New 1 : 1 : 1 periodic solutions in 3–dimensional galactic–type Hamiltonian systems

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Abstract We consider the 3–dimensional galactic Hamiltonians $H = H(q_1,q_2,q_3,p_1,p_2,p_3) = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2}(q_1^2 + q_2^2 + q_3^2) + H_1$, where either $H_1 = -(q_1^2 q_2^2 + q_1^2 q_3^2 + q_2^2 q_3^2)$ or $H_1 = -(q_1 + q_2)q_3^2$. Applying the averaging theory of first (respectively second) order we prove the existence of 18 (respectively 8) families of periodic orbits parameterized by the the energy $H = h$ for the first (respectively second) Hamiltonian.

Keywords periodic orbits, galactic–type Hamiltonian, averaging theory

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1 Introduction and statements of main results

In this work we prove analytically the existence of families of periodic solutions for galactic Hamiltonian systems with 3 degrees of freedom. For a good introduction to the galactic–type Hamiltonians here studied the reader can look at the paper of Caranicolas [4] and at the references therein for a detailed deduction and implications about the importance of these Hamiltonians.

Let

$$H = H(q_1,q_2,q_3,p_1,p_2,p_3) = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2}(q_1^2 + q_2^2 + q_3^2) + H_1, \quad (1)$$

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be a Hamiltonian in resonance 1 : 1 : 1 where H_1 will have two different expressions. First we consider

$$H_1 = H_1(q_1,q_2,q_3,p_1,p_2,p_3) = -(q_1^2 q_2^2 + q_1^2 q_3^2 + q_2^2 q_3^2), \quad (2)$$

with $q_1,q_2,q_3,p_1,p_2,p_3 \in \mathbb{R}$. As Caranicolas said the Hamiltonian (1)–(2) have been used by several authors (see for instance, Van der Aa & Sanders [12]; Martinet & Magnenat [8]; Magnenat [7]; Martinet et al [9]; Hayli [5]) for studying the local dynamical properties of galaxies. Other models for studying the motion in non-axially symmetric galaxies and disk galaxies with non-spherical nuclei has been considered in [18]-[19], respectively.

The Hamiltonian system associated to (1)–(2) is

$$
\begin{aligned}
\dot{q}_1 &= p_1, \\
\dot{q}_2 &= p_2, \\
\dot{q}_3 &= p_3, \\
\dot{p}_1 &= -q_1 + 2q_1(q_2^2 + q_3^2), \\
\dot{p}_2 &= -q_2 + 2q_2(q_1^2 + q_3^2), \\
\dot{p}_3 &= -q_3 + 2q_3(q_1^2 + q_2^2).
\end{aligned}
\quad (3)
$$

As usual the dot denotes derivative with respect to the independent variable $t \in \mathbb{R}$, the time.

In order to apply the averaging theory to system (3) we introduce a small parameter ε doing the rescaling $(q_1,q_2,q_3, p_1,p_2,p_3) = (\sqrt{\varepsilon}X, \sqrt{\varepsilon}Y, \sqrt{\varepsilon}Z, \sqrt{\varepsilon}p_X, \sqrt{\varepsilon}p_Y, \sqrt{\varepsilon}p_Z)$. Since this change of coordinates is ε^{-1}-symplectic (see for more details [10]), the Hamiltonian function (1)–(2) in these new variables writes

$$\mathscr{H} = \frac{1}{2}(p_X^2 + p_Y^2 + p_Z^2) + \frac{1}{2}(X^2 + Y^2 + Z^2) - \varepsilon\left(X^2 Y^2 + X^2 Z^2 + Y^2 Z^2\right), \quad (4)$$

and system (3) becomes

$$\dot{X} = p_X,$$
$$\dot{Y} = p_Y,$$
$$\dot{Z} = p_Z,$$
$$\dot{p}_X = -X + 2\varepsilon X(Y^2 + Z^2),$$
$$\dot{p}_Y = -Y + 2\varepsilon Y(X^2 + Z^2)$$
$$\dot{p}_Z = -Z + 2\varepsilon Z(X^2 + Y^2).$$

(5)

The periodic orbits are the most simple non–trivial solutions of a differential system. Their study is of particular interest because the motion in their neighborhood can be determined by their kind of stability. In general it is very difficult to study analytically the existence of periodic orbits and their kind of stability for a given Hamiltonian system. In this work we use the averaging theory of first order for computing periodic orbits and their kind of stability, see Appendix A for a summary of this theory. The averaging theory allows to find analytically periodic orbits of this galactic model (5) in any positive Hamiltonian level. Roughly speaking, this method reduces the problem of finding periodic solutions of some differential system to the one of finding zeros of some convenient finite dimensional function.

Our main result about the periodic orbits of the Hamiltonian system (5) is summarized as follows.

Theorem 1 *For $\varepsilon > 0$ sufficiently small at every positive Hamiltonian level $H = h$ the galactic-type Hamiltonian system* (5) *has* 18 *periodic solutions* $(X(t,\varepsilon), Y(t,\varepsilon), Z(t,\varepsilon), p_X(t,\varepsilon), p_Y(t,\varepsilon), p_Z(t,\varepsilon))$ *given by*

$$X(t,\varepsilon) = r^* \cos t + O(\varepsilon),$$
$$Y(t,\varepsilon) = \rho^* \cos(t + \alpha^*) + O(\varepsilon),$$
$$Z(t,\varepsilon) = R^* \cos(t + \beta^*) + O(\varepsilon),$$
$$p_X(t,\varepsilon) = r^* \sin t + O(\varepsilon),$$
$$p_Y(t,\varepsilon) = \rho^* \sin(t + \alpha^*) + O(\varepsilon),$$
$$p_Z(t,\varepsilon) = R^* \sin(t + \beta^*) + O(\varepsilon),$$

(6)

where $r^*, \rho^*, R^*, \alpha^*, \beta^*$ *are*

(i) $r^* = 0, \rho^* = \sqrt{h}, R^* = \sqrt{h}, \alpha^* = \pi/3, 4\pi/3, \beta^* = \pi/3,$

(ii) $r^* = \sqrt{h}, \rho^* = 0, R^* = \sqrt{h}, \alpha^* = \pi/3, \beta^* = 0, \pi,$

(iii) $r^* = \sqrt{h}, \rho^* = \sqrt{h}, R^* = 0, \alpha^* = 0, \pi, \beta^* = \pi/3,$

(iv) $r^* = \rho^* = R^* = \sqrt{2h/3}$ *and* (α^*, β^*) *are given by* $(-\pi/3, -2\pi/3), (2\pi/3, -2\pi/3), (-2\pi/3, -\pi/3), (\pi/3, -\pi/3), (-\pi/3, \pi/3), (2\pi/3, \pi/3), (-2\pi/3, 2\pi/3), (\pi/3, 2\pi/3), (0, 0), (\pi, 0), (0, \pi), (\pi, \pi).$

Moreover on each Hamiltonian level $H = h$ the 18 *periodic solutions are unstable.*

Theorem 1 is proved in section 2 using the averaging theory of first order. Note that the change of variables is only a scale transformation, for $\varepsilon > 0$ the original system (3) and the transformed systems (5) have essentially the same phase portrait. Note that system (5) for $\varepsilon > 0$ sufficiently small is close to an integrable one.

Caranicolas in [4] using a semi–numerical method found 4 families of periodic orbits for system (5). These 4 families of periodic orbits are straight line periodic orbits living on the invariant sets $q_1 = \pm q_2 = \pm q_3$. These 4 families correspond to the 4 families $(\alpha^*, \beta^*) \in \{(0,0), (\pi,0), (0,\pi), (\pi,\pi)\}$ of Theorem 1, but we additionally have obtained 12 new families of periodic solutions parameterized by h.

In [4] also was studied the Hamiltonian (1) with

$$H_1 = H_1(q_1, q_2, q_3, p_1, p_2, p_3) = -(q_1 + q_2)q_3^2. \quad (7)$$

The Hamiltonian system associated to (1)–(7) is

$$\dot{q}_1 = p_1,$$
$$\dot{q}_2 = p_2,$$
$$\dot{q}_3 = p_3,$$
$$\dot{p}_1 = -q_1 + q_3^2,$$
$$\dot{p}_2 = -q_2 + q_3^2,$$
$$\dot{p}_3 = -q_3 + 2(q_1 + q_2)q_3.$$

(8)

In order to apply the averaging method of second order to system (8) we introduce a small parameter ε by the change of the variables $(q_1, q_2, q_3, p_1, p_2, p_3) = (\varepsilon X, \varepsilon Y, \varepsilon Z, \varepsilon p_X, \varepsilon p_Y, \varepsilon p_Z)$. Since this change of coordinates is ε^{-2}-symplectic, the Hamiltonian (1)–(7) in these new variables takes the form

$$\mathscr{H} = \frac{1}{2}(p_X^2 + p_Y^2 + p_Z^2) + \frac{1}{2}(X^2 + Y^2 + Z^2) - \varepsilon(X + Y)Z^2,$$

and system (8) becomes

$$\dot{X} = p_X,$$
$$\dot{Y} = p_Y,$$
$$\dot{Z} = p_Z,$$
$$\dot{p}_X = -X + \varepsilon Z^2,$$
$$\dot{p}_Y = -Y + \varepsilon Z^2,$$
$$\dot{p}_Z = -Z + 2\varepsilon(X + Y)Z.$$

(9)

Our main result about the periodic orbits of the Hamiltonian system (9) is summarized as follows.

Theorem 2 *For $\varepsilon > 0$ sufficiently small at every positive Hamiltonian level $H = h$ the galactic–type Hamiltonian system* (9) *has* 8 *periodic solutions* $(X(t,\varepsilon), Y(t,\varepsilon), Z(t,\varepsilon), p_X(t, \varepsilon), p_Y(t,\varepsilon), p_Z(t,\varepsilon))$ *given by* (6) *where* $r^*, \rho^*, R^*, \alpha^*, \beta^*$ *are given by*

(i) $r^* = \sqrt{h/3}, \rho^* = \sqrt{h/3}, R^* = 2\sqrt{h/3}, \alpha^* = 0, \beta^* = 0, \pi;$

(ii) $r^* = \sqrt{(39 + \sqrt{17})h/94}, \rho^* = \sqrt{(57 - 13\sqrt{17})h/94},$

$\quad R^* = \sqrt{2(23 + 3\sqrt{17})h/47}, \alpha^* = 0, \beta^* = 0, \pi;$

(iii) $r^* = \sqrt{(39 - \sqrt{17})h/94}, \ \rho^* = \sqrt{(57 + 13\sqrt{17})h/94},$

$\quad R^* = \sqrt{2(23 - 3\sqrt{17})h/47}, \ \alpha^* = 0, \beta^* = 0, \pi;$

(iv) $r^* = \dfrac{\left(5\sqrt{113\sqrt{89}-729} - \sqrt{9153\sqrt{89}-86153}\right)\sqrt{h}}{308\sqrt{2}},$

$\quad \rho^* = \sqrt{(-729 + 113\sqrt{89})h/242},$

$\quad R^* = \frac{1}{11}\sqrt{(730 - 70\sqrt{87})h}, \alpha^* = \pi, \beta^* = \pm\pi/2.$

Moreover on each Hamiltonian level $H = h$ the two periodic solutions given by conditions (i) are linearly stable; while the 6 ones given by conditions (ii)–(iv) are unstable.

The proof of Theorem 2 is given in Section 3.

Using his semi–numerical method Caranicolas found in [4] two families of rectilinear periodic solutions on the invariant sets $q_1 = q_2 = \pm q_3/2$. These two families correspond to the two families (i) of Theorem 2. Additionally we obtain 6 new families of periodic solutions parameterized by the Hamiltonian level h.

From the statements of Theorems 1 and 2 it follows that the initial conditions for the different periodic orbits in every positive Hamiltonian level $H = h$ found in those theorems change continuously only with h, so in the phase space of the corresponding Hamiltonian systems these periodic orbits live on cylinders parameterized by h.

As we shall see one of the main problems for applying the averaging theory for studying the periodic orbits of a given differential system is to find the changes of variables which allow to write the original differential system in the normal form for applying the averaging theory. For more details in this direction see the book [11].

In [15]-[17] dynamics aspects has been studied in Hamiltonian systems in 2D and 3D, in particular the existence of periodic orbits, escapes and chaos was considered.

2 Proof of Theorem 1

For proving Theorem 1 we shall apply Theorem 3 to the Hamiltonian system (5). Generically the periodic orbits of a Hamiltonian system with more than one degree of freedom are on cylinders filled of periodic orbits parameterized by the value h of Hamiltonian level. Therefore we cannot apply directly Theorem 3 to a Hamiltonian system, because the Jacobian of the function f defined in the Appendix A always is zero. Then we must apply Theorem 3 to every Hamiltonian level where the periodic orbits generically are isolated. For

more details on the families of periodic solutions of Hamiltonian systems see [1].

First we change the Hamiltonian (4) and the equations of motion (5) to convenient generalized polar coordinates in such a way that for $\varepsilon = 0$ we have a pair of harmonic oscillators. Thus we consider the change of variables

$$X = r\cos\theta, \ \ Y = \rho\cos(\theta + \alpha), \ \ Z = R\cos(\theta + \beta),$$
$$p_X = r\sin\theta, \ p_Y = \rho\sin(\theta + \alpha), \ p_Z = R\sin(\theta + \beta), \qquad (10)$$

with $(r, \theta, \rho, \alpha, R, \beta) \in \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{R}^+ \times \mathbb{S}^1$. Note that this is a change of variables when r, ρ and R are positive, and that this change of coordinates is not canonical. So we lose the Hamiltonian structure of the differential equations. Moreover doing this change of variables appear in the system the angular variables θ, α and β. Later on the variable θ will be used for obtaining the periodicity necessary for applying the averaging theory.

The fixed value of the Hamiltonian level $H = h$ in polar coordinates is

$$h = \tfrac{1}{2}(r^2 + \rho^2 + R^2) + \varepsilon \left[-r^2\cos^2(\theta)\left(R^2\cos^2(\beta + \theta) + \right.\right.$$
$$\left.\left. \rho^2\cos^2(\alpha + \theta)\right) - R^2\rho^2\cos^2(\alpha + \theta)\cos^2(\beta + \theta)\right], \qquad (11)$$

and the equations of motion are given by

$$\dot{r} = 2\varepsilon r\sin\theta\cos\theta\left(R^2\cos^2(\beta + \theta) + \rho^2\cos^2(\alpha + \theta)\right),$$

$$\dot{\rho} = 2\varepsilon\rho\sin(\alpha + \theta)\cos(\alpha + \theta)\left(r^2\cos^2(\theta) + R^2\cos^2(\beta + \theta)\right),$$

$$\dot{R} = 2\varepsilon R\sin(\beta + \theta)\cos(\beta + \theta)\left(r^2\cos^2(\theta) + \rho^2\cos^2(\alpha + \theta)\right),$$

$$\dot{\theta} = 1 - \varepsilon\cos^2\theta\left[R^2\cos^2(\beta + \theta) + \rho^2\cos^2(\alpha + \theta)\right],$$

$$\dot{\alpha} = 2\varepsilon\left[\cos^2\theta\left((r^2 - \rho^2)\cos^2(\alpha + \theta) - R^2\cos^2(\beta + \theta)\right) + R^2\cos^2(\alpha + \theta)\cos^2(\beta + \theta)\right],$$

$$\dot{\beta} = 4\varepsilon\left[\rho^2\cos^2(\alpha + \theta)\cos^2(\beta + \theta) - \cos^2\theta\left((R^2 - r^2)\cdot\right.\right.$$
$$\left.\left. \cos^2(\beta + \theta) + \rho^2\cos^2(\alpha + \theta)\right)\right].$$

Note that the derivatives on the left hand side of these equations are with respect to the time t, and that the system is not periodic in t. For $\varepsilon > 0$ sufficiently small in a neighborhood of $r = \rho = R = 0$ we have that $\dot{\theta} > 0$. In such a neighborhood we take θ as the new independent variable, and we denote by a prime the derivative with respect to θ. The angular variables α, β cannot be used as the independent variable because the new differential system would not have the normal form (23) for applying the averaging theory given in Theorem 3.

Fixing the value of the first integral H at h such that $2h - r^2 - R^2 > 0$, solving equation (11) with respect to ρ, and

expanding ρ in Taylor series of ε, we obtain

$$\rho = \sqrt{2h - r^2 - R^2} + \varepsilon \frac{1}{\sqrt{2h - r^2 - R^2}} \left[2r^2 R^2 \cos^2(\theta) \cdot \right.$$
$$\cos^2(\beta + \theta) - \left(-2h + r^2 + R^2 \right) \cos^2(\alpha + \theta)$$
$$\left. \left(r^2 \cos^2(\theta) + R^2 \cos^2(\beta + \theta) \right) \right] + O(\varepsilon^2). \tag{12}$$

We write the differential system $(r', R', \alpha', \beta')$, substitute in it the expression of ρ given in (12), and expanding in Taylor series in ε, we obtain the differential system

$$r' = -\varepsilon r \sin(2\theta) \left[(2h - r^2 - R^2) \cos^2(\alpha + \theta) + \right.$$
$$\left. R^2 \cos^2(\beta + \theta) \right] + O(\varepsilon^2)$$
$$= F_{11} + O(\varepsilon^2),$$
$$R' = -\varepsilon R \sin(2(\beta + \theta)) \left[r^2 \cos^2(\theta) - \right.$$
$$\left. \left(-2h + r^2 + R^2 \right) \cos^2(\alpha + \theta) \right] + O(\varepsilon^2)$$
$$= F_{12} + O(\varepsilon^2),$$
$$\alpha' = -2\varepsilon \left[\cos^2(\theta) \left(-2h + 2r^2 + R^2 \right) \cos^2(\alpha + \theta) + \right.$$
$$\left. R^2 \sin(\alpha) \sin(\alpha + 2\theta) \cos^2(\beta + \theta) \right] + O(\varepsilon^2),$$
$$= F_{13} + O(\varepsilon^2),$$
$$\beta' = 2\varepsilon \left[\cos^2(\beta + \theta) \left(\left(-2h + r^2 + R^2 \right) \cos^2(\alpha + \theta) + \right. \right.$$
$$\left. \left(R^2 - r^2 \right) \cos^2(\theta) \right) +$$
$$\left. \cos^2(\theta) \left(2h - r^2 - R^2 \right) \cos^2(\alpha + \theta) \right]$$
$$= F_{14} + O(\varepsilon^2). \tag{13}$$

Clearly system (13) satisfies the assumptions of Theorem 3, and it has the normal form (23) of the averaging theory with $F_1 = (F_{11}, F_{12}, F_{13}, F_{14})$.

The function F_1 is analytic and 2π–periodic in the variable θ, the independent variable of system (13). In order to apply the averaging theory of first order described in the Appendix A we must calculate the averaged function $f_1 = (f_{11}, f_{12}, f_{13}, f_{14})$ where $f_{1j} = f_{1j}(r, R, \alpha, \beta)$ for $j = 1, 2, 3, 4$. Computing them we obtain

$$f_{11} = \frac{1}{2\pi} \int_0^{2\pi} F_{11} d\theta$$
$$= -\frac{1}{4} r \left[\sin(2\alpha) \left(-2h + r^2 + R^2 \right) - R^2 \sin(2\beta) \right],$$
$$f_{12} = \frac{1}{2\pi} \int_0^{2\pi} F_{12} d\theta \tag{14}$$
$$= \frac{1}{4} \left[(\cos(2\alpha) + 2) \left(2 - 2r^2 - R^2 \right) + \right.$$
$$\left. 2R^2 \sin\alpha \sin(\alpha - 2\beta) \right],$$

$$f_{13} = \frac{1}{2\pi} \int_0^{2\pi} F_{13} d\theta$$
$$= -\frac{1}{4} R \left[\left(-2h + r^2 + R^2 \right) \sin(2(\alpha - \beta)) + r^2 \sin(2\beta) \right],$$
$$f_{14} = \frac{1}{2\pi} \int_0^{2\pi} F_{14} d\theta$$
$$= \frac{1}{4} \left[\left(-2h + r^2 + R^2 \right) \cos(2(\alpha - \beta)) + \right.$$
$$\left. \cos(2\alpha) \left(2h - r^2 - R^2 \right) - \left(r^2 - R^2 \right) (\cos(2\beta) + 2) \right].$$

We compute the real solutions $(r^*, R^*, \alpha^*, \beta^*)$ of the system $f_{1j}(r, R, \alpha, \beta) = 0$ for $j = 1, 2, 3, 4$. It is important to remember that at order 0 in ε we have $\rho = \sqrt{2h - r^2 - R^2}$, so we must to have present that r, R and ρ cannot be simultaneously zero. Solving the first equation of (14) we obtain the following possibilities. First case $r = 0$; second case $\sin(2\alpha) \neq 0$ and $r = \sqrt{\csc(2\alpha) [(2h - R^2) \sin(2\alpha) + R^2 \sin(2\beta)]}$; third case $\sin(2\alpha) = 0$ and $R = 0$; and finally fourth case $\sin(2\alpha) = 0$ and $\sin(2\beta) = 0$.

Case I: $r = 0$. From equation f_{12} of (14) we have $R(R^2 - 2h) \sin(2(\alpha - \beta)) = 0$.

Subcase I.1: $R = 0$. From equation f_{12} of (14) we get $h[\cos(2\alpha) + 2]/2$, so $r = R = 0$ is not a solution.

Subcase I.2: $R = \sqrt{2h}$. From equation of f_{14} in (14) we obtain $h[\cos(2\beta) + 2]/2$, then $r = 0$ and $R = \sqrt{2h}$ is not solution.

Subcase I.3: $\alpha = \beta + k\pi/2$ with $k \in \mathbb{Z}$. Then equations f_{12} and f_{14} of (14) become

$$\left(2h - R^2 \right) \left((-1)^k \cos(2\beta) + 2 \right) + R^2 \left(\cos(2\beta) - (-1)^k \right) = 0,$$
$$-(-1)^k 2 \left(2h - R^2 \right) \sin^2(\beta) + R^2 \cos(2\beta) + 2R^2 = 0.$$

Solving this system we verify that k must be even, i.e. $k = 2l$ and the solutions are $R^* = \sqrt{h}$, $\beta^* = \pm 2\pi/3, \pm \pi/3$, $\alpha^* = l\pi + \beta$. Note that in this case $\rho^* = \sqrt{h}$, and the corresponding Jacobian satisfies

$$|D_{r,R,\alpha,\beta}(f_{11}, f_{12}, f_{13}, f_{14})|_{(r^*, R^*, \alpha^*, \beta^*)}| = \frac{9}{16} h^4 > 0.$$

These families of periodic solutions can be reduced only to two periodic solutions, because they are the same in first order approximation, so we have proved item (i).

Case II: $r = \sqrt{\csc(2\alpha) [(2h - R^2) \sin(2\alpha) + R^2 \sin(2\beta)]}$ and $\sin(2\alpha) \neq 0$. The function f_{13} takes the form

$$\frac{1}{4} R \left[-2h + R^2 \sec(\alpha)(-\cos(\alpha - 2\beta)) + R^2 \right] \sin(2\beta) = 0,$$

where we get $R = 0$, or $-2h + R^2 \sec\alpha(-\cos(\alpha - 2\beta)) + R^2 = 0$, or $\sin(2\beta) = 0$.

Subcase II.1: If we consider the case $R = 0$, then equation f_{12} can be written as $-\frac{1}{2} h(\cos(2\alpha) + 2) = 0$, therefore there is no solutions.

Subcase II.2: The equation $-2h+R^2\sec\alpha(-\cos(\alpha-2\beta))+R^2=0$ has the solution $R=\sqrt{2h/(1-\sec(\alpha)\cos(\alpha-2\beta))}$. Substituting these values of r and R in equations f_{12} and f_{14} we obtain the system

$$-\tfrac{3}{4}h\cot\alpha\csc\beta\sin(\alpha-2\beta)\csc(\alpha-\beta)=0,$$

$$-\tfrac{3}{4}h\csc\alpha\cot\beta\sin(2\alpha-\beta)\csc(\alpha-\beta)=0.$$

Under the restriction $\sin\alpha\sin\beta\sin(\alpha-\beta)\neq 0$, we obtain the solutions $r^*=\rho^*=R^*=\sqrt{2h/3}$, with (α^*,β^*) given by $(-\frac{\pi}{3},-\frac{2\pi}{3})$, $(\frac{2\pi}{3},-\frac{2\pi}{3})$, $(-\frac{2\pi}{3},-\frac{\pi}{3})$, $(\frac{\pi}{3},-\frac{\pi}{3})$, $(-\frac{\pi}{3},\frac{\pi}{3})$, $(\frac{2\pi}{3},\frac{\pi}{3})$, $(-\frac{2\pi}{3},\frac{2\pi}{3})$, $(\frac{\pi}{3},\frac{2\pi}{3})$. It is verified that the Jacobian at these solutions satisfies

$$|D_{r,R,\alpha,\beta}(f_{11},f_{12},f_{13},f_{14})|_{(r^*,R^*,\alpha^*,\beta^*)}|=\frac{h^4}{4}.$$

Therefore Theorem 3 guarantees the existence of 8 periodic solutions and then we have proved part of (iv).

Subcase II.3: Here we have $\beta=k\pi/2$ with $k\in\mathbb{Z}$. Then $r=\sqrt{2h-R^2}$ and substituting this value in equation f_{14} we arrive to the equation $-(h-R^2)((-1)^k+2)/2=0$. So $R=\sqrt{h}$. Next we substitute the value of β and R in equation f_{12} and we obtain that k must be even, i.e. $k=2l$ and then $\alpha=\pm2\pi/3,\pm\pi/3$, $\beta=l\pi$. Therefore $r=\sqrt{h}$. Moreover the Jacobian at these solutions satisfies

$$|D_{r,R,\alpha,\beta}(f_{11},f_{12},f_{13},f_{14})|_{(r^*,R^*,\alpha^*,\beta^*)}|=\frac{9}{8}h^4.$$

These families of periodic solutions can be reduced only to two periodic solutions, because they are the same in first order approximation, so we have proved item (ii).

Case III: $\sin(2\alpha)=0$ and $R=0$, or equivalently, $\alpha=k\pi/2$ with $k\in\mathbb{Z}$. Then equation f_{12} writes $(2h-2r^2)[\cos(2\alpha)+2]/4=0$. So then $r=\sqrt{h}$. Equation for f_{14} implies $-h(-2(-1)^k\sin^2\beta+\cos(2\beta)+2)/4=0$, where we get that k must be even, i.e. $k=2l$ with $l\in\mathbb{Z}$, and $\beta=\pm2\pi/3,\pm\pi/3$ and $\alpha=l\pi$. We verify that the Jacobian at these solutions satisfies

$$|D_{r,R,\alpha,\beta}(f_{11},f_{12},f_{13},f_{14})|_{(r^*,R^*,\alpha^*,\beta^*)}|=\frac{9}{16}h^4.$$

By the same arguments as in the previous cases we conclude the proof of (iii).

Case IV: $\sin(2\alpha)=0$ and $\sin(2\beta)=0$, or equivalently, $\alpha=k\pi/2$ and $\beta=m\pi/2$ with $k,m\in\mathbb{Z}$. Therefore $f_{13}=0$, and f_{12}, f_{14} take the form

$$f_{12}=\frac{1}{4}\left(((-1)^k+2)\left(2h-2r^2-R^2\right)-\right.$$
$$\left.((-1)^k-1)(-1)^mR^2\right),$$

$$f_{14}=\frac{1}{4}\left((-1)^k((-1)^m-1)\left(-2h+r^2+R^2\right)+\right.$$
$$\left.(-(-1)^m-2)(r-R)(r+R)\right).$$

Solving this system we get that $k=2l$, $m=2n$, so $\alpha=l\pi$, $\beta=n\pi$ with $l,n\in\mathbb{Z}$, and $r^*=R^*=\sqrt{2h/3}$. In any of these points we have that the Jacobian at this critical point satisfies

$$|D_{r,R,\alpha,\beta}(f_{11},f_{12},f_{13},f_{14})|_{(r^*,R^*,\alpha^*,\beta^*)}|=-h^4.$$

These periodic solutions can be reduced to only four periodic solutions, so we conclude the proof of(iv).

The proof of the second part of the theorem is as follows. For the family (i) we have that the characteristic polynomial of the linearization of the averaged system associated to (14) at the points $(r^*=0,R^*=\sqrt{h},\rho^*=\sqrt{h},\alpha^*0,\pi,\beta^*=\pi/3)$ is
$$\lambda^4-\frac{1}{2}(h-3)\sqrt{h}\,\lambda^3-\frac{1}{16}h^{3/2}\left(2\sqrt{3}h+15\sqrt{h}-6\sqrt{3}\right)\lambda^2+$$
$$\frac{3}{16}h^{5/2}\left(h-3\sqrt{3}\sqrt{h}-3\right)\lambda+\frac{9}{16}.$$
We verify that one of the roots is $\lambda_1=\sqrt{3}/4h>0$. Thus the two periodic orbits of (i) are unstable.

The characteristic polynomial for any solution of (ii) is
$$\lambda^4+\frac{\sqrt{3}}{4}h\lambda^3+\frac{9}{8}h^2\,\lambda^2+\left[\frac{27}{16}h^{1/2}+\frac{3\sqrt{3}}{8}\right]h^2\,\lambda+\frac{9}{8}h^4.$$
Using the criterium described in [14] we obtain that for each $h>0$ this polynomial has complex roots with positive real part, thus the periodic orbits are unstable.

The characteristic polynomial for any solution in (iii) is
$$p_3(\lambda)=\lambda^4+\left[\frac{\sqrt{3}}{2}-\frac{1}{3}\sqrt{\frac{2}{3}}h\right]\sqrt{h}\,\lambda^3+\left[\frac{\sqrt{3}}{2}-\frac{1}{2}\sqrt{\frac{2}{3}}h\right]h^{5/2}\lambda$$
$$-h^4.$$ Since $p_3(0)<0$ and $p_3(\lambda)\to+\infty$ as $\lambda\to+\infty$, it follows that $p_3(\lambda)$ as at least one positive root, thus the periodic orbits of (iii) are unstable.

The characteristic polynomial for the first 8 solutions of (iv) is

$$p_{4,1}(\lambda)=\lambda^4+\left[\frac{1}{2}\sqrt{\frac{3}{2}}+\frac{h}{3\sqrt{6}}\right]\sqrt{h}\lambda^3+$$
$$\left[\frac{3}{2\sqrt{2}}-\frac{h}{3\sqrt{2}}\right]h^{3/2}\lambda^2-$$
$$\left[\frac{1}{4}\frac{\sqrt{3}}{2}+\frac{h}{6\sqrt{6}}\right]h^{5/2}\,\lambda-\frac{h^4}{4}.$$

Since $p_{4,1}(0)<0$ and $p_{4,1}(\lambda)\to+\infty$ as $\lambda\to+\infty$, it follows that $p_{4,1}(\lambda)$ has at least one positive root, thus these periodic orbits are unstable. For the rest of the solutions in (iv), the characteristic polynomial is

$$p_{4,2}(\lambda)=\lambda^4-\frac{\sqrt{3}}{2}h\,\lambda^3-\frac{3\sqrt{3}}{8}h^3\,\lambda+\frac{9}{16}h^4.$$

Since $p_{4,2}(\lambda)=(\lambda-\sqrt{3}/2h)^2(\lambda+[\sqrt{3}-3i]h/4)(\lambda+[\sqrt{3}+3i]h/4)$, the corresponding periodic orbits are unstable. In short we have proved Theorem1.

3 Proof of Theorem 2

We continue using the polar coordinates given in (10), and we observe that in the Hamiltonian level $H = h$ of the Hamiltonian (1)–(7) in polar coordinates we have

$$h = \frac{1}{2}(r^2 + \rho^2 + R^2) - \varepsilon \cos^2(\beta + \theta) [r \cos \theta + \rho \cos(\alpha + \theta)],$$
(15)

and the equations of motion writes

$$\dot{r} = -\varepsilon R^2 \sin \theta \cos^2(\beta + \theta),$$
$$\dot{\rho} = -\varepsilon R^2 \sin(\alpha + \theta) \cos^2(\beta + \theta),$$
$$\dot{R} = \varepsilon R \sin(2(\beta + \theta))(r \cos \theta + \rho \cos(\alpha + \theta)),$$
$$\dot{\theta} = -1 + \varepsilon \frac{1}{r}[R^2 \cos \theta \cos^2(\beta + \theta)],$$
$$\dot{\alpha} = \varepsilon \frac{1}{r\rho}[R^2 \cos^2(\beta + \theta)(r \cos(\alpha + \theta) - \rho \cos \theta)],$$
$$\dot{\beta} = \varepsilon \frac{1}{r}[\cos^2(\beta + \theta)((2r^2 - R^2) \cos \theta + 2r\rho \cos(\alpha + \theta))].$$

In order to write this system as a periodic differential system, we take the variable θ as the new independent variable, and we use a prime to denote the derivative with respect to θ. The angular variables α and β cannot be used as independent variable because the new differential system would not be in the normal form for applying the averaging theory described in Theorem 3.

Of course the new system has now only five equations because we do not need the $\dot{\theta}$ equation. Writing it in Taylor series of ε, we get

$$r' = -\varepsilon R^2 \sin \theta \cos^2(\beta + \theta) -$$
$$\varepsilon^2 \frac{1}{r} R^4 \sin \theta \cos \theta \cos^4(\beta + \theta) + O(\varepsilon^3),$$
$$\rho' = -\varepsilon R^2 \sin(\alpha + \theta) \cos^2(\beta + \theta) -$$
$$\varepsilon^2 \frac{1}{r} R^4 \cos \theta \sin(\alpha + \theta) \cos^4(\beta + \theta) + O(\varepsilon^3),$$
$$R' = -\varepsilon R \sin(2(\beta + \theta))(r \cos \theta + \rho \cos(\alpha + \theta)) -$$
$$\varepsilon^2 \frac{2}{r} R^3 \cos \theta \sin(\beta + \theta) \cos^3(\beta + \theta)(r \cos \theta +$$
$$\rho \cos(\alpha + \theta)) + O(\varepsilon^3),$$
$$\alpha' = \varepsilon \frac{1}{r\rho} R^2 \cos^2(\beta + \theta)(s \cos \theta - r \cos(\alpha + \theta)) +$$
$$\varepsilon^2 \frac{1}{r^2\rho} R^4 \cos \theta \cos^4(\beta + \theta)(\rho \cos \theta - r \cos(\alpha + \theta)) +$$
$$O(\varepsilon^3),$$
(16)

$$\beta' = \varepsilon \frac{1}{r} \cos^2(\beta + \theta) ((R^2 - 2r^2) \cos \theta - 2r\rho \cos(\alpha + \theta)) +$$
$$\varepsilon^2 \frac{1}{r^2} R^2 \cos \theta \cos^4(\beta + \theta) ((R^2 - 2r^2) \cos \theta -$$
$$2r\rho \cos(\alpha + \theta)) + O(\varepsilon^3).$$

Therefore system (16) is 2π–periodic in the variable θ. In order to apply Theorem 3 we fix the value of the first integral at $h > 0$, and by solving equation (15) for ρ and expanding it in Taylor series of ε we obtain

$$\rho = \sqrt{2h - r^2 - R^2} +$$
$$\varepsilon R^2 \cos^2(\beta + \theta) \left(\frac{r \cos \theta}{\sqrt{2h - r^2 - R^2}} + \cos(\alpha + \theta) \right) -$$
$$\varepsilon^2 \frac{R^4 \cos^4(\beta + \theta)}{2(2h - r^2 - R^2)^{3/2}} \cdot$$
$$[(-2h + r^2 + R^2) \cos^2(\alpha + \theta) + r^2 \cos^2 \theta] + O(\varepsilon^3).$$
(17)

Using this value of ρ in equations (16), we obtain the following 4–dimensional differential system

$$r' = -\varepsilon R^2 \sin \theta \cos^2(\beta + \theta) - \varepsilon^2 \frac{R^4}{r} \sin \theta \cos \theta \cos^4(\beta + \theta)$$
$$+ O(\varepsilon^3)$$
$$= \varepsilon F_{11} + \varepsilon^2 F_{21} + O(\varepsilon^3),$$
$$R' = -\varepsilon R \sin(2(\beta + \theta)) (\sqrt{2h - r^2 - R^2} \cos(\alpha + \theta) +$$
$$r \cos \theta) + \varepsilon^2 \frac{2R^3}{r\sqrt{2h - r^2 - R^2}} \sin(\beta + \theta) \cdot$$
$$\left(\cos \theta \sqrt{2h - r^2 - R^2} + r \cos(\alpha + \theta) \right) \cdot$$
$$\cos^3(\beta + \theta) \left(\sqrt{2h - r^2 - R^2} \cos(\alpha + \theta) +$$
$$r \cos \theta \right) + O(\varepsilon^3)$$
$$= \varepsilon F_{12} + \varepsilon^2 F_{22} + O(\varepsilon^3),$$
$$\alpha' = \varepsilon R^2 \cos^2(\beta + \theta) \left(\frac{\cos \theta}{r} - \frac{\cos(\alpha + \theta)}{\sqrt{2h - r^2 - R^2}} \right) +$$
$$\varepsilon^2 \frac{R^4 \cos^4(\beta + \theta)}{r^2 (2h - r^2 - R^2)^{3/2}} \left(r^2 \sqrt{2h - r^2 - R^2} \cos^2(\alpha + \theta) +$$
$$r \cos \theta \left(-2h + 2r^2 + R^2 \right) \cos(\alpha + \theta) + \cos^2 \theta$$
$$(2h - r^2 - R^2)^{3/2} \right) + O(\varepsilon^3)$$
$$= \varepsilon F_{13} + \varepsilon^2 F_{23} + O(\varepsilon^3),$$
(18)

$$\beta' = \varepsilon \frac{1}{r} \cos^2(\beta + \theta) \left(\left(R^2 - 2r^2\right) \cos \theta - 2r\sqrt{2h - r^2 - R^2} \right.$$

$$\cos(\alpha + \theta)) + \varepsilon^2 R^2 \cos^4(\beta + \theta)$$

$$\left(\frac{2\left(R^2 - 2h\right) \cos \theta \cos(\alpha + \theta)}{r\sqrt{2h - r^2 - R^2}} + \left(\frac{R^2}{r^2} - 2\right) \cos^2 \theta - \right.$$

$$\left. 2\cos^2(\alpha + \theta) \right) + O(\varepsilon^3)$$

$$= \varepsilon F_{14} + \varepsilon^2 F_{24} + O(\varepsilon^3).$$

Using the notation $x = (r, R, \alpha, \beta) \in D = (0, \sqrt{2h}) \times \times (0, \sqrt{2h}) \times \mathbb{R} \times \mathbb{R}$ and $t = \theta$, system (18) has the normal form of the averaging of Theorem 3.

In system (18) r and ρ cannot be zero, so for $r > 0$ and $\rho > 0$ the functions $F_1 = (F_{11}, F_{12}, F_{13}, F_{14})$ and $F_2 = (F_{21}, F_{22}, F_{23}, F_{24})$ are analytical and 2π–periodic in θ, being θ the independent variable of system (18). After some computations and since the components of F_1 are trigonometric polynomials of degree 3 in the variables $\cos \theta$ and $\sin \theta$, we observe that the averaging theory of first order does not apply because the average functions of F_1 are identically zero, i.e.

$$f_1(r, R, \alpha, \beta) = \int_0^{2\pi} (F_{11}, F_{12}, F_{13}, F_{14}) \, d\theta = (0, 0, 0, 0).$$

Now we proceed to calculate the function f_2 of the Appendix A by applying the second order averaging theory. Then

$$f_2(r, R, \alpha, \beta) = \int_0^{2\pi} [D_{rR\alpha\beta} F_1(\theta, r, R, \alpha, \beta) . y_1(\theta, r, R, \alpha, \beta) +$$

$$F_2(\theta, r, R, \alpha, \beta)] \, d\theta,$$

(19)

where

$$y_1(\theta, r, , R, \alpha, \theta) = \int_0^\theta F_1(s, r, R, \alpha, \beta) \, ds.$$

In particular the four components of the vector function y_1 are

$$y_{11} = \int_0^\theta F_{11}(t, r, R, \alpha, \beta) \, dt$$

$$= -\frac{1}{3} R^2 \sin^2(t/2) \left(\cos(2(\theta + \beta)) + 2\cos(\theta + 2\beta) + 3 \right),$$

$$y_{12} = \int_0^\theta F_{12}(t, r, R, \alpha, \beta) \, dt$$

$$= \frac{1}{12r} R^2 \left(3\sin(\theta + 2\beta) + \sin(3\theta + 2\beta) + 6\sin(\theta) - \right.$$

$$4\sin(2\beta)) - \frac{1}{12\sqrt{2h - r^2 - R^2}} R^2 \cdot$$

$$\left(3\sin(\theta - \alpha + 2\beta) + \sin(3\theta + \alpha + 2\beta) + \right.$$

$$6\sin(\theta + \alpha) + 3\sin(\alpha - 2\beta) - \sin(\alpha + 2\beta) -$$

$$\left. 6\sin(\alpha) \right),$$

$$y_{13} = \int_0^\theta F_{13}(t, r, R, \alpha, \beta) \, dt$$

$$= -\frac{1}{6} rR \left(-3\cos(\theta + 2\beta) - \cos(3\theta + 2\beta) + 4\cos(2\beta) \right)$$

$$-\frac{1}{6} R\sqrt{2h - r^2 - R^2} \left(-3\cos(\theta - \alpha + 2\beta) - \right.$$

$$\left. \cos(3\theta + \alpha + 2\beta) + 3\cos(\alpha - 2\beta) + \cos(\alpha + 2\beta) \right),$$

$$y_{14} = \int_0^\theta F_{14}(t, r, R, \alpha, \beta) \, dt$$

$$= -\frac{1}{6} r \left(3\sin(\theta + 2\beta) + \sin(3\theta + 2\beta) + 6\sin\theta - \right.$$

$$4\sin(2\beta)) + \frac{R^2}{12r} \left(3\sin(\theta + 2\beta) + \sin(3\theta + 2\beta) + \right.$$

$$6\sin(\theta) - 4\sin(2\beta)) - \frac{1}{6}\sqrt{2h - r^2 - R^2} \cdot$$

$$\left(3\sin(\theta - \alpha + 2\beta) + \sin(3\theta + \alpha + 2\beta) + \right.$$

$$6\sin(\theta + \alpha) + 3\sin(\alpha - 2\beta) - \sin(\alpha + 2\beta) -$$

$$\left. 6\sin(\alpha) \right).$$

Using Theorem 3 we obtain that the function $f_2 = (f_{21}, f_{22}, f_{23}, f_{24})$ is

$$f_{21} = \frac{1}{24r} R^2 \left(-6r\sqrt{2h - r^2 - R^2} \sin(\alpha - 2\beta) + \right.$$

$$\left. 4r\sin\alpha\sqrt{2h - r^2 - R^2} + 6r^2 \sin(2\beta) + 3R^2 \sin(2\beta) \right),$$

$$f_{22} = \frac{R^2}{48r^2(-2h + r^2 + R^2)} \left(6r\sqrt{2h - r^2 - R^2}(-4h + 4r^2 + \right.$$

$$3R^2) \cos(\alpha - 2\beta) + r\sqrt{2h - r^2 - R^2}$$

$$\left(-16h + 16r^2 + 17R^2 \right) \cos \alpha + 24hr^2 \cos(2(\alpha - \beta))$$

$$-24hr^2 \cos(2\beta) - 12hR^2 \cos(2\beta) - 8hR^2 -$$

$$12r^4 \cos(2(\alpha - \beta)) + 12r^4 \cos(2\beta) - 12r^2 R^2 \cdot$$

$$\cos(2(\alpha - \beta)) + 18r^2 R^2 \cos(2\beta) - r^2 R^2 +$$

$$\left. 6R^4 \cos(2\beta) + 4R^4 \right),$$

(20)

$$f_{23} = -\frac{1}{8r\sqrt{2h-r^2-R^2}}R^3\Big(\sin(\alpha-2\beta)$$

$$\Big(-2r\sqrt{2h-r^2-R^2}-2h+r^2+R^2\Big)\cos\alpha+$$

$$r\sin(2\beta)\sqrt{2h-r^2-R^2}\Big),$$

$$f_{24} = \frac{1}{24r^2}R^2\Big(-5r\sqrt{2h-r^2-R^2}\cos\alpha-15r^2+$$

$$3R^2\cos(2\beta)+2R^2\Big).$$

The next step is to find the zeros $(r^*, R^*, \alpha^*, \beta^*)$ of $f_2(r, R, \alpha, \beta)$, and then we must check that the Jacobian determinant

$$|D_{r,R,\alpha,\beta}f_2(r^*, R^*, \alpha^*, \beta^*)| \neq 0.$$

Since R^2 is a common factor in (20) we must eliminate the case $R = 0$ (because the Jacobian is zero if $R = 0$), and also we must eliminate the case $r = 0$ (because r appears in some denominators). Consequently $\rho \neq \sqrt{2h}$. In order to solve system (20) and due to its complexity we use the Groebner basis. In the expressions of the numerators of the functions f_{2j} for $j = 1, 2, 3, 4$ we use ρ instead of $\sqrt{2h-r^2-R^2}$. Then the system whose zeros we must study becomes polynomial in the variables r, ρ, R, $\cos\alpha$, $\sin\alpha$, $\cos(\beta)$, $\sin(\beta)$, we denote these polynomials by h_j for $j = 1, 2, 3, 4$. Thus we compute the Groebner basis of the functions $h_1, h_2, h_3, h_4, h_5 = \cos^2\alpha + \sin^2\alpha - 1$, $h_6 = \cos^2\beta + \sin^2\beta - 1$ and $h_7 = \rho^2 - (2h-r^2-R^2)$ in the mentioned seven variables. Three of the functions of the Groebner basis are

$$\big(2h-\rho^2\big)\big(3\rho^2-h\big)\big(h+9\rho^2\big)\big(1250h^2-729h\rho^2-$$

$$121\rho^4\big)\big(2h^2-57h\rho^2+47\rho^4\big)\Big(1866240000h^{10}+$$

$$37233561600h^9\rho^2-28399449600h^8\rho^4-$$

$$1496101633920h^7\rho^6+6222678446160h^6\rho^8-$$

$$10345096087400h^5\rho^{10}+9413129629117h^4$$

$$\rho^{12}-6151478550944h^3\rho^{14}+3479008638842h^2\rho^{16}-$$

$$1267934032134h\rho^{18}+134591551875\rho^{20}\Big),$$

(21)

$$\big(2h-\rho^2\big)\big(h+9\rho^2\big)\big(1250h^2-729h\rho^2-121\rho^4\big)$$

$$\Big(1866240000h^{10}+37233561600h^9\rho^2-28399449600h^8\rho^4$$

$$-1496101633920h^7\rho^6+6222678446160h^6\rho^8-$$

$$10345096087400h^5\rho^{10}+9413129629117h^4\rho^{12}-$$

$$6151478550944h^3\rho^{14}+3479008638842h^2\rho^{16}-$$

$$1267934032134h\rho^{18}+134591551875\rho^{20}\Big)\sin\beta,$$

(22)

and the third function is given in the Appendix B due to its extension. Note that is a polynomial in the mentioned variables of degree 2 in the variable $\sin\beta$, and of degree 30 in the variable ρ.

Since $\rho \in (0, \sqrt{2h})$, the zeros of ρ in that interval of the polynomial (21) are

$\rho_1^* = \sqrt{\frac{h}{3}}, \rho_2^* = \sqrt{\frac{(57-13\sqrt{17})h}{94}}, \rho_3^* = \sqrt{\frac{(57+13\sqrt{17})h}{94}},$

$\rho_4^* = \sqrt{\frac{(-729+113\sqrt{89})h}{242}},$

and it is not difficult to check that the polynomial of degree 20 in the variable ρ which appears as factor in (21) has no real roots in the interval $(0, \sqrt{2h})$. From (21) and (22) we obtain that $\sin\beta = 0$, i.e. $\beta = 0, \pi$, for the solutions ρ_j^* for $j = 1, 2, 3$. Now we consider the third polynomial of the Groebner basis given in Appendix B, and we substitute in it the value of ρ_4^* obtaining that $\sin\beta = \pm 1$, i.e. $\beta = \pm\pi/2$.

For each value of ρ_j^* we compute $R = \sqrt{2h-r^2-\rho^2}$, and substituting it in the equation of f_{11} we obtain for $j = 1, 2, 3, 4$ that $\sin\alpha = 0$, i.e. $\alpha = 0, \pi$. Continuing the analysis as in the proof of Theorem 1 we get the solutions (i)–(viii) given in the statement of Theorem 2.

The characteristic polynomial of (i) and (ii) is

$$\frac{6400h^4}{243} + \frac{320h^2}{27}\lambda^2 + \lambda^4,$$

whose roots are $\pm i4\sqrt{5}h/3$ and $\pm i4\sqrt{5/3}h/3$. So the periodic solutions of (i) and (ii) are linearly stable. For (iii)–(vi) the characteristic polynomial is

$$-\frac{25\big(1216497781+295040429\sqrt{17}\big)h^4}{119604096}-$$

$$\frac{5\big(2620033+635097\sqrt{17}\big)h^2}{318096}\lambda^2 + \lambda^4.$$

This polynomial has two real solutions (one positive and one negative) and two pure imaginary pure, therefore the periodic solutions of (iii)–(vi) are unstable.

The characteristic polynomial for (vii) and (viii) is

$$\frac{\big(204565823331\sqrt{89}-1935553558577\big)h^4}{10204191360}+$$

$$\frac{\big(16564885419-1745833457\sqrt{89}\big)h^2}{263538000}\lambda^2 + \lambda^4.$$

Again this polynomial has two real solutions (one positive and one negative) and two pure imaginary pure, therefore the periodic solutions of (vii) and (viii) are unstable. This completes the proof of Theorem 2.

4 Conclusions

The objective of this work is to prove the existence of periodic orbits and its type of stability, in the galactic Hamiltonian of three degrees of freedom with $1 : 1 : 1$ resonance

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2}(q_1^2 + q_2^2 + q_3^2) + H_1,$$

where H_1 is either $-(q_1^2 q_2^2 + q_1^2 q_3^2 + q_2^2 q_3^2)$, or $-(q_1 + q_2)q_3^2$. The families of periodic orbits of these two 3–dimensional galactic–type Hamiltonian systems started to be studied in [4].

We have used an important tool from the area of dynamical systems, the averaging theory for studying the existence of periodic orbits and their stability, and we have applied it for studying the families of periodic solutions of the Hamiltonian systems defined by the two previous Hamiltonians. Our main results are summarized in Theorems 1 and 2.

In Theorem 1 we have recuperated the four families of straight line periodic orbits found by Caranicolas in [4], but we also have obtained 12 new families of periodic solutions parameterized by the value of the Hamiltonian, and we also have proved that these 16 families of periodic orbits are unstable in each Hamiltonian level.

In Theorem 2 we again reobtained the two families of straight line periodic orbits found by Caranicolas, but we have obtained 6 new families of periodic solutions parameterized by the value of the Hamiltonian. Furthermore we have proved that the 2 families of straight line periodic orbits are linearly stable and the other 6 families of periodic orbits are unstable in each Hamiltonian level.

Appendix A: Averaging theory of first and second order

In this section we recall the results of the averaging theory that we have used for proving the results of this paper. For a general introduction to the averaging theory see Chapter 11 of the book [13], and mainly the book [11]. The averaging theory up to second order stated in what follows, with the weak assumptions used, was proved in [3].

Theorem 3 *Consider the differential system*

$$\dot{x}(t) = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 R(t, x, \varepsilon), \qquad (23)$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that the following hypotheses (i) *and* (ii) *hold.*

(i) $F_1(t, \cdot) \in C^1(D)$ *for all $t \in \mathbb{R}$, F_1, F_2, R and $D_x F_1$ are locally Lipschitz with respect to x, and R is differentiable*

with respect to ε. We define $f_1, f_2 : D \to \mathbb{R}^n$ as

$$f_1(z) = \int_0^T F_1(s, z)ds,$$

$$f_2(z) = \int_0^T \left[D_z F_1(s, z) \int_0^s F_1(t, z)dt + F_2(s, z) \right] ds.$$

(ii) For $V \subset D$ an open and bounded set and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \backslash \{0\}$, there exists $a \in V$ such that $f_1(a) + \varepsilon f_2(a) = 0$ and $d_B(f_1 + \varepsilon f_2, V, a) \neq 0$ (the Brouwer degree of $f_1 + \varepsilon f_2$ at a).

Then for $|\varepsilon| > 0$ sufficiently small, there exists a $T-$periodic solution $x(t, \varepsilon)$ of the system such that $x(0, \varepsilon) \to a$ when $\varepsilon \to 0$.

As usual we have denoted by $d_B(f_1 + \varepsilon f_2)$, the Brouwer degree of the function $f_1 + \varepsilon f_2 : V \to \mathbb{R}^n$ at its zero a, for more details on the Brouwer degree see [2]. A sufficient condition for showing that the Brouwer degree of a function f at its zero a is non–zero, is that the Jacobian of the function f at a (when it is defined) is non–zero, see for more details [6].

If the function f_1 is not identically zero, then the zeros of $f_1 + \varepsilon f_2$ are essentially the zeros of f_1 for ε sufficiently small. In this case, Theorem 3 provides the so-called *averaging theory of first order*.

If the function f_1 is identically zero and f_2 is not identically zero, then the zeros of $f_1 + \varepsilon f_2$ are the zeros of f_2. In this case, Theorem 3 provides the so-called *averaging theory of second order*.

In the case of the averaging theory of first order, we consider in D the averaged differential equation

$$\dot{y} = \varepsilon f_1(y), \quad y(0) = x_0, \qquad (24)$$

where

$$f_1(y) = \frac{1}{T} \int_0^T F_1(t, y)dt.$$

Then Theorem 3 gives information about the stability or instability of the periodic solution $x(t, \varepsilon)$. In fact, it is given by the stability or instability of the equilibrium point p of the averaged system (24). In fact, the singular point p has the stability behavior of the Poincaré map associated to the periodic solution $x(t, \varepsilon)$. In the case of the averaging theory of second order, i.e., $f_1 \equiv 0$ and f_2 non-identically zero, we have that the stability and instability of the periodic orbit cycle $\varphi(t, \varepsilon)$ coincide with the type of stability or instability of the equilibrium point p of the averaged system

$$\dot{y} = \varepsilon^2 f_2(y), \quad y(0) = x_0, \qquad (25)$$

i.e., it is the same that the singular point p associated to the Poincaré map of the periodic solution $\varphi(t, \varepsilon)$, see for instance the Chapter 11 of [13].

Appendix B: Third factor in the basis of Groebner

$(\rho^2 - 2h)\big(1324064332683572715589511911202378090166129450778716770776597464834377207450246857188176055806460414406651416999273302109606355649814760897117623649236415079658028357647175680000h^{15}\sin^2\beta$

$-44555537186617939912620430195941644429503175893921844918086392760888236827220901859600972404015904664387462628626205893156715252141296863298919058138545878977619112811671756800h^{15} + 822810490271372290851257689207532675637093879076749056636910042230751453215943733449904331497969500729916164135362840750844433503468383113091155436953552385107506781500752640 0000h^{14}\rho^2 +$

$146854966080133491821703712961604708719153707823042049915475614117699834111028513616431028914848325220078078905066864696653728444338679972872533994990595745505554184894562889312000h^{13}\rho^4 - 830389565452186299875760982111886742597363524398833518083522417704578966304267262423023205553715879159412532982831498530078467160311050828979787335429563222156315577827190545369600h^{12}\rho^6 -$

$421981703706704643673321619113001176040508601944879506259561774235892864373579289034312594843664296350064457735585358787339363759655785180655225225064600608452376150590 2250482159280h^{11}\rho^8 + 436375972404673077012092680727740817691227501193684028882656328020713094097691337630260731391726162536723339034831958544881358844792960366243950797002522037332640281560055635448414 20h^{10}\rho^{10} -$

$147474495164279230137200669235553778926129927328387333085556762050328543051360609080124428548494123341936354261080932481944901739381811878759208187596931559933777156651335313453164700h^{9}\rho^{12} +$

$272435110389721107568385097691385706357784051242653479403910861917138388351955875908303566810399367396413740947217258548965162015871896047237956762614805363387061407655600423917469433h^{8}\rho^{14} -$

$317490082320637496863653826246481582847091290686152918898413816643129053231956050576972821773020978908823342430511061781578431635296162938892579294607219207470955876701507659401336540h^{7}\rho^{16} +$

$562578652477420366637402058990756050408136091824481215639766629626212310973125939332548843914008105429608385300758459273075566544956169750263295062037483219711425173343624165542829228h^{6}\rho^{18} -$

$160844101516042948812327799451394348651603484879597459150552966090736808169966516607096186249468161388025096557620134691155543586430973534122469737974400283841932258677162240211115418h^{5}\rho^{20} +$

$789048334190075277182266970177332825112291443639642468735630398182763102835313715596964024652771571428859225545392869221595782233630479450531379235206860504815404694620147332480777 36h^{4}\rho^{22} -$

$258517930713148746776878167309200203016711864126646425102794415201508217674802430345865302971462459840019810655114951970050837997418536417037162771156730060817391608607185507539811 44h^{3}\rho^{24} +$

$333164274658681138685126734279977222514704139614040707037557303538383278625985698935965568525047516722767847978659493091909691498645615033777949441856012931976143615551275059407335 6h^{2}\rho^{26} +$

$476370614468855621378753496294152748416154369654025625826261421845033227766068139679556599216978735175886125236710331577496764668074679777489590865695157951249950188911056823726 34h\rho^{28} -$

$947428498165410165326992111628162785938676146286352753757508402266044491134564838882538345332207585324614006266606931657279221452457550358817584086984832379804280735306317398331 25\rho^{30}\big).$

Acknowledgments

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