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The Picard–Fuchs equations for the complete hyperelliptic integrals of even order curves, and the actions of the generalized Neumann system

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Abstract

We consider a family of hyperelliptic curves of genus 2 and obtain explicitly the system of linear ODEs for periods of Abelian integrals of first and second kind as functions of the parameters of the curves. These equations are known as Picard–Fuchs equations for the integrals and generalize the hypergeometric equation of the Legendre type.

On the other hand, the periods are linear combinations of the action variables of several algebraic completely integrable systems, in particular the Neumann system. Thus the solutions of the obtained Picard–Fuchs equations can be used to study various properties of the actions of this system.

1 Introduction

Given a family of elliptic curves $\mathcal{E} \subset \mathbb{P}^2$ in the Legendre form

$$w^2 = (1-z^2)(1-k^2z^2),$$

it is well known that the complete elliptic integrals of first kind

$$K(k) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}, \quad K'(k) = \int_1^{1/k} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

as function of the modulus $k \in \mathbb{C}$, give 2 independent solutions of the hypergeometric equation of Legendre type

$$k(1-k^2)\frac{d^2y}{dk^2} - (1+k^2)\frac{dy}{dk} + ky = 0, \tag{1}$$

that is, $K(k) = \frac{\pi}{2}F(\frac{1}{2},\frac{1}{2},1;k^2)$. The equation has singular points $z_{1,2,3} = -1, 0, 1$, which means that the solutions $K(k), K'(k)$ are not single-valued: when k performs a loop around z_i, these functions transform to a linear combination of $K(k), K'(k)$. That is, the solutions $y(k)$ undergo a monodromy. The monodromy group is isomorphic to the homology group $H_1(\mathcal{E}, \mathbb{C})$.

Equivalently, (1) can be rewritten as a system of first order equations for $K(k)$ and the complete integral of the *second kind*[1]

$$\bar{E}(k) = \int_0^1 \frac{z^2 dz}{\sqrt{(1-z^2)(1-k^2z^2)}}\,,$$

namely,

$$\frac{dK}{dk} = \frac{1}{k(1-k^2)}(k^2K - \bar{E}), \qquad \frac{d\bar{E}}{dk} = \frac{k}{1-k^2}(K - \bar{E}) \tag{2}$$

(see e.g., [8]).

The above description can be generalized to the case of curves of higher genus. As an illustration, consider a family of the genus g hyperelliptic curves of odd order

$$\Gamma_h = \{w^2 = (z-a_1)\cdots(z-a_{g+1})(z^g + h_1 z^{g-1} + \cdots + h_{g-1}z + h_g)\} \tag{3}$$

with the parameters $h_1, \dots, h_g \in \mathbb{C}$. Here $a_1, \dots, a_{g+1}$ are constants. For generic vaues of h_i the curves are 2-fold covering of $\mathbb{C} = \{z\}$ ramified at $z = a_1, \dots, a_{g+1}$ and $\rho_1, \cdots, \rho_g$, the roots of the polynomial $P_g(z) = z^g + h_1 z^{g-1} + \cdots + h_{g-1}z + h_g$.

Consider the following canonical basis of g holomorphic differentials and g meromorphic differentials of the second kind on Γ:

$$\omega_i = \frac{z^{i-1}\,dz}{w}, \quad \omega_{g+i} = \frac{z^{g-1+i}\,dz}{w}, \qquad i = 1, \dots, g.$$

Let $\gamma \in H_1(\Gamma, C)$ be a cycle on Γ. Then the periods of the above differentials

$$J_1 = \oint_\gamma \omega_1, \quad \cdots \quad J_{2g} = \oint_\gamma \omega_{2g} \tag{4}$$

also become functions of the parameters $h_1, \dots, h_g$ in (3) or of the roots $\rho_1, \dots, \rho_g$.

Note that J_i are not single-valued functions of h_i: when these parameters vary in such a way that one of the roots, say ρ_1 performs a loop around a_i or $\rho_2, \dots, \rho_g$, each integral J_i becomes a linear combination of $J_1, \dots, J_{2g}$, i.e., undergoes a monodromy.

Following the classical theory of differential equations, the integrals $J_i = J_i(h_1, \dots h_g)$ are solutions of a systems of linear ODEs, with h_i being independent variables, called the *Picard–Fuchs* equations (see, e.g., [2]):

$$\frac{\partial J}{\partial h_k} = M_k(h)\,J, \qquad J = (J_1, \dots, J_{2g})^T, \quad k = 1, \dots, g, \quad M_k \in GL(2g, \mathbb{C}). \tag{5}$$

They are natural generalizations of the Legendre equation (1) or (2)[2].

Due to the monodromy property, some of the components of $M_k(h)$ have poles when one of the roots ρ_i coincides with a_j or with the other roots.

Families of hyperelliptic curves Γ_h often appear in quadratures of integrable systems of classical mechanics and mathematical physics, in particular the Neumann system (see below), whereas certain linear combinations of the integrals $J_i(h)$ give *action* variables $\mathcal{I}_1(h), \dots, \mathcal{I}_g(h)$ of the systems. Knowledge of such functions is important in study of periodic solutions, in quantization, in applications of the KAM theory to perturbations of the integrable systems.

[1]This integral is slightly different from the canonical integral $E(k)$, for this reason we use the notation $\bar{E}(k)$.

[2]More precicely, the original Picard–Fuchs equations are second order equations obtained by eimination of the periods of the meromorphic differentials.

The integrals J_i are transcendental functions of h_j and, as mentioned in several publications, instead of computing them numerically, in some cases it is less expensive to integrate numerically the above Picard–Fuchs equations, at least locally.

Following this idea, the authors of [4] derived differential equations for the periods J_i for any genus g, taking however, as an independent variable one of the roots ρ_i in (3), and not a constant of motion h_k. (Thus, they obtained the *Gauss–Manin* equations.)

A similar approach was followed in [3, 5] to treat the actions of the Kovalevkaya top and the Jacobi problem on geodesics on a triaxial ellipsoid.

For another basis of meromorphic differentials on Γ_h, a similar system of Gauss–Manin equations was obtained in [7].

The ony disadvantage of this approach is the dependence of all the constants h_k on any root ρ_i, which makes it difficult to study the properties of $J_i(h)$ as a function of one h_k, when all the other ones are fixed.

The choice of ρ_i instead of h_k was motivated in [4] by the observation that the Picard–Fuchs equations with the independent variables h_k become highly cumbersome even for the lowest non-trivial case $g = 2$.

The main purpose of our paper is to derive the PF equations for the case of the family of even order genus 2 curves

$$G_h = \{w^2 = (z - a_1)(z - a_2)(z - a_3)(z^3 + h_1 z + h_2)\},$$

which appear in quadratires of an integrable generalization of the Neumann system with a separable quartic potential.

We observe that, in contrast to the curves (3) and equations (5), in our case the order of the PF equations is 5, since they also include an Abelian integral of 3rd kind. To our knowledge, such case was not considered before.

The equations are written in a quite compact and symmetric form, suitable for possible applications.

2 The classical Neumann system and its generalization

Recall that the Neumann system descibes the motion of a point on the unit sphere $S^{n-1} = \{\langle x, x\rangle = 1\}$, $x \in \mathbb{R}^n$ under the action of the quadratic potential $U = \langle x, Ax\rangle/2$, A being diagonal matrix with constant eigenvalues $a_1, a_2, \ldots, a_n$. The Hamiltonian of the problem has the form

$$H(x, y) = \frac{1}{2}(|y|^2|x|^2 - \langle y, x\rangle^2) + \frac{1}{2}\langle x, Ax\rangle,$$

where $p \in T_x S^{n-1}$ is the momentum (see e.g., [9, 10]).

Neumann ([**?**], 1856) considered this problem in the case $n = 3$ and solved it completely in terms of theta-functions of 2 variables.

In the general case, in the the elliptic coordinaates $\lambda_1, \ldots, \lambda_{n-1}$ on S^{n-1} such that

$$x_i^2 = \frac{(a_i - \lambda_1)\cdots(a_i - \lambda_{n-1})}{(a_i - a_1)\cdots(a_i - a_n)},$$

and corresponding conjugated momenta, the Hamiltonian takes Stäckel form, and the system is reduced to the quadratures

$$\frac{\lambda_1^k d\lambda_1}{\sqrt{R(\lambda_1)}} + \cdots + \frac{\lambda_{n-1}^k d\lambda_{n-1}}{\sqrt{R(\lambda_{n-1})}} = \begin{cases} 0 \text{ if } k = 0, 1, \ldots, n-2 \\ dt \text{ if } k = n-1 \end{cases} \qquad k = 0, 1, \ldots, n-2 \quad (6)$$

$$R(\lambda) = \Phi(\lambda) P_{n-1}(\lambda), \quad \Phi(\lambda) = (\lambda - a_1)\cdots(\lambda - a_n),$$

$$P_{n-1}(\lambda) = \lambda^{n-1} + h_1\lambda^{n-2} + \cdots + h_{n-1} = (\lambda - \rho_1)\cdots(\lambda - \rho_{n-1}),$$

where $h_1, \dots, h_{n-1}$ are constants of motion.

Here $\lambda^k d\lambda/\sqrt{R(\lambda)}$ can be regarded as holomorphic differentials on the genus $g = n-1$ hyperelliptic curve $\Gamma_h = \{\mu^2 = \Phi(\lambda) P_{n-1}(\lambda)\}$.

By integrating the quadratures (6) and inverting the integrals, symmetric functions of the elliptic coordinates λ_j and, therefore, the Cartesian coordinates x_i, can be expressed in terms of theta-functions of u_k and, therefore, of the time t (see [11]). The generic real invariant varieties are unions of $n-1$ dimensional tori $\mathbb{T}^{n-1}$. Moreover, the tori are real parts of complex Abelian varieties, which are isogeneous to the Jacobians of the curves, and the system is algebraic integrable [9, 11].

On the other hand, as was shown in several pubications (see e.g., [6, 12]), the Neumann system admits a hierarchy of integrable generalizations, in which the quadratic potential is replaced by polynomial or even rational potentials, which are all separable in the same elliptic coordinates. For all such generalizations, the dimension of the generic invariant tori is the same, $n-1$. For a class of separable polynomial potentials of degree $2N$, the quadratures take the form

$$\frac{\lambda_1^k d\lambda_1}{\sqrt{\mathcal{R}(\lambda_1)}} + \cdots + \frac{\lambda_{n-1}^k d\lambda_{n-1}}{\sqrt{\mathcal{R}(\lambda_{n-1})}} = \begin{cases} 0 \text{ if } k = 0, 1, \dots, n-2 \\ dt \text{ if } k = n-1 \end{cases} \qquad k = 0, 1, \dots, n-2 \qquad (7)$$

$$\mathcal{R}(\lambda) = \Phi(\lambda) \mathcal{P}_{N+1}(\lambda), \quad \Phi(\lambda) = (\lambda - a_1) \cdots (\lambda - a_n),$$
$$\mathcal{P}_{N+1}(\lambda) = \lambda^{N+1} + h_1 \lambda^{n-2} + \cdots + h_{n-1},$$

and include $n-1$ holomorphic differentials on the hyperelliptic curve $\Gamma_h = \{\mu^2 = \Phi(\lambda) \mathcal{P}_{N+1}(\lambda)\}$. of genus $g = [(n+N)/2]$. This implies that for the separable potentials of degree $2N > 4$, the genus of Γ_h is bigger than the dimension of the tori, and one can show that in this case the system is no more algebraic integrable ([1]).

The action variables of the original and generalized Neumann systems are the periods of the Abelian integrals

$$\mathcal{J}_j(h_1, \dots, h_{n-1}) = \frac{1}{2\pi} \oint_{\gamma_j} \frac{(\lambda^{N+1} + h_1 \lambda^{n-2} + \cdots + h_{n-1})\, d\lambda}{\sqrt{\mathcal{R}(\lambda)}}, \qquad j = 1, \dots, n-1,$$

γ_j being certain cycles on the Riemann surface Γ_h. Note that the functions $\mathcal{J}_j(h_1, \dots, h_g)$ are also the frequencies of the angle variables on the tori $\mathbb{T}^{n-1}$. Then a solution to the Neumann system is periodic if and only if the quantities $\mathcal{J}_j$ are *comensurable.* So, knowledge of $\mathcal{J}_j(h)$ is important in describing periodic solutioins of the system.

As follows from the above, the action variables are linear combinations of the periods of the basic g holomorphic and g meromorhic differentials on Γ_h

$$J_k = \oint_\gamma \omega_k, \qquad \omega_s = \frac{\lambda^{s-1} d\lambda}{\sqrt{R(\lambda)}}, \quad \omega_{g+s} = \frac{\lambda^{g+s-1} d\lambda}{\sqrt{R(\lambda)}}, \quad s = 1, \dots, g. \qquad (8)$$

For the classical Neumann system with the quadratic potential ($N = 1$) the above $2g$ differentials satisfy the Picard–Fuch equations (5).

However, for $N > 1$ this is not always true.

For concreteness, below we restrict ourselves to the simplest case $n = 3$ and the quartic separable potential ($N = 2$), which corresponds to genus 2 even order curves

$$w^2 = (\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda^3 + h_1 \lambda + h_2), \quad \text{or} \qquad (9)$$
$$w^2 = (\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda - \rho_1)(\lambda - \rho_2)(\lambda - \rho_3),$$

whose compactifications in $\mathbb{P}^2$ have 2 infinite points, which we denote by ∞_-, ∞_+.

The differentials (8) then are

$$\omega_1 = \frac{d\lambda}{w}, \quad \omega_2 = \frac{\lambda\, d\lambda}{w}, \quad \omega_3 = \frac{\lambda^2\, d\lambda}{w}, \quad \omega_4 = \frac{\lambda^3\, d\lambda}{w}.$$

One observes that, in contrast to ω_4, the differential ω_3 is meromorphic of the 3rd kind, i.e., it has a pair of simple poles at ∞_-, ∞_+, and that the corresponding periods $J_1, \dots, J_4$ do not form a closed system of differential equations with respect to the constants h_1 or h_2. It turns out that in this case the Picard–Fuchs equations must incude also the period J_5 of the differential of the second kind $\omega_5 = \dfrac{\lambda^4\, d\lambda}{w}$.

3 The Picard–Fuchs equations for genus 2 even order curves

To derive the Picard–Fuch equations for the considered case, we first compute the derivatives of the integrals $J_1, \dots, J_5$ with respect to the roots ρ_α in (9). Namely, rewrite the curve in the form

$$y^2 = R(x), \quad R(x) = (x-e_1)(x-e_2)(x-e_3)(x-e_4)(x-e_5)(x-e_6).$$

Like in several other publications (see, e.g., [6]), we will use the folowing key relation

$$A_j^{(k)} \frac{\partial}{\partial e_k}\left(\frac{x^j}{y}\right) = \frac{a_j^{(k)}x^4 + b_j^{(k)}x^3 + c_j^{(k)}x^2 + d_j^{(k)}x + g_j^{(k)}}{y} - \frac{d}{dx}\left(\frac{y}{x-e_k}\right), \qquad (10)$$
$$j = 0,1,\dots,4, \quad k = 1,\dots,6,$$

where $A_j^{(k)}, a_j^{(k)}, \dots, g_j^{(k)}$ are functions of the branch points e_i only. Namely, if we write

$$R'(e_k) = \left.\frac{dR(x)}{dx}\right|_{x=e_k} = e_k^5 + \Delta_1^{(k)}e_k^4 + \Delta_2^{(k)}e_k^3 + \Delta_3^{(k)}e_k^2 + \Delta_4^{(k)}e_k + \Delta_5^{(k)},$$

so that the coefficients $\Delta_i^{(k)}$ are elementary symmetric functions of $\{e_1,\dots,e_6\} \setminus e_k$ of degree i. In particular,

$$\Delta_1^{(1)} = -e_2 - e_3 - e_4 - e_5 - e_6, \quad \Delta_5^{(1)} = -\, e_2e_3e_4e_5e_6,$$

then our calculations give

$$\begin{aligned}
&A_j^{(k)} = \frac{R'(e_k)}{e_k^j},\\
&a_0^{(k)} = a_1^{(k)} = a_2^{(k)} = a_3^{(k)} = a_4^{(k)} = A^{(k)} = 2,\\
&b_0^{(k)} = b_1^{(k)} = b_2^{(k)} = b_3^{(k)} = B^{(k)} = -\frac{1}{2}\left(e_k - 3\Delta_1^{(k)}\right), \quad b_4^{(k)} = B + \frac{R'(e_k)}{2e_k^4}, \qquad (11)\\
&c_0^{(k)} = c_1^{(k)} = c_2^{(k)} = C^{(k)} = -\frac{1}{2}(e_k^2 + e_k\Delta_1^{(k)} - 2\Delta_2^{(k)}), \quad c_3^{(k)} = c_4^{(k)} = C^{(k)} + \frac{R'(e_k)}{2e_k^3},\\
&d_0^{(k)} = d_1^{(k)} = D^{(k)} = -\frac{1}{2}\left(e_k^3 + e_k^2\Delta_1^{(k)} + e_k\Delta_2^{(k)} - \Delta_3^{(k)}\right), \quad d_2^{(k)} = d_3^{(k)} = d_4^{(k)} = D^{(k)} + \frac{R'(e_k)}{2e_k^2},\\
&g_0^{(k)} = G^{(k)} = -\frac{1}{2}\left(e_k^4 + e_k^3\Delta_1^{(k)} + e_k^2\Delta_2^{(k)} + e_k\Delta_3^{(k)}\right), \quad g_1^{(k)} = g_2^{(k)} = g_3^{(k)} = g_4^{(k)} = G^{(k)} + \frac{R'(e_k)}{2e_k}.
\end{aligned}$$

Multipying the both sides of (10) by dx, and using again the notation

$$\omega_1 = \frac{dx}{y}, \quad \omega_2 = \frac{xdx}{y}, \quad \omega_3 = \frac{x^2dx}{y}, \quad \omega_4 = \frac{x^3dx}{y}, \quad \omega_5 = \frac{x^4dx}{y},$$

one gets

$$\frac{\partial}{\partial e_k}\omega_i = \frac{e_k^j}{R'(e_k)}\left(g_j^{(k)}\omega_1 + d_j^{(k)}\omega_2 + c_j^{(k)}\omega_3 + b_j^{(k)}\omega_4 + a_j^{(k)}\omega_5 + dF_k\right), \tag{12}$$
$$F_k = \frac{y}{x-e_k}, \quad j = i-1, \quad i = 1,\ldots,5.$$

Since

$$\frac{\partial}{\partial e_k}\left(\oint_\gamma \omega_i\right) = \oint_\gamma \frac{\partial}{\partial e_k}\omega_i,$$

and since dF_k is a differential of a meromorphic function of Γ_h, from (12) we obtain the following sysyem for the vector of periods $J = (J_1,\ldots,J_5)^t$:

$$2\frac{\partial J}{\partial e_k} = \mathcal{M}_k J, \qquad k = 1,\ldots,6, \tag{13}$$

$$\mathcal{M}_k = \frac{1}{R'(e_k)}\begin{pmatrix} 1 \\ e_k \\ e_k^2 \\ e_k^3 \\ e_k^4 \end{pmatrix}(G^{(k)}\, D^{(k)}\, C^{(k)}\, B^{(k)}\, A^{(k)}) + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ e_k & 1 & 0 & 0 & 0 \\ e_k^2 & e_k & 1 & 0 & 0 \\ e_k^3 & e_k^2 & e_k & 1 & 0 \end{pmatrix}, \tag{14}$$

with $G^{(k)}, D^{(k)}, C^{(k)}, B^{(k)}, A^{(k)}$ defined in (11).

The structure of the matrices $\mathcal{M}_k$ is similar to that of the Picard–Fuchs equations obtained in [4, 7], however, not the same: the system (13) has an odd order.

Now, taking into account (9), we identify the roots ρ_1, ρ_2, ρ_3 with e_1, e_2, e_3, and the parameters a_1, a_2, a_3 with e_4, e_5, e_6, then

$$\begin{aligned} h_1 &= -e_1 - e_2 - e_3 = 0, \\ h_2 &= e_1e_2 + e_1e_3 + e_2e_3, \\ h_3 &= -e_1e_2e_3. \end{aligned} \tag{15}$$

In view of the following relation between the partial derivatives

$$\begin{pmatrix} \dfrac{\partial J_i}{\partial e_1} \\[2ex] \dfrac{\partial J_i}{\partial e_2} \\[2ex] \dfrac{\partial J_i}{\partial e_3} \end{pmatrix} = \begin{pmatrix} 1 & e_2 + e_3 & -e_2e_3 \\[2ex] 1 & e_1 + e_3 & -e_1e_3 \\[2ex] 1 & e_2 + e_1, & -e_1e_2 \end{pmatrix}\begin{pmatrix} \dfrac{\partial J_i}{\partial h_1} \\[2ex] \dfrac{\partial J_i}{\partial h_2} \\[2ex] \dfrac{\partial J_i}{\partial h_3} \end{pmatrix}, \quad i = 1,\ldots,5$$

we have

$$\begin{pmatrix} \dfrac{\partial J_i}{\partial h_1} \\[2ex] \dfrac{\partial J_i}{\partial h_2} \\[2ex] \dfrac{\partial J_i}{\partial h_3} \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} -e_1^2(e_2 - e_3) & e_2^2(e_1 - e_3) & -e_3^2(e_1 - e_2) \\[2ex] e_1(e_2 - e_3) & e_2(e_3 - e_1) & e_3(e_1 - e_2) \\[2ex] e_2 - e_3 & e_3 - e_1 & e_1 - e_2 \end{pmatrix}\begin{pmatrix} \dfrac{\partial J_i}{\partial e_1} \\[2ex] \dfrac{\partial J_i}{\partial e_2} \\[2ex] \dfrac{\partial J_i}{\partial e_3} \end{pmatrix}, \tag{16}$$
$$\Delta = (e_1 - e_2)(e_3 - e_1)(e_3 - e_2).$$

Now combining the above relations with the equations (13), and taking into account (11), (15), we arrive at

Theorem 1. *The vector of periods $J = (J_1, \ldots, J_5)^t$ of the differentials of the even order curve* (9) *satisfies the equations*

$$2\frac{\partial J}{\partial h_2} = \mathcal{U}_2 J, \quad 2\frac{\partial J}{\partial h_3} = \mathcal{U}_3 J, \tag{17}$$

$$\mathcal{U}_2 = \sum_{\alpha=1}^{3} \frac{1}{\Phi(\rho_\alpha)} \frac{\rho_\alpha}{(\rho_\alpha - \rho_\beta)^2(\rho_\alpha - \rho_\gamma)^2} \mathbf{S}_\alpha + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ h_1 & 1 & 0 & 0 & 0 \\ h_2 & h_1 & 1 & 0 & 0 \end{pmatrix},$$

$$\mathcal{U}_3 = \sum_{\alpha=1}^{3} \frac{1}{\Phi(\rho_\alpha)} \frac{1}{(\rho_\alpha - \rho_\beta)^2(\rho_\alpha - \rho_\gamma)^2} \mathbf{S}_\alpha + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ h_1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\Phi(\rho_\alpha) = (\rho_\alpha - a_1)(\rho_\alpha - a_2)(\rho_\alpha - a_3), \quad (\alpha, \beta, \gamma) = (1, 2, 3),$$

$$\mathbf{S}_\alpha = \begin{pmatrix} 1 \\ \rho_\alpha \\ \rho_\alpha^2 \\ \rho_\alpha^3 \\ \rho_\alpha^4 \end{pmatrix} (G^{(\alpha)}\, D^{(\alpha)}\, C^{(\alpha)}\, B^{(\alpha)}\, A^{(\alpha)}).$$

The proof is direct and uses the identities

$$\rho_1^k(\rho_2 - \rho_3) + \rho_2^k(\rho_3 - \rho_1) + \rho_3^k(\rho_1 - \rho_2) = \begin{cases} 0 & s = 1 \\ -(\rho_1 - \rho_2)(\rho_3 - \rho_1)(\rho_3 - \rho_2) & s = 2 \\ (\rho_1 - \rho_2)(\rho_3 - \rho_1)(\rho_3 - \rho_2)h_1 & s = 3 \\ (\rho_1 - \rho_2)(\rho_3 - \rho_1)(\rho_3 - \rho_2)(h_1^2 - h_2) & s = 4. \end{cases}$$

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