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# Hamiltonian linear type centers and nilpotent centers of linear plus cubic polynomial vector fields

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# Contents

# List of Figures

# Chapter 1

# Introduction

In the qualitative theory of real planar polynomial differential systems two of the main problems are the determination of limit cycles and the center-focus problem, i.e. to distinguish when a singular point is either a focus or a center. In this work we provide normal forms for Hamiltonian systems with cubic homogeneous nonlinearities which have a center at the origin, and classify these systems with respect to the topological equivalence of their global phase portraits on the Poincaré disk. This classification will further allow to start the study of how many limit cycles can bifurcate from the periodic orbits of the Hamiltonian centers with only linear and cubic terms when they are perturbed inside the class of all cubic polynomial differential systems. Before going any further we shall talk about some preliminary concepts and definitions that we will use throughout this work. For more details see [14].

## 1.1 Preliminary definitions

Let $A$ be an open set in $\mathbb{R}^2$. We define a *vector field of class* $C^r$ as a $C^2$ map $X : A \to \mathbb{R}^2$ where $X(x, y)$ represents the tip of the vector whose tail is at the point $(x, y) \in A$. The *orbits* of the vector field $X$ are the solutions $\varphi(t) = (x(t), y(t))$ of the differential equation

$$(\dot{x}, \dot{y}) = X(x, y), \tag{1.1}$$

where the dot denotes the sderivative with respect to time $t$. Therefore when we say "vector field $X$" and "differential system (1.1)" we mean the same thing. Here $x$ and $y$ are called the *dependent variables*, and $t$ is called the *independent variable*. An orbit is called a *periodic orbit* if there exists a $c > 0$ such that $\varphi(t) = \varphi(t + c)$ for every $t$. A *limit cycle* is a periodic orbit which has a neighborhood that does not contain any additional periodic orbit.

The *flow* of a vector field is defined as usual, see for instance page 3 of [14]. The union of orbits of the vector field $X$ constitute its *phase portrait*.

A *bifurcation diagram* illustrates how the phase portrait of a vector field depends on its parameters.

A point $(x, y)$ is called a *singular point* (or an *equilibrium point*) if $X(x, y) = 0$. If a singular point has a neighborhood that does not contain any other singular point, then that singular point is called an *isolated singular point*.

We define the *linear part* of $X$ at a point as the Jacobian matrix of $X$ at that point. We say that a singular point is *non–elementary* if both of the eigenvalues of the linear part of the vector field at that point are zero, and *elementary* otherwise. If both of the eigenvalues of the linear part of the vector field at an elementary singular point are real, then the singular point is called *hyperbolic*. A non–elementary singular point is called *degenerate* if the linear part is identically zero, otherwise it is called *nilpotent*.

The notion of center goes back to Poincaré, see [24]. He defined a *center* for a vector field on the real plane as a singular point having a neighborhood filled of periodic orbits with the exception of the singular point. If an analytic system has a center, it is known that after an affine change of variables and a rescaling of the time variable, it can be written in one of the following three forms:

$$\dot{x} = -y + P(x, y), \quad \dot{y} = x + Q(x, y),$$

called a *linear type center*;

$$\dot{x} = y + P(x, y), \quad \dot{y} = Q(x, y), \tag{1.2}$$

called a *nilpotent center*;

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

called a *degenerate center*, where $P(x, y)$ and $Q(x, y)$ are real analytic functions without constant and linear terms, defined in a neighborhood of the origin.

A *saddle* ,a *node*, a *focus* and a *cusp* are defined in the usual way, for more details see for instance [14] pages 7 and 110. A *separatrix of a saddle* is an orbit that tends to that saddle either as $t \to \infty$, or $t \to -\infty$. Clearly a saddle has four separatrices.

We now talk a little about the Poincaré compactification. Let $\mathbb{S}^2$ be the set of points $(s_1, s_2, s_3) \in \mathbb{R}^3$ such that $s_1^2 + s_2^2 + s_3^2 = 1$. We will call this set *the Poincaré sphere*. Given a polynomial vector field

$$X(x, y) = (\dot{x}, \dot{y}) = (P(x, y), Q(x, y)) \tag{1.3}$$

in $\mathbb{R}^2$ of *degree* $d$ (where $d$ is the maximum of the degrees of the polynomials $P$ and $Q$) it can be extended analytically to the Poincaré sphere by projecting each point $x \in \mathbb{R}^2$ identified by $(x_1, x_2, 1) \in \mathbb{R}^3$ onto the Poincaré sphere using the straight line through $x$ and the origin of $\mathbb{R}^3$. In this way we obtain two copies of $X$: one on the *northern hemisphere* $\{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_3 > 0\}$ and another on the *southern hemisphere* $\{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_3 < 0\}$. The *equator* $\mathbb{S}^1 = \{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_3 = 0\}$ corresponds to the infinity of $\mathbb{R}^2$. The *local charts* needed for doing the calculations on the Poincaré sphere are

$$U_i = \{\mathbf{s} \in \mathbb{S}^2 : s_i > 0\}, \quad V_i = \{\mathbf{s} \in \mathbb{S}^2 : s_i < 0\},$$

where $\mathbf{s} = (s_1, s_2, s_3)$, with the corresponding *local maps*

$$\varphi_i(s) : U_i \to \mathbb{R}^2, \quad \psi_i(s) : V_i \to \mathbb{R}^2,$$

such that $\varphi_i(s) = -\psi_i(s) = (s_m/s_i, s_n/s_i)$ for $m < n$ and $m, n \neq i$, for $i = 1, 2, 3$. The expression for the corresponding vector field on $\mathbb{S}^2$ in the local chart $U_1$ is given by

$$\dot{u} = v^d \left[ -uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right) \right], \quad \dot{v} = -v^{d+1} P\left(\frac{1}{v}, \frac{u}{v}\right); \qquad (1.4)$$

the expression for $U_2$ is

$$\dot{u} = v^d \left[ P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right], \quad \dot{v} = -v^{d+1} Q\left(\frac{u}{v}, \frac{1}{v}\right); \qquad (1.5)$$

and the expression for $U_3$ is just

$$\dot{u} = P(u, v), \quad \dot{v} = Q(u, v), \qquad (1.6)$$

where $d$ is the degree of the vector field $X$. The expressions for the charts $V_i$ are those for the charts $U_i$ multiplied by $(-1)^{d-1}$, for $i = 1, 2, 3$. Hence, to study the vector field $X$, it is enough to study its Poincaré compactification restricted to the northern hemisphere plus $\mathbb{S}^1$, which we denote by $\mathbb{D}$ call *the Poincaré disk*. To draw the phase portraits we will consider the projection $\pi(s_1, s_2, s_3) = (s_1, s_2)$ of the Poincaré disk onto the unit disk centered at the origin.

*Finite singular points* of $X$ are the singular points of its compactification which are in $\mathbb{D}^2 \setminus \mathbb{S}^1$, and they can be studied using $U_3$. *Infinite singular points* of $X$ are the singular points of the corresponding vector field on the Poincaré disk $\mathbb{D}$ lying on $\mathbb{S}^1$. Clearly a point $s \in \mathbb{S}^1$ is an infinite singular point if and only if so is $-s \in \mathbb{S}^1$, and the local behavior of one is the same as the other multiplied by $(-1)^{d-1}$. Hence to study the infinite singular points it suffices to look only at $U_1|_{v=0}$ and at the origin of $U_2$.

We say that two vector fields on the Poincaré disk $\mathbb{D}$ are *topologically equivalent* if there exists a homeomorphism $h : \mathbb{D} \to \mathbb{D}$ which sends orbits to orbits preserving or reversing the direction of the flow.

A polynomial differential system (1.3) is called *Hamiltonian* if there exists a nonconstant polynomial $H : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\dot{x} = H_y, \quad \dot{y} = -H_x,$$

where $H_x$ denotes the partial derivative of $H$ with respect to $x$. $H$ is called the *Hamiltonian polynomial.* For a Hamiltonian vector field on the Poincaré disk the *separatrices* are $(i)$ the separatrices of finite and infinite saddles, $(ii)$ the finite and infinite singular points, and $(iii)$ all the orbits at infinity. Let $\Sigma$ be the set of all separatrices, $\Sigma$ is a closed set in $\mathbb{D}$. The open components of $\mathbb{D} \setminus \Sigma$ are called *canonical regions.* The union of $\Sigma$ with an orbit from each canonical region is called a *separatrix configuration.* . The next theorem of Neumann [23] gives a characterization of two topologically equivalent vector fields in the Poincaré disk.

**Theorem 1** (Neumann's Theorem)**.** *Two continuous flows in $\mathbb{D}$ with isolated singular points are topologically equivalent if and only if their separatrix configurations are equivalent.*

This theorem implies that once a separatrix configuration of a vector field in the Poincaré disk is determined, the global phase portrait of that vector field is obtained up to topological equivalence.

Finally we mention without getting into too much detail an important result that classifies the finite singular points of Hamiltonian planar polynomial differential systems. For a detailed definition of the (topological) *index* of a singular point see for instance Chapter 6 of [14], but for our intents and purposes the following theorem known as the *Poincaré Formula* provides enough information for the subject. Similarly a *parabolic sector*, a *hyperbolic sector* and an *elliptic sector* are defined in the standard way, for details see page 18 of [14]. A vector field is said to have the *finite sectorial decomposition property* at a singular point $p$ if either $p$ is a center, a focus or a node, or it has a neighborhood consisting of a finite union of parabolic, hyperbolic or elliptic sectors.

**Theorem 2** (Poincaré Formula)**.** *Let $q$ be an isolated singular point having the finite sectorial decomposition property. Let $e, h,$ and $p$ denote the number of elliptic, hyperbolic, and parabolic sectors of $q$, respectively. Then the index of $q$ is $(e-h)/2+1$.*

For details on Theorem 2 see page 179 of [14].

**Proposition 3.** *Finite singular points of Hamiltonian planar polynomial vector fields are either centers, or have a finite union of an even number of hyperbolic sectors.*

*Proof.* It is known that an analytic planar differential system has the finite sectorial decomposition property, for details see [14]. Moreover, if the system is Hamiltonian, its flow preserves area, see [1]. So a singular point of a Hamiltonian system cannot be a focus, or have elliptic or parabolic sectors. Finally, since the index of a singular point formed by hyperbolic sectors is $1 - h/2$, with $h$ being the number of its hyperbolic sectors, it follows that $h$ is even. For more details about the index, see [14]. □

## 1.2 Background and our main results

An algorithm for the characterization of linear type centers was provided by Poincaré [25] and Lyapunov [20], see also Chazy [6] and Moussu [22]. For an algorithm for the characterization of the nilpotent centers and some class of degenerate centers see the works of Chavarriga *et al.* [5], Giacomini *et al.* [16], Cima and Llibre [8], and Giné and Llibre [17].

The classification of centers for real planar polynomial differential systems started with the classification of centers for quadratic systems, and these results go back mainly to Dulac [13], Kapteyn [18, 19] and Bautin [2]. In [28] Vulpe provides all the global phase portraits of quadratic polynomial differential systems having a center. The bifurcation diagrams of these systems were done by Schlomiuk [26] and Żołądek [31]. There are many partial results for the centers of planar polynomial differential systems of degree larger than two. For instance the linear type centers for cubic systems of the form linear plus homogeneous nonlinearities were characterized by Malkin [21], and Vulpe and Sibirski [29]. We must mention that in this work we do not use their characterization, instead we introduce a different set of normal forms. Some interesting results on some subclasses of cubic systems are those of Rousseau and Schlomiuk [27], and the ones of Żołądek [32, 33]. For polynomial differential systems of the form linear plus homogeneous nonlinearities of degree greater than three the centers at the origin are not characterized, but there are partial results for degrees four and five for the linear type centers, see for instance Chavarriga and Giné [3, 4].

In this work we provide the global phase portraits on the Poincaré disk of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a linear type center or a nilpotent center at the origin, together with their bifurcation diagrams. It is shown in [8] that the degenerate centers of such vector fields are topologically equivalent to 1.18 of Figure 1.1. Hence this work completes the classification of the global phase portraits on the Poincaré disk of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a center at the origin.

We note that the problem of finding the global phase portraits of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a linear type center at the origin has been considered also in [15]. There the authors provide a general algorithm using polar coordinates for finding the phase portraits of Hamiltonian linear type centers with arbitrary $n$–th order homogeneous nonlinearities. In addition, as an application of this algorithm they provide the global phase portraits of the Hamiltonian linear type centers having cubic nonlinearities. However, while it differs from our work in terms of the tools used, there are also some differences in the results obtained. They find the same global phase portraits as us, except for the locations of the cusp points in their phase portraits 21 and 22 (which, in our case, correspond to the phase portraits 1.12 and 1.9 of Figure 1.1, respectively). According to our results, if we perturb linearly the phase portraits 21 and 22 in the same class we should obtain the phase portraits 17 and 20, respectively (which, in our case, correspond to the phase portraits 1.11 and 1.8 of Figure 1.1, respectively). However this clearly is not possible. We remark that for the Hamiltonian nilpotent centers of the form linear plus cubic homogeneous terms there are no previous results.

We now state our main results. We first provide normal forms and the global phase portraits in the Poincaré disk for all the Hamiltonian linear type center or nilpotent centers of linear plus cubic homogeneous planar polynomial vector fields. These results are summarized in Theorems 4 and 5, respectively.

**Theorem 4.** *Any Hamiltonian linear type planar polynomial vector field with linear plus cubic homogeneous terms has a linear type center at the origin if and only if, after a linear change of variables and a rescaling of its independent variable, it can be written as one of the following six classes:*

*(I)* $\dot{x} = ax + by,\ \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3$

*(II)* $\dot{x} = ax + by - x^3,\ \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + 3x^2y,$

*(III)* $\dot{x} = ax + by - 3x^2y + y^3,\ \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + 3xy^2,$

*(IV)* $\dot{x} = ax + by - 3x^2y - y^3,\ \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + 3xy^2,$

*(V)* $\dot{x} = ax + by - 3\mu x^2y + y^3,\ \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3 + 3\mu xy^2,$

*(VI)* $\dot{x} = ax + by - 3\mu x^2y - y^3,\ \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3 + 3\mu xy^2,$

*where* $a, b, \beta, \mu \in \mathbb{R}$ *with* $b \neq 0$ *and* $\beta > 0$. *Moreover, the global phase portraits of these six families of systems are topologically equivalent to the following of Figure 1.1:*

*(a)* 1.1 *or* 1.2 *for systems* (*I*)*;*

*(b)* 1.3 *for systems* $(II)$*;*

*(c)* 1.4, 1.5 *or* 1.6 *for systems* $(III)$*;*

*(d)* 1.1, 1.2, 1.7, 1.8 *or* 1.9 *for systems* $(IV)$*;*

*(e)* 1.3, 1.10, 1.11 *or* 1.12 *for systems* $(V)$*;*

*(f)* 1.13–1.23 *for systems* $(VI)$.

We will prove Theorem 4 in Chapter 2.

**Theorem 5.** *A Hamiltonian planar polynomial vector field with linear plus cubic homogeneous terms has a nilpotent center at the origin if and only if, after a linear change of variables and a rescaling of its independent variable, it can be written as one of the following six classes:*

*(VII)* $\dot{x} = ax + by$, $\dot{y} = -\dfrac{a^2}{b}x - ay + x^3$, *with* $b < 0$,

*(VIII)* $\dot{x} = ax + by - x^3$, $\dot{y} = -\dfrac{a^2}{b}x - ay + 3x^2y$, *with* $a > 0$.

*(IX)* $\dot{x} = ax + by - 3x^2y + y^3$, $\dot{y} = \left(c - \dfrac{a^2}{b+c}\right)x - ay + 3xy^2$, *with either* $a = b = 0$ *and* $c < 0$, *or* $c = 0$, $ab \neq 0$, *and* $a^2/b - 6b > 0$,

*(X)* $\dot{x} = ax + by - 3x^2y - y^3$, $\dot{y} = \left(c - \dfrac{a^2}{b+c}\right)x - ay + 3xy^2$, *with either* $a = b = 0$ *and* $c > 0$, *or* $c = 0$, $a \neq 0$, *and* $b < 0$,

*(XI)* $\dot{x} = ax + by - 3\mu x^2y + y^3$, $\dot{y} = \left(c - \dfrac{a^2}{b+c}\right)x - ay + x^3 + 3\mu xy^2$, *with either* $a = b = 0$ *and* $c < 0$, *or* $c = 0$, $b \neq 0$, *and* $(a^4 - b^4 - 6a^2b^2\mu)/b > 0$,

*(XII)* $\dot{x} = ax + by - 3\mu x^2y - y^3$, $\dot{y} = \left(c - \dfrac{a^2}{b+c}\right)x - ay + x^3 + 3\mu xy^2$, *with either* $a = b = 0$ *and* $c > 0$, *or* $c = 0$, $b \neq 0$, *and* $(a^4 + b^4 + 6a^2b^2\mu)/b < 0$,

*where* $a, b, c, \mu \in \mathbb{R}$. *Moreover the global phase portraits of these six families of systems are topologically equivalent to the following of Figure 1.1:*

*(a)* 1.1 *for systems* $(VII)$ *and* $(X)$*;*

*(b)* 1.3 *for systems* $(VIII)$*;*

*(c)* 1.4, 1.5 *or* 1.6 *for systems* $(IX)$*;*

*(d)* 1.3, 1.10, 1.11 *or* 1.12 *for systems* $(XI)$*;*

*(e)* 1.13, 1.14, 1.15 *or* 1.18 *for systems* $(XII)$.

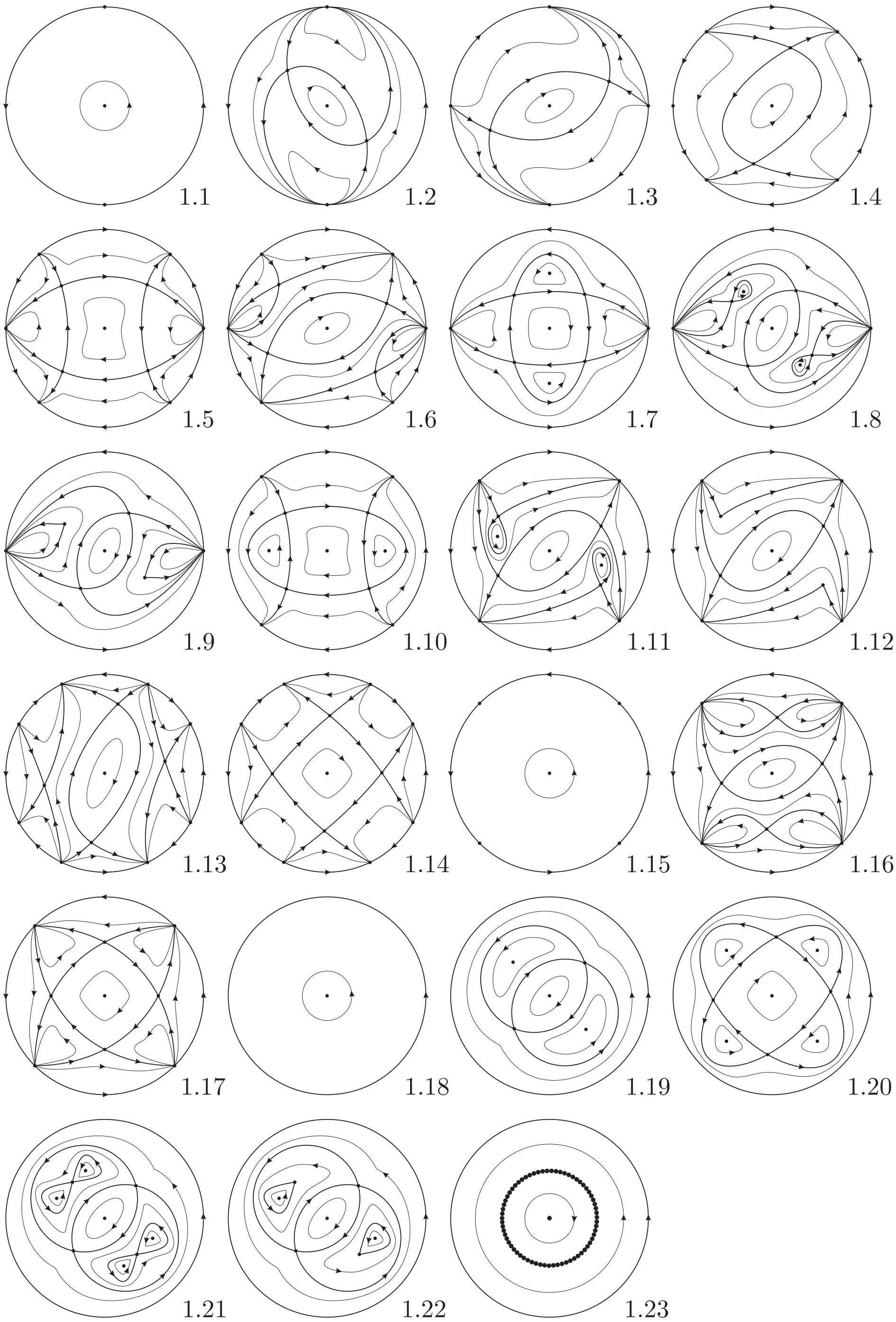



Figure 1.1: Global phase portraits of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a linear type or nilpotent center at the origin. The separatrices are in bold.

We will prove Theorem 5 in Chapter 3.

Second we provide the bifurcation diagrams of the families of vector fields $(I)$–$(XII)$ of Theorems 4 and 5. The bifurcation diagrams for the centers of Theorems 7 and 8 in the particular case when they are reversible were also given in [15].

We note that the parameters of these families can be further simplified, however, at this point such simplifications do not contribute to the proofs. On the other hand, to obtain better and simpler bifurcation diagrams we shall make use of those simplifications. Hence we make the following remark before stating our next results.

**Remark 6.** *Using the change of variables $(u,v) = (x/\sqrt{\beta}, y/\sqrt{\beta})$, the time rescale $d\tau = \beta dt$, and redefining parameters $\bar{a} = a/\beta$ and $\bar{b} = b/\beta$, we can assume $\beta = 1$ in the families of systems $(I)$–$(VI)$. We also note that in the families $(III)$–$(VI)$ the cases with $a < 0$ are obtained from those with $a > 0$ simply by making the change $(t,x) \mapsto (-t,-x)$. Therefore we will assume $a \geq 0$ for these systems.*

*A system in class $(XI)$ with $a = c = 0$ can be transformed to a system inside the same class with $a = b = 0$ and $c \neq 0$ doing the change $(x,y) \mapsto (y,x)$, $c \mapsto b$ and $\mu \mapsto -\mu$. Hence when $c = 0$ we can assume $a \neq 0$. Similarly we can assume $a \neq 0$ in systems $(XII)$ whenever $c = 0$ (in this case the change of variables is $(x,y) \mapsto (-y,x)$).*

*When $a \neq 0$, via the rescaling of the variables $(x,y,t) \mapsto (x/\sqrt{|a|}, y/\sqrt{|a|}, |a|t)$ and the parameter $b \mapsto b/|a|$ we can assume $a = 1$ in the families of systems $(IX) - (XII)$.*

Using Remark 6 we present our results on the bifurcation diagrams in the following two theorems.

**Theorem 7.** *The global phase portraits of Hamiltonian planar polynomial vector fields with linear plus cubic homogeneous terms having a linear type center at the origin are topologically equivalent to the following ones of Figure 1.1 using the notation of Theorem 4.*

*(a) For systems $(I)$ the phase portrait is*

- 1.1 *when $b < 0$;*
- 1.2 *when $b > 0$.*

*(b) For systems $(II)$ the unique phase portrait is* 1.3.

*(c) For systems $(III)$ the phase portrait is*

- 1.4 *when $b < 0$;*
- 1.5 *when $b > 0$ and $a = 0$;*

1.6 *when* $b > 0$ *and* $a > 0$.

*The corresponding bifurcation diagram is shown in Figure 1.2.*

(d) *For systems* $(IV)$ *the phase portrait is*

1.1 *when* $b < 0$;

1.2 *when* $b > 0$, $D = 0$ *and* $a = 0$, *or when* $b > 0$ *and* $D > 0$;

1.7 *when* $b > 0$, $D < 0$ *and* $a = 0$;

1.8 *when* $b > 0$, $D < 0$ *and* $a > 0$;

1.9 *when* $b > 0$, $D = 0$ *and* $a > 0$.

*See* (4.4) *for the definition of* $D$. *The corresponding bifurcation diagram is shown in Figure 1.3.*

(e) *For systems* $(V)$ *we can assume* $b > 0$, *and the phase portrait is*

1.3 *when* $\mu \leq 0$, *or when* $\mu > 0$ *and* $D_4 < 0$, *or when* $\mu > 0$, $D_4 = 0$ *and* $a = 0$;

1.10 *when* $\mu > 0$, $D_4 > 0$ *and* $a = 0$;

1.11 *when* $\mu > 0$, $D_4 > 0$ *and* $a > 0$;

1.12 *when* $\mu > 0$, $D_4 = 0$ *and* $a > 0$.

*See* (4.12) *for the definition of* $D_4$. *The corresponding bifurcation diagram for the case* $\mu > 0$ *is shown in Figure 1.4.*

(f) *For systems* $(VI)$ *we can assume* $b > 0$ *whenever* $\mu < -1/3$, *and the phase portrait is*

1.13 *when* $\mu < -1/3$ *and* $b \neq \sqrt{1+a^2}$;

1.14 *when* $\mu < -1/3$ *and* $b = \sqrt{1+a^2}$;

1.15 *when* $\mu = -1/3$ *and* $b < 0$;

1.16 *when* $\mu = -1/3$, $b > 0$ *and* $b \neq \sqrt{1+a^2}$;

1.17 *when* $\mu = -1/3$ *and* $b = \sqrt{1+a^2}$;

1.18 *when* $\mu > -1/3$ *and* $b < 0$;

1.19 *when* $\mu > -1/3$, $b > 0$, $D_4 < 0$, *or when* $\mu > -1/3$, $b > 0$, $D_4 = D_3 = 0$ *and either* $a \neq 0$ *or* $\mu \neq 1/3$ *or* $b \neq 1$;

1.20 *when* $1/3 - 2a/(3\sqrt{1+a^2}) > \mu > -1/3$, $D_4 > 0$ *and* $b = \sqrt{1+a^2}$, *or when* $\mu > 1/3$, $b > 0$, $D_4 > 0$ *and* $a = 0$;

1.21 *when* $\mu > -1/3$, $b > 0$, $D_4 > 0$ *and* $b \neq \sqrt{1+a^2}$, *or when* $\mu > 1/3 + 2a/(3\sqrt{1+a^2})$, $b = \sqrt{1+a^2}$, $D_4 > 0$ *and* $a \neq 0$;

1.22 *when* $\mu > -1/3$, $b > 0$, $D_4 = 0$ *and* $D_3 \neq 0$;

1.23 *when* $a = 0$, $\mu = 1/3$ *and* $b = 1$.

*See* (4.26) *and* (4.39) *for the definitions of* $D_4$ *and* $D_3$, *respectively. The corresponding bifurcation diagrams are shown in Figures 5–9.*

We will prove Theorem 7 in Chapter 4.

**Theorem 8.** *The global phase portraits of Hamiltonian planar polynomial vector fields with linear plus cubic homogeneous terms having a nilpotent center at the origin are topologically equivalent to the following ones of Figure 1.1 using the notation of Theorem 5.*

*(a) For systems* $(VII)$ *and* $(X)$ *the unique phase portrait is* 1.1.

*(b) For systems* $(VIII)$ *the unique phase portrait is* 1.3.

*(c) For systems* $(IX)$ *the phase portrait is*

1.4 *when* $b < 0$;
1.5 *when* $b = 0$;
1.6 *when* $b > 0$.

*(d) For systems* $(XI)$ *we can assume* $b \geq 0$, *and the phase portrait is*

1.3 *when* $b = 0$ *and* $\mu \leq 0$, *or when* $b > 0$ *and* $D < 0$;
1.10 *when* $b = 0$ *and* $\mu > 0$;
1.11 *when* $b > 0$ *and* $D > 0$;
1.12 *when* $b > 0$ *and* $D = 0$.

*Here* $D = -b^2 - 6b^2\mu + 4(1 - b^4)\mu^3 + 3b^2\mu^4$, *and the corresponding bifurcation diagrams are shown in Figure 1.10.*

*(e) For systems* $(XII)$ *the phase portrait is*

1.13 *when* $\mu > -1/3$ *and* $b \neq 0, 1$;
1.14 *when* $\mu < -1/3$ *and* $b = 0, 1$;
1.15 *when* $\mu = -1/3$;
1.18 *when* $\mu > -1/3$.

*The corresponding bifurcation diagrams are shown in Figure 1.11.*

We will prove Theorem 8 in Chapter 5.

We remark that all the equations controlling the bifurcations of the global phase portraits described in Theorems 7 and 8 are algebraic curves. We must mention that essentially Chapters 2 and 3 are published in the journals J. Differential Equations and Advances in Mathematics, repectively (see

[9] and [10]). The Chapters 4 and 5 are submitted to publication, see [11] and [12].

As we mentioned at the beginning of this chapter, the normal forms, the phase portraits and the bifurcation diagrams provided in Theorems 4, 5, 7 and 8 will lead to new studies in the number of limit cycles that bifurcate from the periodic orbits of the families of differential systems $(I)$–$(XII)$ when they are perturbed inside the class of all cubic polynomial differential systems. This last study was made for the quadratic polynomial differential systems, see the paper [7] and the references quoted therein.

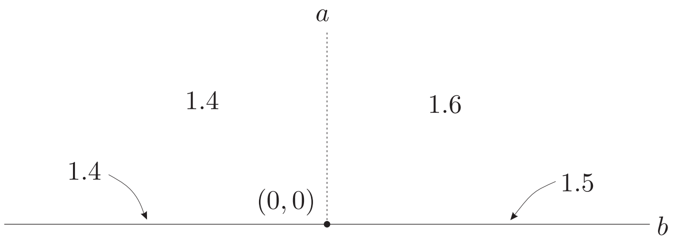



Figure 1.2: Bifurcation diagram for systems $(III)$.

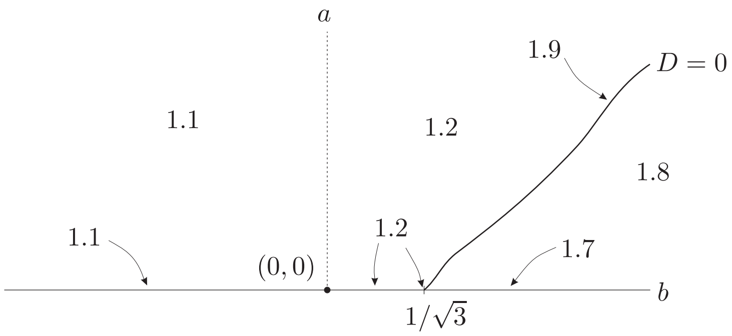



Figure 1.3: Bifurcation diagram for systems $(IV)$.

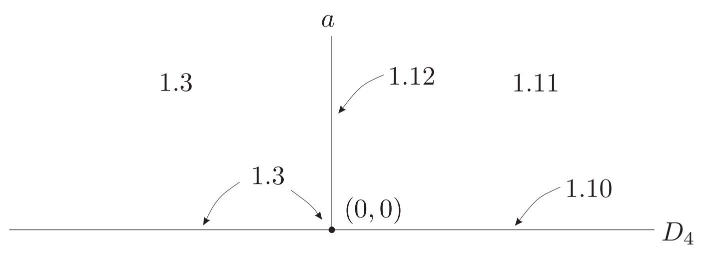



Figure 1.4: Bifurcation diagram for systems $(V)$ with $\mu > 0$.

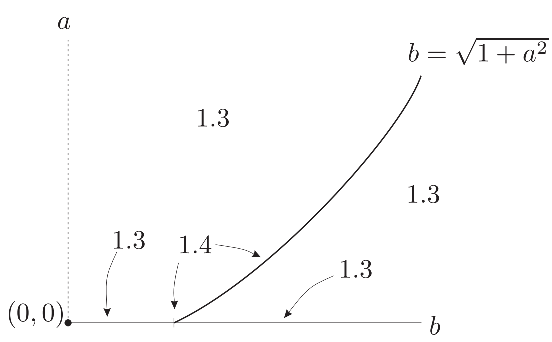



Figure 1.5: Bifurcation diagram for systems $(VI)$ with $\mu < -1/3$ and $b > 0$.

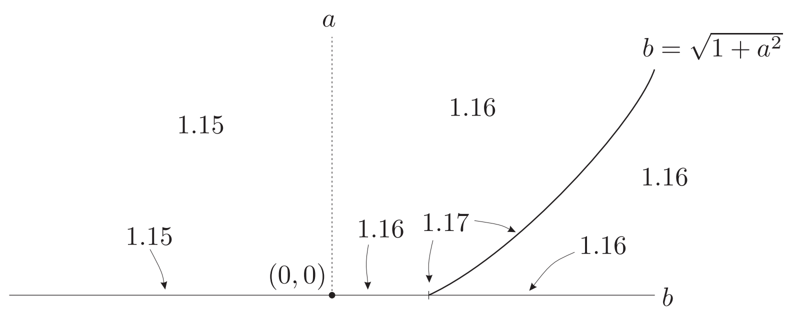



Figure 1.6: Bifurcation diagram for systems $(VI)$ with $\mu = -1/3$.

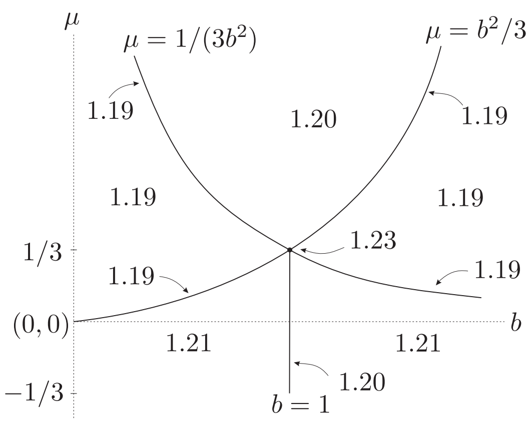



Figure 1.7: Bifurcation diagram for systems $(VI)$ with $\mu > -1/3$ and $a = 0$.

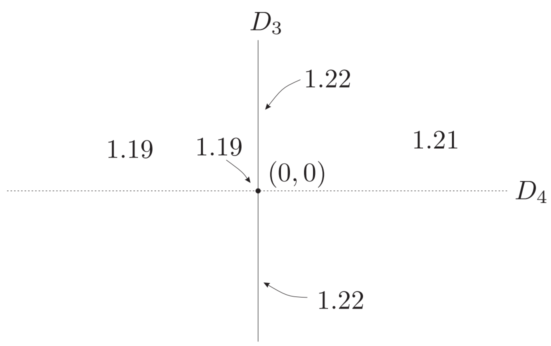



Figure 1.8: Bifurcation diagram for systems $(VI)$ with $\mu > -1/3$, $a > 0$ and $b \neq \sqrt{1+a^2}$.

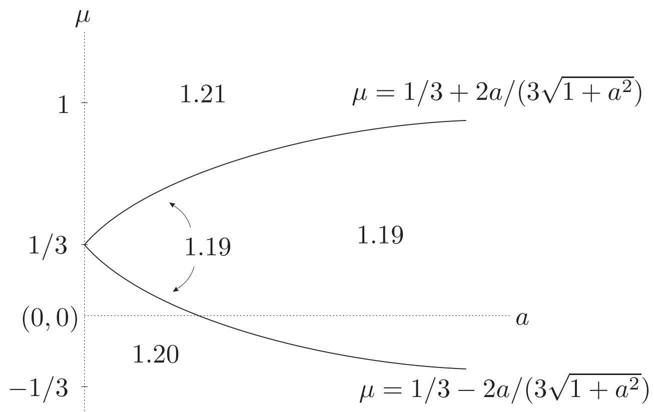



Figure 1.9: Bifurcation diagram for systems $(VI)$ with $\mu > -1/3$, $a > 0$, $b = \sqrt{1+a^2}$ and $D_4 > 0$.

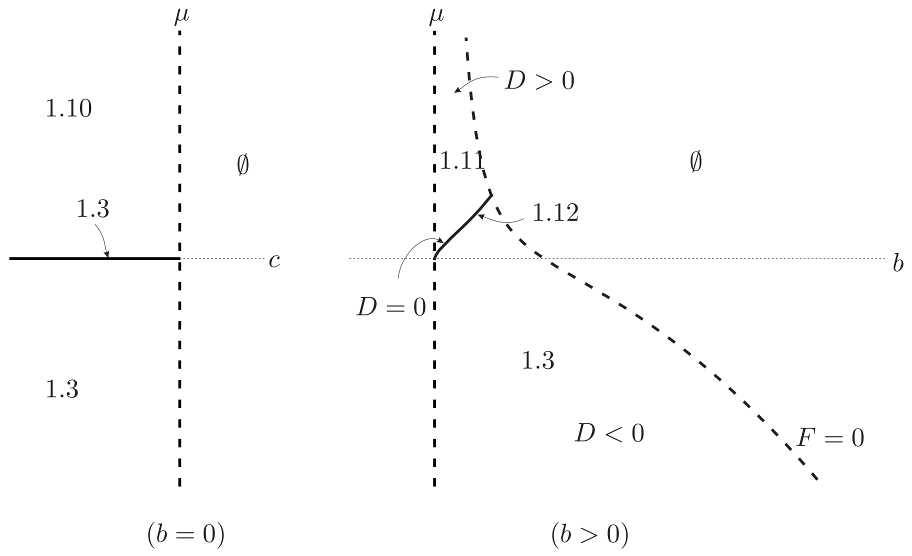



Figure 1.10: The bifurcation diagrams for systems $(XI)$ when $b = 0$ and when $b > 0$. Note that when $b > 0$ we have $c = 0$. In the figure $F = 1 - b^4 - 6b^2\mu$.

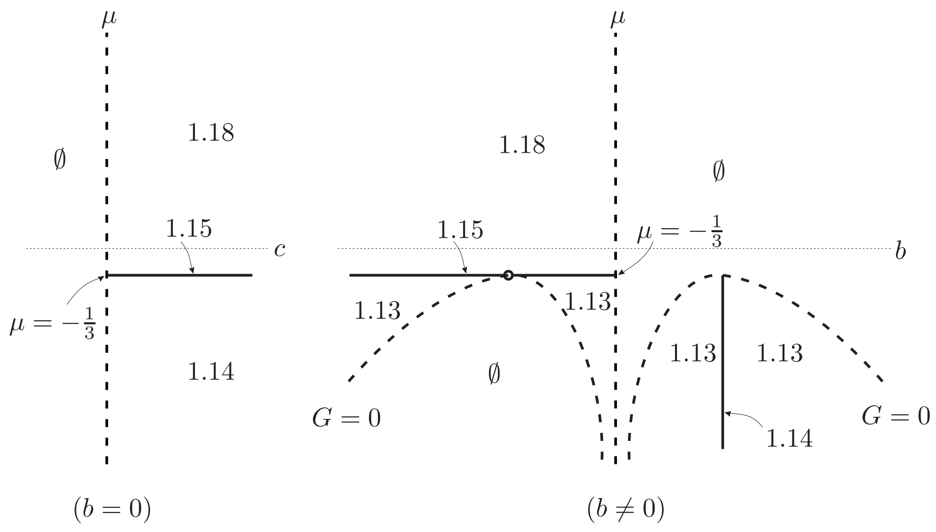



Figure 1.11: The bifurcation diagram for systems $(XII)$ when $b = 0$ and when $b \neq 0$. Note that when $b \neq 0$ we have $c = 0$. In the figure $G = 1 + b^4 + 6b^2\mu$.

# Chapter 2

# Proof of Theorem 4

In this chapter we will prove Theorem 4. In Section 2.1 we will show how we obtain the normal forms given in Theorem 4. The following sections will be the dedicated to finding the global phase portraits of the families of vector fields $(I)$–$(VI)$. We note that throughout this work when we talk about singular points we are talking only about real singular points unless it is stated otherwise.

## 2.1 Obtaining the normal forms given in Theorem 4

Using the normal forms of Theorem 4 it is easier to determine the phase portraits of the Hamiltonian linear type centers at infinity. Then using the information about the infinite singular points and the Poincaré–Hopf theorem we will get information about the finite singular points, and eventually obtain the global phase portraits of these vector fields.

Doing a linear change of variables and a rescaling of the independent variable, planar cubic homogeneous differential systems can be classified into the following ten classes, see [8]:

$$(i) \quad \begin{aligned} \dot{x} &= x(p_1x^2 + p_2xy + p_3y^2), \\ \dot{y} &= y(p_1x^2 + p_2xy + p_3y^2), \end{aligned}$$

$$(ii) \quad \begin{aligned} \dot{x} &= p_1x^3 + p_2x^2y + p_3xy^2, \\ \dot{y} &= \alpha x^3 + p_1x^2y + p_2xy^2 + p_3y^3, \end{aligned}$$

$$(iii) \quad \begin{aligned} \dot{x} &= (p_1 - 1)x^3 + p_2x^2y + p_3xy^2, \\ \dot{y} &= (p_1 + 3)x^2y + p_2xy^2 + p_3y^3, \end{aligned}$$

$$(iv) \quad \begin{aligned} \dot{x} &= p_1x^3 + (p_2 - 3\alpha)x^2y + p_3xy^2, \\ \dot{y} &= p_1x^2y + (p_2 + 3\alpha)xy^2 + p_3y^3, \end{aligned}$$

$$(v) \quad \begin{array}{l} \dot{x} = p_1x^3 + (p_2 - \alpha)x^2y + p_3xy^2 - \alpha y^3, \\ \dot{y} = \alpha x^3 + p_1x^2y + (p_2 + \alpha)xy^2 + p_3y^3, \end{array}$$

$$(vi) \quad \begin{array}{l} \dot{x} = p_1x^3 + (p_2 - 3\alpha)x^2y + p_3xy^2 + y^3, \\ \dot{y} = p_1x^2y + (p_2 + 3\alpha)xy^2 + p_3y^3, \end{array}$$

$$(vii) \quad \begin{array}{l} \dot{x} = p_1x^3 + (p_2 - 3\alpha)x^2y + p_3xy^2 - \alpha y^3, \\ \dot{y} = p_1x^2y + (p_2 + 3\alpha)xy^2 + p_3y^3, \end{array}$$

$$(viii) \quad \begin{array}{l} \dot{x} = p_1x^3 + (p_2 - 3\mu)x^2y + p_3xy^2 + y^3, \\ \dot{y} = x^3 + p_1x^2y + (p_2 + 3\mu)xy^2 + p_3y^3, \end{array} \qquad \mu \in \mathbb{R}.$$

$$(ix) \quad \begin{array}{ll} \dot{x} = p_1x^3 + (p_2 - 3\alpha\mu)x^2y + p_3xy^2 - \alpha y^3, & \mu > -1/3, \\ \dot{y} = \alpha x^3 + p_1x^2y + (p_2 + 3\alpha\mu)xy^2 + p_3y^3, & \mu \neq 1/3, \end{array}$$

$$(x) \quad \begin{array}{l} \dot{x} = p_1x^3 + (p_2 - 3\mu)x^2y + p_3xy^2 - y^3, \\ \dot{y} = x^3 + p_1x^2y + (p_2 + 3\mu)xy^2 + p_3y^3, \end{array} \qquad \mu < -1/3,$$

where $\alpha = \pm 1$. So for studying the cubic planar polynomial vector fields having only linear and cubic terms, it is sufficient to add to the above ten families of systems a linear part. This is due to the fact that the linear changes of variables that are done to obtain the classes $(i)$–$(x)$ are not affine, they are strictly linear. The following propositions define the precise forms of the vector fields that we will study.

**Proposition 9.** *Let $X$ be a cubic planar polynomial vector field having only linear and cubic terms, such that its cubic homogeneous part is given by one of the above ten forms $(i)-(x)$. Then $X$ is Hamiltonian with a Hamiltonian polynomial of degree four if and only if $p_1 = p_2 = p_3 = 0$.*

*Proof.* We will give the proof only for class $(x)$ since the other nine cases can be proved in the same way.

Let $X = (P(x, y), Q(x, y))$ be a differential system in class $(x)$ with some arbitrary linear part, that is

$$\begin{aligned} \dot{x} = P(x, y) &= ax + by + p_1x^3 + (p_2 - 3\mu)x^2y + p_3xy^2 - y^3, \\ \dot{y} = Q(x, y) &= cx + dy + x^3 + p_1x^2y + (p_2 + 3\mu)xy^2 + p_3y^3, \end{aligned}$$

where $a, b, c, d \in \mathbb{R}$. Let $H$ be its Hamiltonian polynomial of degree 4. We have

$$H_x = -Q, \quad H_y = P,$$

where $H_x$ denotes the partial derivative of $H$ with respect to $x$. To find $H$, we first integrate $H_y$ with respect to $y$ and get

$$
\begin{aligned}
H(x,y) &= \int P(x,y)\ dy + f(x) \\
&= axy + \frac{b}{2}y^2 + p_1x^3y + \frac{p_2-3\mu}{2}x^2y^2 + \frac{p_3}{3}xy^3 - \frac{1}{4}y^4 + f(x),
\end{aligned}
$$

for some real polynomial $f$. Then the derivative of $H$ with respect to $x$ is

$$
H_x(x,y) = ay + 3p_1x^2y + (p_2-3\mu)xy^2 + \frac{p_3}{3}y^3 + f'(x),
$$

where $f'(x)$ is the first derivative of the polynomial $f$. Equating $H_x$ to $-Q$ we obtain the three equations

$$
3p_1 = -p_1, \quad p_2 - 3\mu = -p_2 - 3\mu, \quad \frac{p_3}{3} = -p_3,
$$

which hold if and only if $p_1 = p_2 = p_3 = 0$. □

**Remark 10.** We note that when the parameters $p_1, p_2$ and $p_3$ are all zero, systems $(i)$ are not cubic. For this reason, we will restrict our attention to classes $(ii) - (x)$.

**Proposition 11.** *The linearized systems at the origin corresponding to each of the nine classes of Hamiltonian cubic planar polynomial vector fields having only linear and cubic homogeneous terms which have a linear type center at the origin can be chosen as*

$$
\dot{x} = ax + by, \quad \dot{y} = -\frac{a^2+\beta^2}{b}x - ay,
$$

*where* $a, b, \beta \in \mathbb{R}$ *such that* $b \neq 0$ *and* $\beta > 0$.

*Proof.* We will again give the proof only for systems $(x)$ as the remaining cases can be proved in the same way.

Let $X$ be a differential system in class $(x)$ plus a linear part and let it be Hamiltonian. Then, by Proposition 9, $X$ is

$$
\begin{aligned}
\dot{x} &= ax + by - 3\mu x^2y - y^3, \\
\dot{y} &= cx + dy + x^3 + 3\mu xy^2,
\end{aligned}
$$

for some real constants $a, b, c, d$. The eigenvalues of the linear part of system $X$ at the origin are

$$
\lambda_{1,2} = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}.
$$

In order to have a linear type center at the origin, these eigenvalues must be $\pm\beta i$, for some $\beta > 0$, see [14]. So we have

$$
\frac{a + d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} = \pm\beta i. \tag{2.1}
$$

From (2.1) we get that $a + d = 0$, and hence we obtain

$$a^2 + bc = -\beta^2.$$

We see that $b \neq 0$, otherwise the left hand side would be non–negative. Then we can solve for $c$ and get $c = -(a^2 + \beta^2)/b$.

It can be easily shown that with this choice of parameters, $X$ is Hamiltonian with the Hamiltonian

$$H(x, y) = -\frac{1}{4}(x^4 + y^4) - \frac{3\mu}{2}x^2y^2 + \frac{a^2 + \beta^2}{2b}x^2 + \frac{b}{2}y^2 + axy.$$

Since $X$ is Hamiltonian, by Proposition 3, the origin cannot be a focus, and hence it is a center. □

**Remark 12.** In all of the nine vector fields $(ii) - (x)$ we are going to study, we can assume $\alpha = 1$ because the Hamiltonian systems with $\alpha = -1$ can be obtained from those with $\alpha = 1$ simply by the linear transformation $x \mapsto -x$.

In light of the above classification, propositions and remarks, the nine systems that we are going to study become

$$\begin{array}{lll}
(ii') & \dot{x} = ax + by, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3, \\
(iii') & \dot{x} = ax + by - x^3, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + 3x^2y, \\
(iv') & \dot{x} = ax + by - 3x^2y, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + 3xy^2, \\
(v') & \dot{x} = ax + by - x^2y - y^3, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3 + xy^2, \\
(vi') & \dot{x} = ax + by - 3x^2y + y^3, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + 3xy^2, \\
(vii') & \dot{x} = ax + by - 3x^2y - y^3, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + 3xy^2, \\
(viii') & \dot{x} = ax + by - 3\mu x^2y + y^3, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3 + 3\mu xy^2, \\
(ix') & \dot{x} = ax + by - 3\mu x^2y - y^3, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3 + 3\mu xy^2, \\
(x') & \dot{x} = ax + by - 3\mu x^2y - y^3, & \dot{y} = -\dfrac{a^2+\beta^2}{b}x - ay + x^3 + 3\mu xy^2,
\end{array}$$

where $\mu < -1/3$ for systems $(x')$, whereas $\mu > -1/3$ but different from $1/3$ for systems $(ix')$. Hence these last two classes are the same except for the domains of the parameter $\mu$. In fact when $\mu = 1/3$ systems $(ix')$ clearly become systems $(v')$. Additionally in the following proposition we show that when $\mu = -1/3$, systems $(x')$ are transformed into systems $(iv')$.

**Proposition 13.** *When $\mu = -1/3$ systems $(x')$ become systems $(iv')$ via a linear transformation.*

*Proof.* Consider systems $(x')$ with $\mu = -1/3$:

$$\dot{x} = ax + by + x^2y - y^3, \quad \dot{y} = -\frac{a^2+\beta^2}{b}x - ay + x^3 - xy^2.$$

We introduce the new variables $(X, Y)$ obtained by the linear transformation

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (x-y)/\sqrt{2} \\ (x+y)/\sqrt{2} \end{pmatrix}.$$

Hence we have

$$x = (X+Y)/\sqrt{2}, \quad y = (Y-X)/\sqrt{2}.$$

Then we obtain

$$\begin{aligned} \dot{X} =& \frac{a^2-b^2+\beta^2}{2b}X + \frac{(a+b)^2+\beta^2}{2b}Y - 2X^2Y, \\ \dot{Y} =& -\frac{(a-b)^2+\beta^2}{2b}X - \frac{a^2-b^2+\beta^2}{2b}Y + 2XY^2. \end{aligned}$$

Finally, after a time rescale $dT = 2/3\ dt$, and defining

$$A = 3\frac{a^2-b^2+\beta^2}{4b}, \quad B = 3\frac{(a+b)^2+\beta^2}{4b}$$

we get the systems

$$\dot{X} = AX + BY - 3X^2Y, \quad \dot{Y} = -\frac{A^2+\beta^2}{B}X - AY + 3XY^2,$$

which are exactly systems $(iv')$. This ends the proof. □

In short, we can remove the restrictions on the parameter $\mu$ from classes $(ix')$ and $(x')$ so that not only they become the same class, but also they include classes $(iv')$ and $(v')$, and we deduce that Hamiltonian planar polynomial vector fields having only linear and cubic terms which have a linear type center at the origin can be classified into the six vector fields $(I)-(VI)$ given in Theorem 4.

**Remark 14.** Because the right hand sides of each of the differential systems $(I)$–$(VI)$ are odd functions, the global phase portraits of these systems must be symmetric with respect to the origin.

## 2.2 Global phase portraits of systems $(I)$

Systems $(I)$

$$\dot{x} = ax + by, \quad \dot{y} = -\frac{a^2+\beta^2}{b}x - ay + x^3,$$

have the Hamiltonian

$$H_1(x,y) = -\frac{1}{4}x^4 + \frac{a^2+\beta^2}{2b}x^2 + \frac{b}{2}y^2 + axy.$$

We first investigate the infinite singular points of these systems. Using (1.4), we see that in the local chart $U_1$ systems $(I)$ become

$$\begin{aligned} \dot{u} &= -v^2\left(bu^2 + 2au + \frac{a^2+\beta^2}{b}\right) + 1, \\ \dot{v} &= -v^3\left(bu + a\right). \end{aligned}$$

When $v = 0$, there are no singular points on $U_1$.

Next we will check whether the origin of the local chart $U_2$ is a singular point. In $U_2$ we use (1.5) to get

$$\begin{aligned} \dot{u} &= v^2\left(\frac{a^2+\beta^2}{b}u^2 + 2au + b\right) - u^4, \\ \dot{v} &= v^3\left(\frac{a^2+\beta^2}{b}u + a\right) - u^3v, \end{aligned} \tag{2.2}$$

and we see that the origin is a degenerate singular point. We need to do blow-ups to understand the local behavior at this point. We perform the directional blow-up $(u,v) \mapsto (u,w)$ with $w = v/u$ and have

$$\begin{aligned} \dot{u} =& u^2w^2\left(\frac{a^2+\beta^2}{b}u^2 + 2au + b\right) - u^4, \\ \dot{w} =& -uw^3(au+b). \end{aligned}$$

We eliminate the common factor $u$ between $\dot{u}$ and $\dot{w}$, and get the systems

$$\begin{aligned} \dot{u} &= uw^2\left(\frac{a^2+\beta^2}{b}u^2 + 2au + b\right) - u^3, \\ \dot{w} &= -w^3(au+b). \end{aligned} \tag{2.3}$$

When $u = 0$, the only singular point of systems (2.3) is the origin, which is also degenerate. Hence we do another blow-up $(u,w) \to (u,z)$ with $z = w/u$, eliminate the common factor $u^2$, and get

$$\begin{aligned} \dot{u} &= uz^2\left(\frac{a^2+\beta^2}{b}u^2 + 2au + b\right) - u, \\ \dot{z} &= -z^3\left(\frac{a^2+\beta^2}{b}u^2 + 3au + 2b\right) + z. \end{aligned} \tag{2.4}$$

When $u = 0$, the possible singular points of systems (2.4) are the origin and $\left(0, \pm\sqrt{1/2b}\right)$. The sign of the parameter $b$ determines the existence of these points, hence we need to analyze these points in two cases. We note that the linear part of systems (2.4) at any point $(0, z)$ on the $(u, z)$ plane is

$$\begin{pmatrix} bz^2 - 1 & 0 \\ -3az^3 & -6bz^2 + 1 \end{pmatrix}. \tag{2.5}$$

When $b < 0$, the points $\left(0, \pm\sqrt{1/2b}\right)$ are not real. Hence the only singular point is the origin which is a saddle because the eigenvalues of (2.5) at the origin are $\pm 1$. Going back through the change of variables until systems (2.2) as shown in Figure 2.1, we see that locally the origin of $U_2$ consists of two hyperbolic sectors.

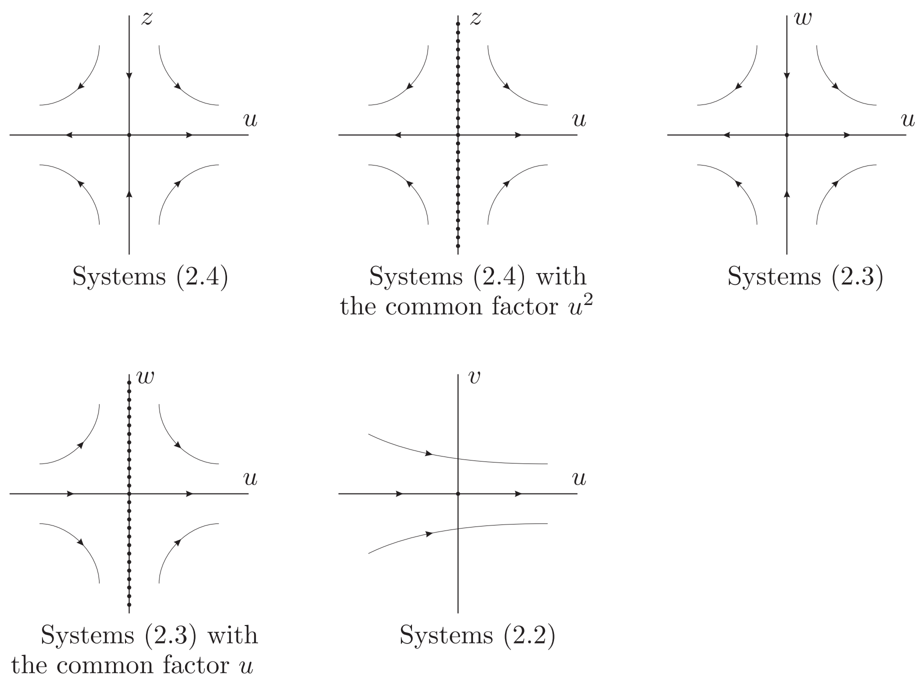
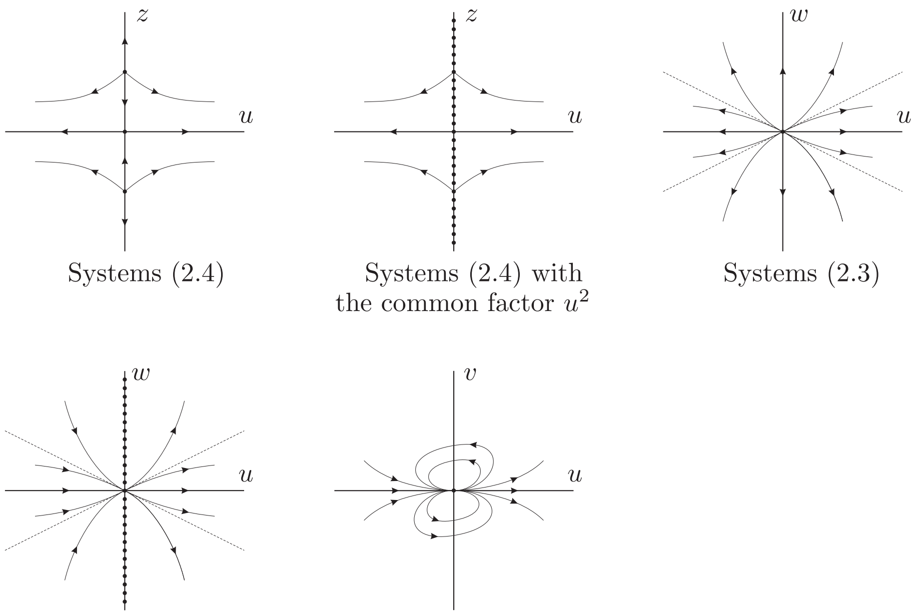



Figure 2.1: Blow-up of the origin of $U_2$ of systems $(I)$ when $b < 0$.

When $b > 0$, however, all three singular points are real. The points $\left(0, \pm\sqrt{1/2b}\right)$ are attracting nodes because the eigenvalues of (2.5) at these points are $-1/2$ and $-2$. Again, tracing back the change of variables to systems (2.2), see Figure 2.2, we see that the origin of $U_2$ has two elliptic and two parabolic sectors.

We now look at the finite singular points of systems $(I)$. Using (1.6) we




Systems (2.4) — Systems (2.4) with the common factor $u^2$ — Systems (2.3)

Systems (2.3) with the common factor $u$ — Systems (2.2)

Figure 2.2: Blow-up of the origin of $U_2$ of systems $(I)$ when $b > 0$.

find that the candidates are

$$(0,0) \quad \text{and} \quad \pm\left(-\frac{\mathrm{a}\beta}{\mathrm{b}^{3/2}}, \frac{\beta}{\sqrt{\mathrm{b}}}\right).$$

We know that the origin is a center. When $b < 0$ we do not have any other finite singular points, and we get the global phase portrait 1.1 in Figure 1.1.

On the other hand when $b > 0$ there are two more finite singular points which are saddles as the eigenvalues of the linear part of systems $(I)$ at these points is $\pm\sqrt{2}\beta$. Since there are no more finite singular points, at least one of the saddles must be on the boundary of the period annulus of the center at the origin, and by symmetry, we conclude that both saddles are on this boundary. Furthermore we observe that the Hamiltonian $H_1$ is quadratic on the $y$-axis. This means that the $y$-axis and the separatrices of a saddle can have at most two intersection points because the Hamiltonian is constant on the separatrices and $H_1|_{x=0} = h$ can have at most two roots for any $h \in \mathbb{R}$. Hence the separatrices passing through the saddles can cross the $y$-axis exactly twice while forming the boundary of the period annulus of the center at the origin. Since there are no singular points on the $y$-axis other than the origin, the global phase portraits of systems $(II)$ when $b > 0$ are topologically equivalent to 1.2 of Figure 1.1.

## 2.3 Global phase portraits of systems $(II)$

Systems $(II)$

$$\dot{x} = ax + by - x^3, \quad \dot{y} = -\frac{a^2+\beta^2}{b}x - ay + 3x^2y,$$

have the Hamiltonian

$$H_2(x,y) = -x^3y + \frac{a^2+\beta^2}{2b}x^2 + \frac{b}{2}y^2 + axy.$$

Note that we can assume $b > 0$ because the linear change $y \mapsto -y$ gives exactly the same systems with $b \mapsto -b$.

Again we will first find the infinite singular points. In the local chart $U_1$ systems $(II)$ are

$$\begin{aligned}\dot{u} &= -v^2\left(bu^2 + 2au + \frac{a^2+\beta^2}{b}\right) + 4u,\\ \dot{v} &= -v^3(bu+a) + v.\end{aligned}$$

When $v = 0$, only the origin of $U_1$ is singular. The eigenvalues of the linear part of systems $(II)$ at the origin are 4 and 1, meaning that it is a repelling node.

Next, we should check the origin of $U_2$, in which systems $(II)$ become

$$\begin{aligned}\dot{u} &= v^2\left(\frac{a^2+\beta^2}{b}u^2 + 2au + b\right) - 4u^3,\\ \dot{v} &= v^3\left(\frac{a^2+\beta^2}{b}u + a\right) - 3u^2v.\end{aligned} \tag{2.6}$$

We see that only the origin is singular and it is degenerate, hence we need blow-up to understand the local behavior at this point. Doing the blow-up $(u,v) \mapsto (u,w)$ with $w = v/u$ and eliminating the common factor $u$ we get the systems

$$\begin{aligned}\dot{u} &= uw^2\left(\frac{a^2+\beta^2}{b}u^2 + 2au + b\right) - 4u^2,\\ \dot{w} &= -w^3(au+b) - uw.\end{aligned} \tag{2.7}$$

When $u = 0$, the only singular point of systems (2.7) is the origin, which is again degenerate. So, we do another blow-up $(u,w) \mapsto (u,z)$ with $z = w/u$, eliminate the common factor $u$, and obtain

$$\begin{aligned}\dot{u} &= u^2z^2\left(\frac{a^2+\beta^2}{b}u^2 + 2au + b\right) - 4u,\\ \dot{z} &= -uz^3\left(\frac{a^2+\beta^2}{b}u^2 + 3au + 2b\right) + 5z.\end{aligned} \tag{2.8}$$

When $u = 0$, the only singular point of systems (2.8) is the origin, and it is a saddle. We trace the change of variables back to systems (2.6) as shown in Figure 2.3, and we find out that the origin of $U_2$ is an attracting node.

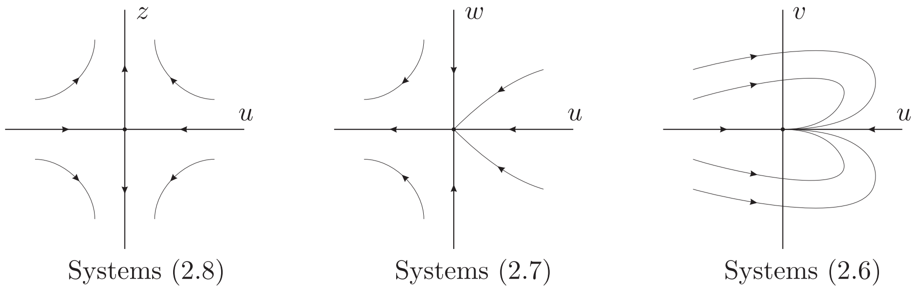



Figure 2.3: Blow-up of the origin of $U_2$ of systems $(II)$.

This finishes the study of the infinite singular points of systems $(II)$, and we now focus on the finite singular points. We know that the origin is singular. In addition systems $(II)$ have the finite singular points

$$\pm\left(\frac{\sqrt{2a+\sqrt{4a^2+3\beta^2}}}{\sqrt{3}}, -\frac{(a-\sqrt{4a^2+3\beta^2})\sqrt{2a+\sqrt{4a^2+3\beta^2}}}{3\sqrt{3}b}\right).$$

The eigenvalues of the linear part of systems $(II)$ at these points is

$$\pm\frac{2\sqrt{4a^2+3\beta^2+2a\sqrt{4a^2+3\beta^2}}}{\sqrt{3}},$$

which means that they are saddles as $\beta > 0$.

Now we will determine the global phase portraits according to this local information. The two saddles must be on the boundary of the period annulus of the center at the origin due to the symmetry of the systems. Also there are no singular points other than the origin on the coordinate axes, on either of which the Hamiltonian $H_2$ is quadratic. Therefore by the same argument used for systems $(I)$, this means that the separatrices passing through saddles cannot cross the coordinate axes anymore. Hence we obtain a global phase portrait topologically equivalent to 1.3 of Figure 1.1.

## 2.4 Global phase portraits of systems $(III)$

Systems $(III)$

$$\dot{x} = ax + by - 3x^2y + y^3, \tag{2.9a}$$

$$\dot{y} = -\frac{a^2+\beta^2}{b}x - ay + 3xy^2, \tag{2.9b}$$

have the Hamiltonian

$$H_3(x,y) = \frac{y^4}{4} - \frac{3}{2}x^2y^2 + \frac{a^2+\beta^2}{2b}x^2 + \frac{b}{2}y^2 + axy.$$

In $U_1$ these systems become

$$\begin{aligned} \dot{u} &= -v^2\left(bu^2 + 2au + \frac{a^2+\beta^2}{b}\right) - u^2(u^2-6), \\ \dot{v} &= -v^3(bu+a) - uv(u^2-3). \end{aligned} \tag{2.10}$$

When $v = 0$, there are three singular points on $U_1$: $(0,0), (\pm\sqrt{6}, 0)$. The linear part of systems (2.10) is

$$\begin{pmatrix} -4u(u^2-3) & 0 \\ 0 & -u(u^2-3) \end{pmatrix}.$$

Hence the singular points $(\sqrt{6}, 0)$ and $(-\sqrt{6}, 0)$ are attracting and repelling nodes, respectively.

At the origin, however, the linear part is zero. Therefore to understand the local behavior we do the blow-up $(u,v) \mapsto (u,w)$ with $w = v/u$. After eliminating the common factor $u$ between $\dot{u}$ and $\dot{w}$, we obtain the systems

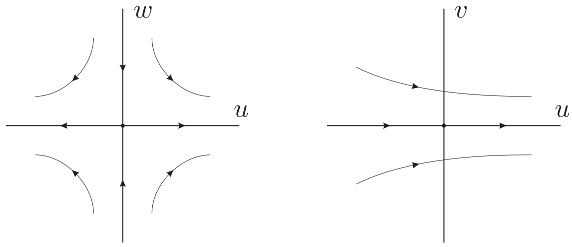
$$\begin{aligned} \dot{u} &= -uw^2\left(bu^2 + 2au + \frac{a^2+\beta^2}{b}\right) - u(u^2-6), \\ \dot{w} &= w^3\left(au + \frac{a^2+\beta^2}{b}\right) - 3w. \end{aligned} \tag{2.11}$$

When $u = 0$, besides the origin systems (2.11) can have the singular points $\left(0, \pm\sqrt{3b/(a^2+\beta^2)}\right)$. The linear part of systems (2.11) at the points $(0,w)$ is

$$\begin{pmatrix} -\frac{a^2+\beta^2}{b}w^2 + 6 & 0 \\ aw^3 & 3\frac{a^2+\beta^2}{b}w^2 - 3 \end{pmatrix}.$$

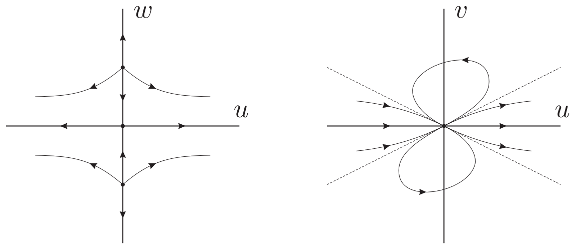
When $b < 0$, we see that $\left(0, \pm\sqrt{3b/(a^2+\beta^2)}\right)$ are not real, hence the only singular point is the origin, which is a saddle. As a result, it is shown in Figure 2.4 that the origin of $U_1$ consists of two hyperbolic sectors.

When $b > 0$, all three singular points are real. In addition to the saddle at the origin, the points $\left(0, \pm\sqrt{3b/(a^2+\beta^2)}\right)$ are repelling nodes. Consequently, this time the origin of $U_1$ has two elliptic sectors and two parabolic sectors, see Figure 2.5.




Systems (2.11) Systems (2.10)

Figure 2.4: Blow-up of the origin of $U_1$ of systems $(III)$ when $b < 0$.




Systems (2.11) Systems (2.10)

Figure 2.5: Blow-up of the origin of $U_1$ of systems $(III)$ when $b > 0$.

We have studied systems $(III)$ on $U_1$, and we now look at the origin of $U_2$, in which systems $(III)$ are written as

$$\begin{aligned} \dot{u} &= v^2 \left( \frac{a^2 + \beta^2}{b} u^2 + 2au + b \right) - 6u^2 + 1, \\ \dot{v} &= v^3 \left( \frac{a^2 + \beta^2}{b} u + a \right) - 3uv. \end{aligned}$$

We see that the origin of $U_2$ is not a singular point. Hence all the infinite singular points are in $U_1$ and $V_1$.

Now we analyze the finite singular points of systems $(III)$. However, contrary to systems $(I)$ and $(II)$, the explicit expressions for the finite singular points of systems $(III)$ in terms of the parameters $a, b, \beta$ are complicated, and therefore it is hard to analyze their existence and their local phase portraits. For this reason we take a different approach, which we will also use in determining the global phase portraits of the rest of the vector fields: We first find the maximum number of finite singular points allowed by these systems. Then, using the Poincaré Formula (see Theorem 2) for the index of a singular point of a planar vector field, we count the indices of the singular points of the systems that we have found up to this point on the Poincaré sphere, namely the origins of the local charts $U_3$ and $V_3$ and the infinite singular points. The next step is to determine the possible number and local phase portraits of the finite singular points of the systems using the *Poincaré-Hopf Theorem* for vector fields in the 2–dimensional sphere.

**Theorem 15** (Poincaré–Hopf Theorem)**.** *For every vector field on the sphere* $\mathbb{S}^2$ *with a finite number of singular points, the sum of the indices of the singular points is* 2*.*

For details about Theorem 15 see page 177 of [14].

We note that singular points with index 0 are hard to detect in our method as they do not contribute to the total index of the singular points of the vector fields on the Poincaré sphere. To overcome this difficulty we present the following lemma, but first we make a remark and give some definitions.

**Remark 16.** Nilpotent singular points of Hamiltonian planar polynomial vector fields are either saddles, centers, or cusps (for more details see Chapters 2 and 3 of [14], specifically Sections 2.6 and 3.5).

From Theorem 2 the following result follows easily.

**Corollary 17.** *The index of a saddle, a center and a cusp are* $-1$*,* 1 *and* 0*, respectively.*

We define *energy levels* of a Hamiltonian vector field as the level curves of its Hamiltonian; and a hyperbolic saddle with a loop and a center inside the loop as in Figure 2.6 will be called a *center–loop*.

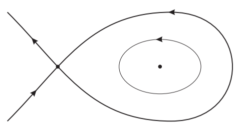


Figure 2.6: A center–loop.

Using the next lemma we will be able to say that the only singular points of systems $(III)$–$(VI)$ which have index zero are cusps, and that if these systems have global phase portraits with cusps, then in a neighborhood of those systems they must have systems with more finite singular points than the ones having cusps. Sometimes these provide contradictions in the sense that those systems will be forced to have more singular points than the ones that they can have.

**Lemma 18.** *Let* $X_0$ *be a real Hamiltonian planar polynomial vector field having only linear and cubic terms. Then* $X_0$ *can be written as*

$$\dot{x} = a_{10}x + a_{01}y + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3,$$
$$\dot{y} = b_{10}x - a_{10}y + b_{30}x^3 - 3a_{30}x^2y - a_{21}xy^2 - \frac{1}{3}a_{12}y^3.$$

*Suppose that $p$ is an isolated singular point of $X_0$ different from the origin. Then there is a perturbation $X_\varepsilon$ of $X_0$ such that $X_\varepsilon$ is also a real Hamiltonian planar polynomial vector field having only linear and cubic terms, and that $p$ is a singular point of $X_\varepsilon$. In addition if $a_{10}^2 + a_{01}b_{10} < 0$, that is if the origin of $X_0$ is a linear type center, then the following statements hold:*

(a) *If $p$ is non–elementary, then it is nilpotent.*

(b) *If $p$ is a non–elementary singular point of $X_0$, then it is an elementary singular point of $X_\varepsilon$ with $\varepsilon \neq 0$.*

(c) *If $p$ is a cusp of $X_0$, then for $\varepsilon \neq 0$ small enough the local phase portrait of $X_\varepsilon$ at $p$ is a center–loop.*

*Proof.* Without loss of generality we can assume that $p = (0, y_0)$, otherwise doing a rotation of the coordinates we can get its $x$-coordinate to be zero. Then we can define $X_\varepsilon$ as

$$\begin{aligned} \dot{x} &= a_{10}x + a_{01}y + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ \dot{y} &= b_{10}x - a_{10}y + b_{30}x^3 - 3a_{30}x^2y - a_{21}xy^2 - \frac{1}{3}a_{12}y^3 + \varepsilon x. \end{aligned} \tag{2.12}$$

It is easy to check that $X_\varepsilon$ is Hamiltonian with the Hamiltonian polynomial

$$H_\varepsilon = \frac{a_{03}y^4 - b_{30}x^4}{4} + a_{30}x^3y + \frac{a_{21}x^2y^2}{2} + \frac{a_{12}xy^3}{3} + \frac{a_{01}y^2 - (b_{10}+\varepsilon)x^2}{2} + a_{10}xy,$$

and that $p = (0, y_0)$ is a singular point of $X_\varepsilon$.

Assume that $a_{10}^2 + a_{01}b_{10} < 0$. Note that this condition implies $a_{01} \neq 0$.

We first prove $(a)$. At $(0, y_0)$ system $X_\varepsilon$ becomes

$$\dot{x} = y_0(a_{01} + a_{03}y_0^2), \quad \dot{y} = -y_0(a_{10} + \frac{1}{3}a_{12}y_0^2),$$

whereas $M_\varepsilon$, the linear part of $X_\varepsilon$ at a point, is

$$\begin{pmatrix} a_{10} + a_{12}y_0^2 & a_{01} + 3a_{03}y_0^2 \\ b_{10} - a_{21}y_0^2 + \varepsilon & -a_{10} - a_{12}y_0^2 \end{pmatrix}. \tag{2.13}$$

Since $y_0 \neq 0$, we have $(0, y_0)$ degenerate only if

$$a_{01} + a_{03}y_0^2 = a_{01} + 3a_{03}y_0^2 = 0, \tag{2.14}$$

which requires $a_{03} = 0$. However, since $a_{01} \neq 0$, equation (2.14) cannot be satisfied. Therefore we conclude that a non–elementary singular point of $X_\varepsilon$ must be nilpotent.

Now we prove $(b)$. Assume that $(0, y_0)$ is a non–elementary singular point of $X_0$. We will prove that the eigenvalues of the linear part of $X_\varepsilon$, with $\varepsilon \neq 0$, are different from zero.

The characteristic polynomial at a singular point of $X_\varepsilon$ is of the form $\lambda^2+\det(M_\varepsilon)$. So the eigenvalues $\lambda$ of the linear part of $X_\varepsilon$ at a singular point are $\pm\sqrt{-\det(M_\varepsilon)}$. Since we have already assumed that the eigenvalues of $M_0$ at $(0, y_0)$ are zero, the only nonzero terms in the determinant of $M_\varepsilon$ at the same point are those having a factor of $\varepsilon$. Hence the eigenvalues of $M_\varepsilon$ at $(0, y_0)$ are

$$\lambda = \pm\sqrt{\varepsilon(3a_{03}y_0^2 + a_{01})}, \tag{2.15}$$

where $\varepsilon \neq 0$. Then the eigenvalues are zero only if we have (2.14), but we have already shown that it is not possible. Therefore $(0, y_0)$ is an elementary singular point of $X_\varepsilon$.

Finally we prove statement $(c)$. Assume that $(0, y_0)$ is a cusp of $X_0$. First we note that for $\varepsilon \neq 0$, due to (2.15), by a proper choice of sign of $\varepsilon$ we can assume that $(0, y_0)$ is a saddle of $X_\varepsilon$. In addition, since $(0, y_0)$, which was a cusp with index 0, is now a saddle having index -1, new singular points must emerge in a neighborhood $W_\varepsilon$ of $(0, y_0)$ to keep the total index of the vector field fixed. Because of the symmetry of the system, there can be at most three new singular points in $W_\varepsilon$ so that the total number of finite singular points does not exceed 9. Since $X_\varepsilon$ is Hamiltonian, these singular points can only be saddles, centers or cusps. Therefore in $W_\varepsilon$ there are additionally to the saddle at $(0, y_0)$ either $(i)$ one center, $(ii)$ one center and one cusp, $(iii)$ one center and two cusps, or $(iv)$ two centers and one saddle. Our claim is that $(i)$ is the only realizable case and that $(0, y_0)$ is the saddle of a center–loop.

Because of the continuity of system $X_\varepsilon$ with respect to $\varepsilon$, the new separatrices of $(0, y_0)$ must be arbitrarily close to $(0, y_0)$ for small $\varepsilon$, therefore they cannot go to any other singular point outside $W_\varepsilon$. Note that in all the possibilities $(i) - (iv)$, there exists a center with $(0, y_0)$ on the boundary of its period annulus. Then we see that $(0, y_0)$ cannot be on the boundary of the period annulus of the center at the origin. Otherwise we could find a straight line $l$ through the origin intersecting the boundary of the period annulus of the new center twice, which would, in fact, have at least three intersection points with the separatrices of $(0, y_0)$, the other being on the boundary of the period annulus of the center at the origin, see Figure 2.7. Then, due to the symmetry of the system with respect to the origin, there would be six points on $l$ all of which are on the same energy level. Clearly this is not possible since the Hamiltonian $H_\varepsilon$ is a quartic polynomial.

If $(0, y_0)$ is not on the boundary of the period annulus of the center at the origin, then there must be other saddles on that boundary. This means that system $X_0$ has at least five finite singular points. This immediately eliminates the possibilities $(iii)$ and $(iv)$, otherwise the number of finite singular points exceeds the maximum of 9. Furthermore, by the same arguments used for $(0, y_0)$, the cusp in case $(ii)$ would also lead to the existence

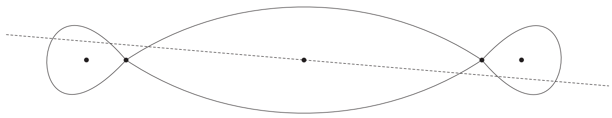


Figure 2.7: The straight line through the origin intersects the separatrices six times.

of more singular points. Therefore we dismiss case (*ii*) also, proving our claim. □

Now we apply this method step by step to systems ($III$) to find their global phase portraits. We first find the maximum number of finite singular points. To do this, we equate (2.9a) to 0, solve for $x$ and get

$$x_{1,2} = \frac{a \pm \sqrt{a^2 + 12by^2 + 12y^2}}{6y}. \tag{2.16}$$

Note that when $y = 0$, (2.9b) is zero if and only if $x = 0$. But since we are looking for finite singular points other than the origin, we can assume that $y \neq 0$. Then, if we substitute (2.16) into (2.9b) we obtain

$$\dot{y}_{1,2} = -\frac{a^3 + 3aby^2 + a\beta^2 \pm (a^2 + \beta^2 - 3by^2)\sqrt{a^2 + 12by^2 + 12y^2}}{6by},$$

where $\dot{y}_1$ and $\dot{y}_2$ denote $\dot{y}$ with $x_1$ and $x_2$ substituted, respectively. Then the maximum number of roots of the product $\dot{y}_1\dot{y}_2$ will give us an upper bound for the number of finite singular points. So we multiply $\dot{y}_1$ and $\dot{y}_2$ and obtain

$$-3y^6 + \frac{2a^2 + 2\beta^2 - 3b^2}{b}y^4 - \frac{(a^2 + \beta^2)(a^2 + \beta^2 - 6b^2)}{3b^2}y^2 - \frac{\beta^2(a^2 + \beta^2)}{3b}. \tag{2.17}$$

We see that (2.17) cannot be identically zero. This means that all the finite singular points of systems ($III$) are isolated, and there are at most six of them. In fact, if we multiply (2.17) by $3b^2$ and replace $y^2$ by $z$, we get the cubic polynomial

$$-9b^2z^3 + 3b(2a^2 + 2\beta^2 - 3b^2)z^2 - (a^2 + \beta^2)(a^2 + \beta^2 - 6b^2)z - b\beta^2(a^2 + \beta^2). \tag{2.18}$$

In order that (2.17) has six real roots, all the roots of (2.18) must be positive. To find the maximum number of positive roots of the polynomial (2.18) we use *Descartes' rule of signs*:

**Theorem 19.** *The number of positive roots of a real polynomial is either equal to the number of sign differences between consecutive nonzero coefficients, or is less than it by a multiple of 2, when the terms of the polynomial are placed in descending or ascending order of exponents.*

The terms of (2.18) are already ordered correctly, so there must be three changes of sign between its coefficients to have all three roots positive. When $b > 0$, since the constant term and the coefficient of $z^3$ are both negative, there cannot be three changes of sign. When $b < 0$, the constant term is positive and the coefficient of $z^3$ is negative. Thus to have three positive roots the coefficients of $z$ and $z^2$ must be negative and positive, respectively. So we must have

$$a^2 + \beta^2 - 6b^2 > 0 \quad \text{and} \quad 2a^2 + 2\beta^2 - 3b^2 < 0,$$

which is not possible since

$$2a^2 + 2\beta^2 - 3b^2 > a^2 + \beta^2 - 6b^2.$$

Therefore polynomial (2.18) cannot have three positive roots, and as a result systems $(III)$ have at most four finite singular points other than the origin.

We continue by counting the indices of the singular points. As we mentioned earlier, singular points with index 0 cannot be detected in the index summation. Furthermore by statement $(a)$ of Lemma 18, Remark 16 and Corollary 17, we know that the only singular points of systems $(III)$ having index 0 are cusps. Therefore we need to obtain some other information about possible cusps of these systems. Since a cusp is a non–elementary singular point, we will check if such singular points exist. We claim that systems $(III)$ have at most two non–elementary singular points. To prove it we need to show that (2.9a), (2.9b) and the determinant of the linear part of systems $(III)$ cannot simultaneously vanish at more than two points. We note that the linear part of systems $(III)$ is

$$M_3 = \begin{pmatrix} a - 6xy & b - 3x^2 + 3y^2 \\ 3y^2 - \dfrac{a^2+\beta^2}{b} & -a + 6xy \end{pmatrix}.$$

We compute the Gröbner basis for these three polynomials and obtain a set of sixteen polynomials, but due to their length we will only present the ones that we shall use. We equate these sixteen polynomials to zero and see that to determine the number of their solutions it is enough to consider only two equations, which are

$$a^2 - 6b^2 + \beta^2 - 9by^2 = 0, \quad (2.19)$$

$$162b^3(3b^2+\beta^2)x - a\big(2(a^2+\beta^2)^2 + b^2(279b^2 - 57a^2 + 24\beta^2)\big)y = 0. \quad (2.20)$$

From equation (2.19) we get that there are at most two solutions for $y$ since $b \neq 0$. In addition, equation (2.20) is only linear in $x$ and its coefficient is different from zero. Therefore we deduce that the determinant of $M_3$ can be zero at most in two singular points of systems $(III)$, proving our claim. As a result systems $(III)$ have at most two cusps.

Having established the above claim, we continue counting the indices of the singular points. Since the infinite singular points depend on the sign of the parameter $b$, we need to investigate those two cases separately.

When $b < 0$, the infinite singular points on the Poincaré sphere are $(\pm\sqrt{6}, 0)$ and $(0, 0)$ in $U_1$, and also the corresponding points on $V_1$. The origins of $U_1$ and $V_1$ consist of two hyperbolic sectors, hence, by Theorem 2, have index 0. The other four infinite singular points are nodes, hence each has index 1. Among the finite singular points we only know that the origins of $U_3$ and $V_3$ are centers with index 1. Hence, the known singular points have total index 6 on the Poincaré sphere. By Theorem 15, the remaining finite singular points, if any, must have total index -4. Thus, on the Poincaré disk, the finite singular points other than the origin must have total index -2. By statement $(a)$ of Lemma 18, the finite singular points are either elementary or nilpotent, hence they are either saddles, centers or cusps due to Remark 16. We remind that systems $(III)$ can have at most four finite singular points other than the origin, and at most two of them are nilpotent. Then according to Corollary 17, there must be either just two saddles, or two saddles and two cusps so that they have total index -2. Now we analyze these two cases.

If there were two saddles and two cusps, due to statement $(c)$ of Lemma 18, after a small perturbation, systems $(III)$ would have four saddles and two centers, which is more than the maximum number of finite singular points allowed by the systems. Therefore, when $b < 0$, systems $(III)$ have only two finite saddles. Due to the symmetry of the vector fields, the saddles are located on the boundary of the period annulus of the center at the origin. On the $x$-axis the Hamiltonian $H_3$ is quadratic, hence the separatrices through the saddles cannot cross the $x$-axis any more, therefore we get the global phase portrait 1.4 of Figure 1.1.

When $b > 0$, on the other hand, the infinite singular points at the origins of $U_1$ and $V_1$ consist of two elliptic and two parabolic sectors, hence each have index 2. Together with the remaining nodes at infinity and the centers at the origins of $U_3$ and $V_3$, they have total index 10 on the Poincaré sphere. So, on the Poincaré disk, the remaining finite singular points must have total index -4. Since there are at most four more singular points, there must be four saddles due to Corollary 17. Then on the boundary of the period annulus of the center at the origin there can be either two or four saddles. Thus we again have two cases to analyze.

Suppose first that all of the four saddles are on the boundary of the period annulus of the center at the origin. Since $b > 0$, the flow around the center at the origin is clockwise. Taking into account the fact that the separatrices through the saddles cannot cross the $x$-axis anymore, the global phase portrait 1.5 shown in Figure 1.1 is obtained. We note that when we set $a = 0$ and $b = \beta = 1$, we get a global phase portrait topologically equivalent to the portrait 1.5 of Figure 1.1.

Second, assume there are only two saddles on the boundary of the period annulus of the center. We claim that these saddles cannot be connected to any of the other saddles. If this were the case, that is if one of these saddles were connected to another saddle $p$ which is not on the boundary of the period annulus of the center, then, on the quadrant of the $xy$-plane where $p$ lies, a straight line $l$ through the origin passing sufficiently close to $p$ would have at least three intersection points with the separatrices which are on the same energy level as $p$ (one with the boundary of the period annulus of the center and at least two with the separatrices of $p$), see Figure 2.8 for an illustration. Taking into account the symmetry of the vector fields with respect to the origin, the straight line $l$ would have six intersection points with the separatrices on the same energy level as $p$, see Figure 2.8. This means that on the straight line $l$, which could be defined by $y = cx$ for some real number $c$, the equation $H_3 = H(p)$ would have six solutions. But this is not possible as $H_3$ is quartic. Therefore the saddles on the boundary of the period annulus of the center has to be connected with the infinite singular points. Due to the fact that the separatrices through these saddles cannot cross the $x$-axis anymore and to the clockwise flow around the origin, we get the global phase portrait 1.6 of Figure 1.1. A phase portrait in this case is realized when $-a = b = \beta = 1$.

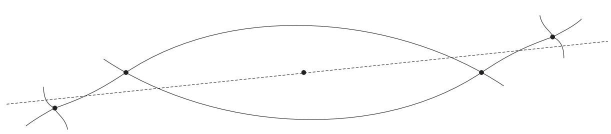


Figure 2.8: The straight line through the origin intersects the separatrices six times.

## 2.5 Global phase portraits of systems $(IV)$

Systems $(IV)$ are defined by the equations

$$\dot{x} = ax + by - 3x^2y - y^3, \tag{2.21a}$$

$$\dot{y} = -\frac{a^2 + \beta^2}{b}x - ay + 3xy^2, \tag{2.21b}$$

and they have the Hamiltonian

$$H_4(x,y) = -\frac{y^4}{4} - \frac{3}{2}x^2y^2 + \frac{a^2+\beta^2}{2b}x^2 + \frac{b}{2}y^2 + axy.$$

In the local chart $U_1$ systems $(IV)$ are written as

$$\begin{aligned} \dot{u} &= -v^2\left(bu^2 + 2au + \frac{a^2+\beta^2}{b}\right) + u^2(u^2+6), \\ \dot{v} &= -v^3(bu+a) + uv(u^2+3). \end{aligned} \tag{2.22}$$

When $v = 0$, the only singular point on $U_1$ is $(0,0)$, which is degenerate. Therefore to study the local behavior at the origin of $U_1$ we do the blow-up $(u,v) \mapsto (u,w)$ with $w = v/u$. Eliminating the common factor $u$ between $\dot{u}$ and $\dot{w}$, we obtain

$$\begin{aligned} \dot{u} &= -uw^2\left(bu^2 + 2au + \frac{a^2+\beta^2}{b}\right) + u(u^2+6), \\ \dot{w} &= w^3\left(au + \frac{a^2+\beta^2}{b}\right) - 3w. \end{aligned} \tag{2.23}$$

When $u = 0$, systems (2.23) have three possible singular points: $(0,0)$, $\left(0, \pm\sqrt{3b/(a^2+\beta^2)}\right)$. Note that the linear part of systems (2.23) at the points $(0,w)$ is

$$\begin{pmatrix} -\frac{a^2+\beta^2}{b}w^2 + 6 & 0 \\ aw^3 & 3\frac{a^2+\beta^2}{b}w^2 - 3 \end{pmatrix}.$$

We see that when $b < 0$, the points $\left(0, \pm\sqrt{3b/(a^2+\beta^2)}\right)$ are not real, hence the only singular point in $U_1$ is the origin, which is a saddle. We see that the blow-up analysis gives the same result as in the case $b < 0$ of systems $(III)$, hence the origin of $U_1$ has two hyperbolic sectors, see Figure 2.4.

When $b > 0$ all three solutions are real. In addition to the saddle at the origin, the points $\left(0, \pm\sqrt{3b/(a^2+\beta^2)}\right)$ are repelling nodes. The blow-up of the origin gives the same information as in the case $b > 0$ of systems $(III)$, and we see that the origin of $U_1$ in this case has two elliptic sectors and two parabolic ones, see Figure 2.5.

In $U_2$ systems $(IV)$ are expressed as

$$\begin{aligned} \dot{u} &= v^2\left(\frac{a^2+\beta^2}{b}u^2 + 2au + b\right) - 6u^2 - 1, \\ \dot{v} &= v^3\left(\frac{a^2+\beta^2}{b}u + a\right) - 3uv. \end{aligned}$$

The origin of $U_2$ is not a singular point. Thus the only infinite singular points on the Poincaré sphere are the origins of $U_1$ and $V_1$.

Now we investigate the finite singular points of these systems. We use the same method that we explained in Section 2.4 in the study of systems $(III)$. We first find the maximum number of finite singular points allowed by the systems. Equating (2.21a) to zero and solving for $x$ gives

$$x_{1,2} = \frac{a \pm \sqrt{a^2 + 12by^2 - 12y^4}}{6y}. \tag{2.24}$$

Note that when $y = 0$ we have $x = 0$ due to (2.21b), so we can assume $y \neq 0$. Then we substitute (2.24) into (2.21b) and obtain

$$\dot{y}_{1,2} = -\frac{a^3 + 3aby^2 + a\beta^2 \pm (a^2 + \beta^2 - 3by^2)\sqrt{a^2 + 12by^2 + 12y^2}}{6by}.$$

Then the product $\dot{y}_1\dot{y}_2$ is

$$3y^6 - \frac{2a^2 + 2\beta^2 + 3b^2}{b}y^4 + \frac{(a^2+\beta^2)(a^2+\beta^2+6b^2)}{3b^2}y^2 - \frac{\beta^2(a^2+\beta^2)}{3b}. \tag{2.25}$$

We see that (2.25) is not identically zero, hence it has at most six real roots. Consequently systems $(IV)$ have at most six finite singular points except the origin.

The next step is to count the indices of the finite and infinite singular points of systems $(IV)$ on the Poincaré sphere. But we need to be careful about the finite singular points which have index 0. Remembering that only non–elementary singular points of systems $(IV)$ can have index 0, we will show that there are at most two non–elementary singular points. Note that the linear part $M_4$ of systems $(IV)$ is

$$M_4 = \begin{pmatrix} a - 6xy & b - 3x^2 - 3y^2 \\ 3y^2 - \dfrac{a^2+\beta^2}{b} & -a + 6xy \end{pmatrix}.$$

As we did for systems $(III)$, we consider the three polynomials (2.21a), (2.21b) and the determinant of $M_4$. We compute their Gröbner basis and obtain sixteen polynomials again. We equate these polynomials to zero and see that the following two are enough for our analysis:

$$a^2 + 6b^2 + \beta^2 - 9by^2 = 0, \tag{2.26}$$

$$162b^3(3b^2 - \beta^2)x - a\big(2(a^2+\beta^2)^2 + b^2(279b^2 + 57a^2 - 24\beta^2)\big)y = 0. \tag{2.27}$$

Observe that the coefficient of $y^2$ in (2.26) is different from zero, therefore there are at most two different $y$–coordinates for the non–elementary singularities. Since (2.27) is linear in $x$, this means that unless the coefficient of

$x$ in (2.27) is zero, there are at most two non–elementary singular points. When the coefficient of $x$ in (2.27) is zero, that is $\beta^2 = 3b^2$, equation (2.27) becomes

$$a\big(2(a^2+\beta^2)^2 + b^2(207b^2+57a^2)\big)y = 0.$$

This yields $a = 0$ because we have $y \neq 0$. Then equations (2.26) and (2.21a) become

$$9b(b-y^2) = 0, \quad y(b-3x^2-y^2) = 0,$$

respectively. This system also has at most two solutions, namely $(0, \pm\sqrt{b})$. This shows that $M_4$ can be zero in at most two points. As a result, by statement $(a)$ of Lemma 18, we conclude that systems $(IV)$ have at most two nilpotent singular points.

We will now determine the number and local phase portraits of the finite singular points by considering the indices of all the singular points of systems $(IV)$ on the Poincaré sphere.

*Case* 1 $(b<0)$*.* When $b<0$, the infinite singular points on the Poincaré sphere are the origins of $U_1$ and $V_1$, each of which consists of two hyperbolic sectors and hence has index 0 due to Theorem 2. Considering the finite singular points, we know that the origins of $U_3$ and $V_3$ are centers, each having index 1. So, for the moment, all the known finite and infinite singular points have total index 2. Then by Theorem 15, the remaining finite singular points on the Poincaré disk should have total index 0. Since the systems have at most six finite singular points in addition to the origin, with at most two of them non–elementary (hence nilpotent due to statement $(a)$ of Lemma 18), there are the following possibilities: $(i)$ no more finite singular points, $(ii)$ two cusps, $(iii)$ two saddles and two centers, or $(iv)$ two saddles, two centers and two cusps.

Case $(iv)$ cannot occur because by Lemma 18, after a small perturbation, each cusp will produce two singular points, leading to an excess of singular points.

Consider case $(iii)$. If we place the two saddles on the Poincaré disk, they will be symmetric with respect to the origin and also be on the boundary of the period annulus of the center at the origin. Moreover, these saddles cannot be on the $x$-axis because when $y=0$, equation (2.21b) is zero only when $x=0$. Since the infinite singular points have only hyperbolic sectors, there are only two possible ways to construct the other two centers, see Figure 2.9:

If the first picture of Figure 2.9 holds, then the $x$-axis intersects the separatrices of the saddles four times, which is not possible because when $y=0$ the Hamiltonian $H_4$ is only quadratic in $x$. If the second picture holds, then a straight line through the origin passing sufficiently close to the saddles will have six intersection points with these separatrices (see Figure 2.7). This also is not possible because $H_4$ is a quartic polynomial on

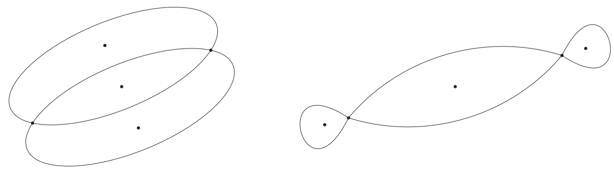


Figure 2.9: Two saddles forming the two centers.

this line, and consequently there can be at most four such points. So case (*iii*) of two saddles and two centers cannot exist.

Note that case (*ii*) is not realizable, otherwise Lemma 18 would require case (*iii*) to be realizable too.

In short, when $b < 0$ there are no finite singular points on the Poincaré disk except the origin, and the global phase portraits of systems $(IV)$ are topologically equivalent to the phase portrait 1.1 of Figure 1.1.

*Case* 2 ($b > 0$). In this case, the infinite singular points, which are the origins of $U_1$ and $V_1$, have two elliptic and two parabolic sectors. and each has index 2. Together with the centers at the origins of $U_3$ and $V_3$, their total index on the Poincaré sphere is 6. This means that in the Poincaré disk, the remaining finite singular points must have total index -2. Then we have the following three possibilities: (*i*) two saddles, (*ii*) two saddles and two cusps, or (*iii*) four saddles and two centers.

Consider case (*i*). The two saddles are of course symmetric with respect to the origin and are on the boundary of the period annulus of the center at the origin. As in the case $b < 0$, they cannot be on the $x$-axis. Since $H_4$ is quadratic on the $x$-axis, the separatrices through these saddles cross the $x$–axis exactly twice, and as a consequence the saddles must be connected with the infinite singular points, and we obtain a global phase portrait which is topologically equivalent to the phase portrait 1.2 of Figure 1.1. We remark that for the values $2a = b = \beta = 1$ a topologically equivalent phase portrait is achieved.

Now consider case (*iii*). There may be either two or four saddles on the boundary of the period annulus of the center at the origin. We analyze these possibilities below.

Assume first that the four saddles are on the boundary of the period annulus of the center at the origin. Because of the symmetry of the systems, two of them will be above the $x$-axis and two will be below because the only singular point lying on the coordinate axes is the origin. Remember that the separatrices through the saddles cannot cross the $x$-axis anymore. So there must be another separatrix connecting the saddles which are on the same side of the $x$-axis, surrounding the centers. Moreover, the remaining separatrices of the two saddles which are on the same side of the $x$-axis must go to different infinite singular points. If they went to the same point, then the $y$-axis would intersect the separatrices of these saddles six times (three below the $x$-axis and three above). However this is not possible since $H_4$ is quartic. Therefore we get the global phase portrait 1.7 shown in Figure 1.1. Such a phase portrait is obtained if $a = 0$ and $b = \beta = 1$.

Assume now that only two of the saddles are on the boundary of the period annulus of the center at the origin. Just as in the case $b > 0$ of systems $(III)$, the saddles on this boundary cannot be connected to other saddles (see Figure 2.8). Then, taking into account that the separatrices of the saddles on this boundary cannot cross the $x$-axis anymore, we see that these saddles must be connected with the infinite singular points, and the separatrices of these saddles are as shown in Figure 2.10. We next claim that the centers must be inside the region enclosed by the separatrices connecting the saddles on the boundary of the period annulus of the center at the origin with the infinite singular points. Suppose this were not the case, i.e. suppose that a center were in the region outside the this area, see Figure 2.10. Because of the flow in these regions, one of the remaining saddles must also be in the same region. Moreover, the saddle must be on the boundary of the period annulus of this center. But then a straight line through the origin passing sufficiently close to the saddle and intersecting this boundary twice would also intersect another separatrix of the saddle because of the flow. So, by symmetry, this straight line would have six intersection points with the separatrices on the same energy level, which is impossible. This proves our claim about the location of the centers. Then again the remaining saddles must be on the boundaries of these centers. Therefore the global phase portraits are equivalent to 1.8 of Figure 1.1. A phase portrait in this case is realized when $a = 2/5$ and $b = \beta = 1$.

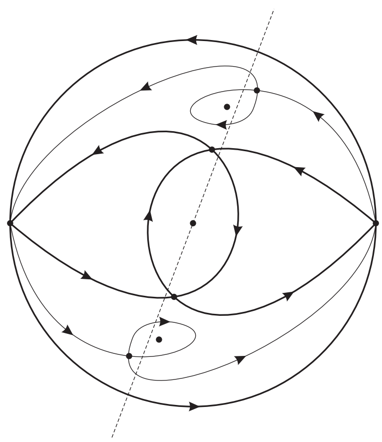


Figure 2.10: The center cannot be outside the region enclosed by the separatrices of the saddles.

**Remark 20.** In fact a center–loop may exist in any one of the vector fields $(I) - (VI)$ only if a straight line $l_1$ passing through the origin and the saddle of the center–loop intersects the separatrices of the saddle exactly at one point, the saddle itself. Otherwise one can find another straight line $l_2$ passing through the origin and sufficiently close to the saddle of the center–loop such that the number of intersection points is at least three, see Figure 2.11. Then, by the symmetry of these systems, $l_2$ would intersect the separatrices on the same energy level at six points, which is impossible because the systems are cubic.

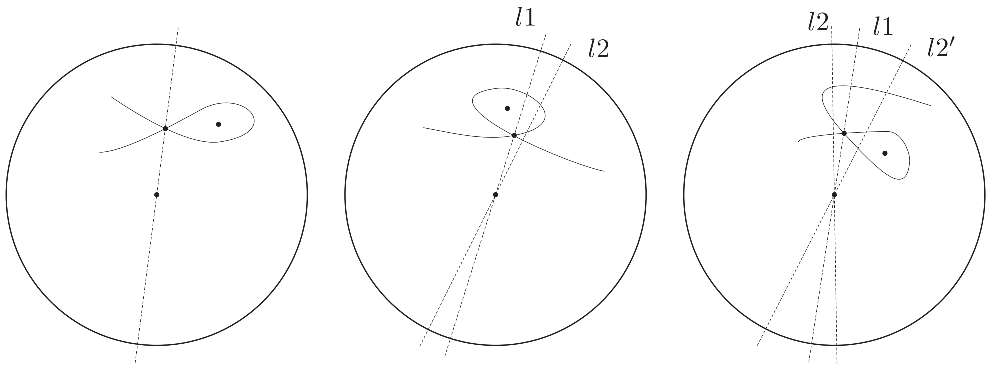



Figure 2.11: Center–loop configuration.

Finally we consider case $(ii)$. Due to Lemma 18, case $(ii)$ will produce a system in case $(iii)$ after a small perturbation. Consequently, the global phase portraits in this case are topologically equivalent to 1.9 of Figure 1.1. We note that as a result of the fact that systems $(III)$ attain the phase portraits 1.2 when $a = 0.5$, $b = \beta = 1$ and 1.8 when $a = 0.4$, $b = \beta = 1$, a global phase portrait in case $(ii)$ exists when $b = \beta = 1$ and $a$ is between 0.4 and 0.5.

## 2.6 Global phase portraits of systems $(V)$

We remind that systems $(V)$ are

$$\dot{x} = ax + by - 3\mu x^2 y + y^3, \tag{2.28a}$$

$$\dot{y} = -\frac{a^2 + \beta^2}{b} x - ay + x^3 + 3\mu x y^2, \tag{2.28b}$$

with the Hamiltonian

$$H_5(x, y) = \frac{y^4 - x^4}{4} - \frac{3\mu}{2} x^2 y^2 + \frac{a^2 + \beta^2}{2b} x^2 + \frac{b}{2} y^2 + axy.$$

As in the previous systems, we first investigate the infinite singular points of systems $(V)$. In the local chart $U_1$ we have

$$\begin{aligned}\dot{u} =& -v^2\left(bu^2+2au+\frac{a^2+\beta^2}{b}\right)-u^4+6\mu u^2+1,\\ \dot{v} =& -v^3(bu+a)-vu(u^2-3\mu).\end{aligned} \tag{2.29}$$

When $v=0$, the singular points are $\big(\pm\sqrt{3\mu+\sqrt{9\mu^2+1}},0\big)$. Note that the linear part of (2.29) when $v=0$ is

$$\begin{pmatrix} -4u(u^2-3\mu) & 0 \\ 0 & -u(u^2-3\mu) \end{pmatrix},$$

thus both of the eigenvalues of the two singular points are negative if $u>0$, and positive if $u<0$. Hence the points $\big(\sqrt{3\mu+\sqrt{9\mu^2+1}},0\big)$ and $\big(-\sqrt{3\mu+\sqrt{9\mu^2+1}},0\big)$ are attracting and repelling nodes respectively.

In $U_2$ systems $(V)$ become

$$\begin{aligned}\dot{u} =& v^2\left(\frac{a^2+\beta^2}{b}u^2+2au+b\right)-u^4-6\mu u^2+1,\\ \dot{v} =& v^3\left(\frac{a^2+\beta^2}{b}u+a\right)-vu(u^2+3\mu),\end{aligned}$$

and we see that the origin is not a singular point. Hence systems $(V)$ have four infinite nodes, all of which are on $U_1$ and $V_1$.

Now we shall discuss the finite singular points. This time we cannot find a better upper bound for the number of finite singular points of systems $(V)$ than the one given by Bezout's theorem which, in particular, says that there are at most nine isolated finite singular points for a cubic planar differential system. To see if there are any non–isolated singularities we compute the resultant of (2.28a) and (2.28b) with respect to $x$ and see that the numerator is

$$\begin{aligned}&b^2(1+9\mu^2)^2y^9+3b(1+9\mu^2)(b^2-2a^2\mu-2\beta^2\mu+3b^2\mu^2)y^7\\ &+3(b^4-4b^2\beta^2\mu+3a^4\mu^2+6b^4\mu^2+6a^2\beta^2\mu^2+3\beta^4\mu^2-18a^2b^2\mu^3\\ &-18b^2\beta^2\mu^3)y^5-b(a^4-b^4+a^2\beta^2-6a^2b^2\mu+6b^2\beta^2\mu-9a^2\beta^2\mu^2\\ &-9\beta^4\mu^2)y^3-a^2b^2\beta^2y,\end{aligned}$$

which cannot be identically zero because the coefficient of $y^9$ is always positive. Hence we conclude that all of the finite singular points, if there are any, are isolated.

We know that the origin is a center, which leaves us with eight more possible singularities. On the Poincaré sphere, the total index of the infinite singular points is 4. Together with the centers at the origins of $U_3$ and $V_3$, the total index becomes 6. Thus, on the Poincaré disk, we need to get a total index of -2 from the possible eight finite singular points.

We claim that systems $(V)$ have at most two nilpotent singularities in the finite region of the Poincaré disk. For the proof it suffices to consider the Gröbner basis of the polynomials (2.28a), (2.28b) and the determinant of the linear part of the systems. A similar analysis that we have done in systems $(III)$ and $(IV)$ proves our claim.

Having established the above claim, we see that the finite singular points of systems $(V)$ other than the origin can be either $(i)$ two saddles, $(ii)$ two saddles and two cusps, $(iii)$ four saddles and two centers, or $(iv)$ four saddles, two centers and two cusps.

Case $(iv)$ cannot occur because due to Lemma 18, it would require another case with two more finite singular points which is not possible.

Before studying the other cases we remark that without loss of generality we can assume $b > 0$. More precisely, if we do the linear transformation $(x, y) \mapsto (-y, -x)$, systems $(V)$ become

$$\begin{aligned} -\dot{y} &= -ay - bx + 3\mu y^2 x - x^3, \\ -\dot{x} &= \frac{a^2 + \beta^2}{b} y + ax - y^3 - 3\mu y x^2, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \dot{x} &= -ax - \frac{a^2 + \beta^2}{b} y + 3\mu x^2 y + y^3, \\ \dot{y} &= bx + ay + x^3 - 3\mu x y^2. \end{aligned} \tag{2.30}$$

After defining $\bar{a} = -a$, $\bar{\mu} = -\mu$, and $\bar{b} = -(a^2 + \beta^2)/b$, we see that systems (2.30) are essentially systems $(V)$. So we assume $b > 0$.

Consider case $(i)$. The two saddles must be on the boundary of the period annulus of the center at the origin. Their remaining separatrices cannot cross the straight lines passing through the origin and the infinite singular points, namely $y = \pm\sqrt{3\mu + \sqrt{9\mu^2 + 1}x}$, because the Hamiltonian $H_5$ is quadratic on these lines. Then, due to the flow at infinity and the clockwise flow around the origin, we get a global phase portrait which is topologically equivalent to 1.3 of Figure 1.1. We remark that this phase portrait is achieved for the values $a = b = \beta = 1$ and $\mu = 0$.

Similar to the previous systems, in case $(iii)$ there are two possibilities.

Assume first that the four saddles are on the boundary of the period annulus of the center at the origin. Since the infinite singular points are nodes, the centers can only be created by connecting two adjacent saddles. Since we have $b > 0$, the flow around the origin clockwise. In addition, the remaining separatrices of any of the saddles must lie on different sides of the straight line passing through that saddle and the origin, otherwise there would exist another straight line through the origin intersecting both separatrices in six intersection points in the same energy level. Then the flow at infinity ensures that the remaining two centers are formed by connecting adjacent saddles which lie on the same side of the $y$-axis. Recalling from case $(i)$ that the remaining separatrices of these saddles cannot cross the lines $y = \pm\sqrt{3\mu + \sqrt{9\mu^2 + 1}}x$ passing through the origin and the infinite singular points, we get the global phase portrait 1.10 of Figure 1.1. A topologically equivalent phase portrait is actually realized when $a = 0$ and $b = \beta = \mu = 1$, for instance.

Assume now that only two of the saddles are on the boundary of the period annulus of the center at the origin. These saddles cannot be connected by a separatrix with each other since their separatrices cannot cross the lines $y = \pm\sqrt{3\mu + \sqrt{9\mu^2 + 1}}x$ anymore. Their remaining separatrices can neither return to the same saddles (see Figure 2.7) nor go to one of the other two saddles (see Figure 2.8). Therefore they must go to the infinite singular points like in case $(i)$. Moreover, because of the symmetry of systems $(V)$ and the flow that we have obtained so far, the only way to have two more saddles and two centers is to have two center–loops. Due to Remark 20 we made earlier, these center–loops can appear only in one of the regions (1) or (2) indicated in Figure 2.12 (see also Figure 2.10). Therefore the global phase portrait in this case must be topologically equivalent to 1.11 of Figure 1.1. A realization of this phase portrait is achieved when $a = b = \beta = \mu = 1$.

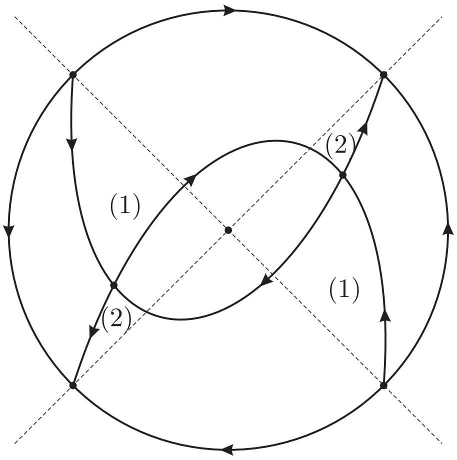



Figure 2.12: Location of the center–loop in systems $(V)$.

Lastly we investigate case $(ii)$. Since, by Lemma 18, case $(ii)$ leads to a system in case (*iii*), we conclude that the global phase portraits of systems $(V)$ in case (*ii*) are topologically equivalent to 1.12 of Figure 1.1. Due to the fact that the global phase portrait 1.3 is obtained when $a = b = \beta = 1$ and $\mu = 0$, and 1.11 when $a = b = \beta = \mu = 1$, a realization of the phase portrait 1.12 is ensured for $a = b = \beta = 1$ and for some $\mu$ between 0 and 1.

## 2.7 Global phase portraits of systems $(VI)$

Systems $(VI)$

$$\dot{x} = ax + by - 3\mu x^2 y - y^3, \tag{2.31a}$$

$$\dot{y} = -\frac{a^2+\beta^2}{b}x - ay + x^3 + 3\mu x y^2, \tag{2.31b}$$

have the Hamiltonian

$$H_6(x,y) = -\frac{y^4+x^4}{4} - \frac{3\mu}{2}x^2y^2 + \frac{a^2+\beta^2}{2b}x^2 + \frac{b}{2}y^2 + axy.$$

In the local chart $U_1$ systems $(VI)$ are written as

$$\begin{aligned} \dot{u} &= -v^2\left(bu^2 + 2au + \frac{a^2+\beta^2}{b}\right) + u^4 + 6u^2\mu + 1, \\ \dot{v} &= -v^3(a+bu) + vu(u^2+3\mu). \end{aligned} \tag{2.32}$$

When $v = 0$, the candidates for singular points of systems (2.32) are $\left(\pm\sqrt{-3\mu \pm \sqrt{9\mu^2-1}}, 0\right)$. Therefore, in $U_1$ there are four singular points if $\mu < -1/3$, two if $\mu = -1/3$, and none if $\mu > -1/3$.

In $U_2$ systems $(VI)$ become

$$\begin{aligned} \dot{u} &= v^2\left(\frac{a^2+\beta^2}{b}u^2 + 2au + b\right) - u^4 - 6u^2\mu - 1, \\ \dot{v} &= v^3\left(a + \frac{a^2+\beta^2}{b}u\right) - vu(u^2+3\mu), \end{aligned}$$

and we see that the origin is not a singular point. Hence all the infinite singular points are in the local charts $U_1$ and $V_1$. The existence of these singular points depend on the parameter $\mu$, so we will investigate the phase portraits of systems $(VI)$ in the corresponding subcases.

We make two remarks here. The first one is that, we can show using Gröbner basis that at most two of the finite singular points of systems $(VI)$ can be nilpotent, just as we did in the previous systems. Second, systems $(VI)$ have non–isolated finite singular points only when $a = 0$, $\mu = 1/3$, and $b = \beta > 0$. The proof is as follows: The numerator of the resultant of (2.31a) and (2.31b) with respect to $x$ is

$$\begin{aligned} &b^2(3\mu-1)^2(1+3\mu)^2y^9 - 3b(3\mu-1)(1+3\mu)(2a^2\mu+2\beta^2\mu+ \\ &+3b^2\mu^2-b^2)y^7 + 3\big(b^4-4b^2\beta^2\mu+(3a^4-6b^4+6a^2\beta^2+3\beta^4)\mu^2 \\ &+(18a^2b^2+18b^2\beta^2)\mu^3\big)y^5 - b(a^4+b^4+a^2\beta^2+6a^2b^2\mu-6b^2\beta^2\mu \\ &+9a^2\beta^2\mu^2+9\beta^4\mu^2)y^3 + a^2b^2\beta^2y. \end{aligned} \tag{2.33}$$

For (2.33) to be identically zero, we need $a = 0$ so that the coefficient of $y$ is zero. Then (2.33) simplifies to

$$\begin{aligned} &-b^2(-1+3\mu)^2(1+3\mu)^2y^9 + 3b(-1+3\mu)(1+3\mu)(-b^2+2\beta^2\mu+3b^2\mu^2)y^7 \\ &-3(b^2-3\mu\beta^2)(-b^2+\beta^2\mu+6b^2\mu^2)y^5 + b(b^2-3\mu\beta^2)^2y^3. \end{aligned}$$

Since $b \neq 0$, coefficients of $y^3$ and $y^9$ imply that we must have $\mu = 1/3$ and $b^2 = \beta^2$. But when $a = 0$ and $\mu = 1/3$ (2.31a) becomes $y(b - x^2 - y^2)$, meaning that $b$ must be positive, hence $b = \beta$. This finishes the proof.

*Case* $\mu < -1/3$. In this case there are the four infinite singular points $\big(\pm\sqrt{-3\mu \pm \sqrt{9\mu^2-1}}, 0\big)$ in $U_1$. The linear part of systems (2.32) on $v = 0$ is

$$\begin{pmatrix} 4u(u^2+3\mu) & 0 \\ 0 & u(u^2+3\mu) \end{pmatrix}.$$

Hence $\big(\sqrt{-3\mu+\sqrt{9\mu^2-1}}, 0\big)$ and $\big(-\sqrt{-3\mu-\sqrt{9\mu^2-1}}, 0\big)$ are repelling nodes, whereas $\big(\sqrt{-3\mu-\sqrt{9\mu^2-1}}, 0\big)$ and $\big(-\sqrt{-3\mu+\sqrt{9\mu^2-1}}, 0\big)$ are attracting ones. We recall that the origins of $U_2$ and $V_2$ are not singular.

We will now study the finite singular points. When $\mu < -1/3$, we claim that, systems $(VI)$ can have at most six finite singular points on $U_3$ other than the origin. To prove our claim we first show that without loss of generality we can assume $b > 0$. If we do the transformation $(x, y) \mapsto \big((x-y)/\sqrt{2}, (x+y)/\sqrt{2}\big) = (X, Y)$, i.e. rotation by $\pi/4$, then systems $(VI)$ become

$$\begin{aligned} \dot{X} &= \frac{a^2-b^2+\beta^2}{2b}X + \frac{(a+b)^2+\beta^2}{2b}Y - \frac{3-3\mu}{2}X^2Y - \frac{1+3\mu}{2}Y^3, \\ \dot{Y} &= -\frac{(a-b)^2+\beta^2}{2b}X - \frac{a^2-b^2+\beta^2}{2b}Y + \frac{3-3\mu}{2}XY^2 + \frac{1+3\mu}{2}X^3. \end{aligned}$$

After the rescale $dT = (1+3\mu)/2\ dt$, which is well defined as $\mu < -1/3$, we finally get the systems

$$\begin{aligned} \dot{X} &= \frac{a^2-b^2+\beta^2}{b(1+3\mu)}X + \frac{(a+b)^2+\beta^2}{b(1+3\mu)}Y - \frac{3-3\mu}{(1+3\mu)}X^2Y - Y^3, \\ \dot{Y} &= -\frac{(a-b)^2+\beta^2}{b(1+3\mu)}X - \frac{a^2-b^2+\beta^2}{b(1+3\mu)}Y + \frac{3-3\mu}{(1+3\mu)}XY^2 + X^3. \end{aligned} \tag{2.34}$$

If we define the variables

$$\bar{a} = \frac{a^2 - b^2 + \beta^2}{b(1+3\mu)}, \quad \bar{b} = \frac{(a+b)^2 + \beta^2}{b(1+3\mu)}, \quad \bar{\mu} = \frac{1-\mu}{1+3\mu},$$

then systems (2.34) can be rewritten as

$$\begin{aligned} \dot{X} &= \bar{a}X + \bar{b}Y - 3\bar{\mu}X^2Y - Y^3, \\ \dot{Y} &= -\frac{\bar{a}^2 + \beta^2}{\bar{b}}X - \bar{a}Y + 3\bar{\mu}XY^2 + X^3. \end{aligned} \tag{2.35}$$

Note that $\bar{\mu}$ is monotone decreasing in $\mu$, and that $\bar{\mu} < -1/3$. Hence systems (2.35) are essentially systems $(VI)$ with $b\bar{b} < 0$, which proves our claim that we can assume $b > 0$.

Now we will determine the maximum number of finite singular points. First suppose that $a = 0$. Then systems $(VI)$ becomes

$$\dot{x} = by - 3\mu x^2 y - y^3, \quad \dot{y} = -\frac{\beta^2}{b}x + x^3 + 3\mu xy^2.$$

If we solve for $y$ in the equation $\dot{x} = 0$, we get either $y = 0$ or $y = \pm\sqrt{b - 3\mu x^2}$. When we substitute either of the latter two values of $y$ into $\dot{y}$ we get

$$\dot{y} = x\left((1-3\mu)(1+3\mu)x^2 + 3b\mu - \frac{\beta^2}{b}\right),$$

which is zero if and only if $x = 0$ since $b > 0$ and $\mu < -1/3$. Substituting $y = 0$ into $\dot{y}$, however, gives $\dot{y} = x(x^2 - \beta^2/b)$, which is zero when $x = 0$ or $x = \pm\sqrt{\beta^2/b}$. Therefore when $a = 0$, there are at most four finite singular points other than the origin, namely $(0, \pm\sqrt{b - 3\mu x^2})$, $(\pm\sqrt{\beta^2/b}, 0)$.

Now assume $a \neq 0$. If we equate (2.31a) to zero and solve for $x$ we obtain

$$x_{1,2} = \frac{a \pm \sqrt{a^2 + 12b\mu y^2 - 12\mu y^4}}{6\mu y}.$$

Note that when $y = 0$, (2.31a) becomes $\dot{x} = ax$ which is zero only if $x = 0$ because $a \neq 0$. So we can assume $y \neq 0$. Since $\mu < -1/3$, both $x_1$ and $x_2$ are well defined. If we substitute these into (2.31b) we get

$$\begin{aligned} \dot{y}_{1,2} =& \frac{1}{216by^3\mu^3}\big(4a^3b - 36ay^2\mu(-b^2 + a^2\mu + \beta^2\mu) - 36aby^4\mu(1+3\mu^2) \\ &+ \sqrt{a^2 + 12by^2\mu - 12y^4\mu}\big(4a^2b + 12by^4\mu(-1+3\mu)(1+3\mu) \\ &+ 12y^2\mu(b^2 - 3a^2\mu - 3\beta^2\mu)\big)\big). \end{aligned}$$

Then the maximum number of roots of the product $\dot{y}_1\dot{y}_2$ will be the maximum number of finite singular points of systems $(VI)$ other than the origin.

The numerator of the product $\dot{y}_1\dot{y}_2$ is

$$\begin{aligned}&b^2y^9(3\mu-1)^2(1+3\mu)^2-3by^7(3\mu-1)(1+3\mu)(2a^2\mu+2\beta^2\mu\\&+3b^2\mu^2-b^2)+3y^5\big(b^4-4b^2\beta^2\mu+\mu^2(3a^4-6b^4+6a^2\beta^2+3\beta^4)\\&+\mu^3(18a^2b^2+18b^2\beta^2)\big)-by^3(a^4+b^4+a^2\beta^2+6a^2b^2\mu-6b^2\beta^2\mu\\&+9a^2\beta^2\mu^2+9\beta^4\mu^2)+a^2b^2y\beta^2.\end{aligned} \tag{2.36}$$

Note that this is exactly the negative of the resultant (2.33). Since $y \neq 0$, we eliminate the common factor $y$, do the change the change $z = y^2$, and rewrite (2.36) as

$$\begin{aligned}&b^2(3\mu-1)^2(1+3\mu)^2z^4-3b(3\mu-1)(1+3\mu)(2a^2\mu+2\beta^2\mu\\&+3b^2\mu^2-b^2)z^3+3\big(b^4-4b^2\beta^2\mu+\mu^2(3a^4-6b^4+6a^2\beta^2+3\beta^4)\\&+\mu^3(18a^2b^2+18b^2\beta^2)\big)z^2-b(a^4+b^4+a^2\beta^2+6a^2b^2\mu-6b^2\beta^2\mu\\&+9a^2\beta^2\mu^2+9\beta^4\mu^2)z+a^2b^2\beta^2.\end{aligned} \tag{2.37}$$

Then the maximum number of roots of (2.36) is equal to the maximum number of positive roots of (2.37), which can be determined by the Descartes' rule of signs. We claim that (2.37) has less than four positive roots if $\mu < -1/3$. We now prove our claim.

The coefficient of $z^4$ is positive. So (2.37) can have four positive roots only if the coefficients of $z^3$ and $z^2$ are negative and positive, respectively. This can happen if and only if

$$A = -b^2+2a^2\mu+2\beta^2\mu+3b^2\mu^2 > 0, \tag{2.38}$$
$$B = -b^4+4b^2\beta^2\mu-3\big((a^2+\beta^2)^2-2b^4\big)\mu^2-18b^2(a^2+\beta^2)\mu^3 < 0, \tag{2.39}$$

because we have $b > 0$ and $\mu < -1/3$. We see that $A$, when considered as a polynomial in $\mu$, has one negative and one positive root. Then, since $\mu < -1/3$ and $\lim_{\mu\to-\infty} A = +\infty$, we have $A > 0$ if and only if $\mu$ is less than the negative root

$$\mu_0 = -\frac{a^2+\beta^2\sqrt{a^4+3b^4+2a^2\beta^2+\beta^4}}{3b^2}.$$

On the other hand, if we apply Descartes' rule of signs to $B$, again when considered as a polynomial in $\mu$, we see that $B$ has only one negative root, say $\mu_1$. So $B > 0$ when $\mu < \mu_1$ because $\lim_{\mu\to-\infty} B = +\infty$. In addition, since $B = -b^4 < 0$ when $\mu = 0$, we have $B < 0$ when $\mu_1 < \mu < 0$. If we evaluate $B$ at $\mu_0$, we get

$$\begin{aligned}&\frac{1}{3b^4}\big(6(a^2+\beta^2)^4+b^4(19a^4+3b^4+34a^2\beta^2+15\beta^4)\\&\quad+2\big(3(a^2+\beta^2)^3+b^4(5a^2+3\beta^2)\big)\sqrt{a^4+3b^4+2a^2\beta^2+\beta^4}\big) > 0.\end{aligned}$$

This means that $\mu_0 < \mu_1$. But then, when $A > 0$, that is when $\mu < \mu_0$, we have $B > 0$, meaning that (2.38) and (2.39) cannot hold together. This proves our claim, which in turn implies that systems $(VI)$ can have at most six finite singular points other than the origin.

Next we get more information about the finite singular points by considering the indices of all the singular points of the systems. We saw that when $\mu < -1/3$, systems $(VI)$ have eight nodes for infinite singular points on the Poincaré sphere. Together with the centers at the origins of the upper and lower hemispheres, their total index is 10. Hence, the total index of the remaining finite singular points on the Poincaré disk must be -4. Therefore there must be either $(i)$ four saddles, or $(ii)$ four saddles and two cusps. But by Lemma 18, case $(ii)$ cannot exist. Hence we are only left with case $(i)$.

We first assume that there are only two saddles on the boundary of the period annulus of the center at the origin. As in systems $(III)$, these saddles cannot be connected with the other saddles, see Figure 2.8. So they must be connected directly with an infinite singular point because there are no other finite singular points. Moreover, the remaining separatrices of these saddles cannot cross the straight lines $y = \big(\pm\sqrt{-3\mu \pm \sqrt{9\mu^2 - 1}x}\big)$ which pass through the origin and the infinite singular points. This is due to the fact that on these lines the Hamiltonian $H_6$ becomes

$$\frac{x^2}{2}\left(\frac{a^2+\beta^2}{b} + b(-3\mu \pm \sqrt{9\mu^2-1}) \pm 2a\sqrt{-3\mu \pm \sqrt{9\mu^2-1}}\right),$$

which is a quadratic polynomial in the variable $x$. In addition, there cannot be any singular points on these lines. More precisely, on $y = -\sqrt{-3\mu - \sqrt{9\mu^2 - 1}x}$ the finite singular points of systems $(VI)$ are given by the equations

$$\begin{aligned}\dot{x} = &\left(a + b\sqrt{-3\mu - \sqrt{9\mu^2-1}}\right)x \\ &+ \sqrt{9\mu^2-1}\sqrt{-3\mu - \sqrt{9\mu^2-1}x^3} = 0\end{aligned}, \qquad (2.40a)$$

$$\begin{aligned}\dot{y} = -&\left(\frac{a^2+\beta^2}{b} + a\sqrt{-3\mu - \sqrt{9\mu^2-1}}\right)x \\ &+ \sqrt{9\mu^2-1}(-3\mu - \sqrt{9\mu^2-1})x^3 = 0\end{aligned}, \qquad (2.40b)$$

If we multiply (2.40a) by $\sqrt{-3\mu - \sqrt{9\mu^2-1}}$ and subtract if from (2.40b) we get

$$-\frac{a^2+\beta^2}{b} + b(3\mu + \sqrt{9\mu^2-1}) = 0,$$

which holds only when $x = 0$ because we have $b > 0$ and $\mu < -1/3$, which implies $y = 0$. Similar calculations give the same result for the straight line $y = \sqrt{-3\mu - \sqrt{9\mu^2 - 1}x}$. Consequently we get a global phase portrait topologically equivalent to 1.13 of Figure 1.1. This phase portrait is realized for the values $a = b = \beta = -\mu = 1$.

Next we assume that all of the saddles are on the boundary of the period annulus of the center at the origin. Because there are no other finite singular points and the remaining separatrices of the saddles cannot cross the straight lines passing through the origin and the infinite singular points, all the saddles must go to the infinite singular points as shown in the global phase portrait 1.14 of Figure 1.1. We note that for $a = 0$ and $b = \beta = -\mu = 1$, one obtains a topologically equivalent phase portrait. This finishes the study of the case $\mu < -1/3$.

*Case* $\mu = -1/3$. The infinite singular points in this case are $(\pm 1, 0)$, and they are degenerate. So we need blow-ups to understand the local behavior at these points. We will show the computations for the study of the local phase portrait of the point $(1, 0)$, and the other point $(-1, 0)$ can be studied in the same way.

First we move $(1, 0)$ to the origin by the shift $u \mapsto u + 1$, and get the systems

$$\begin{aligned} \dot{u} &= u^2(u+2)^2 - v^2\left(bu^2 + 2(a+b)u + 2a + b + \frac{a^2+\beta^2}{b}\right), \\ \dot{v} &= uv(u+1)(u+2) - v^3(bu + a + b). \end{aligned}$$

Now we do the blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$ and eliminate the common factor $u$, we get the systems

$$\begin{aligned} \dot{u} &= u(u+2)^2 - uw^2\left(bu^2 + 2(a+b)u + 2a + b + \frac{a^2+\beta^2}{b}\right), \\ \dot{w} &= -w(u+2) + w^3\left((a+b)u + 2a + b + \frac{a^2+\beta^2}{b}\right). \end{aligned} \tag{2.41}$$

We see that in systems (2.41), when $u = 0$ we have $\dot{w} = 0$ if and only if $w = 0$ or

$$w = \pm \frac{\sqrt{2b}}{\sqrt{(a+b)^2 + \beta^2}}.$$

So, on the $w$–axis, systems (2.41) have one singular point if $b < 0$, and three otherwise. The linear part of systems (2.41) when $u = 0$ is

$$\begin{pmatrix} 4 - \left(2a + b + \dfrac{a^2+\beta^2}{b}\right)w^2 & 0 \\ -w + w^3(a+b) & -2 + 3\left(2a + b + \dfrac{a^2+\beta^2}{b}\right)w^2 \end{pmatrix}.$$

Hence the origin is a saddle, whereas the other two singular points, when they exist (depending on the sign of the parameter $b$), are repelling nodes.

Therefore the local phase portrait at the singular point $(1, 0)$ of systems (2.32), as in Figures 5 and 6, consists of two hyperbolic sectors when $b < 0$, and two parabolic and two elliptic ones when $b > 0$.

Performing the same procedure for the point $(-1, 0)$ reveals that the local phase portrait of this point is the same as that of $(1, 0)$ with the direction of the flow reversed.

Now we analyze the finite singular points. We will show that in this case systems $(VI)$ can have at most six finite singular points other than the origin when $\mu = -1/3$ also. The maximum number of finite singular points with $y \neq 0$ is given by the maximum number of roots of (2.36), which is four because $\mu = -1/3$. When $y = 0$, on the other hand, systems $(VI)$ become

$$\dot{x} = ax,$$
$$\dot{y} = x\left(x^2 - \frac{a^2 + \beta^2}{b}\right),$$

which has at most two finite singular points other than the origin. Therefore we conclude that there can be at most six additional finite singular points.

Next we count the indices of the singular points. Due the fact that infinite singular points have different indices depending on the sign of $b$, we investigate the cases $b < 0$ and $b > 0$ separately.

*Case* 1 ($b < 0$). In this case the infinite singular points and the centers at the origins of $U_3$ and $V_3$ have a total index of 2 on the Poincaré sphere. Hence, in the Poincaré disk, the total index of the remaining possible six finite singular points must be 0. Then, other than the origin, there are either $(i)$ no more singular points, $(ii)$ two cusps, $(iii)$ two saddles and two centers, $(iv)$ two saddles, two centers and two cusps.

We can immediately eliminate case $(iv)$ due to Lemma 18. On the other hand, it is easy to see that the global phase portraits in case $(i)$ are topologically equivalent to the phase portrait 1.15 of Figure 1.1.

Consider case $(iii)$. The two saddles must be on the boundary of the period annulus of the center at the origin. Note that they cannot be located on the straight lines $y = \pm x$ which pass through the origin and the infinite singular points. On $y = x$, for instance, systems $(VI)$ become

$$\dot{x} = (a + b)x,$$
$$\dot{y} = -\left(\frac{a^2 + \beta^2}{b} + a\right)x,$$

and both polynomials are zero only when $x = 0$, which implies that the only finite singular point on the straight line $y = x$ is the origin. The same holds for the straight line $y = -x$. Moreover, the remaining separatrices of these saddles neither can return the same point (see Figure 2.7), nor can cross $y = \pm x$ because on the lines $y = \pm x$ the Hamiltonian $H_6$ becomes

$$H_6 = \frac{(a \pm b)^2 + \beta^2}{2b} x^2,$$

which is only quadratic. This means that those separatrices must go to infinite singular points, yet the infinite singular points only have hyperbolic sectors. Therefore case $(iii)$ is eliminated too.

Since case $(ii)$ cannot exist without case $(iii)$, the global phase portraits of systems $(VI)$ when $\mu = -1/3$ and $b < 0$ are the ones in case $(i)$.

*Case* 1 ($b > 0$). When $b > 0$ the total index of the infinite singular points and the finite centers on the Poincaré sphere is 10. Hence the total index of the remaining finite singular points on the Poincaré disk must be -4. Since there are at most six such points, by Lemma 18 there can only be be four saddles.

We first consider the possibility that only two of the saddles are on the boundary of period annulus of the center at the origin. As always, these saddles cannot be connected with the remaining two. So, they must go to the infinite singular points. Remember that these saddles are not on the straight lines $y = \pm x$, and that their separatrices intersect these lines exactly twice. Then, in accordance with the fact that the infinite singular points have elliptic sectors, the global phase portrait follows, see 1.16 of Figure 1.1. Setting $a = b = \beta = 1$ provides a realization of such a phase portrait.

If, on the other hand, all of the saddles are on the boundary of the center at the origin, then due to the same reasons as in the previous case, the global phase portrait 1.17 of Figure 1.1 is obtained. This phase portrait is achieved for $a = 0$ and $b = \beta = 1$.

*Case* $\mu > -1/3$. Finally, when $\mu > -1/3$, systems $(VI)$ have no infinite singular points. As for the finite singularities, since we already have the centers at the origins of $U_3$ and $V_3$ on the Poincaré sphere, the total index of the remaining possible eight finite singular points on the Poincaré disk must be 0. Hence we have the following possibilities: $(i)$ no singular points, $(ii)$ two cusps, $(iii)$ two saddles and two centers, $(iv)$ two saddles, two centers, and two cusps, $(v)$ four saddles and four centers. Of course if $\mu = 1/3$, there are non–isolated singular points when $a = 0$ and $b = \beta$, but we will study this case separately.

In case $(ii)$, the cusps must be on the boundary of the period annulus of the center at the origin. But by the proof of statement $(c)$ of Lemma 18 we know that this is not possible. As a result we discard case $(ii)$.

Case $(i)$ can occur only if $b < 0$ so that the flow around the origin and the infinity match. Then we easily get the phase portrait 1.18 of Figure 1.1. This phase portrait is realized when $a = -b = \beta = \mu = 1$.

In case $(iii)$ both of the saddles must be on the boundary of the period annulus of the center at the origin as usual. Then, the centers can only be created if different saddles are connected once again, otherwise there would be six points on a straight line through the origin which are on the same energy level, see Figure 2.7. Hence the phase portraits in this case turn out to be topologically equivalent to 1.19 of Figure 1.1. One actually obtains an equivalent phase portrait if $a = b = \beta = 1$ and $\mu = 0$.

Next we consider case $(v)$. Assume first that all four saddles are on the boundary of the period annulus of the center at the origin. The remaining separatrices of any of the saddles must be on different sides of the straight line passing through that saddle and the origin, or else one could find a straight line $l$ through the origin, passing close enough to the saddle and intersecting three of the separatrices of the saddle, which would, taking into account the symmetry, lead to the existence of six points on $l$ which are all in the same energy level. Consequently we obtain the global phase portrait 1.20 of Figure 1.1. If we choose $a = 0$, $b = \beta = 1$ and $\mu = -1/4$, we get a topologically equivalent phase portrait.

Second, assume that only two of the saddles are on the boundary of the period annulus of the center at the origin, name these saddles $p_1$ and $p_2$. These saddles cannot be connected with the other saddles (which we shall name $q_1$ and $q_2$), neither can they be the saddle of a center–loop (see Figure 2.7 and Figure 2.8). Since there are no infinite singular points, their remaining separatrices must coincide symmetrically with respect to the origin. Thus it remains to determine the locations and the separatrices of the remaining two saddles $q_1$ and $q_2$, and the two centers. We claim that they cannot be outside the region enclosed by the outer separatrices of $p_1$ and $p_2$. We now prove this claim.

Assume on the contrary that $q_1$ and $q_2$ are outside the region enclosed by the outer separatrices of $p_1$ and $p_2$. Then there are three possibilities, see Figure 2.13.

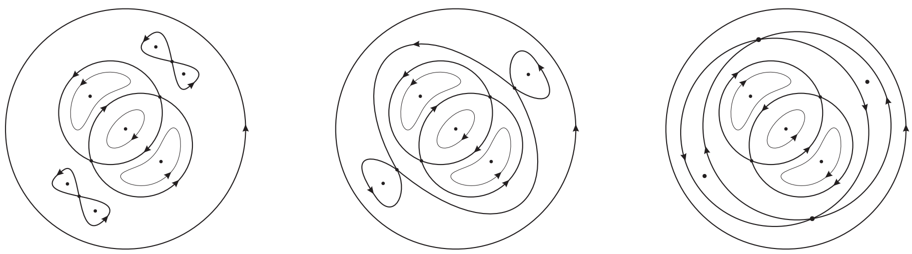


Figure 2.13: Possible separatrix configurations for $q_1$ and $q_2$.

The first picture of Figure 2.13 cannot hold because there are six centers instead of four. The second picture is also discarded due to the fact that a straight line through origin intersects the separatrices of $q_1$ and $q_2$ six times, see Figure 2.7. In the last picture we have $b < 0$ because of the counterclockwise flow at infinity. Then on the positive $y$-axis (2.31a) becomes $\dot{x} = by - y3 < 0$, which contradicts the phase portrait. Hence our claim is proved. Therefore we deduce that the global phase portraits in this case are topologically equivalent to 1.21 of Figure 1.1. The values $a = b = \beta = 1$ and $\mu = -1/4$ provides a realization of such a phase portrait.

Consider case $(iv)$. By Lemma 18 it would lead to the global phase portraits in case $(v)$. Therefore the only possibility is that the global phase portraits in this case are topologically equivalent to 1.22 of Figure 1.1. The facts that the global phase portrait 1.19 is obtained when $a = b = \beta = 1$ and $\mu = 0$, and that 1.21 is achieved if $a = b = \beta = 1$ and $\mu = -1/4$ lead to the conclusion that the phase portrait 1.22 is realized when $a = b = \beta = 1$ for some $\mu$ between 0 and -1/4.

Finally we consider the case $\mu = 1/3$, $a = 0$ and $b = \beta$. In this case systems $(VI)$ become

$$\begin{aligned}\dot{x} =& y(\beta - x^2 - y^2),\\ \dot{y} =& x(-\beta + x^2 + y^2).\end{aligned}$$

We see that other than the origin, the circle $x^2 + y^2 = \beta$ is a set of non–isolated singular points, and that there are no more singularities. Consequently the global phase portraits in this case are topologically equivalent to 1.23 of Figure 1.1 is easily obtained. This completes the proof of Theorem 4.

# Chapter 3

# Proof of Theorem 5

In this chapter we will prove Theorem 5. We will provide the way to obtain the normal forms which appear in Theorem 5 in Section 3.1, and the global phase portraits of the families $(VII)$–$(XII)$ in the remaining sections.

## 3.1 Obtaining the normal forms given in Theorem 5

The cubic terms of the families of the vector fields $(VII)$–$(XII)$ are clearly the same as those of the families $(I)$–$(VI)$. To obtain the complete normal forms we only need to state and prove an analogous result to Proposition 11.

**Proposition 21.** *The linearized systems at the origin corresponding to each of the ten classes of Hamiltonian cubic planar polynomial vector fields having only linear and cubic homogeneous terms which have a nilpotent singular point at the origin can be chosen to be either*

$$\dot{x} = ax + by, \quad \dot{y} = -(a^2/b)x - ay, \tag{3.1}$$

*or*

$$\dot{x} = 0, \quad \dot{y} = cx, \tag{3.2}$$

*where $a, b, c \in \mathbb{R}$ such that $b, c \neq 0$.*

*Proof.* We will give the proof only for systems $(x)$ because the remaining cases can be proved in the same way.

Assume that $X$ is a vector field in class $(x)$ plus a linear part and that it is Hamiltonian. Then by Proposition 9 the polynomial differential system associated to $X$ can be written as

$$\dot{x} = ax + by - 3\mu x^2 y - y^3,$$
$$\dot{y} = cx + dy + x^3 + 3\mu x y^2,$$

for some real constants $a, b, c, d$. Since $X$ is Hamiltonian, we have $d = -a$. The eigenvalues of the linear part of this system at the origin are

$$\lambda_{1,2} = \pm\sqrt{a^2 + bc}.$$

In order for the origin to be nilpotent, these eigenvalues must be equal to zero. So, if $b \neq 0$ we get $c = -(a^2/b)$. If $b = 0$ we have $a = 0$ with $c \neq 0$ because the linear part of the system at the origin cannot be zero. □

In order to write these vector fields in a more compact way, we will write their corresponding linearized systems at the origin as

$$\dot{x} = ax + by, \quad \dot{y} = \left(c - \frac{a^2}{b + c}\right) x - ay,$$

with $b + c \neq 0$ and the condition that either $c = 0$ or $a = b = 0$. We note that when $a = b = 0$ the $y$-axis is invariant for systems $(VII)$ and $(VIII)$, hence their origins are not centers in this case. Therefore we have $c = 0$ in these two classes.

Having determined the forms of the vector fields that we are going to study, using the following proposition we will find necessary and sufficient conditions so that their origins are centers.

**Proposition 22.** *If $P$ and $Q$ are homogeneous polynomials of degree $m$, then systems* (1.2) *have a nilpotent center or a focus at the origin if and only if $m$ is odd and the coefficient of $x^m$ in $Q$ is negative.*

For more details about Proposition 22 and its proof see [5]. We note that this proposition gives necessary and sufficient conditions in order that families of systems $(VII)$–$(XII)$ of Theorem 5 have a nilpotent center at the origin. To be able to determine these conditions we need to apply a linear change of variables provided in the following proposition.

**Proposition 23.** *Systems in classes $(VII)$–$(XII)$ can be written in the form* (1.2) *after applying the change of variables*

$$X = x, \quad Y = ax + by,$$

*or*

$$X = y, \quad Y = cx,$$

*when $b \neq 0$ or $b = 0$, respectively.*

The relations between the parameters $a, b, c$ and $\mu$ given in Theorem 5 are obtained as a result of Proposition 22, and we will prove them for each family as we study their global phase portraits separately. We remark that as the right hand sides of each of the vector fields in $(VII) - (XII)$ are odd functions, their phase portraits are also symmetric with respect to the origin.

## 3.2 Global phase portraits of systems $(VII)$

Systems $(VII)$ have the Hamiltonian

$$H_7 = -\frac{x^4}{4} + \frac{a^2x^2}{2b} + \frac{by^2}{2} + axy.$$

First we will apply Proposition 22 to find the necessary and sufficient conditions for the origin to be a center. After doing the linear change of variables suggested in Proposition 23 systems $(VII)$ become

$$\dot{x} = y, \quad \dot{y} = bx^3.$$

Then by Proposition 22, we see that systems $(VII)$ have a nilpotent center at the origin if and only if $b < 0$.

Now we will find the global phase portraits of systems $(VII)$ under the restriction $b < 0$. We first investigate the infinite singular points of these systems. Using (1.4), we see that in the local chart $U_1$ systems $(VII)$ become

$$\begin{aligned}\dot{u} &= -v^2\left(bu^2 + 2au + a^2/b\right) + 1,\\ \dot{v} &= -v^3\left(bu + a\right).\end{aligned}$$

When $v = 0$, there are no singular points on $U_1$.

Next we will check whether the origin of the local chart $U_2$ is a singular point. In $U_2$ we use (1.5) to get

$$\begin{aligned}\dot{u} &= v^2\left((a^2/b)u^2 + 2au + b\right) - u^4,\\ \dot{v} &= v^3\left((a^2/b)u + a\right) - u^3v,\end{aligned} \tag{3.3}$$

and we see that the origin is a singular point and that its linear part is zero. We need to do blow-ups to describe the local behavior at this point. We perform the directional blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$ and have

$$\begin{aligned}\dot{u} &= u^2w^2\left((a^2/b)u^2 + 2au + b\right) - u^4,\\ \dot{w} &= -\,uw^3(au + b).\end{aligned}$$

We eliminate the common factor $u$ between $\dot{u}$ and $\dot{w}$, and get the vector field

$$\begin{aligned}\dot{u} &= uw^2\left((a^2/b)u^2 + 2au + b\right) - u^3,\\ \dot{w} &= -w^3(au + b).\end{aligned} \tag{3.4}$$

When $u = 0$, because $b < 0$ the only singular point of systems (3.4) is the origin, whose linear part is again zero. Hence we do another blow-up $(u, w) \to (u, z)$ with $z = w/u$, eliminate the common factor $u^2$, and get the vector field

$$\begin{aligned}\dot{u} &= uz^2\left((a^2/b)u^2 + 2au + b\right) - u,\\ \dot{z} &= -z^3\left((a^2/b)u^2 + 3au + 2b\right) + z.\end{aligned} \tag{3.5}$$

When $u = 0$, the unique singular point of systems (3.5) is the origin since $b < 0$. The eigenvalues of the linear part of systems (3.5) at the origin are $\pm 1$, hence it is a saddle. Going back through the changes of variables until systems (3.3) as shown in Figure 3.1, we see that locally the origin of $U_2$ consists of two hyperbolic sectors.

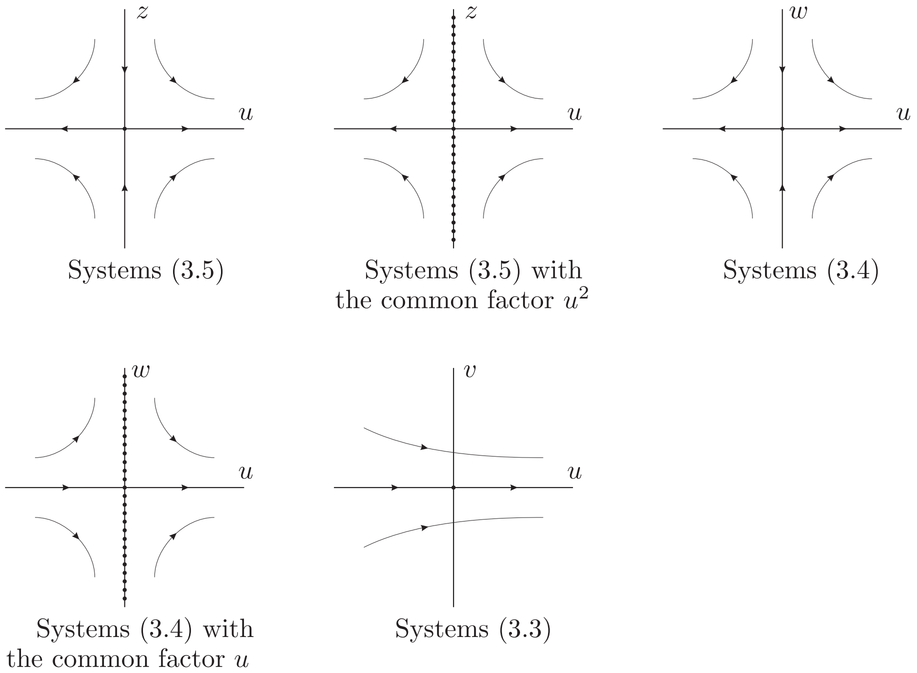



Figure 3.1: Blow-up of the origin of $U_2$ of systems $(VII)$ when $b < 0$.

We now look at the finite singular points of systems $(VII)$ and see that only the origin is singular, which is a center. Therefore the global phase portraits of systems $(VII)$ are topologically equivalent to the phase portrait 1.1 of Figure 1.1.

## 3.3 Global phase portraits of systems $(VIII)$

Systems $(VIII)$ have the Hamiltonian

$$H_8 = -x^3 y + \frac{a^2 x^2}{2b} + \frac{by^2}{2} + axy.$$

Note that for systems $(VIII)$ we can assume $b > 0$ because the linear change $y \mapsto -y$ gives exactly the same systems with the opposite sign of the parameter $b$.

To determine the necessary and sufficient conditions for the origin of systems $(VIII)$ to be a center, using Proposition 23 we rewrite them as

$$\dot{x} = y - x^3, \quad \dot{y} = -4ax^3 + 3x^2 y.$$

Then, by Proposition 22 the origin of systems $(VIII)$ is a center if and only if $a > 0$. Therefore we assume $a > 0$ and begin the study of their phase portraits with finding the infinite singular points.

In the local chart $U_1$ systems $(VIII)$ are

$$\begin{aligned}\dot{u} &= -v^2 \left(bu^2 + 2au + a^2/b\right) + 4u,\\ \dot{v} &= -v^3 (bu + a) + v.\end{aligned}$$

When $v = 0$, only the origin of $U_1$ is singular. The eigenvalues at this point are 4 and 1, meaning that it is a repelling node.

Next, we should check the origin of $U_2$, in which systems $(VIII)$ become

$$\begin{aligned}\dot{u} &= v^2 \left((a^2/b)u^2 + 2au + b\right) - 4u^3,\\ \dot{v} &= v^3 \left((a^2/b)u + a\right) - 3u^2 v.\end{aligned} \tag{3.6}$$

We see that the origin is singular and its linear part is zero. We need to do blow-up for analyzing the local behavior at this point. Doing the blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$ and eliminating the common factor $u$ we get the system

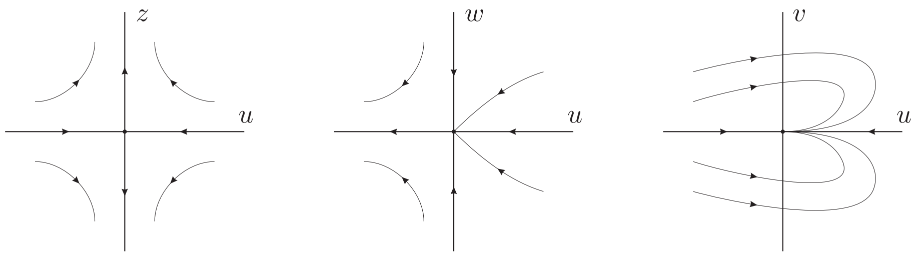
$$\begin{aligned}\dot{u} &= uw^2 \left((a^2/b)u^2 + 2au + b\right) - 4u^2,\\ \dot{w} &= -w^3 (au + b) - uw.\end{aligned} \tag{3.7}$$

When $u = 0$, the only singular point of systems (3.7) is the origin, whose linear part is again zero. So, we do another blow-up $(u, w) \mapsto (u, z)$ with $z = w/u$, eliminate the common factor $u$, and obtain

$$\begin{aligned}\dot{u} &= u^2 z^2 \left((a^2/b)u^2 + 2au + b\right) - 4u,\\ \dot{z} &= -uz^3 \left((a^2/b)u^2 + 3au + 2b\right) + 5z.\end{aligned} \tag{3.8}$$

When $u = 0$, the only singular point of systems (3.8) is the origin, which is a saddle. We trace the changes of variables back to systems (3.6) as shown in Figure 3.2, and we find out that the origin of $U_2$ is an attracting node.

Having determined the infinite singular points of systems $(VIII)$, we now compute their finite singular points, which are $\pm\left(\sqrt{4a/3}, \sqrt{4a^3/27b^2}\right)$ plus the origin. The eigenvalues of the linear part of systems $(VIII)$ at these two points are $\pm 4\sqrt{2}a/\sqrt{3}$, which means that they are saddles since $a > 0$.




Systems (3.8) Systems (3.7) Systems (3.6)

Figure 3.2: Blow-up of the origin of $U_2$ of systems $(VIII)$ when $a, b > 0$.

Now we will determine the global phase portrait according to this local information. The two saddles must be on the boundary of the period annulus of the center at the origin due to the symmetry of the system. Also there are no singular points other than the origin on the axes, on which the Hamiltonian $H_8$ is quadratic and so the equations $H_8|_{x=0} = h$ or $H_8|_{x=0} = h$ have at most two solutions for any $h \in \mathrm{R}$. This means that the separatrices, on which a Hamiltonian is constant, passing through saddles cannot cross the axes anymore. Hence we obtain the global phase portrait 1.3 of Figure 1.1.

## 3.4 Global phase portraits of systems $(IX)$

We first study the case $b = 0$. Then systems $(IX)$ become

$$\dot{x} = -3x^2y + y^3, \quad \dot{y} = cx + 3xy^2, \tag{3.9}$$

where $c \neq 0$, and they have the Hamiltonian

$$H_9^1(x, y) = \frac{y^4}{4} - \frac{3x^2y^2}{2} - \frac{cx^2}{2}.$$

Using Proposition 23 we rewrite systems (3.9) as

$$\dot{x} = y + 3x^2y/c, \quad \dot{y} = cx^3 - 3xy^2/c.$$

Then by Proposition 22 the origin is a center if and only if $c < 0$. Hence we assume that $c < 0$.

In $U_1$ systems (3.9) become

$$\dot{u} = cv^2 - u^2(u^2 - 6), \quad \dot{v} = -uv(u^2 - 3). \tag{3.10}$$

When $v = 0$, there are three singular points on $U_1$: $(0, 0), (\pm\sqrt{6}, 0)$. The linear part of systems (3.10) is

$$\begin{pmatrix} -4u(u^2 - 3) & 0 \\ 0 & -u(u^2 - 3) \end{pmatrix}.$$

Hence the singular points $(\sqrt{6},0)$ and $(-\sqrt{6},0)$ are attracting and repelling nodes, respectively.

At the origin, however, the linear part is zero. Therefore to describe its local behavior we do the blow-up $(u,v)\mapsto(u,w)$ with $w=v/u$. After eliminating the common factor $u$ between $\dot{u}$ and $\dot{w}$, we obtain the system

$$\dot{u}=cuw^2-u(u^2-6),\quad \dot{w}=-w(cw^2+3). \tag{3.11}$$

When $u=0$, systems (3.11) have the singular points $(0,0)$, $\left(0,\pm\sqrt{-3/c}\right)$, all of which are real since $c<0$. The linear part of systems (3.11) at the points $(0,w)$ is

$$\begin{pmatrix} cw^2+6 & 0 \\ 0 & -3cw^2-3 \end{pmatrix}.$$

So, in addition to the saddle at the origin, the points $\left(0,\pm\sqrt{-3/c}\right)$ are repelling nodes. This time we see that the origin of $U_1$ has two elliptic sectors and two parabolic sectors, see Figure 3.3.

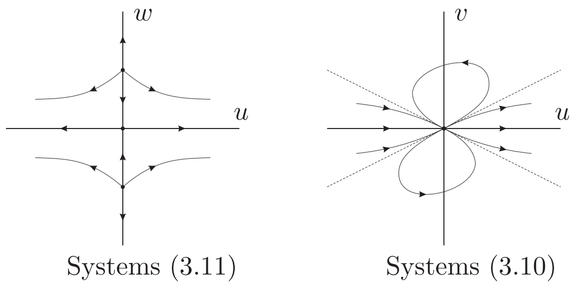



Figure 3.3: Blow-up of the origin of $U_1$ of systems (3.9) when $c<0$.

We now look at the origin of $U_2$, in which systems (3.9) are written as

$$\dot{u}=-cv^2-6u^2+1,\quad \dot{v}=-uv(cv^2+3).$$

We see that the origin of $U_2$ is not a singular point. Hence all the infinite singular points are in $U_1$ and $V_1$.

Besides the origin, systems (3.9) have the four finite singular points $\left(\pm\sqrt{-c/9},\pm\sqrt{-c/3}\right)$ on $U_3$, which are real since $c<0$. The eigenvalues of the linear part of systems (3.9) are $\pm 2c/3$ at each of them, so they are saddles.

We note that the Hamiltonian $H_9^1$ has the same value at all of these saddles since it is an even function. Then we claim that all four of them must be on the boundary of the period annulus of the center at the origin. If there were only two saddles on the mentioned boundary, then a straight line passing through the origin and sufficiently close to the saddles which are not on this boundary would have at least six intersection points with the separatrices of these saddles, which are on the same energy level, see Figure 2.8. But this is impossible because $H_9^1$ is only quartic. Hence the claim is proved.

Moreover, the separatrices of these saddles cross the $x$-axis exactly twice since $H_9^1$ is quadratic in $x$ when $y = 0$. Consequently we conclude that the global phase portrait of systems (3.9) is topologically equivalent to 1.5 in Figure 1.1.

Now we study systems $(IX)$ when $b \neq 0$. In this case systems $(IX)$ become

$$\dot{x} = ax + by - 3x^2y + y^3, \quad \dot{y} = -(a^2/b)x - ay + 3xy^2, \tag{3.12}$$

and they have the Hamiltonian

$$H_9^2(x,y) = \frac{y^4}{4} - \frac{3x^2y^2}{2} + \frac{a^2x^2}{2b} + \frac{by^2}{2} + axy.$$

We use Proposition 23 to rewrite systems (3.12) as

$$\begin{aligned} \dot{x} &= y - \frac{a(a^2-3b^2)x^3}{b^3} + \frac{3(a^2-b^2)x^2y}{b^3} - \frac{3axy^2}{b^3} + \frac{y^3}{b^3}, \\ \dot{y} &= -\frac{a^2(a^2-6b^2)x^3}{b^3} + \frac{3a(a^2-3b^2)x^2y}{b^3} - \frac{3(a^2-b^2)xy^2}{b^3} + \frac{ay^3}{b^3}. \end{aligned}$$

Then by Proposition 22 the origin is a center if and only if

$$a^2/b - 6b > 0 \quad \text{and} \quad a \neq 0. \tag{3.13}$$

Under these restrictions we first investigate the infinite singular points of (3.12). In $U_1$ systems (3.12) become

$$\begin{aligned} \dot{u} &= -v^2\left(bu^2 + 2au + (a^2/b)\right) - u^2(u^2-6), \\ \dot{v} &= -v^3\left(bu + a\right) - uv(u^2-3). \end{aligned} \tag{3.14}$$

When $v = 0$ the singular points are $(0,0), (\pm\sqrt{6}, 0)$. The linear part of systems (3.14) is

$$\begin{pmatrix} -4u(u^2-3) & 0 \\ 0 & -u(u^2-3) \end{pmatrix}.$$

Hence, just like the case $b = 0$, the singular points $(\sqrt{6}, 0)$ and $(-\sqrt{6}, 0)$ are attracting and repelling nodes, respectively.

At the origin, however, the linear part is zero. We do the blow-up $(u,v) \mapsto (u,w)$ with $w = v/u$, eliminate the common factor $u$ between $\dot{u}$ and $\dot{w}$, and obtain the system

$$\begin{aligned} \dot{u} &= -uw^2\left(bu^2 + 2au + (a^2/b)\right) - u(u^2-6), \\ \dot{w} &= w^3\left(au + (a^2/b)\right) - 3w. \end{aligned} \tag{3.15}$$

When $u=0$, systems (3.15) have the singular points $(0,0)$, $\left(0,\pm\sqrt{3b/a^2}\right)$. The linear part of systems (3.15) at the points $(0,w)$ is

$$\begin{pmatrix} -(a^2/b)w^2+6 & 0 \\ aw^3 & 3(a^2/b)w^2-3 \end{pmatrix}.$$

When $b<0$, we see that $\left(0,\pm\sqrt{3b/a^2}\right)$ are not real, hence the only singular point is the origin, which is a saddle. It is shown in Figure 3.4 that the origin of $U_1$ consists of two hyperbolic sectors.

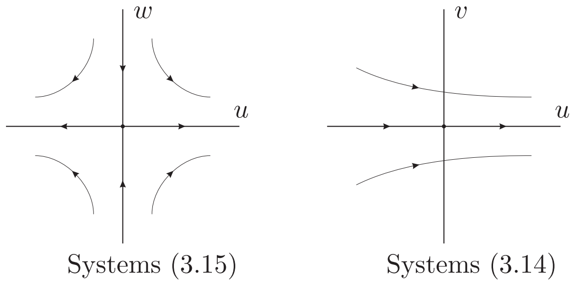



Figure 3.4: Blow-up of the origin of $U_1$ of systems $(IX)$ when $b<0$.

When $b>0$, all three singular points are real. In addition to the saddle at the origin, the points $\left(0,\pm\sqrt{3b/a^2}\right)$ are repelling nodes. This time we see that the origin of $U_1$ has two elliptic sectors and two parabolic sectors, see Figure 3.5.

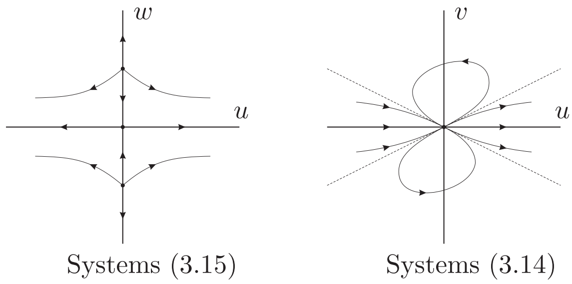



Figure 3.5: Blow-up of the origin of $U_1$ of systems $(IX)$ when $b>0$.

We now look at the origin of $U_2$, in which systems (3.12) are written as

$$\begin{aligned}\dot{u} &= v^2\left((a^2/b)u^2+2au+b\right)-6u^2+1,\\ \dot{v} &= v^3\left((a^2/b)u+a\right)-3uv.\end{aligned}$$

Hence the origin of $U_2$ is not a singular point.

The finite singular points of systems (3.12) other than its origin are

$$p_{1,2} = \pm\left(\frac{(3b-A)\sqrt{B-A}}{6\sqrt{6}a}, \frac{\sqrt{B-A}}{\sqrt{6}}\right),$$
$$p_{3,4} = \pm\left(\frac{(3b+A)\sqrt{B+A}}{6\sqrt{6}a}, \frac{\sqrt{B+A}}{\sqrt{6}}\right),$$

where $A = \sqrt{12a^2 + 9b^2}$ and $B = 2a^2/b - 3b$.

When $b < 0$, we see that the expression

$$(2a^2/b - 3b)^2 - (12a^2 + 9b^2) = 4a^2 b(a^2/b - 6b) \tag{3.16}$$

is negative due to (3.13). Hence $B - A < 0$ and $B + A > 0$. Therefore when $b < 0$ systems (3.12) have only the two finite singular points $p_3$ and $p_4$ in addition to the origin. The eigenvalues of the linear part of systems (3.12) at these points are

$$\pm\frac{\sqrt{4a^4 + 15a^2b^2 + 9b^4 + b(5a^2 - 3b^2)\sqrt{12a^2 + 9b^2}}}{3b}$$

We observe that

$$(4a^4 + 15a^2b^2 + 9b^4)^2 - \left(b(5a^2 - 3b^2)\sqrt{12a^2 + 9b^2}\right)^2 = 4a^2(4a^2 + 3b^2)(a^2 - 6b^2)^2 > 0.$$

Therefore $p_3$ and $p_4$ are saddles. By the symmetry of (3.12) they must be on the boundary of the period annulus of the center at the origin. Since the Hamiltonian $H_9^2$ is quadratic on the $x$-axis, their separatrices can cross the $x$-axis only twice. Consequently we obtain the phase portrait 1.4 of Figure 1.1.

When $b > 0$, on the other hand, (3.16) is positive and

$$2a^2/b - 3b > a^2/b - 6b > 0.$$

Therefore all of the four finite singular points exist. We already saw that $p_3$ and $p_4$ are saddles. Similar computations show that $p_1$ and $p_2$ are also saddles. We observe that only two of these saddles can be on the boundary of the period annulus of the center at the origin. This is due to the fact that the Hamiltonian $H_9^2$ of systems (3.12) is even and that we have

$$H_9^2(p_1) = \frac{2a^4 + 30a^2b^2 - 9b^4 - b(4a^2 + 3b^2)\sqrt{12a^2 + 9b^2}}{72b^2},$$
$$H_9^2(p_3) = \frac{2a^4 + 30a^2b^2 - 9b^4 + b(4a^2 + 3b^2)\sqrt{12a^2 + 9b^2}}{72b^2},$$

which are equal if and only if $b = 0$, which is not the case.

By the same argument used for the case $b < 0$, we see that the saddles which are on the boundary of the period annulus of the center at the origin cannot be connected to any of the other saddles. Therefore the saddles on this boundary have to be connected with the infinite singular points. Due to the fact that the separatrices through these saddles cannot cross the $x$-axis anymore and that the flow around the origin is clockwise, we get the global phase portrait 1.6 shown in Figure 1.1.

## 3.5 Global phase portraits of systems $(X)$

As before we first consider the case $b = 0$. Then systems $(X)$ become

$$\dot{x} = -3x^2y - y^3, \quad \dot{y} = cx + 3xy^2, \tag{3.17}$$

where $c \neq 0$, and they have the Hamiltonian

$$H_{10}^1(x, y) = -\frac{y^4}{4} - \frac{3x^2y^2}{2} - \frac{cx^2}{2}.$$

Using Proposition 23 we rewrite systems (3.17) as

$$\dot{x} = y + 3x^2y/c, \quad \dot{y} = -cx^3 - 3xy^2/c.$$

Then by Proposition 22 the origin is a center if and only if $c > 0$. Hence we will investigate the case $c > 0$.

In $U_1$ systems (3.17) become

$$\dot{u} = cv^2 + u^2(u^2 + 6), \quad \dot{v} = uv(u^2 + 3). \tag{3.18}$$

When $v = 0$, the only singular point is the origin and its linear part is zero. Therefore to study its local behavior we do the blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$. Eliminating the common factor $u$ between $\dot{u}$ and $\dot{w}$, we obtain the system

$$\dot{u} = cuw^2 + u(u^2 + 6), \quad \dot{w} = -w(cw^2 + 3). \tag{3.19}$$

When $u = 0$, systems (3.19) have the unique singular point $(0, 0)$ since $c > 0$ and it is a saddle. Hence we see that the origin of $U_1$ consists of two hyperbolic sectors, see Figure 3.4.

In $U_2$ systems (3.17) are expressed as

$$\dot{u} = -cv^2 - 6u^2 - 1, \quad \dot{v} = -uv(cv^2 + 3).$$

The origin of $U_2$ is not a singular point. Hence the infinite singular points are only the origins of $U_1$ and $V_1$.

The finite singular points of (3.17) are $(0, 0)$ and $\left(\pm\sqrt{-c/9}, \pm\sqrt{-c/3}\right)$ on $U_3$. But because $c > 0$, only the origin is real. As a result we conclude that the global phase portrait of systems (3.17) is topologically equivalent to 1.1 in Figure 1.1.

Now we study systems $(X)$ when $b \neq 0$. In this case systems $(X)$ are written as

$$\dot{x} = ax + by - 3x^2y - y^3, \quad \dot{y} = -(a^2/b)x - ay + 3xy^2, \tag{3.20}$$

and they have the Hamiltonian

$$H_{10}^2(x,y) = -\frac{y^4}{4} - \frac{3x^2y^2}{2} + \frac{a^2x^2}{2b} + \frac{by^2}{2} + axy.$$

We apply Proposition 23 to systems (3.17) and get

$$\begin{aligned}
\dot{x} &= y + \frac{a(a^2+3b^2)x^3}{b^3} - \frac{3(a^2+b^2)x^2y}{b^3} + \frac{3axy^2}{b^3} - \frac{y^3}{b^3},\\
\dot{y} &= \frac{a^2(a^2+6b^2)x^3}{b^3} - \frac{3a(a^2+3b^2)x^2y}{b^3} + \frac{3(a^2+b^2)xy^2}{b^3} - \frac{ay^3}{b^3}.
\end{aligned}$$

Then by Proposition 22 the origin is a center if and only if $b < 0$ and $a \neq 0$. Therefore we impose these two conditions on (3.20) and begin with investigating its infinite singular points.

In $U_1$ we have

$$\begin{aligned}
\dot{u} &= -v^2\left(bu^2 + 2au + (a^2/b)\right) + u^2(u^2+6),\\
\dot{v} &= -v^3\left(bu + a\right) + uv(u^2+3).
\end{aligned} \tag{3.21}$$

When $v = 0$ the only singular point of (3.21) is the origin, at which the linear part of the system is zero. We do the blow-up $(u,v) \mapsto (u,w)$ with $w = v/u$, eliminate the common factor $u$ and obtain the system

$$\begin{aligned}
\dot{u} &= -uw^2\left(bu^2 + 2au + (a^2/b)\right) + u(u^2+6),\\
\dot{w} &= w^3\left(au + (a^2/b)\right) - 3w.
\end{aligned} \tag{3.22}$$

When $u = 0$, the only singular point of systems (3.22) is the origin, which is a saddle. It is shown in Figure 3.4 that the origin of $U_1$ consists of two hyperbolic sectors.

The origin of $U_2$, in which systems (3.20) are written as

$$\begin{aligned}
\dot{u} &= v^2\left((a^2/b)u^2 + 2au + b\right) - 6u^2 - 1,\\
\dot{v} &= v^3\left((a^2/b)u + a\right) - 3uv,
\end{aligned}$$

is clearly not a singular point.

Next we consider the finite singular points of systems (3.20) which are

$$\begin{aligned}
p_{1,2} &= \pm\left(\frac{(3b+A)\sqrt{B-A}}{6\sqrt{6}a}, \frac{\sqrt{B-A}}{\sqrt{6}}\right),\\
p_{3,4} &= \pm\left(\frac{(3b-A)\sqrt{B+A}}{6\sqrt{6}a}, \frac{\sqrt{B+A}}{\sqrt{6}}\right),
\end{aligned}$$

where $A = \sqrt{-12a^2 + 9b^2}$ and $B = 2a^2/b + 3b$. Due to our assumption that $b < 0$, we observe that

$$\begin{aligned} B - A < B + A =& 2a^2/b + 3b + \sqrt{-12a^2 + 9b^2} \\ <& 2a^2/b + 3b + \sqrt{9b^2} \\ =& 2a^2/b < 0 \end{aligned}$$

Therefore the origin is the only finite singular point of systems (3.20) and we easily see that their global phase portrait is topologically equivalent to 1.1 of Figure 1.1.

## 3.6 Global phase portraits of systems $(XI)$

When $b = 0$, systems $(XI)$ become

$$\dot{x} = -3\mu x^2 y + y^3, \quad \dot{y} = cx + x^3 + 3\mu x y^2, \tag{3.23}$$

where $c \neq 0$, and they have the Hamiltonian

$$H_{11}^1(x, y) = \frac{y^4 - x^4}{4} - \frac{3\mu x^2 y^2}{2} - \frac{cx^2}{2}.$$

Using Proposition 23 we rewrite systems (3.23) as

$$\dot{x} = y + 3\mu x^2 y/c + y^3/c, \quad \dot{y} = cx^3 - 3\mu x y^2/c.$$

Then by Proposition 22 the origin is a center if and only if $c < 0$. Therefore we assume $c < 0$.

As in the previous systems, we first investigate the infinite singular points of systems (3.23). In the local chart $U_1$ we have

$$\begin{aligned} \dot{u} =& cv^2 - u^4 + 6\mu u^2 + 1, \\ \dot{v} =& -vu(u^2 - 3\mu). \end{aligned} \tag{3.24}$$

When $v = 0$, the real singular points are $\left(\pm\sqrt{3\mu + \sqrt{9\mu^2 + 1}}, 0\right)$, and the linear part of (3.24) is

$$\begin{pmatrix} -4u(u^2 - 3\mu) & 0 \\ 0 & -u(u^2 - 3\mu) \end{pmatrix},$$

Hence the points $\left(\sqrt{3\mu + \sqrt{9\mu^2 + 1}}, 0\right)$ and $\left(-\sqrt{3\mu + \sqrt{9\mu^2 + 1}}, 0\right)$ are respectively attracting and repelling nodes of systems (3.24).

In $U_2$ systems (3.23) become

$$\begin{aligned} \dot{u} =& -cv^2u^2 - u^4 - 6\mu u^2 + 1, \\ \dot{v} =& -vu(cv^2 + u^2 + 3\mu), \end{aligned}$$

and we see that the origin is not a singular point. Hence systems $(XI)$ have four infinite singular points, all of which are nodes on $U_1$ and $V_1$.

The finite singular points of systems (3.23) are the origin, $\pm\left(\sqrt{-c},0\right)$ and

$$p_i = \pm\left(\pm\sqrt{-c/(1+9\mu^2)}, \sqrt{-3c\mu/(1+9\mu^2)}\right).$$

with $i = 1,2,3,4$. We observe that when $\mu < 0$ the points $p_i$ do not exist, and when $\mu = 0$ they are equal to $\pm\left(\sqrt{-c},0\right)$. As a result, systems (3.23) have three finite singular points when $\mu \leq 0$, and seven when $\mu > 0$. We note that the linear part of systems (3.23) is

$$M_{11}^1 = \begin{pmatrix} -6\mu xy & 3y^2 - 3\mu x^2 \\ c + 3x^2 + 3\mu y^2 & 6\mu xy \end{pmatrix}.$$

We first consider the case $\mu \leq 0$. We know that the origin is a center. The eigenvalues of $M_5^1$ at $\pm\left(\sqrt{-c},0\right)$ are $\pm c\sqrt{-6\mu}$. This means that these points are hyperbolic saddles when $\mu < 0$, but are degenerate when $\mu = 0$. But $M_{11}^1$ is not identically zero at these points since

$$c + 3x^2 + 3\mu y^2 = -2c > 0.$$

So they are nilpotent. Due to the fact that nilpotent singular points of Hamiltonian vector fields are either saddles, centers or cusps, and that the number of singular points is fixed for $\mu \leq 0$, we conclude that the points $\pm\left(\sqrt{-c},0\right)$ are saddles also when $\mu = 0$.

Then the global phase portrait in this case is topologically equivalent to 1.3 of Figure 1.1. This is because the two saddles must be on the boundary of the period annulus of the center at the origin. Their remaining separatrices cannot cross the straight lines passing through the origin and the infinite singular points, namely $y = \pm\sqrt{3\mu + \sqrt{9\mu^2+1}x}$, because the Hamiltonian $H_{11}^1 = -cx^2/2$ is quadratic on these lines. Then taking into account the flow at infinity we obtain the global phase portrait.

Secondly we look at the case $\mu > 0$. In this case the points $\pm\left(\sqrt{-c},0\right)$ are centers. In addition, each $p_i$ is a saddle because the eigenvalues of $M_{11}^1$ at these points are $\pm 2c\sqrt{3\mu}/\sqrt{1+9\mu^2}$. Since the Hamiltonian of systems (3.23) is even, $H_{11}^1(p_i)$ is constant for all $i$. Then, by the same same argument that we used in systems $(IX)$ (see Figure 2.8), we deduce that every $p_i$ must be on the boundary of the period annulus of the center at the origin.

Since the infinite singular points are nodes, the centers can only be created by connecting two adjacent saddles. The flow around the origin is clockwise because $c < 0$. In addition, the remaining separatrices of any of the saddles must lie on different sides of the straight line passing through that saddle and the origin, otherwise there would exist another straight line through the origin intersecting both separatrices in six intersection points in the same energy level. Then the flow at infinity ensures that the remaining two centers are formed by connecting adjacent saddles which lie on the same side of the plane with respect to the $y$-axis. Recalling from the case $\mu \leq 0$ that the remaining separatrices of these saddles cannot cross the straight lines passing through the origin and the infinite singular points, we get the global phase portrait 1.10 in Figure 1.1.

Having established the case $b = 0$, we now investigate the case $b \neq 0$ in which systems $(XI)$ are written as

$$\dot{x} = ax + by - 3\mu x^2 y + y^3, \tag{3.25a}$$
$$\dot{y} = -(a^2/b)x - ay + x^3 + 3\mu xy^2, \tag{3.25b}$$

and they have the Hamiltonian

$$H_{11}^2(x, y) = \frac{y^4 - x^4}{4} - \frac{3\mu x^2 y^2}{2} + \frac{a^2 x^2}{2b} + \frac{by^2}{2} + axy.$$

We remark that without loss of generality we can assume $b > 0$. If we do the linear transformation $(x, y) \mapsto (-y, -x)$, systems $(XI)$ become

$$\begin{aligned} -\dot{y} &= -ay - bx + 3\mu y^2 x - x^3, \\ -\dot{x} &= (a^2/b)y + ax - y^3 - 3\mu yx^2, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \dot{x} &= -ax - (a^2/b)y + 3\mu x^2 y + y^3, \\ \dot{y} &= bx + ay + x^3 - 3\mu xy^2. \end{aligned} \tag{3.26}$$

After defining $\bar{a} = -a$, $\bar{\mu} = -\mu$, and $\bar{b} = a^2/b$, we see that systems (3.26) are basically systems $(XI)$ with $b \mapsto -b$ whenever $a \neq 0$. But when $a = 0$, systems (3.25) are just systems (3.23) with the axes interchanged and $\mu \mapsto -\mu$. Hence we know that they have a center at the origin if and only if $b < 0$ with a global phase portrait topologically equivalent to 1.3 and 1.10 of Figure 1.1 when $\mu \geq 0$ and $\mu < 0$, respectively. Therefore we need to investigate the case $a \neq 0$, and we can assume $b > 0$.

Using Proposition 23 we rewrite systems (3.25) as

$$\begin{aligned} \dot{x} =& y - \frac{a(a^2 - 3b^2\mu)x^3}{b^3} + \frac{3(a^2 - b^2\mu)x^2 y}{b^3} - \frac{3axy^2}{b^3} + \frac{y^3}{b^3}, \\ \dot{y} =& -\frac{(a^4 - b^4 - 6a^2b^2\mu)x^3}{b^3} + \frac{3a(a^2 - 3b^2\mu)x^2 y}{b^3} - \frac{3(a^2 - b^2\mu)xy^2}{b^3} + \frac{ay^3}{b^3}. \end{aligned}$$

Therefore, by Proposition 22, we impose on systems (3.25) the two conditions

$$a^4 - b^4 - 6a^2b^2\mu > 0 \quad \text{and} \quad b > 0. \tag{3.27}$$

The infinite singular points of systems (3.25) are the same as those of (3.23), an attracting node $\left(\sqrt{3\mu+\sqrt{9\mu^2+1}},0\right)$ and a repelling node $\left(-\sqrt{3\mu+\sqrt{9\mu^2+1}},0\right)$ on $U_1$, and the corresponding points on $V_1$. The origin of $U_2$ is not a singular point.

The explicit expressions for the finite singular points in terms of the parameters $a, b, \beta$ are lengthy. For this reason we follow the approach we used for systems $(III)$–$(VI)$. We first find the maximum number of finite singular points allowed by the system. We equate (3.25a) to zero, solve for $x$ and get

$$x_{1,2} = \frac{a \pm \sqrt{a^2+12b\mu y^2+12\mu y^4}}{6\mu y}. \tag{3.28}$$

Note that when $y=0$, (3.25b) is zero if and only if $x=0$ since $a \neq 0$. So we conclude that the only finite singular point on the $x$-axis is the origin, which we know is a center. Hence we can assume that $y \neq 0$. Then it remains to study the case $\mu=0$ separately, which we will consider later on, but we first assume that $\mu \neq 0$.

If we substitute (3.28) into (3.25b) we obtain

$$\begin{aligned}\dot{y}_{1,2} = &-\frac{1}{54b\mu^3y^3}\Big(a\big(a^2b+9\mu(b^2-a^2\mu)y^2+9b\mu(1-3\mu^2)y^4\big)\\ &\pm\sqrt{a^2+12b\mu y^2+12\mu y^4}\big(a^2b+3\mu(b^2-3a^2\mu)y^2+3b\mu(1+9\mu^2)y^4\big)\Big),\end{aligned}$$

where $\dot{y}_1$ and $\dot{y}_2$ denote $\dot{y}$ with $x_1$ and $x_2$ substituted, respectively. Then the maximum number of roots of the product $\dot{y}_1\dot{y}_2$ will give an upper bound for the number of finite singular points. So we multiply $\dot{y}_1$ and $\dot{y}_2$ and obtain

$$\begin{aligned}\frac{-1}{27b^2\mu^3}\big(&b^2(1+9\mu^2)y^6+3b(1+9\mu^2)(b^2-2a^2\mu+3b^2\mu^2)y^4\\ &+3(b^4+3a^4\mu^2+6b^4\mu^2-18a^2b^2\mu^3)y^2\\ &-b(a^4-b^4-6a^2b^2\mu)\big).\end{aligned} \tag{3.29}$$

We see that (3.29) cannot be identically zero because its constant term is different from zero due to (3.27), so it has at most six real roots. This means that for $\mu \neq 0$, all the finite singular points of systems (3.25) are isolated and that there are at most six of them without taking into account the origin.

If $\mu=0$, on the other hand, systems (3.25) are written as

$$\dot{x} = ax+by+y^3, \quad \dot{y} = -(a^2/b)x-ay+x^3.$$

We can easily compute its finite singular points and see that besides the origin it has only two, namely $\pm\big(b^{1/3}\sqrt{(1-b^{4/3})/b}, -\sqrt{(1-b^{4/3})/b}\big)$. Therefore we conclude that systems (3.25) have at most six finite singular points.

Next we will count the indices of the known singular points, both finite and infinite, and then deduce some conditions on the remaining finite singular points of the system. We remind that when determining the finite singular points by considering their total index, those with index zero are hard to detect. To overcome this difficulty we present the following lemma, which is inspired from Lemma 18. The difference between Lemma 18 and Lemma 24 is that the former applies to vector fields with a linear type center at the origin, and the latter applies to those with a nilpotent center. We recall that a center–loop is a center inside the loop as in Figure 2.6.

**Lemma 24.** *Let $X_0$ be a real Hamiltonian planar polynomial vector field having only linear and cubic terms. Then $X_0$ can be written as*

$$\dot{x} = a_{10}x + a_{01}y + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3,$$
$$\dot{y} = b_{10}x - a_{10}y + b_{30}x^3 - 3a_{30}x^2y - a_{21}xy^2 - \frac{1}{3}a_{12}y^3.$$

*Suppose that $p$ is an isolated singular point of $X_0$ different from the origin. Then there is a perturbation $X_\varepsilon$ of $X_0$ such that $X_\varepsilon$ is also a real Hamiltonian planar polynomial vector field having only linear and cubic terms, and that $p$ is a singular point of $X_\varepsilon$. In addition, if $a_{10}^2 + a_{01}b_{10} = 0$ but $a_{01} \neq 0$ then the following statements hold:*

*(a) If $p$ is non–elementary, then it is nilpotent.*

*(b) If $p$ is a non–elementary singular point of $X_0$, then it is an elementary singular point of $X_\varepsilon$ with $\varepsilon \neq 0$.*

*(c) If $p$ is a cusp of $X_0$, then for $\varepsilon \neq 0$ small enough such that $\varepsilon a_{01} < 0$, the origin of $X_\varepsilon$ is a linear type center and the local phase portrait of $X_\varepsilon$ at $p$ is a center-loop.*

*Proof.* As we did in the proof of Lemma 18 without loss of generality we can assume $p = (0, y_0)$, and define $X_\varepsilon$ as (2.12). The proofs of statements $(a)$ and $(b)$ are the same as those of Lemma 18. The proof of statement $(c)$ is almost the same except some details which we point out below. So we assume that $(0, y_0)$ is a cusp of $X_0$ and that $\varepsilon a_{01} < 0$.

First of all, the eigenvalues of the linear part of $X_\varepsilon$ at the origin are $\pm\sqrt{\varepsilon a_{01}}$, which are purely imaginary by assumption. Since $X_\varepsilon$ is Hamiltonian we conclude that the origin is a linear type center.

Second, due to the facts that $(0, y_0)$ is a singular point and $y_0 \neq 0$, we have $a_{01} = -a_{03}y_0^2$. Then the eigenvalues of the linear part of (2.12) at $(0, y_0)$ are

$$\pm\sqrt{\varepsilon(3a_{03}y_0^2 + a_{01})} = \pm\sqrt{\varepsilon(-2a_{01})},$$

see (2.13) and (2.15). Hence $(0, y_0)$ is a saddle by assumption. Then the rest of the proof is exactly the same as the proof of statement $(c)$ of Lemma 18. □

With the help of this lemma, we will be able to detect possible cusps of the vector fields by studying the center–loops of the Hamiltonian linear type centers obtained by the perturbation in statement ($c$) of Lemma 24.

We continue determining the finite singular points of systems (3.25). By Theorem 2, the four nodes at infinity and the centers at the origins of $U_3$ and $V_3$ have total index 6 on the Poincaré sphere. Then by Theorem 15, the remaining finite singular points on the Poincaré disk have to have total index -2.

By statement ($a$) of Lemma 24, these singular points are either elementary or nilpotent. Hence they are either centers, saddles or cusps. We claim that at most two finite singular points of systems (3.25) which are different from the origin are nilpotent. To prove this claim we compute the Gröbner basis for the polynomials (3.25a), (3.25b) and the determinant of the linear part of systems (3.25). Recall that $y \neq 0$. We obtain sixteen polynomials, two of which suffice to prove our claim. One is a quadratic polynomial only in the variable $y$, where the coefficient of $y^2$ is

$$5(1+\mu^2)(-1+3\mu^2)^2(1+9\mu^2)^2, \tag{3.30}$$

and the other is a polynomial in the variables $x$ and $y$, linear in $x$ with the coefficient $a(1+9\mu^2)$.

We know that $a \neq 0$ due to (3.27). Therefore when (3.30) is not zero, systems ($XI$) can have at most two nilpotent singular points. When (3.30) is zero, i.e. $\mu^2 = 1/3$, we see that the Gröbner basis consists only of the two polynomials $y^2$ and $ax+by$. Therefore systems ($XI$) have at most two nilpotent singular points in any case, and this proves our claim.

In short, by Corollary 17, finite singular points of systems (3.25) other than the origin can be either ($i$) two saddles, ($ii$) two saddles and two cusps, or ($iii$) four saddles and two centers.

Consider case ($i$). The two saddles must be on the boundary of the period annulus of the center at the origin. Their remaining separatrices cannot cross the straight lines passing through the origin and the infinite singular points, namely $y = \pm\sqrt{3\mu+\sqrt{9\mu^2+1}}x$, because the Hamiltonian $H_{11}^2$ is quadratic on these lines. Then, due to the flow at infinity, we get a global phase portrait which is topologically equivalent to 1.3 of Figure 1.1. We remark that this phase portrait is achieved for the values $a = 2$, $b = 1$ and $\mu = 1/4$.

In case ($iii$) we claim that there can be at most two saddles on the boundary of the period annulus of the origin. We will prove this by computing the Gröbner basis for the polynomials $\dot{x}$, $\dot{y}$ and $H_{11}^2 - h$ for some $h \in \mathbb{R}$ and showing that $H_{11}^2$ may attain the same value at no more than two singular points.

For future reference, we first consider the cases $h = 0$ and $\mu = 0$. When $h = 0$, one of the polynomials in the Gröbner basis is $y^3(a^4 - b^4 - 6a^2b^2\mu)^2$, which implies that the only singular point where $H_{11}^2$ vanishes is the origin (see (3.27)). So we assume $h \neq 0$.

When $\mu = 0$ the Gröbner basis contains the polynomial

$$9b^5h(a^4 + h^2)y^2 + hb^2(b^8 - 2a^8 + a^4b^4) + h^2(4b^8 - a^8 + 9a^4b^4) \\ +b^2h^3(2a^4 + 11b^4) - b^4h^4,$$

which has at most two roots since the coefficient of $y^2$ is not zero. Then, because (3.25a) is linear in $x$ when $\mu = 0$, we deduce that there can be at most two roots in $\mathbb{R}^2$ Hence we can also assume $\mu \neq 0$.

We now prove our claim but we will not provide every detail because of the length of the polynomials. The Gröbner basis of the previously mentioned three polynomials contains 37 even polynomials that do not contain $x$, of which 23 are quadratic and 12 are quartic. In addition, there is a polynomial which is linear in $x$, and the coefficient of $x$ is $4ah \neq 0$ (see (3.27)). So if we can show that the above mentioned 37 polynomials have at most two roots, then the proof will be finished.

If the coefficient of $y^2$ of at least one of these 23 polynomials is nonzero, then we are done. If not, we obtain two conditions: one from calculating the resultants of pairs of these 23 coefficients with respect to $h$, and the other from equating the coefficient of the simplest of these 23 polynomials to zero and solving for $h$. Then we see that the discriminant of one of the 12 polynomials in the Gröbner basis which are quartic even polynomials in $y$ is zero, and its coefficient of $y^4$ is nonzero. This means that this polynomial can have at most two distinct roots, and the claim is proved.

Having established the property that only two of the saddles can be on the boundary of period annulus of the center at the origin, we observe that the remaining separatrices of these saddles either go to infinity or go back to the one of these saddles in one of the ways shown in Figure 2.9 (see Figure 2.8 also).

The first figure cannot be realized due to the fact that (3.25a) does not change on the positive $y$-axis since $b > 0$. We see that the second figure cannot be achieved either, see Figure 2.7. Hence we conclude that the remaining saparatrices of these saddles go to infinite singular points, of course without crossing the straight lines passing through the origin and the infinite singular points.

Because of the symmetry of systems (3.25) and the flow near infinity, the remaining finite singular points must be symmetric with respect to the origin, and also there must be a saddle on the boundary of the period annulus of each of the centers, creating a center–loop. We observe that a center–loop may exist only if a straight line $l_1$ passing through the origin and the saddle of the center–loop intersects the separatrices of this saddle at exactly on point, namely the saddle itself, see Remark 20 and Figure 2.11. Hence there is only one way, up to topological equivalence, for the center–loops of systems (3.25) to be formed, and we obtain the global phase portrait 1.11 of Figure 1.1. For $a = 5$, $b = 1$ and $\mu = 1/4$ such a phase portrait is a realized.

Finally we consider case (*ii*). By statement (*c*) of Lemma 24, systems (3.25) become Hamiltonian linear type centers which have a center–loop. We already know by our previous work that the global phase portrait of these perturbed systems is topologically equivalent to 1.11 in Figure 1.1 also. Therefore the only possible global phase portrait in this case is the portrait 1.12 of Figure 1.1. The particular global phase portrait examples that we provided for the cases (*i*) and (*iii*) ensure that the phase portrait 1.2 is actually realized when $b = 1$ and $\mu = 1/4$ for some $a \in (2, 5)$.

## 3.7 Global phase portraits of systems $(XII)$

When $b = 0$, systems $(XII)$ become

$$\dot{x} = -3\mu x^2 y - y^3, \quad \dot{y} = cx + x^3 + 3\mu x y^2, \tag{3.31}$$

and they have the Hamiltonian

$$H_{12}^1(x, y) = -\frac{y^4 + x^4}{4} - \frac{3\mu x^2 y^2}{2} - \frac{cx^2}{2}.$$

Using Proposition 22 we see that systems (3.31) have a center at the origin if and only if $c > 0$.

In the local chart $U_1$ systems (3.31) are written as

$$\begin{aligned} \dot{u} &= cv^2 + u^4 + 6u^2\mu + 1, \\ \dot{v} &= uv(u^2 + 3\mu). \end{aligned} \tag{3.32}$$

When $v = 0$, the possible singular points of systems (3.32) are the four points $\big(\pm\sqrt{-3\mu \pm \sqrt{9\mu^2 - 1}}, 0\big)$. Therefore, in $U_1$ there are four singular points if $\mu < -1/3$, two if $\mu = -1/3$, and none if $\mu > -1/3$.

In $U_2$ systems (3.31) become

$$\begin{aligned} \dot{u} &= -cu^2v^2 - u^4 - 6u^2\mu - 1, \\ \dot{v} &= -uv(u^2 + cv^2 + 3\mu), \end{aligned}$$

and we see that the origin is not a singular point. Hence all the infinite singular points are on the local charts $U_1$ and $V_1$. The existence of these singular points depend on the parameter $\mu$, so we will investigate the phase portraits in the corresponding subcases.

The finite singular points of systems (3.31) are the origin, $\pm\left(\sqrt{-c},0\right)$ and

$$p_i = \pm\left(\pm\sqrt{c/9\mu^2-1}), \sqrt{-3c\mu/(9\mu^2-1)}\right). \tag{3.33}$$

with $i=1,2,3,4$, for $\mu \neq \pm 1/3$. The points $\pm\left(\sqrt{-c},0\right)$ are not real since $c>0$. When $\mu > -1/3$ none of the $p_i$ are real, whereas they are all real and are saddles when $\mu < -1/3$.

In addition, when $\mu = -1/3$ systems (3.31) become

$$\dot{x} = y(x^2-y^2), \quad \dot{y} = cx + x(x^2-y^2),$$

hence the only singular point is the origin since $c>0$. Similarly when $\mu = 1/3$ only the origin is a singular point.

In short, systems (3.31) do not have any finite singular points other than the origin if $\mu \geq -1/3$, however, they the points $p_i$ for $i=1,2,3,4$ otherwise.

We first assume that $\mu < -1/3$. Then all of the four singular points on $U_1$ are real such that $\left(\sqrt{-3\mu+\sqrt{9\mu^2-1}},0\right)$ and $\left(-\sqrt{-3\mu-\sqrt{9\mu^2-1}},0\right)$ are repelling nodes, whereas the other two points $\left(\sqrt{-3\mu-\sqrt{9\mu^2-1}},0\right)$ and $\left(-\sqrt{-3\mu+\sqrt{9\mu^2-1}},0\right)$ are attracting ones.

We have $H^1_{12}(p_i) = H^1_{12}(p_{(i+1)})$ for $i=1,2,3$ because the Hamiltonian is an even function. Then all of them must be on the boundary of the period annulus of the center at the origin, see Figure 2.8. The remaining separatrices of the saddles cannot cross the straight lines passing through the origin and the infinite singular points because $H^1_{12} = -cx^2/2$ on these lines. Therefore all these separatrices must go to the infinite singular points as it is shown in the global phase portrait 1.14 of Figure 1.1.

Next we assume that $\mu = -1/3$. The linear part of systems (3.32) at both singular points $(\pm 1,0)$ is zero. So we need blow-ups to understand the local behavior at these points. We will do the computations for the point $(1,0)$, and the other point $(-1,0)$ can be studied in the same way.

First we move $(1,0)$ to the origin by the shift $u \mapsto u+1$, and get the system

$$\dot{u} = cv^2 + u^2(u+2)^2, \quad \dot{v} = uv(u+1)(u+2).$$

Now if we do the blow-up $(u,v) \mapsto (u,w)$ with $w = v/u$ and eliminate the common factor $u$, we get the system

$$\dot{u} = u(u+2)^2 + cuw^2, \quad \dot{w} = -w(u+cw^2+2). \tag{3.34}$$

We see that since $c>0$, the only singular point of systems (3.34) when $u=0$ is the origin, which is a saddle. Therefore the singular point $(1,0)$ of systems (3.32) has two hyperbolic sectors, see Figure 3.4.

Performing the same procedure for the point $(-1, 0)$ reveals that the local behavior around this point is the same as that of $(1, 0)$. Since there are no finite singular points in this case, we get the global phase portrait 1.15 of Figure 1.1.

Finally, when $\mu > -1/3$ systems (3.31) have neither infinite nor finite singular points. Since $c > 0$, we get the global phase portrait 1.18 of Figure 1.1.

We have finished studying systems $(XII)$ in the case $b = 0$, and now we assume that $b \neq 0$. Then systems $(XII)$ are written as

$$\dot{x} = ax + by - 3\mu x^2 y - y^3, \tag{3.35a}$$
$$\dot{y} = -(a^2/b)x - ay + x^3 + 3\mu x y^2, \tag{3.35b}$$

and they have the Hamiltonian

$$H_{12}^2(x, y) = -\frac{y^4 + x^4}{4} - \frac{3\mu x^2 y^2}{2} + \frac{a^2 x^2}{2b} + \frac{by^2}{2} + axy.$$

By Propositions 22 and 23 the origin is a center if and only if

$$\frac{a^4 + b^4 + 6a^2 b^2 \mu}{b} < 0. \tag{3.36}$$

The infinite singular points of systems (3.35) are the same as those of systems (3.31).

We will study the finite singular points in the same way as we did in the case $b \neq 0$ of systems $(XI)$. We first find an upper bound for the number of finite singular points. We equate (3.35a) to zero and solve for $x$ and obtain

$$x_{1,2} = \frac{a \pm \sqrt{a^2 + 12b\mu y^2 - 12\mu y^4}}{6\mu y},$$

if $\mu \neq 0$ (we will handle the case $\mu = 0$ separately). Note that the only singular point of systems (3.35) on the $x$-axis is the origin, so we can assume $y \neq 0$. If we substitute these into (3.35b) and multiply the two respective functions, we obtain a polynomial in $y$ of degree six which cannot be identically zero. This means that systems (3.35) have at most six other finite singular points on the Poincaré disk when $\mu \neq 0$.

When $\mu = 0$ the center condition (3.36) requires that $b < 0$. If we compute the finite singular points of systems (3.35) in this case we see that only the origin is singular. Therefore we conclude that, in any case, systems (3.35) have at most six finite singular points.

We will now determine the global phase portraits of systems (3.35) by considering the different values of $\mu$ that lead to different phase portraits at infinity.

When $\mu < -1/3$, the infinite singular points on $U_1$ are the repelling nodes $\big(\sqrt{-3\mu + \sqrt{9\mu^2 - 1}}, 0\big)$ and $\big(-\sqrt{-3\mu - \sqrt{9\mu^2 - 1}}, 0\big)$, and the attracting nodes $\big(\sqrt{-3\mu - \sqrt{9\mu^2 - 1}}, 0\big)$ and $\big(-\sqrt{-3\mu + \sqrt{9\mu^2 - 1}}, 0\big)$. Then the infinite singular points and the centers at the origins of $U_3$ and $V_3$ have total index 10 in the Poincaré sphere. Hence the remaining finite singular points in the Poincaré disk must have total index -4. As we did for systems $(XI)$, using Gröbner bases and statement $(a)$ of Lemma 24 we see that there can be at most two finite nilpotent singular points. Then the possibilities are: $(i)$ four saddles, and $(ii)$ four saddles and two cusps.

By statement $(c)$ of Lemma 24, case $(ii)$ would require the existence of a Hamiltonian linear type center with eight inifinite singular points and center–loops in the finite region. We know from Theorem 4 that such a vector field does not exist. Therefore we discard case $(ii)$.

In case $(i)$ there can be either two or four saddles on the boundary of the period annulus of the center at the origin. In both cases, the remaining separatrices of these saddles cannot cross the straight lines passing through the origin and the infinite singular points as in the case $b = 0$. Therefore we conclude that the global phase portrait is topologically equivalent to either 1.14 (with $2a = 2b = -\mu = 2$) or 1.13 (with $2a = b = -\mu = 2$) of Figure 1.1.

We now look at the case $\mu = -1/3$. We remark that the center condition (3.36) becomes equivalent to $b < 0$ and $a^2 \neq b^2$. We saw that the infinite singular points $(\pm 1, 0)$ on $U_1$ have two hyperbolic sectors. Therefore the total index of the infinite singular points and the centers at the origins of $U_3$ and $V_3$ is 2 on the Poincaré sphere. Hence, in the Poincaré disk, the total index of the remaining possible six finite singular points must be 0. Then, other than the origin, there are either $(i)$ no more singular points, $(ii)$ two cusps, $(iii)$ two saddles and two centers, $(iv)$ two saddles, two centers and two cusps.

We claim that only case $(i)$, whose global phase portrait is topologically equivalent to 1.15 of Figure 1.1, is realizable. We can immediately eliminate case $(iv)$ by Lemma 24 because a Hamiltonian linear type center with four infinite singular points and center–loops does not exist as it is shown in Theorem 4. Case $(ii)$ is also easily eliminated by Lemma 24, see Figure 2.7. So it only remains to show that case $(iii)$ cannot be realized.

Since the infinite singular points have hyperbolic sectors, the finite region of a phase portrait in case $(iii)$ must be one of the two possibilities shown in Figure 2.9. The first figure is not possible due to the fact that (3.35a) does not change sign on the positive $y$-axis since $b < 0$. The second figure is clearly not possible either, see Figure 2.7. Hence the claim is proved and the case $\mu = -1/3$ is finished.

Finally, when $\mu > -1/3$, systems (3.35) have no infinite singular points. Then the remaining possible six finite singular points on the Poincaré disk must have total index 0. Hence we have the following possibilities: ($i$) no singular points, ($ii$) two cusps, ($iii$) two saddles and two centers, ($iv$) two saddles, two centers, and two cusps. Observe that these are exactly the same possible cases that we had when $\mu = -1/3$.

We claim that (3.35) do not have any additional finite singular points in this case and that its global phase portrait is 1.18 of Figure 1.1. We prove our claim in the next paragraph. We note that the center condition (3.36) is equivalent to $b < 0$ when $\mu > -1/3$.

We can eliminate cases ($ii$) and ($iii$) by the same reasoning that we used for $\mu = -1/3$. Considering case ($iv$) we see that a Hamiltonian linear type center which has no infinite singular points but has center–loops exists, whose global phase portrait is topologically equivalent to the first figure in Figure 3.6. This suggests that a global phase portrait in case ($iv$) may exist only if it is topologically equivalent to the second figure in Figure 3.6. But (3.35a) is strictly negative on the positive $y$-axis since $b < 0$. Hence (3.35) cannot have such a phase portrait. This proves our claim, finishing the proof of Theorem 5.

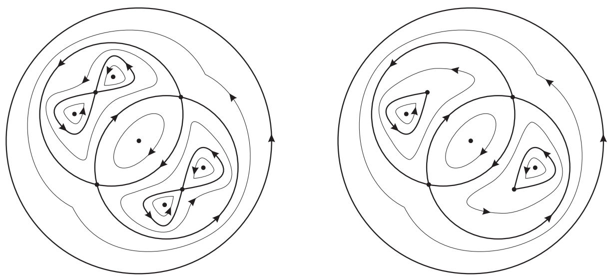


Figure 3.6: A possible cusp for systems $(XII)$ when $b \neq 0$.

# Chapter 4

# Proof of Theorem 7

In this chapter we prove Theorem 7. Statement $(a)$ follows directly from the result obtained in Section 2.2. Furthermore systems $(II)$, up to topological equivalence, have a unique global phase portrait. However Chapter 2 provides very little information to obtain the full bifurcation diagrams for the families $(III)$–$(VI)$. Therefore in this chapter we focus on these families. We note that the explicit expressions of the finite singular points of these families are complicated, making difficult to study their local phase portraits or even their existence on the parameter space. Therefore we will follow different approaches in determining the bifurcation diagrams.

## 4.1 Bifurcation diagram for systems $(III)$

In Section 2.4 it is shown that each phase portrait of systems $(III)$ is topologically equivalent to the phase portrait 1.4 of Figure 1.1 when $b < 0$, and to either 1.5 or 1.6 when $b > 0$. Thus we only need to determine the bifurcation values of parameter $a$ in the case $b > 0$ leading to either the phase portrait 1.5 or the phase portrait 1.6. So we assume $b > 0$. We make the following remark.

**Remark 25.** There can be at most two finite saddles at a fixed energy level in the phase portrait 1.6. Indeed if in the phase portrait 1.6 all four saddles were at the same energy level, then a straight line through the origin passing close enough to the saddles that are not on the boundary of the period annulus of the center at the origin would intersect the separatrices of these saddles six times. Since these separatrices are at the same energy level this clearly cannot happen as $H_3$ is quartic, and $H_3(x, cx) - h$ can have at most four roots for any $h \in \mathbb{R}$.

As a result of Remark 25 we see that the phase portraits 1.5 and 1.6 have four and two finite saddles at a fixed energy level, respectively. We will use this observation to distinguish the two phase portraits.

The number of singular points at the same energy level is equal to the number of solutions $\mathcal{N}$ of the system of equations $\dot{x} = \dot{y} = H_3 - h = 0$ for some $h \in \mathbb{R}$. We note that $h > 0$ at any singular point besides the origin because

$$H_3 - \frac{y\dot{x} - x\dot{y}}{4} = \frac{x^2 + (ax + by)^2}{4b} > 0.$$

To find $\mathcal{N}$ we compute the Gröbner basis of the three polynomials (2.9a), (2.9b) and $H_3 - h$, and obtain a set of 23 polynomials. Due to the size of these polynomials we will only mention in this paper the ones which are enough for our purpose. We remark that since $b > 0$ systems $(V)$ don't have any finite singular points on the coordinate axes other than the origin, so we will assume $xy \neq 0$ in our calculations.

There are two polynomials in the Gröbner basis which do not contain the variable $x$, and they are quadratic in $y$ of the form $my^2 + n$, where $m$ and $n$ are functions of the parameters $a$ and $b$. The coefficient of $y^2$ in these polynomials are $6hp_1$ and $3hp_2$, where

$$\begin{aligned} p_1 &= 1 - 90h + 1728h^2 + 5832b^2h^2 + 13824h^3, \\ p_2 &= 11 + 3a^2 - 18b^2 - 336h - 972b^2h - 2304h^2. \end{aligned}$$

We claim that these coefficients cannot vanish simultaneously. In fact if we calculate the resultant of $p_1$ and $p_2$ with respect to $h$ we obtain

$$4a^6 + a^4(12 - 45b^2) + 3a^2(4 - 3b^2 + 36b^4) + 4(1 + 3b^2)^3, \qquad (4.1)$$

up to a positive constant. If we consider (4.1) as a polynomial in $a^2$ we see that its the discriminant with respect to $a^2$ is

$$-(1 + 3b^2)^3(16 + 39b^2 + 72b^4)^2 < 0.$$

Hence it has a unique real root. In addition it has at least one negative root due to Descartes' rule of signs because $4 - 3b^2 + 36b^4 > 0$. Then the only real root of (4.1), when considered as a polynomial in $a^2$, is negative and consequently it cannot be zero for real $a$. Therefore $p_1$ and $p_2$ cannot be zero at the same time. Since $h > 0$, our claim is proved. Consequently the singular points which are at the same energy level must be on two vertical lines on the real plane.

There is another polynomial in the Gröbner basis which is linear in the variable $x$, and the coefficient of $x$ is $27a(1+a^2)$. So if $a \neq 0$ we have $\mathcal{N} = 2$. If $a = 0$ we can simply calculate the finite singular points besides the origin of systems $(III)$ which are

$$\left(\pm\frac{1}{3}\sqrt{\frac{1 + 3b^2}{b}}, \pm\sqrt{\frac{1}{3b}}\right).$$

Since the Hamiltonian $H_3$ is even these four points are at the same energy level, so we have $\mathcal{N} = 4$.

In short we have shown that when $b > 0$ a global phase portrait of systems $(III)$ is topologically equivalent to the phase portraits 1.5 and 1.6 of Figure 1.1 when $a = 0$ and $a > 0$, respectively. Consequently we obtain the bifurcation diagram shown in Figure 1.2.

## 4.2 Bifurcation diagram for systems $(IV)$

According to Section 2.5 a global phase portrait of systems $(IV)$ is topologically equivalent to the phase portrait 1.1 of Figure 1.1 when $b < 0$. However there are four possibilities when $b > 0$, namely the phase portraits 1.2, 1.7, 1.8 and 1.9 of Figure 1.1. So we will only focus on the case $b > 0$. Note that besides the origin the phase portrait 1.2 has two finite singular points, 1.9 has four, and 1.7 and 1.8 both have six finite singular points. Moreover we observe that there are four saddles at the same energy level in the phase portrait, whereas an argument similar to the one used in Remark 25 proves that there are at most two finite saddles at a fixed energy level in the phase portrait 1.8. We are going to use these two basic properties to distinguish them.

We will first study the case $a = 0$ because it appears as a critical value in our calculations. In this case we can easily calculate the finite singular points of systems $(IV)$ which are the origin, $(0, \pm\sqrt{b})$, and whenever $3b^2 > 1$ the additional four points

$$\left(\pm\frac{1}{3}\sqrt{\frac{3b^2-1}{b}}, \pm\sqrt{\frac{1}{3b}}\right). \tag{4.2}$$

Note that when $3b^2 - 1 = 0$ we get $1/3b = b$, and there are only two distinct singular points.

The linear part of systems $(IV)$ with $a = 0$ is

$$M_4 = \begin{pmatrix} -6xy & b - 3x^2 - 3y^2 \\ -1/b + 3y^2 & 6xy \end{pmatrix},$$

The eigenvalues of $M_4$ at the singular points (4.2) are the same, so they are saddles because there are at most two centers or cusps. Furthermore, since $H_4$ is even these saddles are at the same energy level. Thus a global phase portrait systems $(IV)$ with $a = 0$ and $b > 0$ is topologically equivalent to the phase portrait 1.2 of Figure 1.1 if $b \leq 1/\sqrt{3}$, and to 1.7 if $b > 1/\sqrt{3}$.

We now assume $a > 0$ for the rest of this section. To find the number of finite singular points of systems $(IV)$ we solve for $x$ in the equation $\dot{x} = 0$ and get

$$x_{1,2} = \frac{a \pm \sqrt{a^2 + 12by^2 - 12y^4}}{6y}.$$

Note that when $y = 0$ (2.21a) becomes $ax$, so the only singular point on the $x$–axis is the origin. Since we are looking for singularities other than the origin we assume $y \neq 0$. We substitute $x_1$ and $x_2$ into (2.21b) and obtain $\dot{y}_1$ and $\dot{y}_2$, respectively:

$$\dot{y}_{1,2} = \frac{-a - a^3 - 3aby^2 \mp (1 + a^2 - 3by^2)\sqrt{a^2 + 12by^2 - 12y^4}}{6by}.$$

We claim that the number of distinct real roots $N$ of the product $\dot{y}_1\dot{y}_2$

$$3y^6 - \frac{2 + 2a^2 + 3b^2}{b}y^4 + \frac{(1 + a^2)(1 + a^2 + 6b^2)}{3b^2}y^2 - \frac{1 + a^2}{3b} \tag{4.3}$$

is in one–to–one correspondence with the number of finite singular points $M$ of systems $(IV)$. We now prove our claim.

Let $y_0$ be a real root of (4.3). The corresponding $x$-coordinate $x_0$ is unique depending on whether $y_0$ is a root of $\dot{y}_1$ or $\dot{y}_2$, unless $y_0$ is a common root of $\dot{y}_1$ and $\dot{y}_2$ such that $a^2 + 12by^2 - 12y^4 \neq 0$. But if $y_0$ is a common root of $\dot{y}_1$ and $\dot{y}_2$, then we have

$$\dot{y}_1 + \dot{y}_2 = -\frac{a(1 + a^2 + 3b^2y_0^2)}{3by_0} \neq 0$$

for any $y_0 \in \mathbb{R}$ because $a \neq 0$. Therefore $\dot{y}_1$ and $\dot{y}_2$ cannot have a common root, and we have $M \leq N$.

On the other hand we have $M < N$ only if $a^2 + 12by_0^2 - 12y_0^4 < 0$ so that $x_0$ is complex. If we define

$$\begin{aligned} s_1 =& - a - a^3 - 3aby_0^2, \\ s_2 =& 1 + a^2 - 3by_0^2, \\ s_3 =& a^2 + 12by_0^2 - 12y_0^4, \end{aligned}$$

then $y_0$ is root of (4.3) if and only if $s_1^2 - s_3s_2^2 = 0$. For $x_0$ to be complex we need $s_3 < 0$, which implies $s_1 = s_2 = 0$. But we see that $s_1 - as_2 = 2a(1+a^2) \neq 0$, which is a contradiction. Thus $s_3$ cannot be negative, and as a result we obtain $M = N$, proving the claim. We note that since systems $(IV)$ have at least two finite singular points different from the origin, (4.3) must have at least two distinct real roots.

We will study the root classification of (4.3) using [30], where the author provides in particular the root classification of an arbitrary sextic polynomial of the form

$$x^6 + px^4 + qx^3 + rx^2 + sx^2 + t$$

We first need to compute the "discriminant sequence" $\{D_1, \ldots, D_6\}$ where

$D_1 = 1, \quad D_2 = -p, \quad D_3 = 24rp - 8p^3 - 27q^2,$

$$
\begin{aligned}
D_4 = {} & 32p^4r - 12p^3q^2 + 96p^3t + 324prq^2 - 224r^2p^2 - 288ptr - 120qp^2s \\
& + 300ps^2 - 81q^4 + 324tq^2 - 720qsr + 384r^3, \\
D_5 = {} & -4p^3q^2r^2 - 1344ptr^3 + 24p^4q^2t + 144pq^2r^3 + 1440ps^2r^2 + 162q^4tp \\
& - 5400rts^2 + 1512prtsq + 16p^4r^3 - 192p^4t^2 + 72p^5s^2 - 128r^4p^2 \\
& + 256r^5 + 1875s^4 - 64p^5rt + 592p^3tr^2 + 432rt^2p^2 - 616rs^2p^3 \\
& + 558q^2p^2s^2 + 1080s^2tp^2 - 2400ps^3q - 324pt^2q^2 - 1134tsq^3 \\
& + 648q^2tr^2 + 1620q^2s^2r - 1344qsr^3 + 3240qst^2 + 12p^3q^3s - 1296pt^3 \\
& - 27q^4r^2 + 81q^5s + 1728t^2r^2 - 56p^4rsq - 72p^3tsq + 432r^2p^2sq \\
& - 648rq^2tp^2 - 486prq^3s, \\
D_6 = {} & -32400ps^2t^3 - 3750pqs^5 + 16q^3p^3s^3 - 8640q^2p^3t^3 + 825q^2p^2s^4 \\
& + 108q^4p^3t^2 + 16r^3p^4s^2 - 64r^4p^4t - 4352r^3p^3t^2 + 512r^2p^5t^2 \\
& + 9216rp^4t^3 - 900rp^3s^4 - 17280t^3p^2r^2 - 192t^2p^4s^2 + 1500tp^2s^4 \\
& - 128r^4p^2s^2 + 512r^5p^2t + 9216r^4pt^2 + 2000r^2s^4p + 108s^4p^5 \\
& - 1024p^6t^3 - 4q^2p^3r^2s^2 - 13824t^4p^3 + 16q^2p^3r^3t + 8208q^2p^2r^2t^2 \\
& - 72q^3p^3str + 5832q^3p^2st^2 + 24q^2p^4ts^2 - 576q^2p^4t^2r - 4536q^2p^2s^2tr \\
& - 72rp^4qs^3 + 320r^2p^4qst - 5760rp^3qst^2 - 576rp^5ts^2 + 4816r^2p^3s^2t \\
& - 120tp^3qs^3 + 46656t^3p^2qs - 6480t^2p^2s^2r + 560r^2qp^2s^3 - 2496r^3qp^2st \\
& - 3456r^2qpst^2 - 10560r^3s^2pt + 768sp^5t^2q + 19800s^3rqpt + 3125s^6 \\
& - 46656t^5 - 13824r^3t^3 + 256r^5s^2 - 1024r^6t + 62208prt^4 + 108q^5s^3 \\
& - 874q^4t^3 + 729q^6t^2 + 34992q^2t^4 - 630prq^3s^3 + 3888prq^2t^3 \\
& + 2250rq^2s^4 - 4860prq^4t^2 - 22500rts^4 + 144pr^3q^2s^2 - 576pr^4q^2t \\
& - 8640r^3q^2t^2 + 2808pr^2q^3st + 21384rq^3st^2 - 9720r^2q^2s^2t \\
& - 77760rt^3qs + 43200r^2t^2s^2 - 1600r^3qs^3 + 6912r^4qst - 27540pq^2t^2s^2 \\
& - 27q^4r^2s^2 + 108q^4r^3t - 486q^5str + 162pq^4ts^2 - 1350q^3ts^3 \\
& + 27000s^3qt^2.
\end{aligned}
$$

Then we will determine the "sign list" $[\mathrm{sign}(D_1), \ldots, \mathrm{sign}(D_6)]$ of the discriminant sequence, where the sign function is

$$
\mathrm{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}
$$

And finally we need to construct the associated "revised sign list" $[r_1, \ldots, r_6]$ which will give all the information about the number of real and complex roots of our polynomial. Given any sign list $[s_1, \ldots, s_n]$, the revised sign list $[r_1, \ldots, r_n]$ is obtained as follows:

If $s_k \neq 0$ we write $r_k = s_k$.

If $[s_i, s_{i+1}, \ldots, s_{i+j}]$ is a section of the given sign list such that $s_{i+1} = \cdots = s_{i+j-1} = 0$ with $s_i s_{i+j} \neq 0$, then in place of $[r_{i+1}, \ldots, r_{i+j-1}]$ we write the $(j-1)$-tuple

$$[-s_i, -s_i, s_i, s_i, -s_i, -s_i, s_i, s_i, -s_i, \ldots].$$

Note that in this way there are no zeros between nonzero elements of the revised sign list.

The elements of the discriminant sequence of polynomial (4.3) are

$$D_2 = \frac{A}{3b}, \quad D_3 = \frac{8AB^2}{27b^3}, \quad D_4 = -\frac{32(1+a^2)B^2C}{243b^4},$$
$$D_5 = \frac{16a^2(1+a^2)^2CD}{6561b^6}, \quad D_6 = \frac{64a^4(1+a^2)^3D^2}{531441b^9},$$

where

$$A = 2 + 2a^2 + 3b^2, \quad B = 1 + a^2 - 3b^2,$$
$$C = -2(1-3b^2)^2 + a^2(2+21b^2) + 4a^4,$$
$$D = 4(1-3b^2)^3 + 3a^2(4+3b^2+36b^4) + 3a^4(4+15b^2) + 4a^6. \tag{4.4}$$

We will determine the cases in which (4.3) has six or four distinct real roots, and consequently the case with two distinct real roots will follow.

It is given in [30] that (4.3) has six distinct real roots if and only if the revised sign list of its discriminant sequence is $[1,1,1,1,1,1]$. Since $A > 0$, (4.3) has six distinct real roots if and only $B \neq 0$ and $C, D < 0$. We see that $D \leq 0$ only if $1 - 3b^2 < 0$, in which case

$$D - (a^2 - 2 + 6b^2)C = 9a^2(2 + 2a^2 + 2b^2) > 0.$$

This means that if $D \leq 0$ then $C < 0$ also. In addition when $B = 0$, that is $a = \sqrt{1-3b^2}$ and $1 - 3b^2 > 0$, we have $D > 0$. Consequently we have $B \neq 0$ if $D \leq 0$. Therefore we deduce that (4.3) has six distinct real roots if and only if $D < 0$.

We remind that systems $(IV)$ have two global phase portraits with six finite singular points which are not topologically equivalent, namely the phase portraits 1.7 and 1.8 of Figure 1.1. We will prove below that systems $(IV)$ with $D < 0$ and $a > 0$ cannot have four finite saddles at the same energy level, hence, as we mentioned in the beginning of this section, their phase portraits cannot be topologically equivalent to 1.7.

To determine the number of finite singular points at an energy level we look for the number of solutions of the system of three equations $\dot{x} = \dot{y} = 0$

and $H_4 = h$ for $h \in \mathbb{R}$. As we have shown for systems $(III)$, we have $h > 0$ at finite singular points of systems $(IV)$. We calculate the Gröebner basis of the polynomials $\dot{x}$, $\dot{y}$ and $H_4 - h$ and obtain 23 polynomials. The polynomials and the calculations are almost the same as those for systems $(III)$. Among these 23 polynomials only three are enough for our study: one that is linear in the variable $x$ with the coefficient $27a(1 + a^2) > 0$, and two that do not contain the variable $x$ and they are of the form $my^2 + n$. The coefficients of $y^2$ in these two polynomials are

$$6h(-1 + 90h - 1728h^2 + 5832b^2h^2 - 13824h^3),$$
$$3h(11 + 3a^2 + 18b^2 - 336h + 972b^2h - 2304h^2).$$

We know that $h > 0$. Then we need to check if the remaining non–constant factors can be zero simultaneously. The resultant of these two factors is

$$1253826625536D < 0.$$

Therefore at least one of these polynomials is not identically zero. Taking into account the third polynomial which is linear in $x$, we deduce that this system of equations have at most two solutions. As a result all the global phase portraits of systems $(IV)$ when $D < 0$ and $a > 0$ are topologically equivalent to 1.8 of Figure 1.1.

Now we study when (4.3) has four distinct real roots. According to [30] the revised sing list of the discriminant sequence must be $[1, 1, 1, 1, 0, 0]$ because we have $D_6 \geq 0$. Hence we need $B \neq 0$, $C < 0$ and $D = 0$. We have already seen that $B \neq 0$ and $C < 0$ whenever $D = 0$. Therefore (4.3) has four distinct real roots if and only if $D = 0$.

As a result of the above analysis and the fact that (4.3) has at least two distinct real roots, it follows easily that (4.3) has two distinct real roots if and only if $D > 0$.

We observe that when $a = 0$ we have $D = 4(1 - 3b^2)^3$. Hence we can summarize our results as follows: When $b < 0$ then the global phase portraits of systems $(IV)$ are topologically equivalent to 1.1 of Figure 1.1. When $b > 0$ the systems $(IV)$ have the global phase portrait 1.2 of Figure 1.1 when $D > 0$ or $D = a = 0$, 1.7 if $D < 0$ and $a = 0$, 1.8 if $D < 0$ and $a > 0$, and finally 1.9 if $D = 0$ and $a > 0$. Therefore we obtain the bifurcation diagram shown in Figure 1.3.

## 4.3 Bifurcation diagram for systems $(V)$

Due to Theorem 4 the global phase portraits of systems $(V)$ are topologically equivalent to the phase portraits 1.3, 1.10, 1.11 or 1.12 of Figure 1.1. Note that besides the origin the phase portrait 1.3 has two finite singular points, 1.12 has four, and 1.10 and 1.11 both have six finite singular points. Moreover there are four finite saddles at the same energy level in 1.10, while there can be at most two saddles at a fixed energy level in 1.11 (see Remark 25). We will use these facts to determine the bifurcation points for these phase portraits. We recall that we can assume $b > 0$ for systems $(V)$, see Section 2.6 for details.

As we did for systems $(IV)$ we study the case $a = 0$ separately. In this case the finite singular points besides the origin are $(\pm 1/\sqrt{b}, 0)$, and whenever $3\mu > b^2$ the four points

$$\left( \pm\sqrt{\frac{1+3b^2\mu}{b(1+9\mu^2)}}, \pm\sqrt{\frac{3\mu - b^2}{b(1+9\mu^2)}} \right). \tag{4.5}$$

Note that when $3\mu = b^2$ the four singular points in (4.5) coincide with $(\pm 1/\sqrt{b}, 0)$.

The linear part of systems $(V)$ with $a = 0$ is

$$M_5 = \begin{pmatrix} -6\mu xy & b - 3\mu x^2 + 3y^2 \\ -1/b + 3x^2 + 3\mu y^2 & 6\mu xy \end{pmatrix}.$$

The eigenvalues of $M_5$ at the four singular points in (4.5) are the same. Since there are at most two centers or cusps, these singular points are saddles, and they are at the same energy level because $H_5$ is even. Therefore a global phase portrait of systems $(V)$ with $a = 0$ is topologically equivalent to the phase portrait 1.3 of Figure 1.1 if $3\mu \leq b^2$, and to 1.10 otherwise. This finishes the study of the case $a = 0$ and in the rest of this section we will assume that $a > 0$.

We start by determining the number of finite singular points of systems $(V)$ as a function of the parameters $a, b, \mu$. If we equate (2.28a) to zero, solve for $x$ and substitute both roots into (2.28b) we obtain two functions of $y$. If multiply them we get a polynomial of degree eight instead of six, which was the case for systems $(IV)$. Consequently it is more difficult to study the number of distinct real roots of this polynomial as a guide to determine the number of finite singular points of systems $(V)$. Instead we use the fact that systems $(V)$ are symmetric with respect to the origin, and look for pairs of finite singular points different from the origin which lie on straight lines passing through the origin. Therefore we study systems $(V)$ on the $y$–axis, and on the lines $y = cx$ for $c \in \mathbb{R} \setminus \{0\}$. We can assume $c \neq 0$ due to the fact that when $c = 0$ we have $y = 0$, and (2.28a) becomes $ax$, which means that the only singular point is the origin. We will identify the lines $y = cx$ by the parameter $c$.

On the $y$–axis (2.28b) becomes $-ay$, which means that the only singular point is the origin. So we assume $x \neq 0$ and impose $y = cx$ to rewrite systems $(V)$ as

$$\dot{x} = (a + bc)x + c(c^2 - 3\mu)x^3, \tag{4.6a}$$

$$\dot{y} = -\frac{1 + a^2 + abc}{b}x + (1 + 3\mu c^2)x^3. \tag{4.6b}$$

We equate (4.6a) to zero, solve for $x$ and get

$$x = \pm\sqrt{\frac{-a - bc}{c(c^2 - 3\mu)}}. \tag{4.7}$$

We see that (4.7) are not defined if $\mu > 0$ and $c = \pm\sqrt{3\mu}$. So we will now find out if there are singular points on these lines that are different from the origin.

If $c = \sqrt{3\mu}$ then (4.6a) becomes $(a + b\sqrt{3\mu})x \neq 0$ because $a, b, \mu > 0$ and $x \neq 0$. Thus the only singular point on this line is the origin. If $c = -\sqrt{3\mu}$ then (4.6a) becomes $(a - b\sqrt{3\mu})x$, which is zero if and only if $a = b\sqrt{3\mu}$, in which case equating (4.6b) to zero and solving for $x$ gives

$$x = \pm\frac{1}{\sqrt{b(1 + 9\mu^2)}} = \pm\frac{b^{3/2}}{\sqrt{a^4 + b^4}}, \tag{4.8}$$

which are real and nonzero. Therefore when $c = -\sqrt{3\mu}$ there are singular points other than the origin if and only if $a = b\sqrt{3\mu}$. We will keep this in mind and continue looking for singular points with $c \neq -\sqrt{3\mu}$.

We substitute (4.7) into (4.6b) and obtain

$$\pm\frac{\sqrt{-a - bc}(abc^4 + (1 + a^2 + 3b^2\mu)c^3 + (b^2 - 3(1 + a^2)\mu)c + ab)}{b(c(c^2 - 3\mu))^{3/2}}.$$

This means that at a singular point we must have

$$P_5(c) = abc^4 + (1 + a^2 + 3b^2\mu)c^3 + (b^2 - 3(1 + a^2)\mu)c + ab = 0$$

because $-a - bc = 0$ yields $x = 0$. Moreover in order that $x$ defined in (4.8) are real and nonzero, the roots of $P$ must satisfy

$$Q_5(c) = (-a - bc)c(c^2 - 3\mu) > 0$$

so that (4.7) are real and nonzero. Then each real root of $P_5$ will yield a pair of finite singular points different from the origin. Here the index 5 is a reminder that we are studying systems $(V)$.

We will study the number of distinct real roots of $P_5$ using [30], where the elements of the discriminant sequence of an arbitrary quartic polynomial

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$$

are given as

$$
\begin{aligned}
&D_1 = 1, \quad D_2 = 3a_1^2 - 8a_2a_0, \\
&D_3 = 16a_0^2a_4a_2 - 18a_0^2a_3^2 - 4a_0a_2^3 + 14a_0a_3a_1a_2 - 6a_0a_4a_1^2 + a_2^2a_1^2 \\
&\qquad - 3a_3a_1^3, \\
&D_4 = 256a_0^3a_4^3 - 27a_0^2a_3^4 - 192a_0^2a_3a_4^2a_1 - 27a_1^4a_4^2 - 6a_0a_1^2a_4a_3^2 \\
&\qquad + a_2^2a_3^2a_1^2 - 4a_0a_2^3a_3^2 + 18a_2a_4a_1^3a_3 + 144a_0a_2a_4^2a_1^2 \\
&\qquad - 80a_0a_2^2a_4a_1a_3 + 18a_0a_2a_3^3a_1 - 4a_2^3a_4a_1^2 - 4a_1^3a_3^3 + 16a_0a_2^4a_4 \\
&\qquad - 128a_0^2a_2^2a_4^2 + 144a_0^2a_2a_4a_3^2.
\end{aligned} \tag{4.9}
$$

By using them we will be able to determine the exact number of distinct real roots of $P_5$.

Note that the number of real roots of $Q_5$ are different when $\mu \leq 0$ and $\mu > 0$. We will investigate these separately.

*Case* $\mu \leq 0$. In this case we have $Q_5 > 0$ if and only if $c \in (-a/b, 0)$, see Figure 4.1. On the other hand we have $P_5(0) = ab > 0$ and $P_5(-a/b) = a(3b^2\mu - a^2)/b^3 < 0$, so $P_5$ has at least one root in $(-a/b, 0)$. In fact we observe that $P_5$ has either two or zero negative roots due to Descartes' rule of sign. Additionally it has at least one root in $(-\infty, -a/b)$ because $\lim_{c\to-\infty} P_5 = \infty$. Therefore when $\mu \leq 0$ $P_5$ has exactly one real root in $(-a/b, 0)$, and systems $(V)$ have only two finite singular points other than the origin.

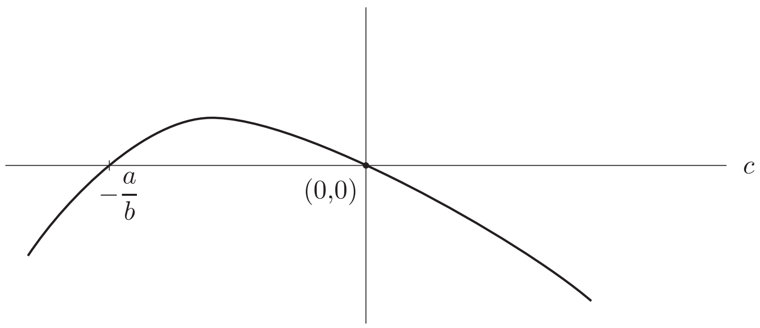



Figure 4.1: A rough graph of $Q_5(c)$ when $\mu \leq 0$.

*Case* $\mu > 0$. Now $Q_5$ has the four roots $c = 0$, $c = -a/b$ and $c = \pm\sqrt{3\mu}$. Moreover

$$
\begin{aligned}
&P_5(-a/b) = \frac{a(3b^2\mu - a^2)}{b^3}, \quad P_5(0) = ab, \\
&P_5(\pm\sqrt{3\mu}) = b(a \pm b\sqrt{3\mu})(1 + 9\mu^2),
\end{aligned} \tag{4.10}
$$

at these points. Since the roots $c = -a/b$ and $c = -\sqrt{3\mu}$ are independent of each other we will investigate this case in three subcases comparing $a/b$ to $\sqrt{3\mu}$.

When $a/b < \sqrt{3\mu}$ we have $Q_5 > 0$ if and only if $c \in (-\sqrt{3\mu}, -a/b) \cup (0, \sqrt{3\mu})$, see Figure 4.2. Thus we will look for the number of real roots $P_5$ in these intervals.

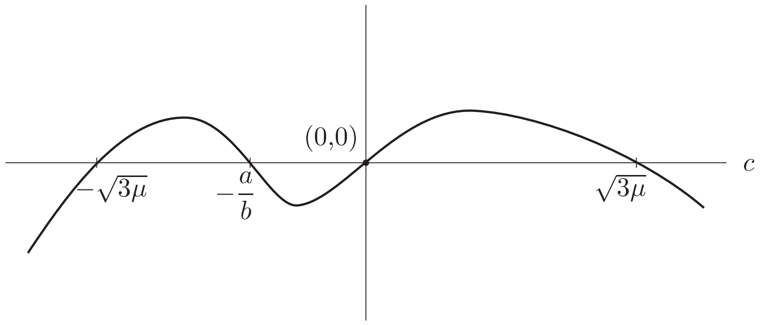



Figure 4.2: A rough graph of $Q_5(c)$ when $\mu > 0$ and $a/b < \sqrt{3\mu}$.

We have $P_5(-\sqrt{3\mu}) < 0$ and $P_5(-a/b) > 0$, see (4.10). Since $P_5$ has at most two negative roots and $\lim_{c\to-\infty} P_5 = \infty$, $P_5$ has exactly one simple root in $(\sqrt{3\mu}, -a/b)$.

On the other hand we have $P_5(0) > 0$ and $P_5(\sqrt{3\mu}) > 0$. We claim that $P_5$ cannot have a real root greater that $\sqrt{3\mu}$. This is due to the fact that the first derivative of $P_5$ with respect to $c$,

$$P_5'(c) = 4abc^3 + 3(1 + a^2 + 3b^2\mu)c^2 + b^2 - 3(1 + a^2)\mu, \tag{4.11}$$

has at most one positive root. If $P_5$ had a real root greater than $\sqrt{3\mu}$ then it would have at least two positive critical points because $P_5(\sqrt{3\mu}) > P_5(0)$ and $\lim_{c\to\infty} P_5 = \infty$. Therefore if $P_5$ has a positive root then it is in $(0, \sqrt{3\mu})$.

In short when $a/b < \sqrt{3\mu}$, $P_5$ has at least two real simple roots (the negative ones), and exactly one of its real roots (the smallest) makes (4.7) complex.

When $a/b > \sqrt{3\mu}$ everything is the same as in the case $a/b < \sqrt{3\mu}$, except that the roles of the roots $c = -a/b$ and $c = -\sqrt{3\mu}$ are exchanged, see Figure 4.3. More precisely we have $Q_5 > 0$ if and only if $c \in (-a/b, -\sqrt{3\mu}) \cup (0, \sqrt{3\mu})$. In addition $P_5(-a/b) < 0$ and $P_5(-\sqrt{3\mu}) > 0$ so that $P_5$ has one negative root in $(-a/b, -\sqrt{3\mu})$, and a smaller one in $(-\infty, -a/b)$. Moreover $P_5(\sqrt{3\mu}) > P_5(0) > 0$, and hence any positive root of $P_5$ is in the interval $(0, \sqrt{3\mu})$ because it has at most one positive critical point, see (4.11). Therefore $P_5$ has at least two real simple roots, and exactly one of them leads to a pair of complex singular points when $a/b > \sqrt{3\mu}$.

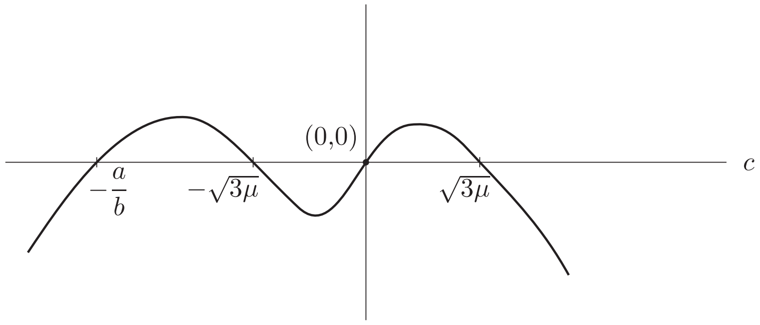



Figure 4.3: A rough graph of $Q_5(c)$ when $\mu > 0$ and $a/b > \sqrt{3\mu}$.

Finally when $a/b = \sqrt{3\mu}$ we see that $Q_5 > 0$ only when $c \in (0, a/b)$, see Figure 4.4. Hence no negative root of $P_5$ satisfies $Q_5 > 0$. We recall that when $a/b = \sqrt{3\mu}$ there *are* extra singular points on the line $y = cx$ with $c = -\sqrt{3\mu} = -a/b$. Also $P_5(-a/b) = 0$, and thus $c = -a/b$ *is* a root of $P_5$. Moreover $P_5'(-\sqrt{3\mu}) = 6\mu + b^2(1 + 9\mu^2) > 0$ so that it is a simple root. Then since $P_5$ has either two or zero negative roots, it has another negative root different from $-a/b$. By the same argument used in the previous two subcases, all the positive roots of $P_5$ are in $(0, \sqrt{3\mu})$, and they satisfy $Q_5 > 0$. Therefore $P_5$ again has at least two simple roots with the property that exactly one of them correspond to complex finite singular points.

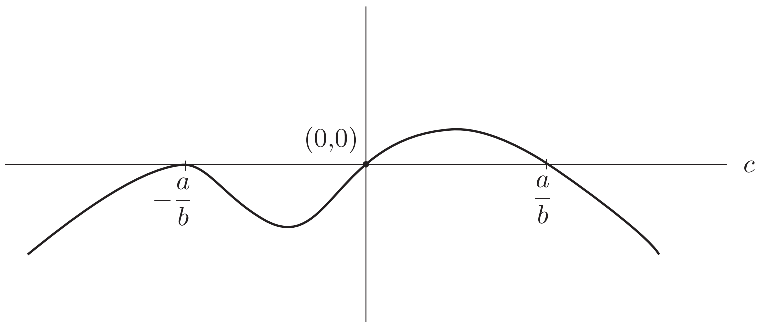



Figure 4.4: A rough graph of $Q_5(c)$ when $\mu > 0$ and $a/b = \sqrt{3\mu}$.

In short we have shown that in any case $P_5$ has at least two simple real roots when $\mu > 0$. Then according to [30] $P_5$ has two, three or four distinct real roots if and only if $D_4 < 0$, $D_4 = 0$ and $D_4 > 0$, respectively, where

$$\begin{aligned}
D_4 = &- b^2(27a^2 + 108a^4 + 162a^6 + 108a^8 + 27a^{10} + 4b^4 + 18a^2b^4 \\
&+ 216a^4b^4 - 54a^6b^4 + 27a^2b^8) + 36b^4(3a^2-1)^2\big((1+a^2)^2 \\
&- b^4\big)\mu - 54b^2(2 + 11a^2 + 24a^4 + 26a^6 + 14a^8 + 3a^{10} - 6b^4 \\
&+ 32a^2b^4 + 50a^4b^4 + 12a^6b^4 + 2b^8 + 3a^2b^8)\mu^2 \\
&+ 108\big((1+a^2)^2 - b^4\big)(1 + 4a^2 + 6a^4 + 4a^6 + a^8 - 8b^4 \\
&+ 8a^2b^4 + 16a^4b^4 + b^8)\mu^3 - 243b^2(-4 - 11a^2 - 4a^4 + 14a^6 \\
&+ 16a^8 + 5a^{10} + 12b^4 + 38a^2b^4 + 40a^4b^4 + 14a^6b^4 - 4b^8 \\
&+ 5a^2b^8)\mu^4 + 2916b^4\big((1+a^2)^2 - b^4\big)(1+a^2)^2\mu^5 \\
&+ 2916b^6(1+a^2)^3\mu^6,
\end{aligned} \tag{4.12}$$

see (4.9). Since there are no additional finite singular points at exactly one simple root of $P_5$, systems $(V)$ have two, four and six additional finite singular points besides the origin whenever $D_4 < 0$, $D_4 = 0$ and $D_4 > 0$, respectively.

Note that the phase portraits 1.10 and 1.11 have the same number of singular points. We observe that there are four finite singular points at a fixed energy level in 1.10. So we will check if the Hamiltonian $H_5$ can attain the same value at four distinct finite singular points.

At a singular point of systems $(V)$ $H_5$ reduces to

$$H_5(x,y) - \frac{y\dot{x} - x\dot{y}}{4} = \frac{x^2 + (ax+by)^2}{4b}. \tag{4.13}$$

If we substitute $y = cx$ in (4.13) we get

$$G_5(c,x) = \frac{\big(1 + (a+bc)^2\big)x^2}{4b}.$$

Then using (4.7) we can rewrite $G_5$ as

$$F_5(c) = -\frac{(a+bc)\big(1+(a+bc)^2\big)}{4bc(c^2 - 3\mu)}.$$

We recall that there are additional singular points on the line $c = -\sqrt{3\mu}$ if and only if $a/b = \sqrt{3\mu}$, and that (4.7) is not well defined at $c = -\sqrt{3\mu}$. Thus to calculate $H_5$ at the additional singular points on the line $c = -\sqrt{3\mu}$ we must use $G_5$, while we can use $F_5$ for all the other singular points. Therefore $H_5$ can attain the same value at four singular points only if one of the following holds:

(*i*) If $F_5$ attains the same value at two distinct real roots of $P_5$ which are different from $-\sqrt{3\mu}$.

(*ii*) If $a/b = \sqrt{3\mu}$ and $F_5(c) = G_5(-a/b, \pm b^{3/2}/\sqrt{a^4+b^4})$ for a real root $c \neq -a/b$ of $P_5$ (see (4.8)).

We claim that none of these two cases holds. The proof is as follows.

To show that (*i*) cannot hold we assume on the contrary that $c_1$ and $c_2$ are two distinct real roots of $P_5$ which satisfy

$$F_5(c_1) - F_5(c_2) = \frac{(c_1 - c_2)E_5(c_1, c_2)}{4bc_1c_2(c_1^2 - 3\mu)(c_2^2 - 3\mu)} = 0,$$

where

$$E_5(c_1, c_2) = a(1 + a^2)(c_2^2 - 3\mu) + c_1c_2\big(a(1 + a^2 + 9b^2\mu) + bc_2(1 + 3a^2 + 3b^2\mu)\big) + c_1^2\big(a(1 + a^2) + bc_2(1 + 3a^2 + 3b^2\mu + 3abc_2)\big)$$

Since $c_1 \neq c_2$ we have $F_5(c_1) = F_5(c_2)$ if and only if $E_5(c_1, c_2) = 0$. To find a necessary condition on the parameters $a, b, \mu$ so that we have $P_5(c_1) = P_5(c_2) = E_5(c_1, c_2) = 0$ we compute the resultant $R(c_2)$ of $P_5(c_1)$ and $E_5(c_1, c_2)$ with respect to $c_1$, and then compute the resultant of $R(c_2)$ and $P_5(c_2)$ with respect to $c_2$. Doing so we obtain

$$-a^{10}b^{10}(1 + 2a^2 + a^4 + b^4)^2(a^2 - 3b^2\mu)^7(1 + 9\mu^2)^2D_4^3, \tag{4.14}$$

which is equal to zero only if $a^2 = 3b^2\mu$ because we have $a, b > 0$ and $D_4 > 0$ when systems $(V)$ have six finite singular points. But when $a^2 = 3b^2\mu$ we have $E_5(c_1, c_2) = 0$ only if either $c_1 = -\sqrt{3\mu} = -a/b$ or $c_2 = -a/b$, which cannot be in case (*i*). More precisely if we set $\mu = a^2/3b^2$ we get

$$P_5(c) = \frac{(a + bc)\bar{P}_5(c)}{b^2}, \quad E_5(c_1, c_2) = -\frac{(a + bc_1)(a + bc_2)\bar{E}_5(c_1, c_2)}{b^2},$$

where

$$\bar{P}_5(c) = -b^3 + a(1 + a^2)c - b(1 + a^2)c^2 - ab^2c^3,$$
$$\bar{E}_5(c_1, c_2) = -a(1 + a^2) + b(1 + a^2)(c_1 + c_2) + 3ab^2c_1c_2.$$

If we impose $c_1, c_2 \neq -a/b$, then $P_5(c_1) = P_5(c_2) = E_5(c_1, c_2) = 0$ if and only if $\bar{P}_5(c_1) = \bar{P}_5(c_2) = \bar{E}_5(c_1, c_2) = 0$. But if we calculate the resultant $\bar{R}(c_2)$ of $\bar{P}_5(c_1)$ and $\bar{E}_5(c_1, c_2)$ with respect to $c_1$, and then the resultant of $\bar{R}(c_2)$ and $\bar{P}_5(c_2)$ with respect to $c_2$ we get

$$-\frac{a^3b^{18}D_4}{(2a^2 + a^4 + b^4)^2} \neq 0.$$

Therefore (4.14) cannot be zero, and hence (*i*) cannot hold.

On the other hand if $(ii)$ holds then substituting $\mu = a^2/3b^2$ gives

$$F_5(c) = \frac{b\big(1 + (a+bc)^2\big)}{4c(a-bc)}, \quad G_5\left(-\frac{a}{b}, \pm\frac{b^{3/2}}{\sqrt{a^4+b^4}}\right) = \frac{b^2}{4(a^4+b^4)} = h.$$

Since a root $c$ of $P_5$ which is different from $-a/b$ must satisfy $F_5(c) = h$, we must have $\bar{P}_5(c) = 0$. However, the resultant of $\bar{P}_5$ and $F_5 - h$ with respect to $c$ is

$$-\frac{b^2(2a^2 + a^4 + b^4)^3(1 + 2a^2 + a^4 + b^4)}{64(a^4+b^4)^3} \neq 0.$$

This disproves $(ii)$. Hence when $a, \mu > 0$ at most two singular points can be at the same energy level, and systems $(V)$ cannot have the phase portrait 1.10 of Figure 1.1. This completes the case $\mu > 0$.

We note that when $a = 0$ we have $D_4 = 4(1 + 3b^2\mu)^3(3\mu - b^2)^3$, so the sign of $D_4$ is enough to determine the phase portraits. Therefore we can summarize our results as follows: when $\mu \leq 0$ a global phase portrait of systems $(V)$ is topologically equivalent to 1.3 of Figure 1.1; when $\mu > 0$ then it is equivalent to 1.3 if $D_4 < 0$ or $D_4 = 0$ and $a = 0$, to 1.10 if $D_4 > 0$ and $a = 0$, to 1.11 if $D_4 > 0$ and $a \neq 0$, and to 1.12 if $D_4 = 0$ and $a \neq 0$. Hence when $\mu \leq 0$ there is a unique phase portrait, and when $\mu > 0$ we obtain the bifurcation diagram shown in Figure 1.4.

## 4.4 Bifurcation diagram for systems $(VI)$

Due to Section 2.7 a global phase portrait of systems $(VI)$ is topologically equivalent to the phase portraits 1.13 and 1.14 of Figure 1.1 if $\mu < -1/3$, to 1.15–1.17 if $\mu = -1/3$, and to 1.18–1.23 if $\mu > -1/3$. Therefore we will determine the bifurcation points of the global phase portraits of systems $(VI)$ when $\mu < -1/3$, $\mu = -1/3$ and $\mu > -1/3$ separately.

*Case* $\mu < -1/3$. First of all we note that without loss of generality we can assume in this case that $b > 0$. Indeed, if we rotate the coordinate axes by $\pi/4$ via the linear transformation $(x, y) \mapsto \big((x-y)/\sqrt{2}, (x+y)/\sqrt{2}\big) = (u, v)$ then systems $(VI)$ become

$$\begin{aligned}\dot{u} &= \frac{a^2 - b^2 + 1}{2b}u + \frac{(a+b)^2 + 1}{2b}v - \frac{3(1-\mu)}{2}u^2v - \frac{1+3\mu}{2}v^3,\\ \dot{v} &= -\frac{(a-b)^2 + 1}{2b}u - \frac{a^2 - b^2 + 1}{2b}v + \frac{3(1-\mu)}{2}uv^2 + \frac{1+3\mu}{2}u^3.\end{aligned}$$

If we further rescale the independent variable by $d\tau = (1 + 3\mu)/2\ dt$ then we get

$$\dot{u} = \frac{a^2 - b^2 + 1}{b(1+3\mu)}u + \frac{(a+b)^2 + 1}{b(1+3\mu)}v - \frac{3(1-\mu)}{1+3\mu}u^2v - v^3,$$

$$\dot{v} = -\frac{(a-b)^2+1}{b(1+3\mu)}u - \frac{a^2-b^2+1}{b(1+3\mu)}v + \frac{3(1-\mu)}{1+3\mu}uv^2 + u^3.$$

An finally after defining the parameters

$$\bar{a} = \frac{a^2-b^2+1}{b(1+3\mu)}, \quad \bar{b} = \frac{(a+b)^2+1}{b(1+3\mu)}, \quad \bar{\mu} = \frac{1-\mu}{1+3\mu},$$

we get the systems

$$\begin{aligned} \dot{u} &= \bar{a}u + \bar{b}v - 3\bar{\mu}u^2v - v^3, \\ \dot{v} &= -\frac{\bar{a}^2+1}{\bar{b}}u - \bar{a}v + 3\bar{\mu}uv^2 + u^3. \end{aligned} \tag{4.15}$$

We see that $d\bar{\mu}/d\mu = -4/(1+3\mu)^2 < 0$ and $\lim_{\mu \to -\infty} \bar{\mu} = -1/3$, hence $\bar{\mu} < -1/3$ whenever $\mu < -1/3$. As a result systems (4.15) are basically systems $(VI)$ with $b \mapsto -b$, proving that we can assume $b > 0$.

We remind that systems $(VI)$ can have two different global phase portraits when $\mu < -1/3$, namely 1.13 and 1.14 of Figure 1.1. Both phase portraits have the same number of singular points. But the difference between them is that there are four finite singular points at the same energy level in 1.14, whereas there are only two in 1.13 because otherwise using the same arguments used in Remark 25 we can find a straight line through the origin that intersects the separatrices of the saddles six times. So we will investigate when there can be four finite singular points at a fixed energy level.

When $a = 0$ the finite singular points of systems $(VI)$ besides the origin are $(\pm 1/\sqrt{b}, 0)$ and $(0, \pm\sqrt{b})$. We also have

$$H_6\left(\pm\frac{1}{\sqrt{b}}, 0\right) = \frac{1}{4b^2}, \quad H_6\left(0, \pm\sqrt{b}\right) = \frac{b^2}{4}. \tag{4.16}$$

Hence the four singular points are on the same energy level if and only if $b = 1$. Therefore a global phase portrait of systems $(VI)$ with $\mu < -1/3$ and $a = 0$ is topologically equivalent to 1.13 of Figure 1.1 if $b \neq 1$, and to 1.14 if $b = 1$.

We now assume $a > 0$ and consider finite singular points of systems $(VI)$ which are different from the origin in pairs lying on the straight lines $y = cx$ with $c \in \mathbb{R} \setminus \{0\}$. We note that there are no finite singular points on the coordinate axes because $a > 0$. We will again identify each line $y = cx$ with its parameter $c$.

We substitute $y = cx$ in systems $(VI)$ and we get

$$\dot{x} = (a+bc)x - c(c^2+3\mu)x^3, \tag{4.17a}$$

$$\dot{y} = -\frac{1+a^2+abc}{b}x + (1+3\mu c^2)x^3. \tag{4.17b}$$

Then we equate (4.17a) to zero and solve for $x$ to obtain

$$x = \pm\sqrt{\frac{a+bc}{c(c^2+3\mu)}}. \tag{4.18}$$

We see that (4.18) are not defined if $c = \pm\sqrt{-3\mu}$. However, when $c = \sqrt{-3\mu}$ we get from (4.17a) that $\dot{x} = (a + b\sqrt{-3\mu})x \neq 0$, and there are no additional singular points on this line. When $c = -\sqrt{-3\mu}$ we have $\dot{x} = (a - b\sqrt{-3\mu})x$, which is zero if and only if $a = b\sqrt{-3\mu}$. But if we substitute $c = -\sqrt{-3\mu}$ and $a = b\sqrt{-3\mu}$ in (4.17b) then the roots of $\dot{y}$ become

$$x = \pm 1/\sqrt{b(1-9\mu^2)} \tag{4.19}$$

which are complex because $\mu < -1/3$. Therefore (4.18) are well–defined at the singular points.

We proceed as we did for systems $(V)$. We can substitute (4.18) into (4.17b) to get

$$\pm\frac{\sqrt{a+bc}\big(abc^4 + (1+a^2-3b^2\mu) + (3(1+a^2)\mu - b^2)c - ab\big)}{b\big(c(c^2+3\mu)\big)^{3/2}}.$$

Therefore at a singular point different from the origin we must have

$$P_6(c) = abc^4 + (1+a^2-3b^2\mu) + (3(1+a^2)\mu - b^2)c - ab = 0 \tag{4.20}$$

and

$$Q_6(c) = (a+bc)c(c^2+3\mu) > 0. \tag{4.21}$$

Now that we know the necessary and sufficient conditions for a point to be a finite singular point of systems $(VI)$ with $\mu < -1/3$, we check if four singular points can be at the same energy level. We follow the same way we used for systems $(V)$.

At a singular point $H_6$ can be written exactly as (4.13). Then we substitute $y = cx$ and obtain

$$G_6(c,x) = \frac{\big(1+(a+bc)^2\big)x^2}{4b}, \tag{4.22}$$

If we further substitute (4.18) in $G_6$ we get

$$F_6(c) = \frac{(a+bc)\big(1+(a+bc)^2\big)}{4bc(c^2+3\mu)}. \tag{4.23}$$

Thus systems $(VI)$ with $\mu < -1/3$ have four finite singular points at the same energy level if and only if $P_6$ has two distinct real roots $c_1$ and $c_2$ such that $F_6(c_1) = F_6(c_2)$ and $Q_6(c_{1,2}) > 0$. We now prove that this is possible if and only if $b = \sqrt{1+a^2}$.

Assume that $P_6(c_1) = P_6(c_2) = 0$ but $c_1 \neq c_2$. We have

$$F_6(c_1) - F_6(c_2) = \frac{(c_1 - c_2)E_6(c_1, c_2)}{4bc_1c_2(c_1^2 + 3\mu)(c_2^2 + 3\mu)} = 0,$$

if and only if $E_6(c_1, c_2) = 0$ where

$$\begin{aligned} E_6(c_1, c_2) = {} & a(1 + a^2)(c_2^2 + 3\mu) + c_1c_2\big(a(1 + a^2 - 9b^2\mu) \\ & + bc_2(1 + 3a^2 - 3b^2\mu)\big) + c_1^2\big(a(1 + a^2) \\ & + bc_2(1 + 3a^2 - 3b^2\mu + 3abc_2^2)\big). \end{aligned} \tag{4.24}$$

If we calculate the resultant $R(c_2)$ of $P_6(c_1)$ and $E_6(c_1, c_2)$ with respect to $c_1$, and then the resultant of $P_6(c_2)$ and $R(c_2)$ we obtain

$$a^{10}b^{10}(1 - 9\mu^2)^2(a^2 + 3b^2\mu)^7\big(b^4 - (1 + a^2)^2\big)^2 D_4^3, \tag{4.25}$$

where

$$\begin{aligned} D_4 = & - b^2(27a^2 + 108a^4 + 162a^6 + 108a^8 + 27a^{10} - 4b^4 \\ & - 18a^2b^4 - 216a^4b^4 + 54a^6b^4 + 27a^2b^8) - 36b^4(3a^2 - 1)^2 \\ & \big((1 + a^2)^2 + b^4\big)\mu + 54b^2(2 + 11a^2 + 24a^4 + 26a^6 + 14a^8 \\ & + 3a^{10} + 6b^4 - 32a^2b^4 - 50a^4b^4 - 12a^6b^4 + 2b^8 + a^2b^8)\mu^2 \\ & - 108\big((1 + a^2)^2 + b^4\big)(1 + 4a^2 + 6a^4 + 4a^6 + a^8 + 8b^4 \\ & - 8a^2b^4 - 16a^4b^4 + b^8)\mu^3 - 242b^2(-4 - 11a^2 - 4a^4 + 14a^6 \\ & + 16a^8 + 5a^{10} - 12b^4 - 38a^2b^4 - 40a^4b^4 - 14a^6b^4 - 4b^8 \\ & + 5a^2b^8)\mu^4 - 2916b^4(1 + a^2)^2\big((1 + a^2)^2 + b^4\big)\mu^5 \\ & + 2916b^6(1 + a^2)^3\mu^6. \end{aligned} \tag{4.26}$$

We remark that we denote (4.26) by $D_4$ because it coincides with the fourth element of the discriminant sequence of $P_6(c)$, see (4.9). Since $a, b > 0$ we see that (4.25) is zero only if $(i)$ $D_4 = 0$, $(ii)$ $\mu = -a^2/(3b^2)$, or $(iii)$ $b = \sqrt{1 + a^2}$. We now analyze these three cases.

We first prove that $(i)$ cannot hold, that is $D_4 \neq 0$. We observe that $D_4$ is equal to the "standard" discriminant of $P_6(c)$. Hence we can prove that $(i)$ cannot hold by showing that all the roots of $P_6$ are simple. The roots of $Q_6$ are 0, $-a/b$ and $\pm\sqrt{-3\mu}$. If we evaluate $P_6$ at these points we get

$$\begin{aligned} & P_6(0) = -ab, \quad P_6(-a/b) = -a(a^2 + 3b^2\mu)/b^3, \\ & P_6(\pm\sqrt{-3\mu}) = b(a \pm b\sqrt{-3\mu})(9\mu^2 - 1). \end{aligned} \tag{4.27}$$

If $a/b < \sqrt{-3\mu}$ then $Q_6$ is positive if and only if $c \in (-\infty, -\sqrt{-3\mu}) \cup (-a/b, 0) \cup (\sqrt{-3\mu}, \infty)$, see Figure 4.5. Since $a, b > 0$ we have $P_6(-\sqrt{-3\mu}) < 0$, $P_6(-a/b) > 0$, $P_6(0) < 0$ and $P_6(\sqrt{-3\mu}) > 0$, see (4.27). Since $P_6$

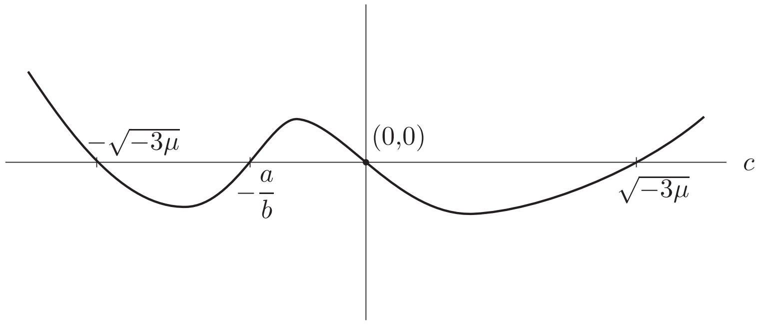



Figure 4.5: A rough graph of $Q_6(c)$ when $\mu < 0$ and $a/b < \sqrt{-3\mu}$.

is a quartic polynomial in $c$ it has four simple roots, two of which satisfy $Q_6 > 0$ as expected because the global phase portraits have four finite singular points.

If $a/b > \sqrt{-3\mu}$ then we have $Q_6(c) > 0$ if and only if $c \in (-\infty, -a/b) \cup (-\sqrt{-3\mu}, 0) \cup (\sqrt{-3\mu}, \infty)$, see Figure 4.6. In this case we have $P_6(-a/b) < 0$, $P_6(-\sqrt{-3\mu}) > 0$, $P_6(0) < 0$ and $P_6(\sqrt{-3\mu}) > 0$. Hence $P_6$ has four simple roots again, and exactly two of its roots are in the region where $Q_6 > 0$.

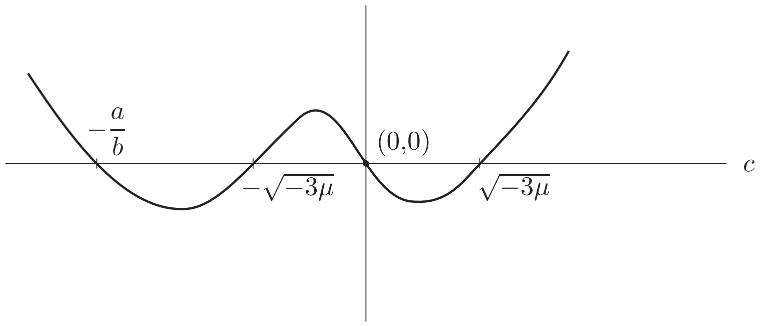



Figure 4.6: A rough graph of $Q_6(c)$ when $\mu < 0$ and $a/b > \sqrt{-3\mu}$.

Finally if $a/b = \sqrt{-3\mu}$, we have $Q_6(c) > 0$ unless $c \in \{-a/b\} \cup [0, \sqrt{-3\mu}]$, see Figure 4.7. We see that $P_6$ has at least one positive root with $Q_6 < 0$. In addition $P_6(-a/b) = 0$, and thus at least two distinct roots of $P_6$ do not satisfy $Q_6 > 0$. But since we know that $P_6$ has exactly two distinct roots with $Q_6 > 0$, we conclude that each root of $P_6$ is simple.

In any case $P_6$ has four simple real roots, therefore $D_4 \neq 0$ and $(i)$ does not hold.

We now consider $(ii)$. Note that the conditions $P_6(c_1) = P_6(c_2) = 0$ and $E_6(c_1, c_2) = 0$ do not imply $Q_6(c_{1,2}) > 0$. We claim that when $\mu = -a^2/(3b^2)$ the resultant (4.25) vanishes only if $Q_6(c_1)Q_6(c_2) = 0$. We now prove this

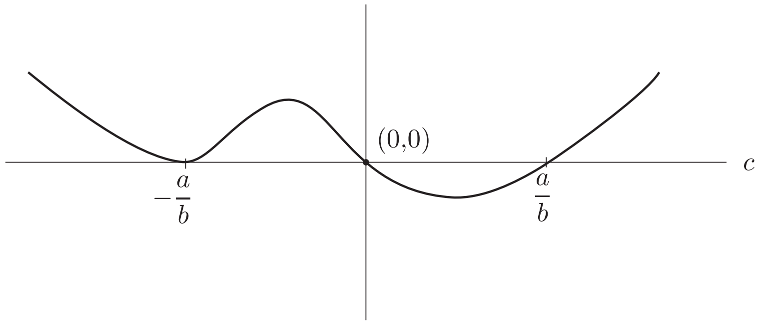



Figure 4.7: A rough graph of $Q_6(c)$ when $\mu < 0$ and $a/b = \sqrt{-3\mu}$.

claim by showing that if we do not allow $Q_6(c_1)Q_6(c_2) = 0$ then (4.25) cannot be zero. The proof is as follows.

If we substitute $\mu = -a^2/(3b^2)$ in $P_6$ and $E_6$ we obtain

$$\begin{aligned} P_6(c) &= -\frac{(a+bc)\bar{P}_6(c)}{b^2}, \\ E_6(c_1,c_2) &= -\frac{(a+bc_1)(a+bc_2)\bar{E}_6(c_1,c_2)}{b^2}, \end{aligned} \tag{4.28}$$

where

$$\begin{aligned} \bar{P}_6(c) &= b^3 + a(1+a^2)c_2 - b(1+a^2)c_2^2 - ab^2c^3, \\ \bar{E}_6(c_1,c_2) &= -a(1+a^2) + b(1+a^2)(c_1+c_2) + 3ab^2c_1c_2. \end{aligned} \tag{4.29}$$

Since we have $Q_6(-a/b) = 0$ (see Figure 4.7), systems $(VI)$ have four finite singular points at the same energy level if and only if $\bar{P}_6(c_1) = \bar{P}_6(c_2) = \bar{E}_6(c_1,c_2) = 0$. But if, as we did above, calculate first the resultant $\bar{R}(c_2)$ of $\bar{P}_6(c_1)$ and $\bar{E}_6(c_1,c_2)$ with respect to $c_1$, and then the resultant of $\bar{P}_6(c_2)$ and $\bar{R}(c_2)$ with respect to $c_1$ we get

$$\begin{aligned} -a^3b^{12}(a^2 + 8a^4 + 18a^6 + 16a^8 + 5a^{10} + 4b^4 + 30a^2b^4 + 48a^4b^4 \\ +22a^6b^4 - 27a^2b^8)^3, \end{aligned} \tag{4.30}$$

which is different from zero because when $\mu = -a^2/(3b^2)$ we have

$$\begin{aligned} D_4 = \frac{(2a^2+a^4-b^4)^2}{b^6}&(a^2 + 8a^4 + 18a^6 + 16a^8 + 5a^{10} + 4b^4 \\ &+ 30a^2b^4 + 48a^4b^4 + 22a^6b^4 - 27a^2b^8) \neq 0 \end{aligned} \tag{4.31}$$

This finishes the proof of our claim, and hence there cannot be four finite singular points at the same energy level when $(ii)$ holds.

Finally we consider (*iii*). We remark that in this case we have $a^2/b^2 = 1 - 1/b^2 < 1$ whereas $-3\mu > 1$, and thus cases (*ii*) and (*iii*) are disjoint. If we substitute $b = \sqrt{1+a^2}$ in (4.20) we get

$$P_6(c) = \sqrt{1+a^2}(c^2-1)\left(a + \sqrt{1+a^2}(1-3\mu)c + ac^2\right). \tag{4.32}$$

Since we have $a/b < 1 < \sqrt{-3\mu}$, the roots $c = \pm 1$ of (4.32) make $Q_6 < 0$, see Figure 4.5. In addition we know that $P_6$ has two distinct roots with $Q_6 > 0$, so they must be the remaining two roots, which are

$$c_{1,2} = \frac{\sqrt{1+a^2}(3\mu-1) \pm \sqrt{(1+a^2)(1-3\mu)^2 - 4a^2}}{2a}. \tag{4.33}$$

Indeed, substituting (4.33) into (4.24) gives $E_6 = 0$. Therefore we conclude that systems $(VI)$ with $\mu < -1/3$ have four singular points at the same energy level if and only if $1 + a^2 = b^2$.

We observe that when $a = 0$ the condition $1 + a^2 = b^2$ translates into $b = 1$, which is the unique case in which systems $(VI)$ have four finite singular points at a fixed energy level. Consequently we have proved that systems $(VI)$ with $\mu < -1/3$ have the global phase portrait 1.14 of Figure 1.1 if $b = \sqrt{1+a^2}$, and the phase portrait 1.13 otherwise. Therefore we obtain the bifurcation diagram shown in Figure 1.5.

*Case* $\mu = -1/3$. In [9] it is shown that if $b < 0$ the unique phase portrait is 1.15 of Figure 1.1. So we study the case $b > 0$, in which a global phase portrait of systems $(VI)$ is topologically equivalent to either 1.16 or 1.7 of Figure 1.1. As in the case $\mu < -1/3$, these two phase portraits differ in the sense that in 1.17 there exists an energy level at which there are four finite singular points, whereas 1.16 has at most two finite singular points at a fixed energy level. This follows from applying the argument used in Remark 25. Therefore we will study the number of finite singular points at a fixed energy level of systems $(VI)$ with $\mu = -1/3$.

When $a = 0$ the finite singular points of systems are the same as in the case $\mu < -1/3$ and they are at the same energy level if and only if $b = 1$, see (4.16). Thus a global phase portrait is topologically equivalent to 1.17 of Figure 1.1 if $b = 1$, and to 1.16 otherwise.

Assume now that $a > 0$. Following the same way we used in the case $\mu < -1/3$, we rewrite systems $(VI)$ by substituting $\mu = -1/3$ in (4.17) and we get

$$\dot{x} = (a + bc)x - c(c^2-1)x^3, \tag{4.34a}$$

$$\dot{y} = -\frac{1 + a^2 + abc}{b}x - (c^2-1)x^3. \tag{4.34b}$$

In addition (4.18) becomes

$$x = \pm\sqrt{\frac{a+bc}{c(c^2-1)}}. \tag{4.35}$$

It is easy to check that on the lines $c = \pm 1$ the only singular point is the origin. Since we are looking for singular points other than the origin, we suppose $c \neq \pm 1$ and substitute (4.35) into (4.34b) to obtain

$$\mp \frac{\sqrt{a+bc}(c^2-1)\big(ab+(1+a^2+b^2)c+abc^2\big)}{b\big(c(c^2-1)\big)^{3/2}}. \tag{4.36}$$

Since we want $x \neq 0$, the roots of (4.36) we are interested in are

$$c_{1,2} = -\frac{1+a^2+b^2 \pm \sqrt{(1+a^2+b^2)^2-4a^2b^2}}{2ab}. \tag{4.37}$$

We notice that $c_1$ and $c_2$ in (4.37) are real and distinct. Due to the fact that systems (4.34) have at least four finite singular points, (4.37) must make (4.35) real and nonzero. Now we will check if these four singular points can be at the same energy level.

When $\mu = -1/3$ we see that (4.23) becomes

$$F_6(c) = \frac{(a+bc)\big(1+(a+bc)^2\big)}{4bc(c^2-1)}$$

Then we have

$$F_6(c_1) - F_6(c_2) = -\frac{(1+a^2-b^2)\sqrt{(1+a^2+b^2)^2-4ab^2}}{4b^2},$$

which is zero if and only if $b = \sqrt{1+a^2}$. We remind that when $a = 0$ the four singular points are at the same energy level if and only if $b = 1$, which coincides with $b = \sqrt{1+a^2}$.

In short we have the following result: When $b < 0$ systems $(VI)$ with $\mu = -1/3$ have the global phase portrait 1.15 of Figure 1.1, and when $b > 0$ their global phase portraits are topologically equivalent to 1.17 if $b = \sqrt{1+a^2}$, and to 1.16 otherwise. Thus the bifurcation diagram shown in Figure 1.6 is obtained.

*Case* $\mu > -1/3$. Due to [9] a global phase portrait of systems $(VI)$ in this case is topologically equivalent to one of the phase portraits 1.18–1.23 of Figure 1.1. Due to the direction of the flow at infinity the unique global phase portrait when $b < 0$ is clearly 1.18, so we only need to study systems $(VI)$ with $b > 0$. It is also shown in [9] that the global phase portrait 1.23 is obtained if and only if $a = 0$, $b = 1$ and $\mu = 1/3$. Hence we will focus on the phase portraits 1.19–1.22.

In order to distinguish these phase portraits, we will use the properties that allowed us to distinguish the phase portraits of the previous families of systems. More precisely, the phase portrait 1.19 has four finite singular points, 1.22 has six, and 1.20 and 1.21 both have eight finite singular points besides the origin. Moreover 1.20 has four finite saddles at some fixed energy energy level, whereas 1.21 has at most two. This is again due to the same argument used in Remark 25.

As we did for the previous systems we will study the cases $a = 0$ and $a > 0$ separately. We note that this time the case $a = 0$ is a little more complicated so we will further divide the case $\mu > -1/3$ into the two corresponding subcases.

*Subcase* $a = 0$. The finite singular points of systems $(VI)$ other than the origin are $(\pm 1/\sqrt{b}, 0)$ and $(0, \pm\sqrt{b})$, and the additional four points

$$\left(\sqrt{\frac{1-3b^2\mu}{b(1-9\mu^2)}}, \sqrt{\frac{b^2-3\mu}{b(1-9\mu^2)}}\right) \tag{4.38}$$

if $\mu < 1/3$, $1 - 3b^2\mu > 0$ and $b^2 - 3\mu > 0$, or if $\mu > 1/3$, $1 - 3b^2\mu < 0$ and $b^2 - 3\mu < 0$. Note that when $(1 - 3b^2\mu)(b^2 - 3\mu) = 0$ the singular points in (4.38) coincide with $(\pm 1/\sqrt{b}, 0)$ or $(0, \pm\sqrt{b})$. We also point out that if $1 - 3b^2\mu = b^2 - 3\mu = 0$ then we have $b = 1$ and $\mu = 1/3$, and thus there are infinitely many singular points (see the phase portrait 1.23).

We see that when $a = 0$ systems $(VI)$ with $\mu > -1/3$ have either four or eight finite singular points besides the origin. When it has four finite singular points, clearly their phase portraits are topologically equivalent to 1.19. When it has eight finite singular points, there are two possibilities, namely the phase portraits 1.20 and 1.21. We now analyze these two possible cases.

The eigenvalues of the linear part of systems $(VI)$ with $a = 0$ at each of the singular points (4.38) are

$$\pm\sqrt{\frac{4(1-3b^2\mu)(b^2-3\mu)}{b^2(9\mu^2-1)}}.$$

Hence they are centers if $\mu < 1/3$, and saddles if $\mu > 1/3$. Consequently the remaining finite singular points $(\pm 1/\sqrt{b}, 0)$ and $(0, \pm\sqrt{b})$ are saddles if $\mu < 1/3$, and centers if $\mu > 1/3$. Now we shall check if the Hamiltonian $H_6$ can attain the same value at all the saddles.

When $\mu < 1/3$ we have $H_6(\pm 1/\sqrt{b}, 0) = 1/4b^2$ and $H_6(0, \pm\sqrt{b}) = b^2/4$ at the saddles. Therefore we have the phase portrait 1.20 if and only if $b = 1$, and 1.21 otherwise.

When $\mu > 1/3$, at each of the the singular points (4.38) the Hamiltonian becomes

$$\frac{1 + b^4 - 6b^2\mu}{4b^2(1 - 9\mu^2)},$$

hence we only have the phase portrait 1.20.

In short when $a = 0$ we obtain the bifurcation diagram shown in (1.7).

*Subcase* $a > 0$. The calculations in this case are very similar to the case $\mu < -1/3$, so we will often refer to the ones in the case $\mu < -1/3$.

Since $a > 0$ there are no additional finite singular points on the $y-axis$, so we substitute $y = cx$ with $c \neq 0$ as usual and obtain systems (4.17). We solve for $x$ by equating (4.17a) to zero and obtain (4.18). This time, however, we see that there are additional singular points on the line $c = -\sqrt{-3\mu}$ if and only if $a = b\sqrt{-3\mu}$ because (4.19) are real when $-1/3$. Therefore similar to what we did for systems $(V)$ we will keep this in mind and look for singular points with $c \neq \sqrt{-3\mu}$. Then (4.18) are well-defined, and we can substitute them in (4.17b) to see that we must have $P_6 = 0$ (see (4.20)) and $Q_6 > 0$ (see (4.21)) at the finite singular points.

The roots of $Q_6$ in this case are $0$, $-a/b$, and additionally $\pm\sqrt{-3\mu}$ whenever $\mu < 0$. At these points $P_6$ becomes as in (4.27). Moreover the graph of $Q_6$ is roughly the one shown in Figure 4.8 if $\mu \geq 0$, the one in Figure 4.5 if $\mu < 0$ and $a < b\sqrt{-3\mu}$, the one in Figure 4.6 if $\mu < 0$ and $a > b\sqrt{-3\mu}$, and the one in Figure 4.7 if $\mu < 0$ and $a = b\sqrt{-3\mu}$. We will show now that any root of $P_6$ that is different from $c = -\sqrt{-3\mu}$ satisfy $Q_6 > 0$. We recall that $c = -\sqrt{-3\mu}$ is a root only when $a = b\sqrt{-3\mu}$. As a result we will conclude that the number of additional finite singular points of systems $(VI)$ in this case is equal to the number of real distinct roots of $P_6$.

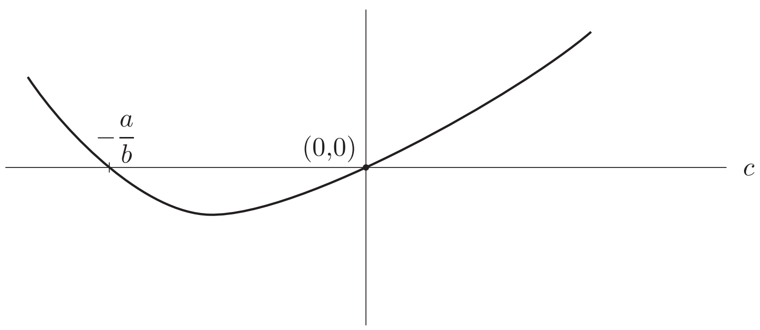



Figure 4.8: A rough graph of $Q_6(c)$ when $\mu \geq 0$.

We see that $P_6(\sqrt{-3\mu}) < 0$, and that $P_6$ has exactly one positive real root. Since $\lim_{c\to\infty} P_6 = \infty$, this positive root is greater than $\sqrt{-3\mu}$, and hence satisfies $Q_6 > 0$. Therefore it remains to show that $P_6$ does not have a negative root (*i*) in $(-a/b, 0)$ when $\mu \geq 0$,

(*ii*) in $(-\sqrt{-3\mu}, -a/b)$ when $\mu < 0$ and $a < b\sqrt{-3\mu}$,

(*iii*) in $(-a/b, -\sqrt{-3\mu})$ when $\mu < 0$ and $a > b\sqrt{-3\mu}$.

To determine if a polynomial has a root in some interval we will use the following lemma.

**Lemma 26.** *If a polynomial $p(x)$ of degree $n$ has a real root in an interval $(\alpha, \beta)$, then the polynomial $(1+x)^n(p \circ h)(x)$ with $h(x) = (\alpha + \beta x)/(1+x)$ has a positive root.*

*Proof.* Clearly $(1+x)^n(p \circ h)(x)$ is a polynomial. We have $h(0) = a$ and $\lim_{x\to\infty} h(x) = b$. In addition $h'(x) = (b-a)/(1+x^2) > 0$ so that $h$ is bijective. Therefore if $p(x) = 0$ for some point $x_0 \in (\alpha, \beta)$ then $p(h(x_1)) = 0$ for $x_1 = h^{-1}(x_0) > 0$. □

To study roots of type (*i*) we define $h(c) = (-a/b)/(1+c)$. Since the degree of $P_6$ is four we have

$$\begin{aligned}(1+c)^4 P_6(h(c)) = -\frac{a}{b^3}\big(&a^2 + 3b^2\mu + (a^2 + a^4 + b^4 + 9b^2\mu + 6a^2b^2\mu)c \\ &+ 3b^2(b^2 + 3\mu + 3a^2\mu)c^2 + 3b^2(b^2 + \mu + a^2\mu)c^3 + b^4c^4\big),\end{aligned}$$

which does not have a positive root due to Descartes' rule of signs. Hence $P_6$ does not have a root of type (*i*).

We remark that in Lemma 26 although we chose $h(x) = (\alpha+\beta x)/(1+x)$, we could as well choose $h(x) = (\beta + \alpha x)/(1+x)$. Thus $P_6$ has a root of type (*ii*) if and only if it has a root of type (*iii*), so we only study (*ii*). To simplify notation will write $m = \sqrt{-3\mu}$. Hence we have $0 < m < 1$, $a > bm$ and

$$(1+c)^4 P_6(h(c)) = -\frac{a - bm}{b^3} S(c),$$

where

$$\begin{aligned}S(c) = &b^4(1 - m^4) + b^2(3b^2 + 2m^2 + 2a^2m^2 - 4abm^3 - b^2m^4)c \\ &+ 3b(b^3 + am + a^3m + bm^2 - a^2bm^2 - ab^2m^3)c^2 \\ &+ (a^2 + a^4 + b^4 + 4abm + b^2m^2 - 2a^2b^2m^2)c^3 \\ &+ a(a + bm)c^4\end{aligned}$$

We claim that $S$ does not have a positive root, and we now prove this claim by showing that the sign of the coefficients of the monomials in $S$ are all positive. The constant term and the coefficient of $c^4$ are clearly positive, so we look at the coefficients of $c$, $c^2$ and $c^3$.

The coefficient of $c^3$, which we denote by $k_3$, is

$$\begin{aligned} a^2 + a^4 + b^4 + 4abm + b^2m^2 - 2a^2b^2m^2 &> a^2 + a^4 + b^4 - 2a^2b^2 \\ &= a^2 + (a^2 - b^2)^2 > 0 \end{aligned}$$

because $0 < m < 1$.

The coefficient of $c^2$, denoted by $k_2$, has exactly one positive root due to Descartes' rule of signs when considered as a polynomial in $m$. Moreover that root is greater than 1 due to the facts that when $m = 1$ we have

$$\begin{aligned} k_2 &= 3b(b^3 + a + a^3 + b - a^2b - ab^2) \\ &= 3b\big((a+b)(a^2 - ab + b^2) + a + b - ab(a+b)\big) \\ &= 3b(a+b)\big(1 + (a-b)^2\big) > 0, \end{aligned}$$

and that it is negative for large $m > 0$. Therefore $k_2$ is positive for $0 < m < 1$.

Finally the coefficient $k_1$ of $c$ is also positive because of the same reasons: it has a exactly one positive root when considered as a polynomial in $m$; it is negative for large $m > 0$; and when $m = 1$ it becomes $2\big(1 + (a-b)^2\big) > 0$. This proves that $S$ does not have a positive root, which in turn implies that $P_6$ has no root of types $(ii)$ or $(iii)$.

In short the number of finite singular points other than the origin is double the number of distinct real roots of $P_6$. Since we know that the phase portraits 1.19–1.22 have at least four finite singular points additional to the origin, $P_6$ must have at least two real distinct roots. Therefore, according to [30], a global phase portrait of systems $(VI)$ in this case is topologically equivalent to 1.19 of Figure 1.1 if $D_4 < 0$ or $D_4 = D_3 = 0$, to 1.22 if $D_4 = 0$ and $D_3 \neq 0$, and to 1.20 or 1.21 if $D_4 > 0$, where $D_4$ is given in (4.26) and

$$\begin{aligned} D_3 = {} & 3b^2(+1 + 5a^2 + 7a^4 + 3a^6 - 6a^2b^4) - 9(1 + a^2)(1 + 3a^2 + 3a^4 \\ & + a^6 + 3b^4 - 5a^2b^4)\mu + 27b^2(3 + 3a^2 - 3a^4 - 3a^6 + 3b^4 \\ & + 5a^2b^4)\mu^2 - 81b^4(3 + 6a^2 + 3a^4 + b^4)\mu^3 + 243(1 + a^2)b^6\mu^4, \end{aligned} \tag{4.39}$$

see (4.9).

Note that when $D_4 > 0$ there are two phase portraits. We know that there are four saddles at a fixed energy level in 1.20, but there are at most two in 1.21. Hence following the exact same steps that we used in distinguishing the phase portraits 1.13 and 1.14 of systems $(VI)$ with $\mu < -1/3$, we deduce that the phase portrait 1.20 is achieved only if (4.25) is zero. So we should investigate whether (4.25) can be zero.

We have $D_4 \neq 0$. When $\mu = 1/3$ we have

$$P_6(c) = (1 + c^2)(-ab + c + a^2c - b^2c + abc^2),$$

and hence the phase portrait is topologically equivalent to 1.19. So it remains to study the cases $\mu = -a^2/(3b^2)$ and $b = \sqrt{1+a^2}$.

When $\mu = -a^2/(3b^2)$, we have $-1/3 < \mu < 0$. Moreover, due to the results obtained in the case $\mu < -1/3$, if there are four finite singular points at the same energy level then two of these singular points must be on the line $c = -a/b$ (see (4.28), (4.29) and (4.30)). The Hamiltonian $H_6$ at the singular points when $c = -a/b$ is given by (4.22). The $x$–coordinates of the singular points when $c = -a/b$ are (4.19), and at these singular points the Hamiltonian becomes

$$G_6(c) = \frac{b^2}{4(b^4 - a^4)} = h.$$

Note that $0 < a^2/b^2 = -3\mu < 1$ so that $b^4 - a^4 > 0$. Now we should check if there are other singular points at which $H_6 = h$. For the singular points that are not on the line $c = -a/b$ we have $H_6 = F_6$ (see (4.23)), and these singular points satisfy $\bar{P}_6 = 0$ (see (4.29)). Hence we are looking for points that satisfy $\bar{P}_6 = F_6 - h = 0$. If we calculate the resultant of $\bar{P}_6$ and $F_6 - h$ we obtain

$$-\frac{b^2\big(b^4 - (1+a^2)^2\big)(2a^2 + a^4 - b^4)}{(64(b^4 - a^4)^3)},$$

which is zero if and only if $b = \sqrt{1+a^2}$, see (4.31). Note that this is the last condition that makes (4.25) zero, so now we will study this final case.

If we substitute $b = \sqrt{1+a^2}$ in (4.20) we get (4.32). We have seen that the roots (4.33) are at the same energy level. What remains to be done is to determine when these points are saddles and when they are centers. For reasons of simplicity we study the local phase portraits of the singular point on the lines $c = \pm 1$. If we evaluate (4.24) at $(1, -1)$ we obtain

$$-4a(1+a^2)(1+3\mu) \neq 0 \tag{4.40}$$

because $a > 0$. So the singular points on these lines cannot be at the same energy level. Hence we will deduce that if these singular points are centers then (4.33) are saddles and we have the phase portrait 1.20 of Figure 1.1, and if they are saddles then we have 1.21.

The linear part of systems (4.17) when $b = \sqrt{1+a^2}$ is

$$\begin{pmatrix} a - 6\mu c x^2 & \sqrt{1+a^2} - 3c^2x^2 - 3\mu x^2 \\ -\sqrt{1+a^2} + 3x^2 + 3\mu c^2 x^2 & -a + 6\mu c x^2 \end{pmatrix}. \tag{4.41}$$

When $c = \pm 1$ the $x$–coordinates of the singular points are obtained by (4.18). Then we see that the determinant of (4.41) is

$$d_1 = \frac{4\big(1 - 3a(a + \sqrt{1+a^2})(1-\mu) - 3\mu)\big)}{1 + 3\mu}$$

when $c = 1$, and it is

$$d_2 = \frac{4\big(1 - 3a(a - \sqrt{1+a^2})(1-\mu) - 3\mu)\big)}{1+3\mu}$$

when $c = -1$. If we multiply them we get

$$d_1 d_2 = \frac{16\big((1-3\mu)^2 - 3a^2(1-\mu)(1+3\mu)\big)}{(1+3\mu)^2}.$$

We observe that if we substitute $b = \sqrt{1+a^2}$ in $D_4$ we obtain

$$4(1+a^2)^3\big((1-3\mu)^2 - 3a^2(1-\mu)(1+3\mu)\big)^3. \qquad (4.42)$$

Since we assume $D_4 > 0$ we have $d_1 d_2 > 0$, so meaning that they are different from zero and they have the same sign. Due to the fact that the eigenvalues of the linear part $M$ of a Hamiltonian system is of the form $\pm\sqrt{-\det(M)}$, where $\det(M)$ denotes the determinant of $M$, we deduce that the singular points that are on the lines $c = \pm 1$ are saddles if $d_1 < 0$, and are centers if $d_1 > 0$.

Since $d_1$ is linear in $\mu$ we can solve $d_1 = 0$ and get

$$\mu_0 = 1/3 + 2a/(3\sqrt{1+a^2}).$$

So we have $d_1 > 0$ and $d_1 < 0$ for $\mu < \mu_0$ and $\mu > \mu_0$, respectively. On the other hand if we equate (4.42) to zero and solve for $\mu$ we get

$$\mu_{1,2} = 1/3 \mp 2a/(3\sqrt{1+a^2}),$$

Note that $\mu_0 = \mu_2$ and $-1/3 < \mu_1 < \mu_0$. Hence (4.42) is positive if and only if $\mu < \mu_1$ or $\mu > \mu_0$. Thus whenever $D_4 > 0$ we have $d_1 > 0$ if $\mu < \mu_1$, and $d_1 < 0$ if $\mu > \mu_0$. Therefore we get the global phase portrait 1.20 if $\mu < \mu_1$, and 1.21 if $\mu > \mu_0$. This finishes the analysis of the subcase $a > 0$.

Before summarizing our results for the case $\mu > -1/3$, we comment on the relation between the subcases $a = 0$ and $a > 0$. When $a = 0$ we have $\mu_0 = \mu_1 = 1/3$, and the condition $b = \sqrt{1+a^2}$ becomes $b = 1$. So for $\mu < \mu_1$ the conditions to have the phase portrait 1.20 when $a = 0$ can be obtained by substituting $a = 0$ in those when $a > 0$. However, for $\mu > \mu_0$, (4.40) becomes zero if $a = 0$, meaning that the saddles are at the same energy level as well as the centers, and we get the phase portrait 1.20 again. On the other hand, when $a = 0$ we have

$$D_4 = 4(b^2 - 3\mu)^3(1 - 3b^2\mu)^3,$$
$$D_3 = 3(b^2 - 3\mu)(1 - 3b^2\mu)^3.$$

So the conditions for the phase portrait 1.19 when $a = 0$ can also be obtained by substituting $a = 0$ in those when $a > 0$. And finally the conditions for the phase portrait 1.22 when $a > 0$ can also be extended to $a = 0$ due the fact that when $a = 0$ we do not have $D_4 = 0$ and $D_3 \neq 0$, and also we do not have the phase portrait 1.22.

In short we obtain when $b < 0$ a global phase portrait is topologically equivalent to 1.18 of Figure 1.1. When $b > 0$ a global phase portrait is topologically equivalent to 1.19 if $D_4 < 0$, or $D_4 = D_3 = 0$ but either $a \neq 0$, $\mu \neq 1/3$ or $b \neq 1$; to 1.20 if $D_4 > 0$, $b = \sqrt{1 + a^2}$ and $\mu < \mu_1$, or $D_4 > 0$, $a = 0$ and $\mu > 1/3$; to 1.21 if $D_4 > 0$ and $b \neq \sqrt{1 + a^2}$, or $D_4 > 0$, $b = \sqrt{1 + a^2}$, $a \neq 0$ and $\mu > \mu_0$; to 1.22 if $D_4 = 0$ but $D_3 \neq 0$; and to 1.23 if $a = 0$, $\mu = 1/3$ and $b = 1$. Therefore we obtain the bifurcation diagrams shown in Figures 5–9.

# Chapter 5

# Proof of Theorem 8

In this chapter we prove Theorem 8. Observe that each of the classes $(VII)$, $(VIII)$ and $(X)$ have a unique global phase portrait. Also the bifurcation diagram of the phase portraits of systems $(IX)$ is easy and it follows directly from Section 3.4. Consequently it remains to prove the last two statements of Theorem 8, and we will prove them in the following sections.

## 5.1 Bifurcation diagram for systems $(XI)$

Recall that for systems $(XI)$ we have $b \geq 0$. According to Section 3.6 when $a = b = 0$ systems $(XI)$ have the global phase portraits (up to topological equivalence) 1.3 and 1.10 of Figure 1.1 when $\mu \leq 0$ and $\mu > 0$ respectively. On the other hand, when $b > 0$ there are three possible phase portraits: 1.3, 1.11 and 1.12 of Figure 1.1. However, the information obtained in Chapter 3 is not enough to determine exactly when each phase portrait is achieved by these systems.

We see that each of these three phase portraits has a different number of finite singular points, hence we will use this property to distinguish them. The explicit expressions of the finite singular points are complicated, and since we are only interested in the number of finite singular points we will make use of Yang's work [30] on the number of real roots of polynomials depending on their coefficients.

When $b > 0$, by Remark 6 systems $(XI)$ can be written as (3.25). Moreover the center condition (3.27) becomes

$$1 - b^4 - 6b^2\mu > 0. \tag{5.1}$$

We equate (3.25a) to zero, solve for $x$ and get

$$x_{1,2} = \frac{1 \pm \sqrt{1 + 12b\mu y^2 + 12\mu y^4}}{6\mu y}. \tag{5.2}$$

We see that (5.2) is not defined when $\mu y = 0$, so we need to address this case separately.

When $y = 0$ we have (3.25a) equal to zero if and only if $x = 0$, so we can assume $y \neq 0$ because we are not interested in the origin. On the other hand, when $\mu = 0$ we can easily calculate the finite singular points of systems (3.25) and see that other than the origin there are only two, namely $\pm\big(b^{1/3}\sqrt{(1-b^{4/3})/b}, -\sqrt{(1-b^{4/3})/b}\big)$. Note that these points are real because when $\mu = 0$ inequality (5.1) yields $b < 1$. Therefore when $\mu = 0$ the global phase portrait of systems (3.25) is topologically equivalent to 1.5 of Figure 1.1.

Now we can assume $\mu y \neq 0$, substitute (5.2) into (3.25b) and obtain

$$\begin{aligned}\dot{y}_{1,2} =& \frac{1}{54b\mu^3y^3}\Big(b + 9\mu(b^2-\mu)y^2 + 9b\mu(1-3\mu^2)y^4 \\ &\pm \sqrt{1+12b\mu y^2+12\mu y^4}\big(b+3\mu(b^2-3\mu)y^2+3b\mu(1+9\mu^2)y^4\big)\Big),\end{aligned}$$

where $\dot{y}_1$ and $\dot{y}_2$ denote $\dot{y}$ with $x$ substituted by $x_1$ and $x_2$ respectively. Each root of $\dot{y}_1$ and $\dot{y}_2$ will be paired with at most one $x$ by (5.2). Therefore the number of roots of $\dot{y}_1$ and $\dot{y}_2$ provides important information on the number of finite singular points of systems (3.25). So we compute the product $\dot{y}_1\dot{y}_2$ and obtain the sextic polynomial

$$\begin{aligned}-\frac{1}{27b^2\mu^3}&\big(b^2(1+9\mu^2)^2y^6 + 3b(1+9\mu^2)(b^2-2\mu+3b^2\mu^2)y^4 \\ &+ 3(b^4+3\mu^2+6b^4\mu^2-18b^2\mu^3)y^2 - b(1-b^4-6b^2\mu)\big).\end{aligned} \tag{5.3}$$

Then we study the relation between the number of roots of (5.3) and the number of finite singular points of systems (3.25).

First we claim that the number of finite singular points cannot be less than the number of roots of (5.3). Now we prove our claim. If we define

$$\begin{aligned}s_1 =& b + 3\mu(b^2-3\mu)y^2 + 3b\mu(1+9\mu^2)y^4, \\ s_2 =& b + 9\mu(b^2-\mu)y^2 + 9b\mu(1-3\mu^2)y^4, \\ s_3 =& 1 + 12b\mu y^2 + 12\mu y^4,\end{aligned}$$

then we have $\dot{y}_{1,2} = (s_2 \pm \sqrt{s_3}s_1)/54b\mu^3y^3$, and polynomial (5.3) can be rewritten as

$$\frac{1}{2916b^2\mu^6y^6}(s_2^2 - s_3s_1^2). \tag{5.4}$$

The number of finite singular points are less than the number of roots of (5.3) only when $s_3 < 0$ because then (5.2) become complex. If $s_3 < 0$ then (5.4) is zero if and only if $s_1 = s_2 = 0$. But if we subtract $s_2$ from $s_1$ we obtain

$$6b\mu\big(-b+(9\mu^2-1)y^2\big)y^2. \tag{5.5}$$

Since $b > 0$ and $\mu y \neq 0$, (5.5) is zero if and only if $y = \pm\sqrt{b/(9\mu^2 - 1)}$, where $9\mu^2 - 1$ must be positive. Then we substitute these $y$ into $s_1$ and $s_2$, and see that they are roots of these two polynomials provided that we have

$$1 - 9\mu^2 + 54b^2\mu^3 = 0. \tag{5.6}$$

We note that equation (5.6) further requires $\mu > 0$ because we have $9\mu^2 - 1 > 0$. But this means that $s_3 > 0$, and this is a contradiction to the assumption that $s_3 < 0$. Hence we conclude that $s_3 \geq 0$, so $x_{1,2}$ are real whenever $y$ is a root of (5.3), and this proves the claim.

Second we consider the case in which the number of finite singular points could be greater than the number of roots of (5.3). This is only possible when $\dot{y}_1$ and $\dot{y}_2$ have common roots which produce distinct $x$ in (5.2), so we must have $s_3 > 0$ and $s_1 = s_2 = 0$ for a common root. We have seen that this occurs if and only if equation (5.6) is satisfied with $\mu > 1/3$. Moreover in this case the number of common roots of $s_1$ and $s_2$ is two due to the fact that they have the same constant terms whereas their second order terms are different since $\mu \neq 0$.

In short the number of finite singular points of systems (3.25) is equal to the number of real roots of polynomial (5.3) unless $1 - 9\mu^2 + 54b^2\mu^3 = 0$ and $\mu > 1/3$, in which case there are two more singular points.

As we mentioned earlier we will determine the number of roots of (5.3) following [30], as we did for systems $(IV)$ in Section 4.2. We compute the discriminant sequence of the polynomial (5.3) and get

$$\begin{aligned} D_2 =& \frac{3A}{b(1+9\mu^2)}, \quad D_3 = \frac{216AB}{b^3(1+9\mu^2)^3}, \quad D_4 = \frac{2596BC}{b^4(1+9\mu^2)^6} \\ D_5 =& \frac{3888CDE^2}{b^6(1+9\mu^2)^{10}}, \quad D_6 = \frac{46656D^2E^4F}{b^9(1+9\mu^2)^{14}}, \end{aligned}$$

where

$$\begin{aligned} A =& -b^2 + 2\mu - 3b^2\mu^2, \\ B =& (-4b^2 + \mu + 6b^2\mu^2 + 9b^4\mu^3)\mu, \\ C =& -b^2 + 2(1-4b^4)\mu - 27b^2\mu^2 - 18(1-2b^4)\mu^3 + 9b^2(7+2b^4)\mu^4 \\ &+ 54(2+9b^4)\mu^5 - 81b^2(7-2b^4)\mu^6 - 486b^4\mu^7, \\ D =& -b^2 - 6b^2\mu^2 + 4(1-b^4)\mu^3 + 3b^2\mu^4, \\ E =& 1 - 9\mu^2 + 54b^2\mu^3, \\ F =& 1 - b^4 - 6b^2\mu. \end{aligned}$$

Observe that we have $F > 0$ due to (5.1), and $b(1 + 9\mu^2) > 0$. Hence the sign list of this discriminant sequence is determined only by the signs of $A$, $B$, $C$, $D$ and $E$. Note that $D_6 \geq 0$.

We have seen that systems (3.25) have six finite singular points other than the origin when

($i$) polynomial (5.3) has six real distinct roots,

($ii$) it has four real distinct roots provided that $E = 0$ and $\mu > 1/3$, see (5.5).

According to [30] the only revised sign list in case ($i$) is $[1, 1, 1, 1, 1, 1]$. Hence we need $D_i > 0$ for all $i = 1, \ldots, 6$. Since $A$ must be positive we get $\mu, B, C, D > 0$. We also have $E \neq 0$. We plot the graphs of $A = 0$, $B = 0$, $C = 0$, $D = 0$, $E = 0$ and $F = 0$ in Figure 5.1 in the first quadrant of the $(b, \mu)$–plane in order to study these inequalities. It is not difficult to prove that the curve $F = 0$ does not intersect $E = 0$, and that it intersects each of the remaining curves only once. Also note that $\mu > 1/3$ when $E = 0$.

Since we are only interested in the case $F > 0$, which is to the left of the curve $F = 0$ in Figure 5.1, we are not interested in the component of the curve $C = 0$ which does not pass through the origin. We see that $D$ is positive on the left and negative on the right of the curve $D = 0$ in Figure 5.1. Moreover we have $A, B, C > 0$ whenever $D, F > 0$. Therefore case ($i$) characterized by the conditions $D > 0$ and $E \neq 0$.

Due to [30] the unique revised sign list in case ($ii$) is $[1, 1, 1, 1, 0, 0]$, so we need $A, B, C > 0$. Figure 5.1 shows that these three inequalities and the equality $E = 0$ are satisfied only when $D > 0$. Hence case ($ii$) is characterized by the conditions $D > 0$ and $E = 0$.

We have shown that systems (3.25) have six finite singular points other than the origin independent of $E$, and that their global phase portraits are topologically equivalent to 1.11 of Figure 1.1 if and only if $D > 0$.

Now we study systems (3.25) having four finite singular points different from the origin. This can be achieved if and only if

($iii$) either (5.3) has four real distinct roots provided that $E \neq 0$,

($iv$) or (5.3) has two real distinct roots and $E = 0$.

Hence the possible revised sign lists that we need to study are $[1, 1, 1, 1, 0, 0]$ and $[1, 1, 0, 0, 0, 0]$ corresponding to cases ($iii$) and ($iv$) respectively.

In case ($iii$) we need $A, B, C > 0$ and $D = 0$. We again have $\mu > 0$ because $A > 0$. From Figure 5.1 we see that when $F > 0$ and $D = 0$ we have $A, B, C > 0$ unless $\mu = \sqrt{5/27}$. On the other hand case ($iv$) requires $B = E = 0$, which is possible only when $\mu = \sqrt{5/27}$, at which we have $D = 0$. Therefore whether $\mu = \sqrt{5/27}$ or not, systems (3.25) have four finite singular points additional to the origin if and only if $D = 0$. Consequently the global phase portrait is 1.12 of Figure 1.1 if and only if $D = 0$.

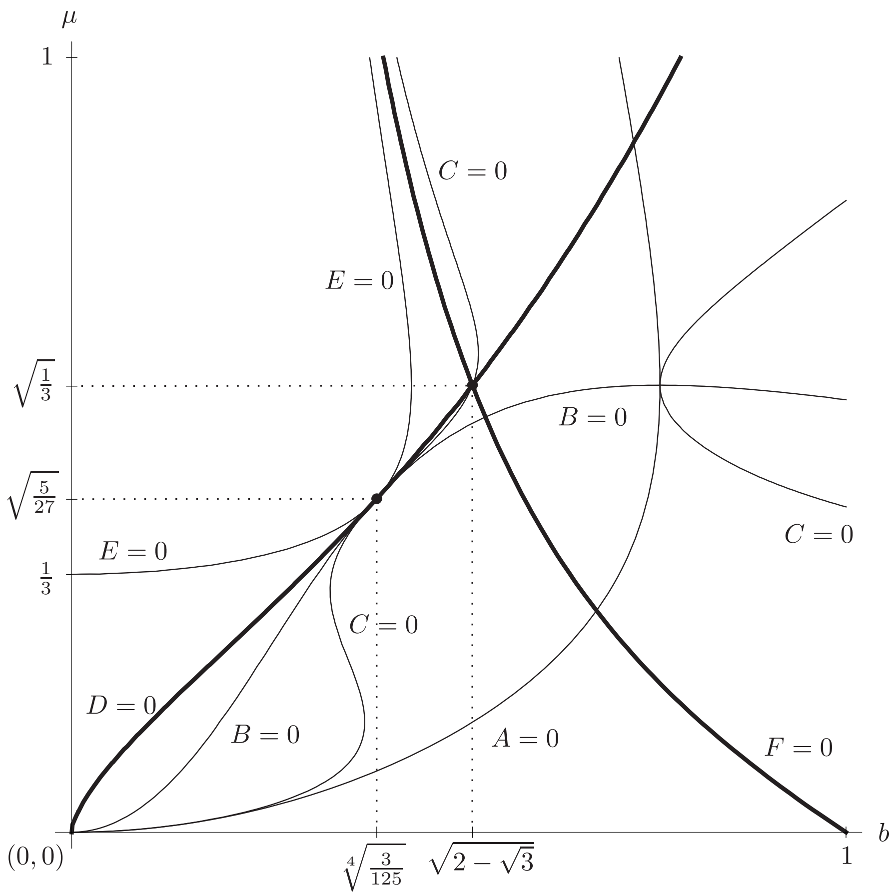



Figure 5.1: The graphs of $A = 0$, $B = 0$, $C = 0$, $D = 0$ and $E = 0$ on the $(b, \mu)$–plane.

Finally it only remains to study systems (3.25) having only two additional finite singular points, in which case their global phase portraits are topologically equivalent to 1.3 of Figure 1.1. But as a trivial result of the above study this case can be realized if and only if $D < 0$.

In light of all the information that we obtained, for systems $(XI)$ we get the bifurcation diagram given in Figure 1.10.

## 5.2 Bifurcation diagram for systems $(XII)$

In Section 3.7 it is shown that for $\mu = -1/3$ and $\mu > -1/3$, the global phase portraits of systems $(XII)$ are topologically equivalent to 1.15 and 1.18 of Figure 1.1, respectively. When $\mu < -1/3$, the unique global phase portrait is 1.14 of Figure 1.1 if $b = 0$, but there are two possibilities if $b \neq 0$: 1.14 and 1.13. We are going to distinguish these last two phase portraits using the facts that systems $(XII)$ are Hamiltonian and that there are four finite singular points on the same energy level in the former but only two in the latter, see Remark 25.

When $b \neq 0$, due to Remark 6 systems $(XII)$ are written as (3.35), and the center condition (3.36) becomes

$$\frac{1 + b^4 + 6b^2\mu}{b} < 0, \tag{5.7}$$

We remind that systems (3.35) have the Hamiltonian

$$H_{12}^2(x, y) = -\frac{x^4 + y^4}{4} - \frac{3\mu x^2 y^2}{2} + \frac{x^2}{2b} + \frac{by^2}{2} + xy.$$

Assume $\mu < -1/3$. We are going to look for the number $\mathcal{N}$ of distinct real solutions of the three equations $\dot{x} = 0$, $\dot{y} = 0$ and $H_{12}^2 - h = 0$, where $h \in \mathbb{R} \setminus \{0\}$. Note that $h \neq 0$ because the only singular point of systems (3.35) at which $H_{12}^2 = 0$ is the origin. Indeed, evaluating $H_{12}^2$ at a singular point $(x_0, y_0)$ of systems (3.35) we get

$$H_{12}^2(x_0, y_0) = H_{12}^2(x_0, y_0) - \frac{y_0\dot{x} - x_0\dot{y}}{4} = \frac{(x_0 + by_0)^2}{4b} = h,$$

due to the fact that $\dot{x} = \dot{y} = 0$ at $(x_0, y_0)$. Then we have $h = 0$ if and only if $x_0 + by_0 = 0$. But when $x_0 = -by_0$ we obtain

$$\begin{aligned} \dot{x} &= -(1 + 3b^2\mu)y_0^3 = 0, \\ \dot{y} &= -b(b^2 + 3\mu)y_0^3 = 0. \end{aligned}$$

If $y_0 \neq 0$, then, since $b \neq 0$, we need to have $1 + 3b^2\mu = 0 = b^2 + 3\mu$. Hence we get $b^2 = -3\mu$ and $1 - 9\mu^2 = 0$, which is not possible because $\mu < -1/3$. So we have $y_0 = x_0 = 0$.

In order to simplify our calculations we multiply $H_{12}^2$ by 4 and calculate the Gröbner basis of the three polynomials $\dot{x}$, $\dot{x}$ and $4H_{12}^2 - h$. We see that it consists of 27 polynomials in the variables $x$ and $y$. Due to the length of these polynomials we cannot present all of them here but we provide all the necessary information that we get from them. First of all there are 21 polynomials that do not contain $x$, and they are of degrees varying between two and six in $y$. In particular, 7 of these polynomials are of the form $py^2+q$, where $p$ and $q$ are constants in terms of the parameters $b$ and $\mu$. Second, there is another polynomial that is linear in $x$ such that the coefficient of $x$ is $809238528h$, which is different from zero. This means that whenever $p \neq 0$ in one of the 7 polynomials of the form $py^2+q$ we have $\mathcal{N} \leq 2$, and therefore at most two singular points of systems (3.35) are on the same energy level.

We pick 4 of these 7 polynomials and call them $P_1$, $P_2$, $P_3$ and $P_4$. Due to the length of these polynomials, we will only provide the coefficients of their quadratic terms:

$$
\begin{aligned}
p_1 = {} & h(b^4-1)^2\big(-27b^2(b^4-1)^2 + 16(b^4+1)^3h - 48b^2(b^4+1)^2h^2 \\
& + 48b^4(b^4+1)h^3 - 16b^6h^4\big), \\
p_2 = {} & h(b^4-1)\big(10368(b^4-1)^3 + 24(192\mu + 1399b^2 - 846b^6 - 233b^{10} \\
& - 256b^{14})h + 4(b^4-1)(520 + 5759b^4 + 4345b^8)h^2 - 2(155b^2 - 5046b^6 \\
& + 6139b^{10} + 864b^{14} + 6336\mu)h^3 + (b^4-1)(1592 + 13109b^4 + 3731b^8)h^4 \\
& + (5577b^2 - 466b^6 - 3399b^{10} + 880b^{14} + 7776\mu)h^5 - 2(b^4-1)(721 \\
& + 2240b^4 + 1151b^8)h^6 - 4(656b^2 + 203b^6 - 861b^{10} + 152b^{14} + 450\mu)h^7 \\
& + 4(b^4-1)(153 + 230b^4 + 445b^8)h^8 + 16(74b^2 + 43b^6 - 118b^{10} + 4b^{14} \\
& + 9\mu)h^9 - 32(b^4-1)(b^4-2)(6b^4-1)h^{10} + 64(b^4-1)b^2(3b^4+2)h^{11} \\
& - 64(b^4-1)b^4h^{12}\big), \\
p_3 = {} & h(b^4-1)\big(96(-149 + 328b^4 - 369b^8 + 126b^{12} - 192b^2\mu) + 2(24419b^2 \\
& - 15526b^6 - 2429b^{10} - 3584b^{14} + 8640\mu)h - (5136 - 14383b^4 \\
& + 26286b^8 - 29327b^{12} - 36864b^2\mu)h^2 + (6053b^2 + 10182b^6 - 20571b^{10} \\
& - 8144b^{14} - 37440\mu)h^3 + 2(b^4-1)(4923 + 21104b^4 + 11621b^8)h^4 \\
& + 4(4688b^2 + 2065b^6 - 6775b^{10} + 520b^{14} + 1494\mu)h^5 - 4(b^4-1) \\
& (523 - 494b^4 + 1527b^8)h^6 - 16(254b^2 + 145b^6 - 402b^{10} + 12b^{14} \\
& + 27\mu)h^7 + 32(b^4-1)(6 - 47b^4 + 18b^8)h^8 - 192(b^4-1)b^2(2+3b^4)h^9 \\
& + 192(b^4-1)b^4h^{10}\big), \\
p_4 = {} & h\big(254016(b^4-1)(119b^2 - 78b^6 + 23b^10 + 192\mu) - 96(13289 + 46216b^4 \\
& - 329710b^8 + 282592b^{12} - 37475b^16 - 479808b^2\mu + 329280b^6\mu \\
& - 225792\mu^2)h + 2(5152135b^2 - 3602421b^6 + 34453b^{10} + 2916057b^{14} \\
& - 2091776b^{18} + 13070016\mu - 5844672b^4\mu)h^2 + (7140592 + 9690755b^4
\end{aligned}
$$

$$
\begin{aligned}
&- 240715936^8 - 1663991b^{12} + 10710573b^{16} - 3612672b^2\mu \\
&- 27095040\mu^2)h^3 - (12340399b^2 - 118144936^6 - 7892771b^{10} \\
&+ 9122145b^{14} - 551056b^{18} + 9119808\mu - 5507136b^4\mu)h^4 \\
&- 2(1165263 - 456814b^4 - 917952b^8 - 250482b^{12} + 761041b^{16} \\
&- 2709504\mu^2)h^5 - 4(b^4 - 1)(683024b^2 - 241779b^6 - 455259b^{10} \\
&+ 45544b^{14} + 94590\mu)h^6 + 4(b^4 - 1)^2(45471 - 694b^4 + 137435b^8)h^7 \\
&+ 16(b^4 - 1)(22918b^2 + 11021b^6 - 33866b^{10} - 292b^{14} - 657\mu)h^8 \\
&+ 32(b^4 - 1)^2(146 + 6131b^4 + 438b^8)h^9 - 4672(b^4 - 1)^2b^2(2 + 3b^4)h^{10} \\
&+ 4672(b^4 - 1)^2b^4h^{11}\big),
\end{aligned}
$$

where $p_i$ is the coefficient of the quadratic term of the polynomial $P_i$.

We compute the resultant of $p_1$ and $p_2$ with respect to $h$, remove the nonzero constant and the repeating factors, and obtain

$$
\begin{aligned}
r_1 =&b(b^4 - 1)(1 + 2b^2 - b^4)(1 - 2b^2 - b^4)(32 - 155b^2 + 138b^4 - 155b^6 + 32b^8) \\
&(32 + 155b^2 + 138b^4 + 155b^6 + 32b^8)(128 + 87b^4 + 128b^8) \\
&\big(b^2 - 6b^2\mu^2 + 4(1 + b^4)\mu^3 - 3b^2\mu^4\big),
\end{aligned}
$$

When $r_1 \neq 0$, due to the properties of the resultant we know that the coefficients $p_1$ and $p_2$ cannot be zero simultaneously, and as a result $\mathcal{N} \leq 2$. Therefore we are going to study the number $\mathcal{N}$ when $r_1 = 0$.

Since $b \neq 0$, we begin with $b^4 - 1 = 0$. If $b = 1$, systems (3.35) become

$$
\dot{x} = x + y - 3\mu x^2 y - y^3, \quad \dot{y} = -x - y + x^3 + 3\mu x y^2. \tag{5.8}
$$

Then we can explicitly calculate their finite singular points and we get

$$
(0,0), \ \pm(\sqrt{M_1}M_2(1 - 3\mu), \sqrt{M_1}), \text{ and } \pm(\sqrt{M_2}M_1(1 - 3\mu), \sqrt{M_2}),
$$

where

$$
M_{1,2} = 1 \pm \frac{\sqrt{3(3\mu^2 - 2\mu - 1)}}{1 - 3\mu}.
$$

Observe that

$$
3\mu^2 - 2\mu - 1 > 3\left(-\frac{1}{3}\right)^2 - 2\left(-\frac{1}{3}\right) - 1 = 0
$$

and

$$
3(3\mu^2 - 2\mu - 1) = 9\mu^2 - 6\mu - 3 < 9\mu^2 - 6\mu + 1 = (1 - 3\mu)^2
$$

whenever $\mu < -1/3$, hence $M_{1,2}$ are positive. In addition, it is easy to check that we have $H_{12}^2 = (3\mu + 1)/\big(4(3\mu - 1)\big)$ at the finite singular points other than the origin. This means that systems (5.8) have four finite singular points which are on the same energy level, and therefore their global phase portraits are topologically equivalent to 1.14 of Figure 1.1.

If $b = -1$ then we have

$$\frac{1 + b^4 + 6b^2\mu}{b} = -2 - 6\mu > 0$$

whenever $\mu < -1/3$, which means that systems (3.35) cannot have a center at the origin (see (5.7)). So we have $b \neq -1$.

Now we study the case $1 + 2b^2 - b^4 = 0$. Solving for $b$ yields $b = \pm\sqrt{1+\sqrt{2}}$. When we substitute these values into $p_1$ (note that $p_1$ is an even polynomial in the variable $b$), equate it to zero and solve for $h$, we obtain $h = 1/\sqrt{2}$. Then we substitute both of these $b$ and $h$ into $p_4$ and get

$$4741632(\mu^2 - 2\sqrt{2}\mu - 1),$$

which is greater than zero for $\mu < -1/3$. This means that when $1+2b^2-b^4 = 0$ we have $p_4 \neq 0$, hence $\mathcal{N} \leq 2$.

If $1 - 2b^2 - b^4 = 0$, we can show by repeating the same calculations that we did in the case with $1 + 2b^2 - b^4 = 0$ that $\mathcal{N} \leq 2$.

Next is the case $32 - 155b^2 + 138b^4 - 155b^6 + 32b^8 = 0$. Following the same steps as in the last two cases is a little cumbersome here due to the higher degree of this polynomial in $b$. Instead we calculate the resultant of $p_1$ and $p_3$ with respect to $h$, and see that the only factor that does not appear in $r_1$ is

$$\begin{aligned} r_2 = {} & 4218421248 - 204309374976b^4 + 32568253073355b^8 - 59437602175975b^{12} \\ & - 18532611271775b^{16} - 3733079567170415b^{20} + 1715045088159217b^{24} \\ & - 2179298014880247b^{28} + 357602721621501b^{32} - 81806891966683b^{36} \\ & + 3902515292160b^{40} - 354375696384b^{44} + 72477573125b^{48}. \end{aligned}$$

Then we calculate the resultant of $32 - 155b^2 + 138b^4 - 155b^6 + 32b^8$ and $r_2$ with respect to $b$ and see that it is not zero. Therefore in this case even if $p_1 = p_2 = 0$, we have $p_3 \neq 0$, and consequently we have $\mathcal{N} \leq 2$.

The next two factors in $r_1$ cannot be zero for real $b$, so it only remains to study the case $w = b^2 - 6b^2\mu^2 + 4(1 + b^4)\mu^3 - 3b^2\mu^4 = 0$. However, this case is not possible because for $\mu < -1/3$ we have

$$w < b^2 - 6b^2\left(-\frac{1}{3}\right)^2 + 4(1+b^4)\left(-\frac{1}{3}\right)^3 - 3b^2\left(-\frac{1}{3}\right)^4 = -\frac{4}{27}(b^4-1)^4 \leq 0.$$

As a result of the above analysis we conclude that when $\mu < -1/3$ systems (3.35) have the global phase portrait 1.14 of Figure 1.1 if and only if $b = 1$. Therefore we obtain the bifurcation diagram for systems $(XII)$ as shown in Figure 1.11.

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# Index