copyright^{\tau}, \copyright^{\tau}; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN,$ *(if* $\kappa < 0,\ W_2 < 0,\ \mathcal{T}_4 = 0,\ \eta = 0$ *);* $(Cf_6)$
- $n,\ \copyright^{\tau}, \copyright^{\tau}; \overline{\binom{1}{1}}SN, \copyright, \copyright$ *(if* $\kappa < 0,\ W_2 > 0,\ \eta < 0$*);* $(Cf_7)$
- $n,\ \copyright^{\tau}, \copyright^{\tau}; \overline{\binom{1}{1}}SN, S, N^{\infty}$ *(if* $\kappa < 0,\ W_2 > 0,\ \eta > 0$*);* $(Cf_8)$
- $n,\ \copyright^{\tau}, \copyright^{\tau}; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN,$ *(if* $\kappa < 0,\ W_2 > 0,\ \eta = 0$*);* $(Cf_9)$
- $n^d,\ \copyright^{\tau}, \copyright^{\tau}; \overline{\binom{1}{1}}SN, \copyright, \copyright$ *(if* $\kappa < 0,\ W_2 = 0,\ \eta < 0$ *);* $(Cf_{10})$
- $n^d,\ \copyright^{\tau}, \copyright^{\tau}; \overline{\binom{1}{1}}SN, S, N^{\infty}$ *(if* $\kappa < 0,\ W_2 = 0,\ \eta > 0$ *);* $(Cf_{11})$
- $n^d,\ \copyright^{\tau}, \copyright^{\tau}; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN,$ *(if* $\kappa < 0,\ W_2 = 0,\ \eta = 0$ *);* $(Cf_{12})$
- $s,\ \copyright^{\tau}, \copyright^{\tau}; \overline{\binom{1}{1}}SN, N^f, N^f$ *(if* $\kappa > 0,\ \mathcal{T}_4 \neq 0$ *);* $(Cf_{13})$
- $s^{(1)},\ \copyright^{\tau}, \copyright^{\tau}; \overline{\binom{1}{1}}SN, N^f, N^f$ *(if* $\kappa > 0,\ \mathcal{T}_4 = 0,\ \mathcal{F}_1 \neq 0$ *);* $(Cf_{14})$
- $s^{(2)},\ \copyright^{\tau}, \copyright^{\tau}; \overline{\binom{1}{1}}SN, N^f, N^f$ *(if* $\kappa > 0,\ \mathcal{T}_4 = 0,\ \mathcal{F}_1 = 0$ *);* $(Cf_{15})$

*Proof:* For systems (17) we calculate:

$$\tau_{1,2} = c^2 - 16am - 4(b-h)^2 \pm 4i(2ae - bc + ch) = 0.$$

Hence we get the conditions $c^2 - 16am - 4(b-h)^2 = 0 = 2ae - bc + ch$ and we shall consider two possibilities: $a \neq 0$ and $a = 0$.

**A. The possibility** $a \neq 0$ Then we may assume $a = 1$ due to a time rescaling and we calculate $e = c(b-h)/2$, $m = \left[c^2 - 4(b-h)^2\right]/16$. So we get the family of systems

$$\begin{aligned} \dot{x} &= 1 + cx + 2hxy + y^2, \\ \dot{y} &= b + c(b-h)x/2 + \left[c^2 - 4(b-h)^2\right]xy/8 + by^2, \end{aligned} \tag{18}$$

possessing the complex singular points $M_{1,2}(0, \pm i)$ and one real elemental singularity $M_3$. For these systems we calculate:

$$\begin{aligned} \mu_0 =& 0,\ \ \mu_1 = -\frac{c\,\omega_1\omega_2}{64}\,y, \quad \tilde{K} = \frac{\omega_1}{4}\,y^2, \quad \kappa = -\frac{\omega_1\omega_3^2}{32}, \\ G_9 =& \frac{\omega_1\omega_3^2}{4096}, \quad W_4 = W_3 = 0, \quad W_2 = \frac{c^2\omega_1^2\omega_2^2\omega_3^2}{2^{18}}\,\tau_3, \\ \mathcal{T}_4 =& \frac{c\omega_1\omega_2\omega_3}{512}\,\rho_1\rho_2\rho_3, \quad \mathcal{T}_3 = \frac{c\omega_1\omega_2\omega_3}{512}\,(\rho_1\rho_2 + \rho_1\rho_3 + \rho_2\rho_3), \end{aligned}$$

where

$$\omega_1 = 4(b+h)^2 - c^2, \quad \omega_2 = 4(b^2 - h^2) - c^2, \quad \omega_3 = 4(b-h)^2 - c^2.$$

3.1.2. *The case* $\kappa < 0$. Then $\tilde{K} > 0$ and according to [6] (see the Table 1, lines 124-135) the elemental real singular point is an anti-saddle and its type is governed by the invariant polynomial $W_2$, as in the case under consideration we have $G_9 \neq 0$ and $W_4 = W_3 = 0$.

3.1.2.1. The subcase $W_2 < 0$. In this case we obtain $\tau_3 < 0$, i.e. the real singularity of systems (18) is a focus, which could be either strong or weak.

3.1.2.1.1. The possibility $\mathcal{T}_4 \neq 0$. Then the focus is strong and we arrive at the following three configurations:

$(Cf_1)$ if $\eta < 0$: $(b = 1,\ c = 1/2,\ h = 0)$;

$(Cf_2)$ if $\eta > 0$: $(b = 1,\ c = 1/5,\ h = 4/5)$;

$(Cf_3)$ if $\eta = 0$: $(b = 1,\ c = 1/5,\ h = 7/10)$.

3.1.2.1.2. *The possibility* $\mathcal{T}_4 = 0$. As $\tau_1 = \tau_2 = 0$ it is clear that $\rho_1\rho_2 \neq 0$, otherwise we get degenerate systems. Therefore $\rho_3 = 0$ (i.e. focus is weak) and in this case $\mathcal{T}_3 \neq 0$ due to $\rho_2\rho_3 \neq 0$.

On the other hand according to [1] we have the next result.

**Remark 6.** *In the class of quadratic systems with a weak focus of order two the condition* $\mu_0 = 0 = W_4$ *implies the existence of exactly three distinct real singularities.*

So by this remark, systems (18) could not have a weak focus of the second order. Therefore in the case considered, we arrive at the following three configurations:

$(Cf_4)$ if $\eta < 0$: $(b = 1,\ c = 2/\sqrt{7},\ h = 0)$;

$(Cf_5)$ if $\eta > 0$: $(b = 1,\ c = \frac{2}{5}\sqrt{(76 - \sqrt{5209})/7},\ h = 4/5)$;

$(Cf_6)$ if $\eta = 0$: $(b = 1,\ c = \sqrt{(\sqrt{41} - 5)/8},\ h = (\sqrt{41} - 1)/8)$.

3.1.2.2. *The subcase* $W_2 > 0$. Then $\tau_3 > 0$, i.e. the real singular point is a generic node. In this case we get the next three configurations:

$(Cf_7)$ if $\eta < 0$: $(b = 1,\ c = 1/5,\ h = 0)$;

$(Cf_8)$ if $\eta > 0$: $(b = 1,\ c = 1/5,\ h = -4/5)$;

$(Cf_9)$ if $\eta = 0$: $(b = 1,\ c = 1/5,\ h = -7/10)$;

3.1.2.3. *The subcase* $W_2 = 0$. Then $\tau_3 = 0$ and the real singularity is a node with coinciding eigenvalues. We claim that we could not have a star node in this case.

Indeed, assume that the singular point $M_3$ is a star node. According to [3, Lemma 5] the conditions either *(i)* or *(ii)* of this lemma must hold. Suppose first that the conditions *(i)* are satisfied, i.e. $U_1 \neq 0,\ U_2 \neq 0,\ U_3 = Y_1 = 0$. Then we must have

$$Y_1 = c[c^2(b - 3h) - 4(b - h)^2(b + h)]/32 = 0.$$

We observe that the condition $c = 0$ implies $U_2 = 0$. So, $c \neq 0$ and then clearly the condition $(b-3h)(b+h) \geq 0$ must hold. Obviously we could assume that both factors are non-negative and we set new parameters $u$ and $v$ as follows: $(b - 3h) = u^2$ and $(b + h) = v^2$. So we get $b = (u^2 + 3v^2)/4$ and $h = (v^2 - u^2)/4$ and we calculate

$$Y_1 = c(cu - u^2v - v^3)(cu + u^2v + v^3)/32 = 0.$$

As the third factor can be obtained by the second one replacing $v$ by $-v$ we may assume $cu - u^2v - v^3 = 0$. If $u = 0$ then $v = 0$ and we obtain

$$U_3 = -27c^4y(c^2x^2 + 16y^2)^2/2^{12}.$$

Since $c \neq 0$ we get $U_3 \neq 0$, i.e. the conditions *(i)* of the lemma are not satisfied.

Assume $u \neq 0$. Then $c = v(u^2 + v^2)/u$ and we calculate

$$\begin{aligned} U_2 =&(u - v)^2(u + v)^2(u^2 + v^2)v^3\left[(u^2 + v^2)^3x^2 - 8u^2(u^2 + v^2)xy + 16u^2y^2\right]/(16u^5),\\ U_3 =&3(u - v)^2(u + v)^2(u^2 + v^2)^2\left[(u^2 + v^2)^3x^2 - 8u^2(u^2 + v^2)xy + 16u^2y^2\right]\times\\ &\left[(u^2 + v^2)^2(u^2 + 3v^2)x - 4(u^2 - 2uv + 3v^2)(u^2 + 2uv + 3v^2)y\right]/(2^{14}u^8) \end{aligned}$$

Since $u \neq 0$ the condition $U_3 = 0$ implies $(u - v)(u + v) = 0$ and this yields $U_2 = 0$.

Thus the conditions *(i)* of Lemma 5 from [3] cannot be satisfied.

Assume now that the conditions *(ii)* are satisfied, i.e. $U_1 = U_4 = U_5 = U_6 = 0$, $Y_2 \neq 0$. We have Coefficient$[U_6, y^2] = -4h^2$ and hence the condition $U_6 = 0$ implies $h = 0$. Then we calculate Coefficient$[U_5, y^2] = -12b = 0$, i.e. $b = 0$. Therefore we obtain

$$U_6 = -c^4x^2/64, \quad Y_2 = 3c^2$$

and hence the condition $U_6 = 0$ implies $Y_2 = 0$. So the conditions *(ii)* of of Lemma 5 from [3] also cannot be satisfied and this completes the proof of our claim.

Thus we get a node $n^d$ and we obtain the three configurations:

$(Cf_{10})$ if $\eta < 0$: $(b = 1,\ c = 2(4 - \sqrt{7})/9,\ h = 0)$;

$(Cf_{11})$ if $\eta > 0$: $(b = 1,\ c = 2(4 + \sqrt{7})/9,\ h = 0)$;

$(Cf_{12})$ if $\eta = 0$: $(b = 1,\ \ c = -(\alpha^{1/3} - 1)^2/(2\alpha^{1/3}),\ \ h = (\alpha^{4/3} + 1)/(4\alpha^{2/3});\ \ \alpha = 2 - \sqrt{3})$.

3.1.3. *The case* $\kappa > 0$. Then $\tilde{K} < 0$ and according to [6] (see the Table 1, line 123) the elemental real singular point is a saddle. In this case the total index of the infinite singularities must be $+2$ and hence, at infinity besides a saddle-node we have two nodes (i.e. $\eta > 0$).

3.1.3.1. *The subcase* $\mathcal{T}_4 \neq 0$. Then the saddle is strong and we get the configuration

$(Cf_{13})$: $(b = 1,\ c = 2,\ h = -1)$.

3.1.3.2. *The subcase* $\mathcal{T}_4 = 0$. Then $\rho_3 = 0$ and the saddle is weak and its order is governed by the invariant polynomial $\mathcal{F}_1$ (see [31, Main Theorem]).

3.1.3.2.1. *The possibility* $\mathcal{F}_1 \neq 0$. In this case the we have a weak saddle of the first order:

$(Cf_{14})$: $(b = 1,\ c = \sqrt{(256 + 4\sqrt{1009})/175},\ h = -2/5)$.

3.1.3.2.2. *The possibility* $\mathcal{F}_1 = 0$. In this case we have a weak saddle of order two and it could not be of order three. Indeed as it follows from [20] the condition $\mu_0 = 0$ implies $\mathbf{D} < 0$, i.e. we could not have one real and two complex singular points. So we arrive at the configuration

$(Cf_{15})$: $\Big(h = 1, (b, c) = (b_0, c_0) \approx (-1.98345, 2.06329) \in \big\{\mathcal{T}_4^{-1}(0), \mathcal{F}_1^{-1}(0)\big\}\Big)$.

**B. The possibility** $a = 0$ In this case we for systems (17) we get $\tau_{1,2} = -(2b - 2h + ic)^2 = 0$ that implies $c = 0$ and $h = b$. Hence we get the family of systems

$$\dot{x} = 2bxy, \quad \dot{y} = b + ex + 2mxy + by^2, \tag{19}$$

for which we have $\mu_1 = -4eb^3y \neq 0$ and therefore we can consider $e = 1 = b$ due to the rescaling $(x, y, t) \mapsto (bx/e, y, t/b)$ and then we calculate

$$\kappa = -128m^2, \quad \mathcal{T}_4 = 256m^2, \quad W_2 = 256m^2(m^2 - 2), \quad \eta = 4m^2.$$

Since $\kappa \neq 0$ we get $\kappa < 0$, $\eta > 0$, $\mathcal{T}_4 \neq 0$ and $\operatorname{sign}(W_2) = \operatorname{sign}(m^2 - 2)$. So we could only have the configurations $(Cf_2)$, $(Cf_8)$ and $(Cf_{11})$, which are already detected in the case $a \neq 0$.

As all the cases are considered, the lemma is proved. ∎

3.2. **Systems with one double and one simple real finite singularities.** Assume that quadratic systems (2) possess one double and one simple finite singularities. Then according to [31] via an affine transformation and time rescaling these systems could be brought to the form:

$$\dot{x} = cx + cy - cx^2 + 2hxy, \quad \dot{y} = ex + ey - ex^2 + 2mxy \tag{20}$$

possessing the double singular point $M_{1,2}(0, 0)$ and the elemental singularity $M_3(1, 0)$. For these systems calculations yield

$$\mu_0 = 0, \quad \mu_1 = -4(cm - eh)^2x, \quad \kappa = 128h^2(cm - eh).$$

As for systems above we have $\mu_1 \neq 0$ (i.e. $gm - hl \neq 0$) we observe that the condition $\kappa = 0$ is equivalent to $h = 0$.

3.2.1. **The case $\kappa \neq 0$.** Then $h \neq 0$ and we may assume $h = 1$ due to a time rescaling. So we get the systems

$$\dot{x} = cx + cy - cx^2 + 2xy, \quad \dot{y} = ex + ey - ex^2 + 2mxy, \tag{21}$$

possessing the singular points $M_{1,2}(0,0)$ and $M_3(1,0)$. For these singularities we have the following values of the traces $\rho_i$, of the determinants $\Delta_i$ and of the discriminants $\tau_i$:

$$\rho_1 = \rho_2 = c + e, \quad \Delta_1 = \Delta_2 = 0, \quad \tau_1 = \tau_2 = (c+e)^2;$$
$$\rho_3 = e - c + 2m, \quad \Delta_3 = 2(e - cm), \quad \tau_3 = (e - c + 2m)^2 + 8(cm - e).$$

Then for the above systems we have

$$\begin{gathered}\kappa = -64\Delta_3, \quad \widetilde{K} = 2\Delta_3 x^2, \quad E_1 = -\rho_1\Delta_3^4/2, \quad G_9 = \Delta_3/2,\\ \mathcal{T}_4 = -2\rho_1^2\rho_3\Delta_3^2, \quad \mathcal{T}_3 = 2\rho_1(\rho_1 + 2\rho_3)\Delta_3^2,\\ \eta = 4\big[(c-2m)^2 - 4\Delta_3\big], \quad W_4 = 4\rho_1^4\Delta_3^4\tau_3,\\ \widetilde{M} = -8\big[(c+2m)^2 - 6e\big]x^2 + 16(c+2m)xy - 32y^2.\end{gathered} \tag{22}$$

**Remark 7.** *We observe that $\widetilde{M} \neq 0$ and the condition $\kappa \neq 0$ implies $\widetilde{K}G_9 \neq 0$. Moreover we have* $\operatorname{sign}(\widetilde{K}) = -\operatorname{sign}(\kappa)$.

3.2.1.1. *The subcase $\kappa < 0$.* This implies $\widetilde{K} > 0$ and according to [6] (see Table 1, lines 138 - 144) the elemental singular point is an anti-saddle and its type is governed by the invariant polynomials $W_4$ and $W_2$ (as in the considered case we have $G_9 \neq 0$).

3.2.1.1.1. *The possibility $W_4 < 0$.* Considering (22) we have $\tau_3 < 0$, i.e. the elemental singular point is a focus. On the other hand as the condition $W_4 \neq 0$ implies $E_1 \neq 0$ and $G_9 \neq 0$, according to [6] (see Table 1, line 140) the double point is a semi-elemental saddle-node.

**1)** *Assume first $\mathcal{T}_4 \neq 0$.* Then the focus is strong and considering Lemma 1 and $\widetilde{M} \neq 0$ from (22) we arrive at the following three configurations of singularities:

- $f,\ \overline{sn}_{(2)};\ \overline{\binom{1}{1}}SN, ©, ©$ : *Example* $\Rightarrow$ $(c = 2,\ e = 1,\ m = 0)$ (if $\eta < 0$);
- $f,\ \overline{sn}_{(2)};\ \overline{\binom{1}{1}}SN, S, N^{\infty}$ : *Example* $\Rightarrow$ $(c = 2,\ e = 5/12,\ m = 0)$ (if $\eta > 0$);
- $f,\ \overline{sn}_{(2)};\ \overline{\binom{0}{2}}SN,\ \overline{\binom{1}{1}}SN$ : *Example* $\Rightarrow$ $(c = 2,\ e = 1/2,\ m = 0)$ (if $\eta = 0$).

**2)** *Suppose now $\mathcal{T}_4 = 0$.* Then considering (22) we get $\rho_3 = 0$, i.e. the focus is weak. In this case we have $e = c - 2m$ and we calculate

$$\begin{gathered}\mathcal{T}_4 = 0, \quad \mathcal{T}_3 = 32(c-m)^2(c-2m-cm)^2, \quad \mathcal{F} = -2(c-m)^2(c-2m-cm),\\ \mathcal{F}_1 = 4(c + 2c^2 - 2m - 2cm)(c - 2m - cm), \quad W_4 = -8192(c-m)^4(c-2m-cm)^5.\end{gathered} \tag{23}$$

So the condition $W_4 < 0$ gives $\mathcal{T}_3\mathcal{F} < 0$ and by [31] the order of weak focus is governed by invariant polynomial $\mathcal{F}_1$.

***a)*** *The case $\mathcal{F}_1 \neq 0$.* Then the simple singular point of systems (21) is a first order weak focus and considering Lemma 1 we arrive at the following three global configurations of singularities:

- $f^{(1)}, \overline{sn}_{(2)};\ \overline{\binom{1}{1}}SN, ©, ©$ : *Example* $\Rightarrow$ $(c = 0,\ e = 2,\ m = -1)$ (if $\eta < 0$);
- $f^{(1)}, \overline{sn}_{(2)};\ \overline{\binom{1}{1}}SN, S, N^{\infty}$ : *Example* $\Rightarrow$ $(c = 0,\ e = 10,\ m = -5)$ (if $\eta > 0$);
- $f^{(1)}, \overline{sn}_{(2)};\ \overline{\binom{0}{2}}SN,\ \overline{\binom{1}{1}}SN$ : *Example* $\Rightarrow$ $(c = 0,\ e = 8,\ m = -4)$ (if $\eta = 0$).

***b)*** *The case $\mathcal{F}_1 = 0$.* By (23) the condition $\mathcal{F}_1 = 0$ gives $c + 2c^2 - 2m - 2cm = 0$ and $c + 1 \neq 0$, otherwise we get $\mathcal{F}_1 = -4(1+m) \neq 0$ due to the condition $\kappa = 128(1+m) \neq 0$. So $c + 1 \neq 0$ and we obtain

$m = c(1+2c)/(2(1+c))$. Then we calculate

$$\mathcal{F}_2 = -\frac{c^8(3+2c)^2(6+5c)}{(1+c)^5}, \quad \eta = 4\frac{c^2(12+20c+9c^2)}{(1+c)^2},$$
$$W_4 = 16\frac{c^{14}(3+2c)^5}{(1+c)^9}, \quad \kappa = 64\frac{c^2(3+2c)}{1+c}$$

and we observe that due to $\kappa \neq 0$ the condition $\eta > 0$ holds. Moreover, we get $\mathcal{F}_2 = 0$ for $c = -6/5$ and in this case the weak focus is of order 3 as $\mathcal{F}_3\mathcal{F}_4 = 2^{23}3^{21}5^{-13} \neq 0$ (see [31], the Main Theorem). Thus considering Lemma 1 we get the following two global configurations of singularities:

- $f^{(2)}, \overline{sn}_{(2)}; \overline{\binom{1}{1}}SN, S, N^\infty$ : *Example* $\Rightarrow (c = -4/3,\ e = 16/3,\ m = -10/3)$ (if $\mathcal{F}_2 \neq 0$);
- $f^{(3)}, \overline{sn}_{(2)}; \overline{\binom{1}{1}}SN, S, N^\infty$ : *Example* $\Rightarrow (c = -6/5,\ e = 36/5,\ m = -21/5)$ (if $\mathcal{F}_2 = 0$).

3.2.1.1.2. *The possibility* $W_4 > 0$. Then the anti-saddle is a generic node and we get the next three global configurations of singularities:

- $n,\ \overline{sn}_{(2)}; \overline{\binom{1}{1}}SN, ©, ©$ : *Example* $\Rightarrow (c = 0,\ e = 1,\ m = 1)$ (if $\eta < 0$);
- $n,\ \overline{sn}_{(2)}; \overline{\binom{1}{1}}SN, S, N^\infty$ : *Example* $\Rightarrow (c = 0,\ e = 1/3,\ m = 1)$ (if $\eta > 0$);
- $n,\ \overline{sn}_{(2)}; \overline{\binom{0}{2}}SN,\ \overline{\binom{1}{1}}SN$ : *Example* $\Rightarrow (c = 0,\ e = 1/2,\ m = 1)$ (if $\eta = 0$).

3.2.1.1.3. *The possibility* $W_4 = 0$. Then as $\Delta_3 \neq 0$ by (22) we have $\rho_1\tau_3 = 0$ and clearly $\rho_1 = 0$ if and only if $W_4 = \mathcal{T}_4 = 0$.

**1)** *Assume first* $\mathcal{T}_4 \neq 0$. In this case $\rho_1 \neq 0$ and then the condition $W_4 = 0$ gives $\tau_3 = 0$. So in this case we have a node with coinciding eigenvalues. We claim that this node will be a star node if and only if $U_3 = 0$.

Indeed the linearization matrix corresponding to $M_3(1,0)$ is $\mathcal{M}_3 = \begin{pmatrix} -c & 2+c \\ -e & e+2m \end{pmatrix}$ and clearly we have a star node if and only if $e = 0$, $c = -2$ and $m = 1$. In this case a straightforward computation gives us $U_3 = 0$.

Conversely, assume that $U_3 = 0$. As Coefficient$[U_3, y^5] = 12c^2(2+c)$ we have $c(2+c) = 0$. We claim that we could not have $c = 0$, otherwise we get Coefficient$[U_3, xy^4] = 24e^2 = 0$ and this contradicts $\kappa \neq 0$. So we have $c = -2$ and then we calculate Coefficient$[U_3, xy^4] = 24(e-2+2m)(e+2m) = 0$ and as $\kappa = -128(e+2m) \neq 0$ we get $e = 2(1-m)$. Finally we calculate Coefficient$[U_3, x^2y^3] = 288(m-1) = 0$ and we have $m = 1$ and this implies $U_3 = 0$. So we arrive at the conditions above (i.e. the elemental singularity is a star node) and this proves our claim.

On the other hand when we have a star node we obtain $\eta = 0$. So considering Lemma 1 we arrive at the following four global configurations of singularities:

- $n^d, \overline{sn}_{(2)}; \overline{\binom{1}{1}}SN, ©, ©$ : *Example* $\Rightarrow (c = 1,\ e = 3,\ m = 1)$ (if $U_3 \neq 0$, $\eta < 0$);
- $n^d, \overline{sn}_{(2)}; \overline{\binom{1}{1}}SN, S, N^\infty$ : *Example* $\Rightarrow (c = -3,\ e = -1,\ m = 1)$ (if $U_3 \neq 0$, $\eta > 0$);
- $n^d, \overline{sn}_{(2)}; \overline{\binom{0}{2}}SN,\ \overline{\binom{1}{1}}SN$ : *Example* $\Rightarrow (c = 6,\ e = 8,\ m = 1)$ (if $U_3 \neq 0$, $\eta = 0$).
- $n^*, \overline{sn}_{(2)}; \overline{\binom{0}{2}}SN,\ \overline{\binom{1}{1}}SN$ : *Example* $\Rightarrow (c = -2,\ e = 0,\ m = 1)$ (if $U_3 = 0$).

**2)** *Suppose now* $\mathcal{T}_4 = 0$. As it was mention above in this case we have $\rho_1 = 0$ and then $E_1 = 0$. As $\widetilde{K} > 0$ and $G_9 \neq 0$ according to [6] (see Table 1, lines 142 and 144) the double point is a nilpotent cusp, whereas the type of the anti-saddle in this case is governed by the invariant polynomial $W_2$. Setting $e = -c$ for the elemental singular point we have

$$\mathcal{M}_3 = \begin{pmatrix} -c & 2+c \\ c & 2m-c \end{pmatrix}, \quad \rho_3 = 2(m-c), \quad \Delta_3 = -2c(1+m), \quad \tau_3 = 4(c^2+2c+m^2) \tag{24}$$

and calculations yield

$$\begin{aligned} &\kappa = -64\Delta_3, \quad E_1 = \mathcal{T}_4 = \mathcal{T}_3 = 0, \quad G_9 = \Delta_3/2, \quad \mathcal{T}_2 = 2\rho_3\Delta_3^2, \\ &\mathcal{T}_1 = 2\Delta_3^2, \quad \mathcal{F} = (1+c-m)\rho_3\Delta_3/2, \quad \mathcal{F}_1 = -2c(1+c-m)\Delta_3, \\ &\eta = 4\big[(c+2m)^2 + 8c\big], \quad W_2 = 4\Delta_3^4\tau_3. \end{aligned} \tag{25}$$

***a)*** *The case* $W_2 < 0$. This gives $\tau_3 < 0$ and the elemental singularity is a focus.

$\boldsymbol{a}_1$) *The subcase* $\mathcal{T}_2 \neq 0$. Then $\rho_3 \neq 0$ and the focus is strong. So considering Lemma 1 we arrive at the next three global configurations of singularities:

- $f,\ \widehat{cp}_{(2)};\ \overline{\binom{1}{1}}SN, ©, ©$ : *Example* $\Rightarrow (c=-1/12,\ e=1/12,\ m=-9/25)$ (if $\eta<0$);
- $f,\ \widehat{cp}_{(2)};\ \overline{\binom{1}{1}}SN, S, N^\infty$ : *Example* $\Rightarrow (c=-1/13,\ e=1/13,\ m=-9/25)$ (if $\eta>0$);
- $f,\ \widehat{cp}_{(2)};\ \overline{\binom{0}{2}}SN,\ \overline{\binom{1}{1}}SN$ : *Example* $\Rightarrow (c=-2/25,\ e=2/25,\ m=-9/25)$ (if $\eta=0$).

$\boldsymbol{a}_2$) *The subcase* $\mathcal{T}_2 = 0$. We have a weak focus and the condition $\rho_3 = 0$ gives $m = c$. Then by (25) we obtain

$$\begin{aligned} &\mathcal{T}_2 = \mathcal{F} = 0, \quad \mathcal{T}_1 = 8c^2(1+c)^2, \quad \mathcal{F}_1 = 4c^2(1+c), \\ &\mathcal{B} = -2c^2, \quad \mathcal{H} = 4c(1+c), \quad \kappa = 128c(1+c). \end{aligned} \tag{26}$$

So the condition $\kappa \neq 0$ yields $\mathcal{T}_1\mathcal{F}_1 \neq 0$ (moreover $\kappa < 0$ implies $\mathcal{H} < 0$) and according to [31] (see the statement ($d_2$) of Main Theorem) the weak focus could only be of order one. So we get the next three global configurations of singularities:

- $f^{(1)},\ \widehat{cp}_{(2)};\ \overline{\binom{1}{1}}SN, ©, ©$ : *Example* $\Rightarrow (c=-1/2,\ e=1/2,\ m=-1/2)$ (if $\eta<0$);
- $f^{(1)},\ \widehat{cp}_{(2)};\ \overline{\binom{1}{1}}SN, S, N^\infty$ : *Example* $\Rightarrow (c=-17/18,\ e=17/18,\ m=-17/18)$ (if $\eta>0$);
- $f^{(1)},\ \widehat{cp}_{(2)};\ \overline{\binom{0}{2}}SN,\ \overline{\binom{1}{1}}SN$ : *Example* $\Rightarrow (c=-8/9,\ e=8/9,\ m=-8/9)$ (if $\eta=0$).

***b)*** *The case* $W_2 > 0$. Then we have $\tau_3 > 0$ and the elemental singularity is a generic node:

- $n,\ \widehat{cp}_{(2)};\ \overline{\binom{1}{1}}SN, ©, ©$ : *Example* $\Rightarrow (c=-2,\ e=2,\ m=2)$ (if $\eta<0$);
- $n,\ \widehat{cp}_{(2)};\ \overline{\binom{1}{1}}SN, S, N^\infty$ : *Example* $\Rightarrow (c=-1,\ e=1,\ m=2)$ (if $\eta>0$);
- $n,\ \widehat{cp}_{(2)};\ \overline{\binom{0}{2}}SN,\ \overline{\binom{1}{1}}SN$ : *Example* $\Rightarrow (c=-8/9,\ e=8/9,\ m=16/9)$ (if $\eta=0$).

***c)*** *The case* $W_2 = 0$. Considering (25) we have $\tau_3 = 0$ and systems (21) with $e = -c$ possess a node with coinciding eigenvalues. As the linearization matrix has the form given in (24) and $c \neq 0$ (due to $\kappa \neq 0$) this node could not be a star node. So considering Lemma 1 we arrive at the following three global configurations of singularities:

- $n^d,\ \widehat{cp}_{(2)};\ \overline{\binom{1}{1}}SN, ©, ©$ : *Example* $\Rightarrow (c=-9/5,\ e=9/5,\ m=-3/5)$ (if $\eta<0$);
- $n^d,\ \widehat{cp}_{(2)};\ \overline{\binom{1}{1}}SN, S, N^\infty$ : *Example* $\Rightarrow (c=-1/5,\ e=1/5,\ m=-3/5)$ (if $\eta>0$);
- $n^d,\ \widehat{cp}_{(2)};\ \overline{\binom{0}{2}}SN,\ \overline{\binom{1}{1}}SN$ : *Example* $\Rightarrow (c=-32/25,\ e=32/25,\ m=-24/25)$ (if $\eta=0$).

3.2.1.2. *The subcase* $\kappa > 0$. This implies $\widetilde{K} < 0$ and according to [6] (see Table 1, lines 136, 137) the elemental singular point is a saddle. At the same time the type of the double point depends on the value of the invariant polynomial $E_1$.

On the other hand considering (22) we observe that the condition $\kappa > 0$ implies $\eta > 0$ and according to Lemma 1 at infinity there could be the unique configuration of singularities: $\overline{\binom{1}{1}}SN, N^f, N^f$.

3.2.1.2.1. *The possibility* $\mathcal{T}_4 \neq 0$. By (22) this implies $\rho_1\rho_3 \neq 0$ and then $E_1 \neq 0$, i.e. the double point is semi-elemental saddle-node. On the other hand as $\rho_3 \neq 0$ the saddle is strong and we obtain the following global configuration of singularities:

- $s,\ \overline{sn}_{(2)};\ \overline{\binom{1}{1}}SN, N^f, N^f$ : *Example* $\Rightarrow (c=0,\ e=-1,\ m=0)$.

3.2.1.2.2. *The possibility* $\mathcal{T}_4 = 0$. Considering (22) we observe that $\rho_1 = 0$ if and only if $\mathcal{T}_4 = \mathcal{T}_3 = 0$.

**1)** *Assume first* $\mathcal{T}_3 \neq 0$. In this case we get $\rho_1 \neq 0$ (then $E_1 \neq 0$) and $\rho_3 = 0$. We get $e = c - 2m$ and by (23) and [31] we obtain a weak saddle the order of which is governed by the invariant polynomial $\mathcal{F}_1$.

***a)*** *The case* $\mathcal{F}_1 \neq 0$. Clearly we arrive at the configuration:

- $s^{(1)},\ \overline{sn}_{(2)};\ \overline{\binom{1}{1}}SN, N^f, N^f$ : *Example* $\Rightarrow$ $(c = 0,\ e = -2,\ m = 1)$.

***b)*** *The case* $\mathcal{F}_1 = 0$. It was shown above (see subsection 3.2.1.1.1) that in this case the condition $c + 1 \neq 0$ holds and we obtain $m = c(1 + 2c)/(2(1 + c))$. Then calculations yield

$$\mathcal{F}_2 = -\frac{c^8(3 + 2c)^2(6 + 5c)}{(1 + c)^5}, \qquad \kappa = 64\frac{c^2(3 + 2c)}{1 + c}$$

and due to $\kappa > 0$ the condition $\mathcal{F}_2 \neq 0$ holds. So we could only have a weak saddle of order two and this leads to the configuration:

- $s^{(2)},\ \overline{sn}_{(2)};\ \overline{\binom{1}{1}}SN, N^f, N^f$ : *Example* $\Rightarrow$ $(c = 1,\ e = -1/2,\ m = 3/4)$.

**2)** *Suppose now* $\mathcal{T}_3 = 0$. In this case we have $\rho_1 = 0$ and then $E_1 = 0$. As $\widetilde{K} < 0$ according to [6] (see Table 1, line 137) besides the saddle we have a nilpotent cusp.

***a)*** *The case* $\mathcal{T}_2 \neq 0$. Considering (25) we get $\rho_3 \neq 0$ and therefore we have a strong saddle:

- $s,\ \widehat{cp}_{(2)};\ \overline{\binom{1}{1}}SN, N^f, N^f$ : *Example* $\Rightarrow$ $(c = 1,\ e = -1,\ m = 0)$.

***b)*** *The case* $\mathcal{T}_2 = 0$. Then $m = c$ and taking into account (26) we conclude, that the weak saddle could be only of the first order:

- $s^{(1)},\ \widehat{cp}_{(2)};\ \overline{\binom{1}{1}}SN, N^f, N^f$ : *Example* $\Rightarrow$ $(c = 1,\ e = -1,\ m = 1)$.

3.2.2. **The case $\kappa = 0$.** In this case $h = 0$ and then $c \neq 0$, otherwise systems (20) become degenerate. So we may assume $c = 1$ due to a time rescaling and we obtain the 2-parameter family of systems

$$\dot{x} = x + y - x^2, \quad \dot{y} = ex + ey - ex^2 + 2mxy, \tag{27}$$

for the singular points $M_{1,2}(0, 0)$ and $M_3(1, 0)$ of which we have the following linearization matrices, traces, determinants and discriminants:

$$\begin{aligned} &\mathcal{M}_1 = \begin{pmatrix} 1 & 1 \\ e & e \end{pmatrix}, \quad \rho_1 = \rho_2 = 1 + e, \quad \Delta_1 = \Delta_2 = 0, \quad \tau_1 = \tau_2 = (1 + e)^2; \\ &\mathcal{M}_3 = \begin{pmatrix} -1 & 1 \\ -e & e + 2m \end{pmatrix}, \quad \rho_3 = e - 1 + 2m, \quad \Delta_3 = -2m, \quad \tau_3 = \rho_3^2 - 4\Delta_3. \end{aligned} \tag{28}$$

For the systems above calculations yield

$$\begin{gathered} \mu_0 = 0, \quad \mu_1 = -\Delta_3^2 x, \quad \widetilde{K} = 2\Delta_3 x^2, \quad E_1 = -\rho_1\Delta_3^4/2, \quad G_9 = 0, \\ \mathcal{T}_i = 0,\ i = 1, 2, 3, 4, \quad \sigma = \rho_1 + 2(m - 1)x, \quad W_7 = 3\rho_1^4\Delta_3^6\tau_3/16, \\ \mathcal{B}_1 = \rho_1^2\rho_3\Delta_3, \quad \mathcal{B}_2 = -\rho_1(m - 1)^2(3e - 1 + 4m)\Delta_3. \end{gathered} \tag{29}$$

We observe that the condition $\mu_1 \neq 0$ implies $\widetilde{K} \neq 0$. According to [6] and [31] the polynomials above are responsible for the types of the finite singularities of systems (27). In order to describe the infinite singularities considering Lemma 1 we calculate the additional invariant polynomials:

$$\begin{gathered} \kappa = \eta = 0, \quad \widetilde{M} = -8(1 + 2m)^2x^2, \quad C_2 = ex^3 - (1 + 2m)x^2y, \\ \tilde{L} = 8(1 + 2m)x^2, \quad \widetilde{N} = -4m(1 + m)x^2. \end{gathered} \tag{30}$$

So considering Lemma 1 we have the next remark.

**Remark 8.** *Systems* (27) *could possess at infinity only one of the following seven configurations of singularities:*

$$\begin{array}{rcl}
\widetilde{M}\neq 0,\ \widetilde{K}<0 & \Rightarrow & \widehat{\binom{1}{2}}\,\widehat{P}_{\curlywedge}E\widehat{P}_{\curlywedge}-H,\ N^f;\\
\widetilde{M}\neq 0,\ \widetilde{K}>0,\ \tilde{L}<0,\ \widetilde{N}<0 & \Rightarrow & \widehat{\binom{1}{2}}\,\widehat{P}_{\curlywedge}E\widehat{P}_{\curlywedge}-H,\ S;\\
\widetilde{M}\neq 0,\ \widetilde{K}>0,\ \tilde{L}<0,\ \widetilde{N}>0 & \Rightarrow & \widehat{\binom{1}{2}}\,\widehat{P}_{\curlywedge}H\widehat{P}_{\curlywedge}-E,\ S;\\
\widetilde{M}\neq 0,\ \widetilde{K}>0,\ \tilde{L}<0,\ \widetilde{N}=0 & \Rightarrow & \widehat{\binom{1}{2}}\,H-E,\ S;\\
\widetilde{M}\neq 0,\ \widetilde{K}>0,\ \tilde{L}>0 & \Rightarrow & \widehat{\binom{1}{2}}\,H_{\curlywedge}HH_{\curlywedge}-H,\ N^{\infty};\\
\widetilde{M}=0,\ C_2\neq 0 & \Rightarrow & \widehat{\binom{1}{3}}\,H_{\curlywedge}H\widehat{P}_{\curlywedge}-\widehat{P};\\
\widetilde{M}=C_2=0 & \Rightarrow & [\infty;\,\emptyset].
\end{array}$$

3.2.2.1. *The subcase* $\widetilde{K}<0$. Then $\Delta_3<0$ and the elemental singularity is a saddle.

3.2.2.1.1. *The possibility* $\mathcal{B}_1\neq 0$. In this case we have $\rho_1\rho_3\neq 0$ and therefore the double point is a semi-elemental saddle-node and the saddle is strong. We observe that the condition $\widetilde{K}<0$ implies $m>0$ and then $\widetilde{M}\neq 0$. So considering the above remark we arrive at the global configurations of singularities

- $s,\ \overline{sn}_{(2)};\ \widehat{\binom{1}{2}}\,\widehat{P}_{\curlywedge}E\widehat{P}_{\curlywedge}-H,\ N^f$ : *Example* $\Rightarrow(e=0,\ m=1)$.

3.2.2.1.2. *The possibility* $\mathcal{B}_1=0$. Then $\rho_1\rho_3=0$ and we shall consider two cases: $\mathcal{B}_2\neq 0$ and $\mathcal{B}_2=0$

**1)** *Assume first* $\mathcal{B}_2\neq 0$. Considering (29) we have $\rho_1\neq 0$ and then $\rho_3=0$ (i.e. $e=1-2m$) and the saddle is weak. We calculate

$$\mathcal{B}_1=\mathcal{H}=0,\quad \mathcal{B}_2=8m(m-1)^4,\quad \sigma=2(m-1)(x-1),\quad \mathcal{F}_1=8m(m-1)\tag{31}$$

and clearly the conditions $\mathcal{B}_2\neq 0$ and $\widetilde{K}<0$ imply $\sigma\mathcal{F}_1\neq 0$ and $\mathcal{B}_2>0$. So according to [31] (see the statement $(e_1)$ of the Main Theorem) the weak saddle is of order one. Considering Remark 8 we get the configuration

- $s^{(1)},\ \overline{sn}_{(2)};\ \widehat{\binom{1}{2}}\,\widehat{P}_{\curlywedge}E\widehat{P}_{\curlywedge}-H,\ N^f$ : *Example* $\Rightarrow(e=0,\ m=1/2)$.

**2)** *Suppose now* $\mathcal{B}_2=0$.

***a)*** *The case* $\sigma\neq 0$. We claim that in this case the condition $\mathcal{B}_1=\mathcal{B}_2=0$ is equivalent to $\rho_1=0$. Indeed suppose the contrary. Then $\rho_3=0$ and since $\widetilde{K}\neq 0$, considering (31) the condition $\mathcal{B}_2=0$ implies $\sigma=0$. This contradiction proves our claim. So $\rho_1=0$ (i.e. $e=-1$ and then $E_1=0$) and by [6] (see Table 1, line 137) the double singular point is a nilpotent cusp. In this case $\rho_3=2(m-1)\neq 0$ due to $\sigma=2(m-1)(x-1)\neq 0$. By Remark 8 we get the global configurations of singularities

- $s,\ \widehat{cp}_{(2)};\ \widehat{\binom{1}{2}}\,\widehat{P}_{\curlywedge}E\widehat{P}_{\curlywedge}-H,\ N^f$ : *Example* $\Rightarrow(e=-1,\ m=2)$.

***b)*** *The case* $\sigma=0$. Then $e=-1$, $m=1$ and the system (27) is Hamiltonian, i.e. the saddle is integrable:

- $\$,\ \widehat{cp}_{(2)};\ \widehat{\binom{1}{2}}\,\widehat{P}_{\curlywedge}E\widehat{P}_{\curlywedge}-H,\ N^f$ : *Example* $\Rightarrow(e=-1,\ m=1)$.

3.2.2.2. *The subcase* $\widetilde{K}>0$. In this case according to [6] (see Table 1, lines 139 - 143) the elemental singular point is an anti-saddle and its type is governed by the invariant polynomial $W_7$ (as in the considered case we have $G_9=0$).

**Remark 9.** *Considering* (30) *we observe that the condition* $\tilde{L}=0$ *implies* $\widetilde{M}=0$. *Moreover as in this case* $W_7=3e(e-4)(1+e)^4/16$ *we obtain that the condition* $W_7\neq 0$ *yields* $C_2\neq 0$.

3.2.2.2.1. *The possibility* $W_7<0$. By (29) we have $\rho_1\neq 0$ and $\tau_3<0$ and we conclude, that the double point is a semi-elemental saddle-node, whereas the simple point is a focus.

**1)** *Assume first* $\mathcal{B}_1\neq 0$. Then $\rho_3\neq 0$ and the focus is strong. Considering Remarks 8 and 9 we arrive at the next five global configurations of singularities

- $f,\ \overline{sn}_{(2)};\ \widehat{\binom{1}{2}}\,\widehat{P}_{\curlywedge}E\widehat{P}_{\curlywedge}-H,\ S$ : *Example* $\Rightarrow(e=1,\ m=-3/2)$ (if $\tilde{L}<0,\ \widetilde{N}<0$);

- $f,\ \overline{sn}_{(2)};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} H \widehat{P}_{\curlywedge} - E,\ S:$ *Example* $\Rightarrow (e=1,\ m=-2/3)$ (if $\tilde{L}<0,\ \widetilde{N}>0$);
- $f,\ \overline{sn}_{(2)};\ \widehat{\binom{1}{2}}\ H - E,\ S:$ *Example* $\Rightarrow (e=1,\ m=-1)$ (if $\tilde{L}<0,\ \widetilde{N}=0$);
- $f,\ \overline{sn}_{(2)};\ \widehat{\binom{1}{2}}\ H_{\curlywedge} H H_{\curlywedge} - H,\ N^{\infty}:$ *Example* $\Rightarrow (e=1,\ m=-1/3)$ (if $\tilde{L}>0$);
- $f,\ \overline{sn}_{(2)};\ \widehat{\binom{1}{3}}\ H_{\curlywedge} H \widehat{P}_{\curlywedge} - \widehat{P}:$ *Example* $\Rightarrow (e=1,\ m=-1/2)$ (if $\tilde{L}=0$).

**2)** *Suppose now* $\mathcal{B}_1=0$. As $W_7\neq 0$ (i.e. $\rho_1\neq 0$) we have $\rho_3=0$ and this gives $e=1-2m$. Then we obtain $W_7=1536(m-1)^4m^7$ and considering (31) we observe that the condition $W_7<0$ implies $\sigma\mathcal{F}_1\neq 0$ and $\mathcal{B}_2<0$. According to [31] (see the statement $(e_2)$ of the Main Theorem) we have a first order weak. Considering Remarks 8 and 9 we get the following five global configurations of singularities:

- $f^{(1)},\ \overline{sn}_{(2)};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ S:$ *Example* $\Rightarrow (e=4,\ m=-3/2)$ (if $\tilde{L}<0,\ \widetilde{N}<0$);
- $f^{(1)},\ \overline{sn}_{(2)};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} H \widehat{P}_{\curlywedge} - E,\ S:$ *Example* $\Rightarrow (e=7/3,\ m=-2/3)$ (if $\tilde{L}<0,\ \widetilde{N}>0$);
- $f^{(1)},\ \overline{sn}_{(2)};\ \widehat{\binom{1}{2}}\ H - E,\ S:$ *Example* $\Rightarrow (e=3,\ m=-1)$ (if $\tilde{L}<0,\ \widetilde{N}=0$);
- $f^{(1)},\ \overline{sn}_{(2)};\ \widehat{\binom{1}{2}}\ H_{\curlywedge} H H_{\curlywedge} - H,\ N^{\infty}:$ *Example* $\Rightarrow (e=5/3,\ m=-1/3)$ (if $\tilde{L}>0$);
- $f^{(1)},\ \overline{sn}_{(2)};\ \widehat{\binom{1}{3}}\ H_{\curlywedge} H \widehat{P}_{\curlywedge} - \widehat{P}:$ *Example* $\Rightarrow (e=2,\ m=-1/2)$ (if $\tilde{L}=0$).

3.2.2.2.2. *The possibility* $W_7>0$. By (29) we have $\rho_1\neq 0$ and $\tau_3>0$ and we conclude, that the double point is a semi-elemental saddle-node, whereas the simple point is a generic node. So by Remark 8 we arrive at the next five global configurations of singularities

- $n,\ \overline{sn}_{(2)};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ S:$ *Example* $\Rightarrow (e=-2,\ m=-3/2)$ (if $\tilde{L}<0,\ \widetilde{N}<0$);
- $n,\ \overline{sn}_{(2)};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} H \widehat{P}_{\curlywedge} - E,\ S:$ *Example* $\Rightarrow (e=-2,\ m=-2/3)$ (if $\tilde{L}<0,\ \widetilde{N}>0$);
- $n,\ \overline{sn}_{(2)};\ \widehat{\binom{1}{2}}\ H - E,\ S:$ *Example* $\Rightarrow (e=-2,\ m=-1)$ (if $\tilde{L}<0,\ \widetilde{N}=0$);
- $n,\ \overline{sn}_{(2)};\ \widehat{\binom{1}{2}}\ H_{\curlywedge} H H_{\curlywedge} - H,\ N^{\infty}:$ *Example* $\Rightarrow (e=-2,\ m=-1/3)$ (if $\tilde{L}>0$);
- $n,\ \overline{sn}_{(2)};\ \widehat{\binom{1}{3}}\ H_{\curlywedge} H \widehat{P}_{\curlywedge} - \widehat{P}:$ *Example* $\Rightarrow (e=-2,\ m=-1/2)$ (if $\tilde{L}=0$).

3.2.2.2.3. *The possibility* $W_7=0$. By (29) we have $\rho_1\tau_3=0$.

**1)** *Assume first* $\mathcal{B}_1\neq 0$. Then we obtain $\rho_1\neq 0$ and $\tau_3=0$. So the simple singularity is a node with coinciding eigenvalues. On the other hand considering the linearization matrix $\mathcal{M}_3$ from (28) we observe that we could not have a star node.

Thus considering Remark 8 we get the next six global configurations of singularities

- $n^d,\ \overline{sn}_{(2)};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ S:$ *Example* $\Rightarrow (e=1,\ m=-2)$ (if $\tilde{L}<0,\ \widetilde{N}<0$);
- $n^d,\ \overline{sn}_{(2)};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} H \widehat{P}_{\curlywedge} - E,\ S:$ *Example* $\Rightarrow (e=1/25,\ m=-18/25)$ (if $\tilde{L}<0,\ \widetilde{N}>0$);
- $n^d,\ \overline{sn}_{(2)};\ \widehat{\binom{1}{2}}\ H - E,\ S:$ *Example* $\Rightarrow (e=3+2\sqrt{2},\ m=-1)$ (if $\tilde{L}<0,\ \widetilde{N}=0$);
- $n^d,\ \overline{sn}_{(2)};\ \widehat{\binom{1}{2}}\ H_{\curlywedge} H H_{\curlywedge} - H,\ N^{\infty}:$ *Example* $\Rightarrow (e=1/4,\ m=-1/8)$ (if $\tilde{L}>0$);
- $n^d,\ \overline{sn}_{(2)};\ \widehat{\binom{1}{3}}\ H_{\curlywedge} H \widehat{P}_{\curlywedge} - \widehat{P}:$ *Example* $\Rightarrow (e=4,\ m=-1/2)$ (if $\tilde{L}=0,\ C_2\neq 0$);
- $n^d,\ \overline{sn}_{(2)};\ [\infty;\ \emptyset]:$ *Example* $\Rightarrow (e=0,\ m=-1/2)$ (if $\tilde{L}=0,\ C_2=0$).

**2)** *Suppose now* $\mathcal{B}_1=0$. Then $\rho_1\rho_3=0$ and this gives $\rho_1=0$, otherwise we get $\rho_3=\tau_3=0$ and $M_3$ becomes a non-elemental singularity. So $\rho_1=0$ (i.e. $e=-1$ and then $E_1=0$) and the double point is a cusp. As in this case we have $\tau_3=4(1+m^2)>0$, the simple singular point is a generic node. So as $e\neq 0$ (i.e. $C_2\neq 0$) we get the five configurations of singularities:

- $n,\ \widehat{cp}_{(2)};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ S:$ *Example* $\Rightarrow (e=-1,\ m=-3/2)$ (if $\tilde{L}<0,\ \widetilde{N}<0$);
- $n,\ \widehat{cp}_{(2)};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} H \widehat{P}_{\curlywedge} - E,\ S:$ *Example* $\Rightarrow (e=-1,\ m=-2/3)$ (if $\tilde{L}<0,\ \widetilde{N}>0$);

- $n,\ \widehat{cp}_{(2)};\ \widehat{\binom{1}{2}}\ H\!-\!E,\ S:$ *Example* $\Rightarrow (e=-1,\ m=-1)$ (if $\tilde{L}<0,\ \widetilde{N}=0$);
- $n,\ \widehat{cp}_{(2)};\ \widehat{\binom{1}{2}}\ H_{\curlywedge}HH_{\curlywedge}\!-\!H,\ N^{\infty}:$ *Example* $\Rightarrow (e=-1,\ m=-1/3)$ (if $\tilde{L}>0$);
- $n,\ \widehat{cp}_{(2)};\ \widehat{\binom{1}{3}}\ H_{\curlywedge}H\widehat{P}_{\curlywedge}-\widehat{P}:$ *Example* $\Rightarrow (e=-1,\ m=-1/2)$ (if $\tilde{L}=0$).

Thus we obtain a total of 62 geometrically distinct configurations of singularities for the family of quadratic systems with one double and one simple finite singularities.

3.3. **Systems with one triple real finite singularity.** Assume that quadratic systems (2) possess a triple finite singularity. In this case according to [31] we shall consider the family of systems

$$\dot{x}=gy+gx^2+2hxy,\quad \dot{y}=ly+lx^2+2mxy \tag{32}$$

possessing only one finite singularity: the triple point $M_{1,2,3}(0,0)$. For these systems calculations yield

$$\mu_0=0,\quad \mu_1=4(gm-hl)^2x,\quad \kappa=128h^2(hl-gm).$$

As for the above systems we have $\mu_1\neq 0$ (i.e. $gm-hl\neq 0$) we observe that the condition $\kappa=0$ is equivalent to $h=0$.

3.3.1. *The case* $\kappa\neq 0$. Then $h\neq 0$ and we may assume $h=1$ due to a time rescaling. Moreover we can consider $m\in\{0,1\}$ due to the rescaling $(x,y,t)\mapsto(mx,\ m^2y,t/m^2)$ in the case $m\neq 0$. So we get the systems

$$\begin{aligned}\dot{x}&=gy+gx^2+2xy,\quad gm-l\neq 0,\\ \dot{y}&=ly+lx^2+2mxy,\quad m\in\{0,1\},\end{aligned} \tag{33}$$

for which we calculate

$$\begin{gathered}\kappa=128(l-gm),\quad \widetilde{K}=4(gm-l)x^2,\quad G_{10}=l^3(gm-l)^3,\quad \mathcal{T}_4=-8l^3(gm-l)^2,\\ \eta=4\big[(g-2m)^2+8l\big],\ \widetilde{M}=-8\big[(g-2m)^2+6l\big]x^2-16(g-2m)xy-32y^2.\end{gathered} \tag{34}$$

**Remark 10.** *We observe that the condition* $G_{10}=0$ *is equivalent to* $\mathcal{T}_4=0$ *and in this case we get* $\eta\geq 0$. *Moreover we have* $\operatorname{sign}(\widetilde{K})=-\operatorname{sign}(\kappa)$.

3.3.1.1. *The subcase* $\kappa<0$. This implies $\widetilde{K}>0$ and according to to [6] (see Table 1, lines 146, 147) the triple finite singular point is a semi–elemental node if $G_{10}\neq 0$ and it is a nilpotent elliptic saddle if $G_{10}=0$ (in this case $l=0$ due to $\kappa\neq 0$).

As $\widetilde{M}\neq 0$ considering Lemma 1 we arrive at the following five configurations of singularities:

- $\overline{n}_{(3)};\ \overline{\binom{1}{1}}SN,$ ©, © : *Example* $\Rightarrow (g=1,\ l=-1,\ m=1)$ (if $\mathcal{T}_4\neq 0,\ \eta<0$);
- $\overline{n}_{(3)};\ \overline{\binom{1}{1}}SN,S,N^{\infty}:$ *Example* $\Rightarrow (g=1,\ l=1/2,\ m=1)$ (if $\mathcal{T}_4\neq 0,\ \eta>0$);
- $\overline{n}_{(3)};\ \overline{\binom{0}{2}}SN,\ \overline{\binom{1}{1}}SN:$ *Example* $\Rightarrow (g=1,\ l=-1/8,\ m=1)$ (if $\mathcal{T}_4\neq 0,\ \eta=0$);
- $\widehat{es}_{(3)};\ \overline{\binom{1}{1}}SN,S,N^{\infty}:$ *Example* $\Rightarrow (g=1,\ l=0,\ m=1)$ (if $\mathcal{T}_4=0,\ \eta>0$);
- $\widehat{es}_{(3)};\ \overline{\binom{0}{2}}SN,\ \overline{\binom{1}{1}}SN:$ *Example* $\Rightarrow (g=2,\ l=0,\ m=1)$ (if $\mathcal{T}_4=0,\ \eta=0$).

3.3.1.2. *The subcase* $\kappa>0$. In this case we have $\widetilde{K}<0$ and according to to [6] (see Table 1, line 145) the triple finite singular point is saddle. As its eigenvalues are $\lambda_1=0$ and $\lambda_2=l$ we conclude that this triple saddle is semi-elemental if $l\neq 0$ and it is a nilpotent saddle if $l=0$.

We claim that the condition $\kappa>0$ implies $\eta>0$. Indeed, as $\kappa=128(l-gm)>0$ then we can set $l=gm+\varepsilon$ ($\varepsilon>0$) and then we obtain

$$4\big[(g-2m)^2+8l\big]=4\big[(g+2m)^2+8\varepsilon\big]>0.$$

Thus considering Lemma 1 we obtain the following two configurations of singularities:

- $\overline{s}_{(3)};\ \overline{\binom{1}{1}}SN,N^f,N^f:$ *Example* $\Rightarrow (g=0,\ l=1,\ m=1)$ (if $\mathcal{T}_4\neq 0$);
- $\widehat{s}_{(3)};\ \overline{\binom{1}{1}}SN,N^f,N^f:$ *Example* $\Rightarrow (g=-1,\ l=0,\ m=1)$ (if $\mathcal{T}_4=0$).

3.3.2. *The case* $\kappa = 0$. Then $h = 0$ and $gm \neq 0$ due to $\mu_1 = 4g^2m^2x \neq 0$. So we may assume $g = 1$ due to the rescaling $(x, y) \mapsto (x/g,\, y/g^2)$ and we get the family of systems

$$\dot{x} = y + x^2, \quad \dot{y} = ly + lx^2 + 2mxy, \quad m \neq 0, \tag{35}$$

for which we calculate

$$\begin{gathered} \mu_0 = \kappa = \eta = 0, \quad \mu_1 = 4m^2x, \quad \widetilde{K} = 4mx^2, \quad G_{10} = l^3m^3, \\ \mathcal{F}_1 = 6lm, \quad \tilde{L} = 8(1-2m)x^2, \quad \widetilde{N} = 4m(1-m)x^2, \\ \widetilde{M} = -8(2m-1)^2x^2, \quad C_2 = -lx^3 + (1-2m)x^2y, \end{gathered} \tag{36}$$

smallskip

3.3.2.1. *The subcase* $\widetilde{K} < 0$. According to to [6] (see Table 1, line 145) the triple finite singular point is saddle. As it was mentioned earlier this triple saddle is semi-elemental if $G_{10} \neq 0$ and it is a nilpotent saddle if $G_{10} = 0$, which is equivalent to $\mathcal{F}_1 = 0$.

On the other hand the condition $\widetilde{K} < 0$ implies $m < 0$ and then $\widetilde{M} \neq 0$. Therefore considering Lemma 1 we obtain the following two configurations of singularities:

- $\overline{s}_{(3)};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ N^f$ : *Example* $\Rightarrow (l = 1,\ m = -1)$ (if $\mathcal{F}_1 \neq 0$);
- $\widehat{s}_{(3)};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ N^f$ : *Example* $\Rightarrow (l = 0,\ m = -1)$ (if $\mathcal{F}_1 = 0$).

smallskip

3.3.2.2. *The subcase* $\widetilde{K} > 0$. As we have mentioned earlier the triple finite singular point is a semi–elemental node if $G_{10} \neq 0$ and it is a nilpotent elliptic saddle if $G_{10} = 0$.

3.3.2.2.1. *The possibility* $\widetilde{M} \neq 0$. Then $2m - 1 \neq 0$ and as $m > 0$ (due to $\widetilde{K} > 0$) we have

$$\operatorname{sign}(\tilde{L}) = \operatorname{sign}(1-2m) \quad \operatorname{sign}(\widetilde{N}) = \operatorname{sign}(1-m).$$

Therefore according to Lemma 1 we arrive at the following 8 configurations of singularities:

- $\overline{n}_{(3)};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ S$ : *Example* $\Rightarrow (l = 1,\ m = 2)$ (if $\mathcal{F}_1 \neq 0,\ \tilde{L} < 0,\ \widetilde{N} < 0$);
- $\overline{n}_{(3)};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} H \widehat{P}_{\curlywedge} - E,\ S$ : *Example* $\Rightarrow (l = 1,\ m = 2/3)$ (if $\mathcal{F}_1 \neq 0,\ \tilde{L} < 0,\ \widetilde{N} > 0$);
- $\overline{n}_{(3)};\ \widehat{\binom{1}{2}}\ H - E,\ S$ : *Example* $\Rightarrow (l = 1,\ m = 1)$ (if $\mathcal{F}_1 \neq 0,\ \tilde{L} < 0,\ \widetilde{N} = 0$);
- $\overline{n}_{(3)};\ \widehat{\binom{1}{2}}\ H_{\curlywedge} H H_{\curlywedge} - H,\ N^{\infty}$ : *Example* $\Rightarrow (l = 1,\ m = 1/3)$ (if $\mathcal{F}_1 \neq 0,\ \tilde{L} > 0$);
- $\widehat{es}_{(3)};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ S$ : *Example* $\Rightarrow (l = 0,\ m = 2)$ (if $\mathcal{F}_1 = 0,\ \tilde{L} < 0,\ \widetilde{N} < 0$);
- $\widehat{es}_{(3)};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} H \widehat{P}_{\curlywedge} - E,\ S$ : *Example* $\Rightarrow (l = 0,\ m = 2/3)$ (if $\mathcal{F}_1 = 0,\ \tilde{L} < 0,\ \widetilde{N} > 0$);
- $\widehat{es}_{(3)};\ \widehat{\binom{1}{2}}\ H - E,\ S$ : *Example* $\Rightarrow (l = 0,\ m = 1)$ (if $\mathcal{F}_1 = 0,\ \tilde{L} < 0,\ \widetilde{N} = 0$);
- $\widehat{es}_{(3)};\ \widehat{\binom{1}{2}}\ H_{\curlywedge} H H_{\curlywedge} - H,\ N^{\infty}$ : *Example* $\Rightarrow (l = 0,\ m = 1/3)$ (if $\mathcal{F}_1 = 0,\ \tilde{L} > 0$).

3.3.2.2.2. *The possibility* $\widetilde{M} = 0$. Then $m = 1/2$ and we obtain $C_2 = -lx^3$. So the condition $\mathcal{F}_1 = 0$ is equivalent to $C_2 = 0$ and considering Lemma 1 we get the following two global configurations of singularities:

- $\overline{n}_{(3)};\ \widehat{\binom{1}{3}} H_{\curlywedge} H \widehat{P}_{\curlywedge} - \widehat{P}$ : *Example* $\Rightarrow (l = 1,\ m = 1/2)$ (if $\mathcal{F}_1 \neq 0$);
- $\widehat{es}_{(3)};\ [\infty;\ \emptyset]$ : *Example* $\Rightarrow (l = 0,\ m = 1/2)$ (if $\mathcal{F}_1 = 0$).

Thus we obtained that a quadratic system with a triple finite singularity possesses only one of the 19 global configurations of singularities given above.

As all the cases are examined, we have constructed all 155 possible configurations for the family of quadratic systems with $m_f = 3$ possessing at most two real finite singularities. Therefore our Main Theorem is completely proved.

## References


[1] J. C. Artés, J. Llibre and D. Schlomiuk, *The geometry of quadratic differential systems with a weak focus of second order.* International J. of Bifurcation and Chaos **16** (2006), 3127–3194.

[2] J. C. Artés, J. Llibre and D. Schlomiuk, N. Vulpe, *From topological to geometric equivalence in the classification of singularities at infinity for quadratic vector fields.* To appear in Rocky Mountain J. of Math.

[3] J. C. Artés, J. Llibre and D. Schlomiuk, N. Vulpe, *Configurations of singularities for quadratic differential systems with total finite multiplicity lower than 2.* Bul. Acad. Stiinte Repub. Mold. Mat. **1** (2013), 72–124.

[4] J. C. Artés, J. Llibre and D. Schlomiuk, N. Vulpe, *Algorithm for determining the global geometric configurations of singularities of total finite multiplicity 2 for quadratic differential systems.* Preprint, UAB Barcelona, 2013.

[5] J. C. Artés, J. Llibre and D. Schlomiuk, N. Vulpe, *Geometric configurations of singularities for quadratic differential systems with three distinct real simple finite singularities* . To appear in J. Fixed Point Theory Appl.

[6] J. C. Artés, J. Llibre and N. I. Vulpe, *Singular points of quadratic systems: A complete classification in the coefficient space* $\mathbb{R}^{12}$. International J. of Bifurcation and Chaos **18** (2008), 313–362.

[7] J. C. Artés, J. Llibre, N. Vulpe, *Complete geometric invariant study of two classes of quadratic systems.* Electron. J. Differential Equations, **2012** (2012) No. 9, 1–35.

[8] V. Baltag, *Algebraic equations with invariant coefficients in qualitative study of the polynomial homogeneous differential systems.* Bul. Acad. Stiinte Repub. Mold. Mat. **2** (2003), 31–46

[9] V. A. Baltag and N. I. Vulpe, *Affine-invariant conditions for determining the number and multiplicity of singular points of quadratic differential systems.* Izv. Akad. Nauk Respub. Moldova Mat. **1** (1993), 39–48.

[10] V. A. Baltag and N. I. Vulpe, *Total multiplicity of all finite critical points of the polynomial differential system.* Planar nonlinear dynamical systems (Delft, 1995), Differ. Equ. Dyn. Syst. **5** (1997), 455–471.

[11] N. N. Bautin, *On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type.* Mat. Sb. (N.S) **30** (7)2 (1952), 181–196 (Russian), Transl. Amer. Math. Soc. Ser. (1) **100** (1954), 397–413.

[12] D. Bularas, Iu. Calin. L. Timochouk and N. Vulpe, *T–comitants of quadratic systems: A study via the translation invariants.* Delft University of Technology, Faculty of Technical Mathematics and Informatics, Report no. 96-90, 1996; (URL: `ftp://ftp.its.tudelft.nl/publications/tech- reports/1996/DUT-TWI-96- 90.ps.gz`).

[13] Cai, Sui Lin, *The weak saddle and separatrix cycle of a quadratic system.* Acta Math. Sinica, **30** (1987), 553–559 (Chinese).

[14] Iu. Calin, *On rational bases of* $GL(2,\mathbb{R})$*-comitants of planar polynomial systems of differential equations.* Bul. Acad. Stiinte Repub. Mold. Mat. **2** (2003), 69–86.

[15] B. Coll, *Qualitative study of some classes of vector fields in the plane.* Ph. D., Universitat Autónoma de Barcelona, (1987), pp. 5–34.

[16] F. Dumortier, J. Llibre and J. C. Artés, *Qualitative Theory of Planar Differential Systems.* Universitext, Springer–Verlag, New York–Berlin, 2008.

[17] E. A. Gonzalez Velasco, *Generic properties of polynomial vector fields at infinity.* Trans. Amer.Math. Soc. **143** (1969), 201–222.

[18] J. H. Grace and A. Young, *The algebra of invariants.* Stechert, New York, 1941.

[19] Q. Jiang and J. Llibre, *Qualitative classification of singular points.* Qual. Theory Dyn. Syst. **6** (2005), 87–167.

[20] J. Llibre and D. Schlomiuk, *Geometry of quadratic differential systems with a weak focus of third order.* Canad. J. of Math. **56** (2004), 310–343.

[21] P. J. Olver, *Classical Invariant Theory.* London Math. Soc. Student Texts **44**, Cambridge University Press, 1999.

[22] I. Nikolaev and N. Vulpe, *Topological classification of quadratic systems at infinity.* J. London Math. Soc. **2** (1997), 473–488.

[23] J. Pal and D. Schlomiuk, *Summing up the dynamics of quadratic Hamiltonian systems with a center.* Canad. J. Math. **56** (1997), 583–599.

[24] M. N. Popa, *Applications of algebraic methods to differential systems.* Romania, Piteşti Univers., The Flower Power Edit., 2004.

[25] D. Schlomiuk, *Algebraic particular integrals, integrability and the problem of the center.* Trans. Amer. Math. Soc. **338** (1993), 799–841.

[26] D. Schlomiuk *Algebraic and Geometric Aspects of the Theory of Polynomial vector fields*, in Bifurcations and periodic Orbits of Vector Fields, Dana Schlomiuk Editor, NATO ASI Series, Series C: Vol.408, Kluwer Academic Publishers, 1993, pp. 429–467.

[27] D. Schlomiuk and J. Pal, *On the geometry in the neighborhood of infinity of quadratic differential phase portraits with a weak focus.* Qual. Theory Dyn. Syst. **2** (2001), 1–43.

[28] D. Schlomiuk and N. I. Vulpe, *Geometry of quadratic differential systems in the neighborhood of infinity.* J. Differential Equations **215** (2005), 357–400.

[29] D. Schlomiuk and N. I. Vulpe, *The full study of planar quadratic differential systems possessing a line of singularities at infinity.* J. Dynam. Differential Equations **20** (2008), 737–775.

[30] K. S. Sibirskii, *Introduction to the algebraic theory of invariants of differential equations*. Translated from the Russian. Nonlinear Science: Theory and Applications. Manchester University Press, Manchester, 1988.

[31] N. Vulpe, *Characterization of the finite weak singularities of quadratic systems via invariant theory.* Nonlinear Analysis **74** (2011), 6553–6582.

[32] N. I. Vulpe, *Affine–invariant conditions for the topological discrimination of quadratic systems with a center*. Differential Equations **19** (1983), 273–280.

[33] N. I. Vulpe, *Polynomial bases of comitants of differential systems and their applications in qualitative theory.* "Ştiinţa", Kishinev, 1986, (in Russian).

[34] N. Vulpe and M. Lupan, *Quadratic systems with dicritical points*. Bul. Acad. Stiinte Repub. Mold. Mat. **3** (1994), 52–60.