FAMILIES OF NESTED COMPLETELY REGULAR CODES AND DISTANCE-REGULAR GRAPHS

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ABSTRACT. In this paper infinite families of linear binary nested completely regular codes are constructed. They have covering radius ρ equal to 3 or 4, and are $1/2^i$th parts, for $i \in \{1, \ldots, u\}$ of binary (respectively, extended binary) Hamming codes of length $n = 2^m - 1$ (respectively, 2^m), where $m = 2u$. In the usual way, i.e., as coset graphs, infinite families of embedded distance-regular coset graphs of diameter D equal to 3 or 4 are constructed. In some cases, the constructed codes are also completely transitive codes and the corresponding coset graphs are distance-transitive. This gives antipodal covers of some distance-regular and distance-transitive graphs.

1. INTRODUCTION

Let $\mathbb{F}_q$ denote the finite field with $q \geq 2$ elements, q being a prime power. For a vector $\mathbf{x} \in \mathbb{F}_q^n$ denote by $\text{wt}(\mathbf{x})$ its Hamming weight (i.e., the number of its nonzero positions). For every two vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ from $\mathbb{F}_q^n$ denote by $d(\mathbf{x}, \mathbf{y})$ the Hamming distance between $\mathbf{x}$ and $\mathbf{y}$ (i.e., the number of positions i, where $x_i \neq y_i$). We use the standard notation $[n, k, d]$ for a binary

linear code C of length n, dimension k and minimum distance d over the binary field $\mathbb{F}_2$.

The automorphism group $\mathrm{Aut}(C)$ of C consists of all $n \times n$ binary permutation matrices M, such that $\mathbf{c}M \in C$ for all $\mathbf{c} \in C$. Note that the automorphism group $\mathrm{Aut}(C)$ coincides with the subgroup of the symmetric group S_n consisting of all $n!$ permutations of the n coordinate positions which send C into itself. $\mathrm{Aut}(C)$ acts in a natural way over the set of cosets of C: $\pi(C + \mathbf{v}) = C + \pi(\mathbf{v})$ for every $\mathbf{v} \in \mathbb{F}_2^n$ and $\pi \in \mathrm{Aut}(C)$.

For any $\mathbf{v} \in \mathbb{F}_2^n$ its *distance* to the code C is $d(\mathbf{v}, C) = \min_{\mathbf{x} \in C}\{d(\mathbf{v}, \mathbf{x})\}$ and the *covering radius* of the code C is $\rho = \max_{\mathbf{v} \in \mathbb{F}_2^n}\{d(\mathbf{v}, C)\}$. Let $I = \{1, 2, \ldots, n\}$ be the set of coordinate positions of vectors from $\mathbb{F}_2^n$. Denote by $\mathrm{Supp}(\mathbf{x})$ the *support* of the vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}_2^n$, i.e., $\mathrm{Supp}(\mathbf{x}) = \{i \in I : x_j \neq 0\}$. Say that two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{F}_2^n$ are *neighbors* if $d(\mathbf{x}, \mathbf{y}) = 1$ and also say that vector $\mathbf{x}$ *covers* vector $\mathbf{y}$ if $\mathrm{Supp}(\mathbf{y}) \subseteq \mathrm{Supp}(\mathbf{x})$.

For a given binary code C with the zero codeword and with covering radius $\rho = \rho(C)$ define

$$C(i) = \{\mathbf{x} \in \mathbb{F}_2^n : d(\mathbf{x}, C) = i\}, \quad i = 1, 2, \ldots, \rho,$$

and

$$C_i = \{\mathbf{c} \in C : \mathrm{wt}(\mathbf{c}) = i\}, \quad i = 0, 1, \ldots, n.$$

Definition 1.1. A binary code C with covering radius $\rho = \rho(C)$ is completely regular, if for all $l \geq 0$ every vector $x \in C(l)$ has the same number c_l of neighbors in $C(l-1)$ and the same number b_l of neighbors in $C(l+1)$. Also, define $a_l = n - b_l - c_l$ and note that $c_0 = b_\rho = 0$.

Alternatively, C is completely regular if and only if the weight distribution of any coset $C + \mathbf{v}$ of weight i, for $i \in \{0, \ldots, \rho\}$, is uniquely determined by the minimum weight i of $C + \mathbf{v}$.

For a completely regular code, define $(b_0, \ldots, b_{\rho-1}; c_1, \ldots, c_\rho)$ as the intersection array of C.

Definition 1.2. [16] A binary linear code C with covering radius ρ is completely transitive if $\mathrm{Aut}(C)$ has $\rho + 1$ orbits when acts on the cosets of C.

Since two cosets in the same orbit have the same weight distribution, it is clear that any completely transitive code is completely regular.

Existence and enumeration of completely regular and completely transitive codes are open hard problems (see [5, 7, 9, 13, 16] and references there). The purpose of this paper is to construct nested infinite families of completely regular codes with covering radius $\rho = 3$ ($\rho = 4$ for the extended codes). When m is growing the length of the chain of these nested codes (with constant covering radius) is also growing. For length $n = 2^m - 1$, where $m = 2u$, each family is formed by $u + 1$ nested completely regular codes of length n with the same covering radius $\rho = 3$. The last code in the nested family, so the code with the smallest cardinality is a $1/2^u$th part of a Hamming code of length n. These last codes are known to be completely regular codes due to Calderbank and Goethals [6, 12]. These nested families of completely regular codes and their extended codes induce infinite families of embedded distance-regular coset graphs with diameters 3 and 4, respectively, which also give interesting families of embedded covering graphs. We point out that in some cases such completely regular codes are also completely transitive and hence the corresponding coset graphs are also distance-transitive.

2. PRELIMINARY RESULTS

Definition 2.1. Let C be a binary code of length n and let ρ be its covering radius. We say that C is *uniformly packed* in the wide sense, i.e., in the sense of [1], if there exist rational numbers $\beta_0, \ldots, \beta_\rho$ such that for any $\mathbf{v} \in \mathbb{F}_2^n$

$$(1) \qquad \sum_{k=0}^{\rho} \beta_k \, \alpha_k(\mathbf{v}) \; = \; 1,$$

where $\alpha_k(\mathbf{v})$ is the number of codewords at distance k from $\mathbf{v}$.

Let C be a linear code. Denote by s the number of nonzero weights in its dual code $C^{\perp}$. Following [7], we call s the *external distance* of C.

Lemma 2.2. *Let C be a code with covering radius ρ and external distance s. Then:*

(i) [7]: $\rho \leq s$.

(ii) [2]: $\rho = s$ *if and only if C is uniformly packed in the wide sense.*

(iii)[5]: *If C is completely regular, then it is uniformly packed in the wide sense.*

Lemma 2.3. [15] *Let C be a linear completely regular $[n, k, d]$ code with covering radius ρ and intersection array $(b_0, \ldots, b_{\rho-1}; c_1, \ldots c_\rho)$. Let μ_i denote the number of cosets of C of weight i, where $i = 0, 1, \ldots, \rho$. Then the following equality holds:*

$$b_i \mu_i = c_{i+1} \mu_{i+1}, \; i = 0, \ldots, \rho - 1.$$

Next, following [5], we give some facts on distance-regular graphs. Let Γ be a finite connected simple graph (i.e., undirected, without loops and multiple edges). Let $d(\gamma, \delta)$ be the distance between two vertices γ and δ (i.e., the number of edges in the shortest path between γ and δ). The *diameter D* of Γ is its largest distance. Two vertices γ and δ from Γ are *neighbors* if $d(\gamma, \delta) = 1$. Say

$$\Gamma_i(\gamma) = \{\delta \in \Gamma : \; d(\gamma, \delta) = i\}.$$

An *automorphism* of a graph Γ is a permutation π of the vertex set of Γ such that, for all $\gamma, \delta \in \Gamma$ we have $d(\gamma, \delta) = 1$ if and only if $d(\pi\gamma, \pi\delta) = 1$. Let Γ_i be the graph with the same vertices of Γ, where an edge (γ, δ) is defined when the vertices γ, δ are at distance i in Γ. Clearly, $\Gamma_1 = \Gamma$. The graph Γ is called *primitive* if Γ and all Γ_i ($i = 2, \ldots, D$) are connected. Otherwise, Γ is called *imprimitive*. A graph is called *complete* (or a *clique*) if any two of its vertices are adjacents, i.e. neighbors.

A connected graph Γ with diameter $D \geq 3$ is called *antipodal* if the graph Γ_D is a disjoint union of cliques [5]. Such a graph is imprimitive by definition. In this case, the *folded graph, or antipodal quotient* of Γ is defined as the graph $\hat{\Gamma}$, whose vertices are the maximal cliques (which are called *fibres*) of Γ_D, with two adjacent if and only if there is an edge between them in Γ. If, in addition, each vertex $\gamma \in \Gamma$ has the same valency as its image under folding, then Γ is called an *antipodal covering graph* of $\hat{\Gamma}$. If, moreover, all fibres of Γ_D have the same size r, then Γ is also called an *r-fold antipodal cover* of $\hat{\Gamma}$.

Definition 2.4. [5] A simple connected graph Γ is called *distance-regular* if it is regular of valency k, and if for any two vertices $\gamma, \delta \in \Gamma$ at distance i apart, there are precisely c_i neighbors of δ in $\Gamma_{i-1}(\gamma)$ and b_i neighbors of δ in $\Gamma_{i+1}(\gamma)$. Furthermore, this graph is called *distance-transitive*, if for any pair of vertices γ, δ at distance $d(\gamma, \delta)$ there is an automorphism π from $\mathrm{Aut}(\Gamma)$ which moves this pair (γ, δ) to any other given pair γ', δ' of vertices at the same distance $d(\gamma, \delta) = d(\gamma', \delta')$.

The sequence $(b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D)$, where D is the diameter of Γ, is called the *intersection array* of Γ. The numbers c_i, b_i, and a_i, where $a_i = k - b_i - c_i$, are called *intersection numbers*. Clearly $b_0 = k$, $b_D = c_0 = 0$, $c_1 = 1$.

Let C be a linear completely regular code with covering radius ρ and intersection array $(b_0, \ldots, b_{\rho-1}; c_1, \ldots c_\rho)$. Let $\{B\}$ be the set of cosets of C. Define the graph Γ_C, which is called the *coset graph of C*, taking all different cosets $B = C + \mathbf{x}$ as vertices, with two vertices $\gamma = \gamma(B)$ and $\gamma' = \gamma(B')$ adjacent if and only if the cosets B and B' contain neighbor vectors, i.e., there are $\mathbf{v} \in B$ and $\mathbf{v}' \in B'$ such that $d(\mathbf{v}, \mathbf{v}') = 1$.

Lemma 2.5. [5, 14] *Let C be a linear completely regular code with covering radius ρ and intersection array $(b_0, \ldots, b_{\rho-1}; c_1, \ldots c_\rho)$ and let Γ_C be the coset graph of C. Then Γ_C is distance-regular of diameter $D = \rho$ with the same intersection array. If C is completely transitive, then Γ_C is distance-transitive.*

Definition 2.6. A set T of vectors $\mathbf{v} \in \mathbb{F}_2^n$ of weight w is a *t-design*, denoted by $T(n, w, t, \lambda)$, if for any vector $\mathbf{z} \in \mathbb{F}_2^n$ of weight t, $1 \le t \le w$, there are precisely λ vectors $\mathbf{v}_i$, $i = 1, \ldots, \lambda$ from $T(n, w, t, \lambda)$, each of them covering $\mathbf{z}$.

The following well known fact directly follows from the definition of completely regular code.

Lemma 2.7. *Let C be a completely regular code with minimum distance d and containing the zero codeword. Then, the set C_w (of codewords of C of weight w), $d \le w \le n$ forms a t-design, if it is not empty, where $t = e$, if $d = 2e + 1$ and $t = e + 1$, if $d = 2e + 2$.*

Let H_m denote a binary matrix of size $m \times n$, where $n = 2^m - 1$, whose columns are all different nonzero binary vectors of length m, i.e., H_m is a parity check matrix of a binary (Hamming) $[n, n - m, 3]$-code, denoted by $\mathcal{H}_m$.

Given a binary code C with minimum distance $d = 2e + 1$, denote by C^* the extended code, i.e., the code obtained from C by adding an overall parity checking position. In [2] it has been shown when an extension of an uniformly packed code is again uniformly packed. If this happens the extended code C^* has the following property.

Lemma 2.8. [2] *Let C be a uniformly packed binary code of length n with odd minimum distance d and let C_d be a t-design $T(n, d, t, \lambda)$. If the extended code C^* is uniformly packed, then the set C^*_{d+1} is a $(t+1)$-design $T(n+1, d+1, t+1, \lambda)$.*

Now we give a lemma, which is an strengthening of a result from [3].

Lemma 2.9. *Let C be a completely regular linear binary code of length $n = 2^m - 1$, minimum distance $d = 3$, covering radius $\rho = 3$ and intersection array $(n, b_1, 1; 1, c_2, n)$. Let the dual code $C^\perp$ have nonzero weights w_i, $i = 1, 2, 3$. Then the extended code C^* is completely regular with covering radius $\rho^* = 4$ if and only if*

$$(2) \qquad\qquad w_1 + w_3 \;=\; 2\,w_2 \;=\; n + 1$$

and, in this case, the intersection array is $(n+1, n, b_1, 1; 1, c_2, n, n+1)$.

Proof. Let C be given by a parity check matrix H. The parity check matrix H^* of the extended code C^* is obtained from H by adding the zero column and then the all-one vector. From the condition $w_1 + w_3 = 2\,w_2 = n + 1$, we conclude that the external distance s^* of C^* equals $s^* = s + 1 = 4$. Since $\rho^* = \rho + 1 = 4$ (Lemma 2.2, (i)), we deduce that $s^* = \rho^*$ and C^* is uniformly packed (Lemma 2.2, (ii)). If the equalities $w_1 + w_3 = 2\,w_2 = n + 1$ are not satisfied we will have $s^* > 4$, and the code C^* is not uniformly packed. Hence, C^* is uniformly packed if and only if (2) is satisfied.

To complete the proof, it is enough to see that the parameters $(b_0^*, b_1^*, b_2^*, b_3^*; c_1^*, c_2^*, c_3^*, c_4^*)$ of the intersection array of C^* are computable.

By definition $b_0^* = n + 1$, $c_4^* = n + 1$ and since C^* has distance $d^* = 4$ we have $b_1^* = b_0 = n$, $c_1^* = c_1 = 1$.

Since codewords of weight 3 of C form a design $T(n, 3, 1, \lambda)$ (Lemma 2.7) we have that $b_2^* = b_1 = n - 1 - 2\,\lambda$ (Theorem 1 in [3]). Now, we show that $c_2^* = c_2$. Let $\mathbf{x} \in C(2)$. The number c_2 is the number of cases when the vector $\mathbf{y}$ of weight 3, at distance one from $\mathbf{x}$, is covered by some codewords $\mathbf{c} \in C$ of weight 4. Consider C^* and see that the vector $\mathbf{x}^* = (0, \mathbf{x})$ is also in $C^*(2)$. Since the set of codewords of weight 4 of C with zero parity check position is not changed, we conclude that, for this vector $\mathbf{x}^*$, we have $c_2^*(\mathbf{x}^*) = c_2$. Now, for the case when $\mathbf{x}^* = (1, \mathbf{x})$ is of weight 2, we obtain the same value $c_2^*(\mathbf{x}^*) = c_2$, because the codewords of C^* of weight 4

form a 2-design, i.e., the number of vectors $\mathbf{y}$, at distance one from $\mathbf{x}$, covering by some words from C_4^*, does not depend on the choice of $\mathbf{x}^*$.

Evidently $b_3^* = 1 = b_2$ and hence $c_3^* = c_3 = n$, $c_4^* = n+1$, finishing the proof. $\square$

3. COMPLETELY REGULAR AND COMPLETELY TRANSITIVE NESTED CODES

Recall that $\mathcal{H}_m$ is a binary Hamming code of length $n = 2^m - 1$. Assume that m is an even number $m = 2u$. Let $q = 2^u$, $r = 2^u + 1$ and $\bar{r} = 2^u - 1$. We can think of the parity check matrix H_m of $\mathcal{H}_m$ as the binary representation of $[\alpha^0, \alpha^1, \ldots, \alpha^{n-1}]$, where $\alpha \in \mathbb{F}_{2^m}$ is a primitive element.

We can present the elements of $\mathbb{F}_{2^m}$ as elements in a quadratic extension of $\mathbb{F}_{2^u}$. Let $\beta = \alpha^r$ be a primitive element of $\mathbb{F}_{2^u}$ and let $\mathbb{F}_{2^m} = \mathbb{F}_{2^u}[\alpha]$.

Every element $\gamma \in \mathbb{F}_{2^m}$ can be presented as $\gamma = \gamma_1 + \gamma_2 \alpha \in \mathbb{F}_{2^u}[\alpha]$, where $\gamma_1, \gamma_2 \in \mathbb{F}_{2^u}$. The matrix H_m can also be written as the binary matrix of size $(2u \times n)$, where the columns are binary presentations of $[\gamma_i, \gamma_j]$ with $\gamma_i, \gamma_j \in \{0, \beta^1, \ldots, \beta^{q-1}\}$.

Definition 3.1. Given $a = \gamma_1 + \gamma_2 \alpha$ and $b = \gamma_1' + \gamma_2' \alpha$ from $\mathbb{F}_{2^u}[\alpha]$, define the determinant of a, b in $\mathbb{F}_{2^u}$ as

$$\det{}_u(a, b) = \det \begin{bmatrix} \gamma_1 & \gamma_1' \\ \gamma_2 & \gamma_2' \end{bmatrix} = \gamma_1 \gamma_2' + \gamma_1' \gamma_2.$$

The above definition is the usual definition of determinant. For a homomorphism $g : \mathbb{F}_{2^u}^2 \longrightarrow \mathbb{F}_{2^u}^2$ and any two elements a and b from $\mathbb{F}_{2^u}[\alpha]$ we have $\det(g(a), g(b)) = \det(g) \det_u(a, b)$, where $\det(g)$ is the determinant of the matrix defining this homomorphism. So, if

$$(3) \qquad\qquad g = \begin{bmatrix} g_1 & g_1' \\ g_2 & g_2' \end{bmatrix},$$

then $\det(g) = g_1 g_2' + g_1' g_2$.

Let E_m be the binary representation of the matrix $[\alpha^{0r}, \alpha^r, \ldots, \alpha^{(n-1)r}]$. Take the matrix P_m as the vertical join of H_m and E_m.

It is well known [6] that the code with parity check matrix P_m, denoted by $C^{(u)}$, is a cyclic binary completely regular code with covering radius $\rho = 3$, minimum distance $d = 3$ and dimension $n - (m + u)$. The generator polynomial of $C^{(u)}$ is $g(x) = m_\alpha(x) m_{\alpha^r}(x)$, where $m_{\alpha^i}(x)$ means the minimal polynomial associated to α^i.

For $i \in I = \{1, \ldots, n\}$, denote by $\mathbf{e}_i$ the vector with only one nonzero coordinate of value 1 in the position ith. Binary vectors $\mathbf{v} \in \mathbb{F}_2^n$ can be written as $\mathbf{v} = \sum_{i \in I_\mathbf{v}} \mathbf{e}_i$, where $I_\mathbf{v} = \mathrm{Supp}(\mathbf{v})$. Finite fields $\mathbb{F}_{2^m}$ and $\mathbb{F}_{2^u}[\alpha]$ are isomorphic and so the positions of vectors in $\mathbb{F}_2^n$ can be enumerated by using the nonzero elements in $\mathbb{F}_{2^m}$ which, in turn, can be seen as elements in $\mathbb{F}_{2^u}[\alpha]$ by substituting any $\alpha^i \in \mathbb{F}_{2^m}$ with the corresponding $\alpha^i = \gamma_{i1} + \gamma_{i2}\alpha \in \mathbb{F}_{2^u}[\alpha]$, where $\gamma_{i1}, \gamma_{i2} \in \mathbb{F}_{2^u}$.

For $\mathbf{v} \in \mathbb{F}_2^n$, we have that $H_m \mathbf{v}^T \in \mathbb{F}_2^m$, however we can consider the representation of it as an element of $\mathbb{F}_{2^u}[\alpha]$. Clearly we have

$$H_m \mathbf{v}^T = H_m \Big(\sum_{i \in I_\mathbf{v}} \mathbf{e}_i \Big)^T = \sum_{i \in I_\mathbf{v}} \alpha^i = \sum_{i \in I_\mathbf{v}} (\gamma_{i1} + \gamma_{i2}\alpha).$$

For any $\mathbf{v} = \sum_{i \in I_\mathbf{v}} \mathbf{e}_i \in \mathbb{F}_2^n$, denote $S(\mathbf{v}) = \sum_{i \in I_\mathbf{v}} \gamma_{i1}\gamma_{i2} \in \mathbb{F}_{2^u}$. The next lemma gives a new description for the code $C^{(u)}$.

Lemma 3.2. *The code $C^{(u)}$ consists of elements $\mathbf{v} \in \mathbb{F}_2^n$, such that $H_m \mathbf{v}^T = 0$ and $S(\mathbf{v}) = 0$.*

Proof. By definition, a binary vector $\mathbf{v}$ belongs to $C^{(u)}$ if and only if $P_m \mathbf{v}^T = 0$, implying $H_m \mathbf{v}^T = 0$ and $E_m \mathbf{v}^T = 0$. Taking the vector $\mathbf{v} = \sum_{i \in I_\mathbf{v}} \mathbf{e}_i$, we are going to prove that conditions $H_m \mathbf{v}^T = 0$ and $E_m \mathbf{v}^T = 0$ (i.e., $\sum_{i \in I_\mathbf{v}} (\alpha^i)^r = 0$) lead to $S(\mathbf{v}) = 0$.

From the first condition we have $0 = H_m \mathbf{v}^T = \sum_{i \in I_\mathbf{v}} \gamma_{i1} + \gamma_{i2}\alpha$, implying that $\sum_{i \in I_\mathbf{v}} \gamma_{i1} = 0$ and $\sum_{i \in I_\mathbf{v}} \gamma_{i2} = 0$. It also gives $\sum_{i \in I_\mathbf{v}} \gamma_{i1}^2 = 0$ and $\sum_{i \in I_\mathbf{v}} \gamma_{i2}^2 = 0$.

Now consider the second one:

$$E_m \mathbf{v}^T = \sum_{i \in I_\mathbf{v}} \alpha^{ir} = \sum_{i \in I_\mathbf{v}} (\gamma_{i1} + \gamma_{i2}\alpha)^r.$$

Since $r = 2^u + 1$ and $\gamma_{ik}^{2^u} = \gamma_{ik}$ for $k = 1, 2$, we obtain

$$
\begin{aligned}
E_m \mathbf{v}^T &= \sum_{i \in I_\mathbf{v}} (\gamma_{i1} + \gamma_{i2}\alpha)^{2^u} (\gamma_{i1} + \gamma_{i2}\alpha) \\
&= \sum_{i \in I_\mathbf{v}} (\gamma_{i1} + \gamma_{i2}\alpha^{2^u})(\gamma_{i1} + \gamma_{i2}\alpha) \\
&= \sum_{i \in I_\mathbf{v}} (\gamma_{i1}^2 + \gamma_{i2}^2\beta + \gamma_{i1}\gamma_{i2}(\alpha + \alpha^{r-1}))
\end{aligned}
$$

(recall that $\beta = \alpha^r$) and, since $H_m \mathbf{v}^T = 0$ and $\alpha + \alpha^{r-1} \neq 0$, we finally obtain $E_m \mathbf{v}^T = 0$ if and only if

$$S(\mathbf{v}) = \sum_{i \in I_{\mathbf{v}}} \gamma_{i1} \gamma_{i2} = 0.$$

$\square$

The code $C^{(u)}$ is a binary $[n = 2^m - 1, k = n - m - u]$ code and it is a subcode of the $[2^m - 1, n - m]$ Hamming code $\mathcal{H}_m$. Now we show, that $C^{(u)}$ is not only completely regular [6], but also completely transitive.

An isomorphism $\Phi : \mathbb{F}_{2^u}^2 \longrightarrow \mathbb{F}_{2^u}^2$ is given by a (2×2)-matrix over $\mathbb{F}_{2^u}$,

$$\Phi = \begin{bmatrix} a & a' \\ b & b' \end{bmatrix},$$

with nonzero determinant $\det(\Phi) = ab' + a'b \neq 0$, such that

$$\Phi(\gamma_{i1}, \gamma_{i2})^T = (a\gamma_{i1} + a'\gamma_{i2}, \ b\gamma_{i1} + b'\gamma_{i2})^T = (\gamma_{j1}, \gamma_{j2})^T.$$

The above isomorphism Φ induces a permutations of columns, denoted by φ, where the column $\alpha^i = \gamma_{i1} + \gamma_{i2}\alpha$ is moved under the action of φ to the column $\alpha^j = \gamma_{j1} + \gamma_{j2}\alpha$, i.e., $\varphi((\gamma_{i1}, \gamma_{i2})^T) = (\gamma_{j1}, \gamma_{j2})^T$.

The above presentation, looking at the elements in $\mathrm{GL}_2(2^u)$ as elements in S_n, implies that the general linear group $\mathrm{GL}_2(2^u)$ stabilizes $C^{(u)}$.

Proposition 3.3. *The automorphism group of $C^{(u)}$ contains the general linear group of degree 2 over $\mathbb{F}_{2^u}$, so $GL_2(2^u) \subseteq \mathrm{Aut}(C^{(u)})$.*

Proof. Let $\Phi \in \mathrm{GL}_2(2^u)$ and, as we said before, consider the associated permutation $\varphi \in \mathrm{S}_n$. We want to see that $\varphi \in \mathrm{Aut}(C^{(u)})$.

Let $\mathbf{v} = \sum_{i \in I_{\mathbf{v}}} \mathbf{e}_i \in C^{(u)}$, hence from Lemma 3.2, $H_m \mathbf{v}^T = 0$ and $S(\mathbf{v}) = 0$. Thus,

$$\sum_{i \in I_{\mathbf{v}}} \gamma_{i1} = 0, \quad \sum_{i \in I_{\mathbf{v}}} \gamma_{i2} = 0 \ \text{ and } \ \sum_{i \in I_{\mathbf{v}}} \gamma_{i1}\gamma_{i2} = 0,$$

where for $i \in \{0, \ldots, n-1\}$ we have $\alpha^i = \gamma_{i1} + \gamma_{i2}\alpha \in \mathbb{F}_{2^u}[\alpha]$. Also we have $\sum_{i \in I_{\mathbf{v}}} \gamma_{i1}^2 = 0$ and $\sum_{i \in I_{\mathbf{v}}} \gamma_{i2}^2 = 0$.

Now we have to prove that $H_m(\varphi(\mathbf{v}))^T = S(\varphi(\mathbf{v})) = 0$. We obtain

$$H_m(\varphi(\mathbf{v}))^T = \sum_{i \in I_\mathbf{v}} \Phi(\gamma_{i1}, \gamma_{i2})^T = \sum_{i \in I_\mathbf{v}} (a\gamma_{i1} + a'\gamma_{i2}) + (b\gamma_{i1} + b'\gamma_{i2})\alpha = 0.$$

and

$$
\begin{aligned}
S(\varphi(\mathbf{v})) &= \sum_{i \in I_\mathbf{v}} (a\gamma_{i1} + b\gamma_{i2})(a'\gamma_{i1} + b'\gamma_{i2}) \\
&= \sum_{i \in I_\mathbf{v}} aa'\gamma_{i1}^2 + bb'\gamma_{i2}^2 + (ab' + a'b)\gamma_{i1}\gamma_{i2} \\
&= \det(\Phi) \sum_{i \in I_\mathbf{v}} \gamma_{i1}\gamma_{i2} = 0.
\end{aligned}
$$

$\square$

Proposition 3.4. *The automorphism group of $C^{(u)}$ gives four orbits on the cosets of $C^{(u)}$ in $\mathbb{F}_2^n$ and so $C^{(u)}$ is a completely transitive code.*

Proof. Denote the syndrome of any vector $\mathbf{v} \in \mathbb{F}_2^n$ as $[h, e]$, where $e = S(\mathbf{v}) \in \mathbb{F}_{2^u}$ and $h = H_m\mathbf{v}^T$. Since $C^{(u)}$ has covering radius $\rho = 3$ we have four different classes of cosets of $C^{(u)}$ depending on their weight. The coset of weight 0 coincides with $C^{(u)}$, so its vectors have syndrome $[0, 0]$. The cosets of weight 1 are those with syndrome $[h, e]$, where $h = H_m\mathbf{v}^T$ for some vector $\mathbf{v}$ of weight one such that $e = S(\mathbf{v})$, hence a total of $2^m - 1 = r\bar{r}$ cosets. Since $\mathrm{GL}_2(\mathbb{F}_{2^u})$ is transitive over the set $\{\mathbf{e}_i : 1 \le i \le n\}$ the orbit of a vector in a coset of weight one covers all cosets of weight one.

The cosets of weight 3 are those with syndrome $[h, e]$, where $h = H_m\mathbf{v}^T = 0$ and $e = S(\mathbf{v}) \ne 0$. As we saw in the preamble of Lemma 3.2, $e \in \mathbb{F}_{2^u}$, so it has $\bar{r}$ possible values and there are a total of $\bar{r}$ cosets of weight 3. Like for the above case when the cosets are of weight 1, the orbit of a vector in a coset of weight 3 contains all cosets of weight 3. Indeed, from Proposition 3.3 there exists an automorphism with the appropriate determinant which takes e to any other possible e'.

Regarding the cosets of weight 2 note that their leaders are vectors as $\mathbf{e}_{i_1} + \mathbf{e}_{i_2}$, for $i_1 \ne i_2$ in $\{1, \ldots, n\}$, in such a way that if we take $\mathbf{e}_{i_3}$ with $H_m(\mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3})^T = 0$ we should have $S(\mathbf{e}_{i_1} + \mathbf{e}_{i_2}) \ne S(\mathbf{e}_{i_3})$. Otherwise, $\mathbf{e}_{i_3}$ would be the leader, which is impossible in a coset of weight two. Call $a = (a_1, a_2)$ and $b = (b_1, b_2)$, where $H_m\mathbf{e}_{i_1}^T = a_1 + a_2\alpha \in \mathbb{F}_{2^u}[\alpha]$ and $H_m\mathbf{e}_{i_2}^T = b_1 + b_2\alpha \in \mathbb{F}_{2^u}[\alpha]$. We will say that the pair a, b, where $a = (a_1, a_2) \in \mathbb{F}_{2^u}^2$, $b = (b_1, b_2) \in \mathbb{F}_{2^u}^2$, is a representative of the coset. We

have $S(\mathbf{e}_{i_1}+\mathbf{e}_{i_2}) = a_1a_2+b_1b_2$ and $H_m\mathbf{e}_{i_3}^T = H_m\mathbf{e}_{i_1}^T+H_m\mathbf{e}_{i_2}^T = (a_1+b_1)+(a_2+b_2)\alpha$. Hence $S(\mathbf{e}_{i_3}) = (a_1+b_1)(a_2+b_2) = S(\mathbf{e}_{i_1}+\mathbf{e}_{i_2})+\det_u(a,b)$. In conclusion, the cosets of weight two are those with representative pairs a,b such that $\det_u(a,b) \neq 0$. Given two pairs a,b and c,d, with $\det_u(a,b) \neq 0$ and $\det_u(c,d) \neq 0$, from Proposition 3.3 we always can find an isomorphism of $\mathbb{F}_{2^u}^2$ taking a,b to c,d and so, an automorphism of $C^{(u)}$ sending the coset with representative pair a,b to the coset with representative pair c,d. $\qquad\square$

As we already know, the number of cosets $C^{(u)} + \mathbf{v}$, of weight three, is $\bar{r}$. Indeed, their syndromes $S(\mathbf{v})$ are the nonzero elements of $\mathbb{F}_{2^u}$. For $i \in \{0,\ldots,u\}$, taking $u-i$ cosets $C^{(u)} + \mathbf{v}_1,\ldots,C^{(u)} + \mathbf{v}_{u-i}$ with independent syndromes $S(\mathbf{v}_1),\ldots,S(\mathbf{v}_{u-i})$ (independent, means that they are independent binary vectors in $\mathbb{F}_2^u$) we can generate a linear binary code $C^{(i)} = \langle C^{(u)}, \mathbf{v}_1,\ldots\mathbf{v}_{u-i}\rangle$. Let A_{u-i} be the linear subspace of $\mathbb{F}_2^u$ generated by the syndromes $S(\mathbf{v}_1),\ldots,S(\mathbf{v}_{u-i})$.

The dimension of code $C^{(i)}$ is $\dim(C^{(i)}) = u-i+\dim(C^{(u)})$, where $\dim(C^{(u)}) = n-m-u$. Note that the maximum number of independent syndromes we can take is u, so the biggest code we can obtain is of dimension $u + \dim(C^{(u)}) = n - m$, which is the Hamming code $C^{(0)} = \mathcal{H}_m$. All the constructed codes contain $C^{(u)}$ and, at the same time, they are contained in the Hamming code $C^{(0)}$.

The number of codes $C^{(u-i)}$ equals the number of subspaces of dimension i we can take in $\mathbb{F}_2^u$, so the Gaussian binomial coefficient

$$|\{C^{(u-i)}\}| = \binom{u}{i}_2 = \frac{(2^u - 1)(2^u - 2)\cdots(2^u - 2^{i-1})}{(2^i - 1)(2^i - 2)\cdots(2^i - 2^{i-1})}.$$

Taking all possibilities, we are able to construct several nested families of codes between $C^{(u)}$ and $C^{(0)} = \mathcal{H}_m$. In fact, it is easy to compute that there are

$$\prod_{i=0}^{u-1}(2^{u-i} - 1)$$

different families.

The following property was stated in [6] for code $C^{(u)}$, but it can be extended to all codes $C^{(i)}$, for $i \in \{1,\ldots,u\}$.

Proposition 3.5. *For $i \in \{1,\ldots,u\}$ the cosets of weight three of $C^{(i)}$ are at distance three from each other and $C^{(i)} \cup C^{(i)}(3)$ is the Hamming code.*

Proof. Let A_{u-i} be the linear subspace of $\mathbb{F}_{2^u}$ generated by $S(\mathbf{v}_1), \ldots, S(\mathbf{v}_{u-i})$ and $C^{(i)} = \langle C^{(u)}, \mathbf{v}_1, \ldots \mathbf{v}_{u-i} \rangle$. The cosets $C^{(i)} + \mathbf{v}$, where $\mathbf{v}$ is a representative of minimum weight three are those that $H_m \mathbf{v}^T = 0$ and $S(\mathbf{v}) \notin A_{u-i}$. On the other hand, the elements $\mathbf{v} \in C^{(i)}$ are those that $H_m \mathbf{v}^T = 0$ and $S(\mathbf{v}) \in A_{u-i}$. Hence, the elements $\mathbf{v} \in C^{(i)} \cup C^{(i)}(3)$ are those with $H_m \mathbf{v}^T = 0$ and so they are the elements in the Hammimg code $C^{(0)} = \mathcal{H}_m$. This proves the second part of the statement.

For the first part, take two cosets $C^{(i)} + \mathbf{v} \neq C^{(i)} + \mathbf{w}$ of minimum weight three. The distance from one to the other is the minimum weight of the coset $C^{(i)} + (\mathbf{v} + \mathbf{w})$. We know that $H_m(\mathbf{v} + \mathbf{w}) = H_m(\mathbf{v}) + H_m(\mathbf{w}) = 0$, so $C^{(i)} + (\mathbf{v} + \mathbf{w})$ has minimum weight zero or three. In the first case, we should have $\mathbf{w} \in C^{(i)} + \mathbf{v}$ and the two cosets would be equal, against the hypothesis. Hence, the distance from one to the other coset is three, which proves the first part of the statement. $\square$

All codes $C^{(i)}$ are completely regular as we show later in Theorem 3.8. We have seen that $C^{(u)}$ and $C^{(0)}$ are completely transitive and, in addition we show that $C^{(1)}$ is also a completely transitive code.

Proposition 3.6. *The automorphism group of $C^{(1)}$ induces 4 orbits on the cosets of $C^{(1)}$ in $\mathbb{F}_2^n$ and so $C^{(1)}$ is a completely transitive code.*

Proof. Let $\mathbb{F}_2^u = A_{u-1} \oplus A_1$ be the decomposition of the binary linear space $\mathbb{F}_2^u$ as a direct sum of subspaces and let $S_1(\mathbf{v})$ be the projection of $S(\mathbf{v})$ over A_1. By definition of $C^{(1)}$ the elements $\mathbf{v} \in C^{(1)}$ are those that $H_m \mathbf{v}^T = 0$ and $S(\mathbf{v})$ belongs to a subspace $A_{u-1} \subset \mathbb{F}_2^u$ of dimension $u - 1$ over $\mathbb{F}_2$. Therefore, the elements of $C^{(1)}$ can be characterized by the syndrome $h = H_m \mathbf{v}^T = 0$ and $e = S_1(\mathbf{v}) = 0$. Following the same argumentation and computations as in Proposition 3.3, we easily obtain that $\mathrm{SL}_2(2^u) \subset \mathrm{Aut}(C^{(1)})$, where $\mathrm{SL}_2(2^u)$ is the special linear group of automorphisms, so the normal subgroup of the general linear group $\mathrm{GL}_2(2^u)$, consisting of those matrices Φ with determinant $\det(\Phi) = 1$.

The cosets of $C^{(1)}$ of weight 1 are those with syndrome $[h, e]$, where $h = H_m \mathbf{u}^T \in \mathbb{F}_{2^u}[\alpha] \backslash \{0\}$ for some vector $\mathbf{u} \neq 0$ of weight one such that $e = S_1(\mathbf{u})$, hence a total of $2^m - 1 = r\bar{r}$ cosets. Since $\mathrm{SL}_2(2^u)$ is transitive over $\{\mathbf{e}_i : 1 \leq i \leq n\}$ the orbit of a vector in a coset of weight one contains all cosets of weight one.

There is only one coset of weight 3, say $C^{(1)} + \mathbf{u}$, where $H_m \mathbf{u}^T = 0$ and $S_1(\mathbf{u}) \neq 0$. Hence, there is nothing to prove, automorphisms of $C^{(1)}$ act transitively over this unique coset.

The cosets of weight 2 are $C^{(1)} + \mathbf{u} + \mathbf{v}$, where $C^{(1)} + \mathbf{u}$ is the coset of weight three and $\mathbf{v}$ is of weight one. The syndrome of these cosets is $[h, e]$, where $h = H_m(\mathbf{u} + \mathbf{v})^T = H_m \mathbf{v}^T \neq 0$ and $e = S_1(\mathbf{u} + \mathbf{v}) \neq S_1(\mathbf{v})$. We have a total of $2^m - 1 = r\bar{r}$ cosets of weight two. Like for the cosets of weight one, since $\mathrm{SL}_2(2^u)$ is transitive over $\{\mathbf{e}_i : 1 \leq i \leq n\}$ the orbit of a vector in a coset of weight two cover all cosets of weight two. $\qquad \square$

Finally, we can prove that all codes $C^{(i)}$ are completely regular.

Lemma 3.7. *Let $C_3^{(i)}$ be the set of all codewords in $C^{(i)}$ of weight three. Then $C_3^{(i)}$ is a $T(n, 3, 1, \lambda_i)$ design, where $\lambda_i = 2^{m-i-1} - 1$ and $n = 2^m - 1$.*

Proof. From the construction of codes $C^{(i)}$ we know that the codewords $\mathbf{v}$ of weight three are those that $H_m \mathbf{v}^T = 0$ and $S(\mathbf{v})$ belongs to a fixed subspace $A_{u-i} \subset \mathbb{F}_2^u$ of dimension $u - i$ over $\mathbb{F}_2$. Hence, taking a fixed nonzero element $\gamma = \gamma_1 + \gamma_2 \alpha \in \mathbb{F}_{2^m}$ every codeword of weight three covering this element γ is defined giving $\gamma' = \gamma_1' + \gamma_2' \alpha \in \mathbb{F}_{2^m}$ such that $\det_u(\gamma, \gamma') \in A_{u-i}$. Indeed, if $\mathbf{v} \in \mathbb{F}_2^n$ is of weight three let

$$\{\gamma = \gamma_1 + \gamma_2 \alpha, \ \gamma' = \gamma_1' + \gamma_2' \alpha, \ \gamma'' = \gamma_1'' + \gamma_2'' \alpha\}$$

be its support. Then, since $H_m \mathbf{v}^T = 0$ we have $\gamma'' = \gamma + \gamma'$ and so

$$S(\mathbf{v}) = \gamma_1 \gamma_2 + \gamma_1' \gamma_2' + \gamma_1'' \gamma_2'' = \det_u(\gamma, \gamma').$$

Now we want to count how many codewords of weight three cover a fixed nonzero element $\gamma \in \mathbb{F}_{2^m}$. We begin by counting how many $\gamma' \in \mathbb{F}_{2^m}$ gives $\det_u(\gamma, \gamma') \in A_{u-i}$. Recall that $\det_u(\gamma, \gamma')$ is an element of $\mathbb{F}_{2^u}$, considered as a binary vector. For any nonzero element $\gamma = \gamma_1 + \gamma_2 \alpha \in \mathbb{F}_{2^m}$ there are $2^u - 2$ nonzero values $\gamma' = \beta^i \gamma \neq \gamma$, where $i \in \{1, \ldots, 2^u - 2\}$ such that $\det_u(\gamma, \gamma') = 0$ and there are 2^u values γ' giving $\det_u(\gamma, \gamma') = \beta^j \in \mathbb{F}_{2^u}$, for a fixed $j \in \{0, \ldots, 2^u - 2\}$. There are $2^{u-i} - 1$ nonzero vectors in A_{u-i}. Hence, summing up, we conclude that there are $2^u - 2 + 2^u(2^{u-i} - 1) = 2^{m-i} - 2$ values γ', such that $\det(\gamma, \gamma') \in A_{u-i}$. However, the codeword with support $\{\gamma, \gamma', \gamma'' = \gamma + \gamma'\}$ is counted twice, once as γ' and

again as γ''. Hence, finally, the number λ_i of codewords of weight three covering γ is $2^{m-i-1} - 1$. $\square$

Theorem 3.8. *For $i \in \{0, \ldots, u\}$, the code $C^{(i)}$ is completely regular with intersection array $(2^m - 1, 2^m - 2^{m-i}, 1; 1, 2^{m-i}, 2^m - 1)$.*

Proof. For each $i \in \{0, \ldots, u\}$, we have that $C^{(i)}$ is completely regular if the parameters $(b_0, b_1, b_2; c_1, c_2, c_3)$ of the intersection array are computable. Since the minimum distance in $C^{(i)}$ and in $C^{(i)}(\rho) = C^{(i)}(3)$ is 3, it is obvious that $b_0 = c_3 = n = 2^m - 1$ and $b_2 = c_1 = 1$. Let $\mathbf{x} \in C^{(i)}(1)$, we count the number of neighbors of $\mathbf{x}$ in $C^{(i)}(1)$. Without loss of generality, we assume that $\mathbf{x}$ has weight one. Therefore, $\mathbf{x}$ has $n - 1$ neighbors of weight two. By Lemma 3.7, twice λ_i of these neighbors are covered by minimum weight codewords of $C^{(i)}$. As the result does not depend on the choice of x, we conclude that $a_1 = 2\lambda_i$. Hence, $b_1 = n - c_1 - a_1 = n - 1 - 2\lambda_i = 2^m - 2^{m-i}$. A similar argument shows that any vector in $C^{(i)}(2)$ has a fixed number of neighbors in $C^{(i)}(2)$. Therefore, c_2 is also calculable. Applying Lemma 2.3, we have that $\mu_1 b_1 = \mu_2 c_2$. From

$$\mu_0 = 1; \; \mu_1 = n; \; \mu_3 = 2^i - 1 \; \text{ and } \; \mu_0 + \mu_1 + \mu_2 + \mu_3 = 2^i(n+1),$$

we deduce $\mu_2 = (2^i - 1)n$ and $c_2 = \mu_1 b_1 / \mu_2 = b_1 / (2^i - 1) = 2^{m-i}$. $\square$

Corollary 3.9. *For $i \in \{0, \ldots, u\}$, the extended code $C^{(i)*}$ is completely regular with intersection array $(2^m, 2^m - 1, 2^m - 2^{m-i}, 1; 1, 2^{m-i}, 2^m - 1, 2^m)$.*

Proof. By Theorem 3.8 any code $C^{(i)}$ is completely regular. In particular, this means (Lemma 2.2), that for any such code $C^{(i)}$ the external distance $s(C^{(i)})$ equals the covering radius $\rho(C^{(i)})$, i.e. $s(C^{(i)}) = \rho(C^{(i)})$. Since $\rho(C^{(i)}) = 3$, we conclude that $s(C^{(i)}) = 3$ for any $i \in \{0, \ldots, u\}$. As it was shown in [6], the dual code of $C^{(u)}$ has the following values in the weight spectrum:

$$2^{m-1}, \; 2^{m-1} \pm 2^{u-1}.$$

But any code $C^{(i)}$ contains the code $C^{(u)}$ as a subcode, implying that the dual $(C^{(i)})^\perp$ is contained in $(C^{(u)})^\perp$. This, in turn, implies that any such code $(C^{(i)})^\perp$ has the same values for the weights of its codewords as the code $(C^{(u)})^\perp$. Now the result follows from Lemma 2.9. $\square$

The next proposition generalizes the previous Proposition 3.5 to the extended codes $C^{(i)*}$.

Proposition 3.10. *For $i \in \{1, \ldots, u\}$ the cosets of weight four of $C^{(i)*}$ are at distance four from each other and $C^{(i)*} \cup C^{(i)*}(4)$ is the extended Hamming code.*

Proof. Code $C^{(i)*}$ is the extension of $C^{(i)}$ by an overall parity check coordinate, which we assume is the 0th coordinate. The number of cosets of $C^{(i)*}$ is twice the cosets of $C^{(i)}$. If $C^{(i)} + \mathbf{v}$ is a coset of $C^{(i)}$, where $\mathbf{v}$ is a representative vector of minimum weight then $C^{(i)*} + (0|\mathbf{v})$ and $C^{(i)*} + (1|\mathbf{v})$ are cosets of $C^{(i)*}$. The cosets of $C^{(i)*}$ of weight 4 are of the form $C^{(i)*} + (1|\mathbf{v})$, where $C^{(i)} + \mathbf{v}$ is a coset of weight 3 of $C^{(i)}$.

Take two different cosets of $C^{(i)*}$ of weight 4, say $C^{(i)*} + (1|\mathbf{v})$ and $C^{(i)*} + (1|\mathbf{w})$, where $C^{(i)} + \mathbf{v}$ and $C^{(i)} + \mathbf{w}$ are cosets of weight 3 of $C^{(i)}$. The distance from $C^{(i)*} + (1|\mathbf{v})$ to $C^{(i)*} + (1|\mathbf{w})$ is the minimum weight of the coset $C^{(i)*} + (1|\mathbf{v}) + (1|\mathbf{w}) = C^{(i)*} + (0|\mathbf{v} + \mathbf{w})$. From Proposition 3.5 we know that the coset $C^{(i)} + \mathbf{v} + \mathbf{w}$ is of weight three and so, there is a vector $\mathbf{r}$ of weight three such that $\mathbf{v} + \mathbf{w} \in C^{(i)} + \mathbf{r}$. Now, extending these vectors we obtain $(0|\mathbf{v} + \mathbf{w}) \in C^{(i)*} + (1|\mathbf{r})$ proving the first assertion.

The second part of the statement is straightforward. $\qquad\square$

The next theorem shows that the extended codes $C^{(i)*}$ are not only completely regular, but completely transitive whenever $C^{(i)}$ is completely transitive.

Theorem 3.11. *For $i \in \{0, \ldots, u\}$, the automorphism group of the extended code $C^{(i)*}$ is $\mathrm{Aut}(C^{(i)*}) = \mathrm{Aut}(C^{(i)}) \ltimes \mathbb{F}_2^m$. Code $C^{(i)*}$ is completely transitive when $C^{(i)}$ is completely transitive.*

Proof. Let $\mathbb{F}_2^u = A_{u-i} \oplus A_i$ be the decomposition of the binary linear space $\mathbb{F}_2^u$ as a direct sum of subspaces of dimension $u - i$ and i, respectively. Let $S_i(\mathbf{v})$ be the projection of $S(\mathbf{v})$ over A_i. By definition, the elements of $\mathbf{v} \in C^{(i)}$ can be characterized by the syndrome $H_m \mathbf{v}^T = 0$ and $S_i(\mathbf{v}) = 0$.

Code $C^{(i)*}$ is the extension of $C^{(i)}$ by an overall parity check coordinate, which we assume is the 0th coordinate. Codewords in $C^{(i)*}$ have $n + 1 = 2^m$ components and we can associate, at random and once for all, the ith coordinate with a vector $\mathbf{w}_i \in \mathbb{F}_2^m$. Any vector $\mathbf{w} \in \mathbb{F}_2^m$ defines a permutation $\pi_{\mathbf{w}} \colon \{0, \ldots, n\} \longrightarrow \{0, \ldots, n\}$

such that $\pi_{\mathbf{w}}(i) = j$, where $\mathbf{w}_j = \mathbf{w} + \mathbf{w}_i$. Let $T = \{\pi_{\mathbf{w}} : \mathbf{w} \in \mathbb{F}_2^m\}$ be the set of all these permutations and note that T has a group structure isomorphic to the additive structure of $\mathbb{F}_2^m$. For each $\mathbf{w} \in \mathbb{F}_2^m$, set $\mathbf{w} = \gamma_1 + \gamma_2\alpha \in \mathbb{F}_{2^u}[\alpha]$.

As all codewords in $C^{(i)*}$ have even weight it is clear that T is a subgroup of $\text{Aut}(C^{(i)*})$. Indeed, let $\mathbf{a} = (a_0, \ldots, a_n) \in C^{(i)*}$, this means that $\mathbf{a}$ has an even number of nonzero components ($\sum_{i=0}^{n} a_i = 0$); $H_m\mathbf{a}^T = \sum_{\mathbf{w}_i \in \mathbb{F}_2^m} a_i\mathbf{w}_i = \mathbf{0}$ and $S(\mathbf{a}) \in A_{u-i}$. Now take $\pi_{\mathbf{w}}(\mathbf{a}) = \mathbf{a}' = (a_0', a_1', \ldots, a_n')$, where $a_j' = a_i$, such that $\mathbf{w}_j = \mathbf{w} + \mathbf{w}_i$ and compute:

$$\sum_{j=0}^{n} a_j' = \sum_{i=0}^{n} a_i = 0;$$

$$H_m\mathbf{a}'^T = \sum_{\mathbf{w}_j \in \mathbb{F}_2^m} a_j'\mathbf{w}_j = \sum_{\mathbf{w}_i \in \mathbb{F}_2^m} a_i(\mathbf{w} + \mathbf{w}_i) = \mathbf{w}\sum_{i=0}^{n} a_i = \mathbf{0}$$

$$S(\mathbf{a}') = \sum_j a_j'(\gamma_{j1}\gamma_{j2}) = \sum_i a_i(\gamma_1 + \gamma_{i1})(\gamma_2 + \gamma_{i2}) =$$

$$\sum_i a_i\gamma_{i1}\gamma_{i2} + \gamma_1\sum_i a_i\gamma_{i1} + \gamma_2\sum_i a_i\gamma_{i2} + \gamma_1\gamma_2\sum_i a_i = S(\mathbf{a}) \in A_{u-i}.$$

Hence, $\pi_{\mathbf{w}}(\mathbf{a}) \in C^{(i)*}$.

Further, T is a normal subgroup in $\text{Aut}(C^{(i)*})$. Indeed, for any $\varphi \in \text{Aut}(C^{(i)*})$ we have that $\varphi\pi_{\mathbf{w}}\varphi^{-1}$ is again a translation $\pi_{\mathbf{z}}$, where $\mathbf{z} = \varphi(\mathbf{w})$. For any $\varphi \in \text{Aut}(C^{(i)*})$, it is clear that we can find $\varphi' \in \text{Aut}(C^{(i)})$ fixing the extended coordinate and a vector $\mathbf{y} \in \mathbb{F}_2^m$, such that $\varphi = \varphi'\pi_{\mathbf{y}}$. Therefore, we have $\text{Aut}(C^{(i)*})/T \cong \text{Aut}(C^{(i)})$ and so $\text{Aut}(C^{(i)*})$ is the semidirect product of $\mathbb{F}_2^m$ and $\text{Aut}(C^{(i)})$ (obviously, we can identify T with $\mathbb{F}_2^m$). The first statement is proven.

Let us assume that $C^{(i)}$ is completely transitive. To prove that $C^{(i)*}$ is completely transitive we show that all cosets of $C^{(i)*}$ in $\mathbb{F}_2^{2^m}$ with the same minimum weight are in the same orbit by the action of $\text{Aut}(C^{(i)*})$.

As we said before, if $C^{(i)} + \mathbf{v}$ is a coset of $C^{(i)}$, where $\mathbf{v}$ is a representative vector of minimum weight then $C^{(i)*} + (0|\mathbf{v})$ and $C^{(i)*} + (1|\mathbf{v})$ are cosets of $C^{(i)*}$. The cosets of $C^{(i)*}$ of weight 4 are of the form $C^{(i)*} + (1|\mathbf{v})$, where $C^{(i)} + \mathbf{v}$ is a coset of weight 3 of $C^{(i)}$. Since the cosets of weight 3 of $C^{(i)}$ are in the same $\text{Aut}(C^{(i)})$-orbit and $\text{Aut}(C^{(i)}) \subset \text{Aut}(C^{(i)*})$, it follows that all the cosets of weight 4 of $C^{(i)*}$ are in the same $\text{Aut}(C^{(i)*})$-orbit.

Now consider the cosets of $C^{(i)*}$ of weight $r \in \{1, 2, 3\}$. They are of the form $C^{(i)*}+(0|\mathbf{v})$, where $C^{(i)}+\mathbf{v}$ is a coset of weight r of $C^{(i)}$ and of the form $C^{(i)*}+(1|\mathbf{v})$, where $C^{(i)}+\mathbf{v}$ is a coset of weight $r-1$ of $C^{(i)}$. Cosets of the same minimum weight in $C^{(i)}$ can be moved among them by $\mathrm{Aut}(C^{(i)})$ and so, we need only to show that there exists an automorphism in $\mathrm{Aut}(C^{(i)*})$ moving $C^{(i)*} + (0|\mathbf{v})$ to $C^{(i)*} + (1|\mathbf{v}')$, where $\mathbf{v}, \mathbf{v}'$ are at distance r and $r-1$ from $C^{(i)}$, respectively. Without loss of generality, we further assume that $\mathrm{Supp}(\mathbf{v}') \subset \mathrm{Supp}(\mathbf{v})$ and so, $\mathrm{Supp}(\mathbf{v}) = \mathrm{Supp}(\mathbf{v}') \cup \{j\}$, for some index $j \in \{1, \ldots, 2^m - 1\}$. The automorphism $\pi_{\mathbf{w}_j}$ moves $C^{(i)*} + (0|\mathbf{v})$ to $C^{(i)*} + (1|\mathbf{v}'')$, where $\mathrm{Supp}(\mathbf{v}'') = \{k \; : \; \mathbf{w}_k = \mathbf{w}_j + \mathbf{w}_s; s \in \mathrm{Supp}(\mathbf{v}')\}$ and, finally, by using an automorphism from $\mathrm{Aut}(C^{(i)})$ we can move from $C^{(i)*} + (1|\mathbf{v}'')$ to $C^{(i)*} + (1|\mathbf{v}')$. $\qquad\square$

4. Nested antipodal distance-regular graphs and distance-transitive graphs of diameter 3 and 4

Denote by $\Gamma^{(i)}$ (respectively, $\Gamma^{(i)*}$) the coset graph obtained from the code $C^{(i)}$ (respectively, $C^{(i)*}$) by Lemma 2.5.

From Proposition 3.5 (respectively, Proposition 3.10) we know that all cosets of weight 3 (respectively, of weight 4) of the code $C^{(i)}$ (respectively, of the extended code $C^{(i)*}$) belong to the Hamming code $\mathcal{H}_m$ (respectively, to the extended Hamming code $\mathcal{H}_m^*$). Therefore, we conclude that all graphs $\Gamma^{(i)}$ (respectively, $\Gamma^{(i)*}$) are antipodal. This means that for $i > 0$ all graphs $\Gamma^{(i)}$ and $\Gamma^{(i)*}$ are imprimitive.

We need the following statement from [10].

Lemma 4.1. *Let Γ be an antipodal distance-regular graph of diameter three. Then Γ is a r-fold covering graph of K_n, for some r and n and recall that c_2 is the number of common neighbors of two vertices in Γ at distance two. Then the intersection array of Γ is $(n-1, (r-1)c_2, 1; 1, c_2, n-1)$.*

As a direct result of Lemma 4.1, Theorem 3.8 and taking into account [14] we obtain the following distance-regular and distance-transitive coset graphs. As far as we know, the presentation of these graphs as coset graphs is new.

Theorem 4.2. *For any even $m = 2u$, $m \geq 4$ there exists a family of embedded antipodal distance-regular coset graphs $\Gamma^{(i)}$ with 2^{2u+i} vertices and diameter 3, for $i = 1, \ldots, u$. Specifically:*

- *The graph $\Gamma^{(0)}$ has diameter 1, i.e., it is a complete graph $K_{n+1}, n = 2^m - 1$.*
- *$\Gamma^{(i)}$, $i = 1, \ldots, u$ has intersection array*

$$(2^m - 1, 2^m - 2^{m-i}, 1; 1, 2^{m-i}, 2^m - 1).$$

- *$\Gamma^{(i)}$ is a subgraph of $\Gamma^{(i+1)}$ for all $i = 0, 1, \ldots, u - 1$.*
- *$\Gamma^{(i)}$ covers $\Gamma^{(j)}$, $j \in \{0, 1, \ldots, i - 1\}$ with size of the fibre $r_{i,j} = 2^{i-j}$. In particular, $\Gamma^{(i)}$ is 2^i-fold antipodal cover graph of K_{n+1}.*
- *The graphs $\Gamma^{(i)}$ are distance-transitive for $i \in \{0, 1, u\}$ when $m \geq 8$ and for $i \in \{0, 1, 2, 3\}$ when $m = 6$.*

We have seen that the graphs $\Gamma^{(i)}$ are antipodal covering graphs of the complete graph K_{n+1}, where $n = 2^m - 1$. For the graphs $\Gamma^{(i)*}$ we give the following properties.

Lemma 4.3. *Let Γ^* be an antipodal bipartite distance-regular graph of diameter $D = 4$ and valency $n + 1$. Then Γ^* is a r-fold covering graph of the complete bipartite graph $K_{n+1,n+1}$ for some r. The intersection array of Γ^* is*

$$\{n + 1, n, (n + 1)(r - 1)/r, 1; 1, (n + 1)/r, n, n + 1\}.$$

Proof. Since Γ^* is distance-regular, we have that the size of each fibre, or antipodal class, is constant. Then, the result follows by observing that the antipodal quotient graph $\hat{\Gamma}^*$ (taking as vertices the fibres) is a bipartite graph with diameter 2 and, clearly, it is regular of valency $n + 1$. The intersection array follows from [8, Prop. 5.9]. $\qquad\square$

Proposition 4.4. *The graph $\Gamma^{(i)*}$ is an antipodal bipartite distance-regular graph for any $i \in \{0, 1, \ldots, u\}$. Specifically, $\Gamma^{(0)*}$ is the complete bipartite graph $K_{n+1,n+1}$.*

Proof. We already know that $\Gamma^{(i)*}$ is antipodal and distance-regular. $C^{(i)*}$ and the cosets of weight two and four form one of the parts and the cosets of weight one and three form the other part. Hence, $\Gamma^{(i)*}$ is bipartite. $\qquad\square$

Finally, from Lemma 2.5, Theorem 3.11, Corollary 3.9, Lemma 4.3 and Proposition 4.4, we can establish the following results for the coset graphs coming from the extended codes $C^{(i)*}$.

Theorem 4.5. *For any even $m = 2u$, $m \geq 4$ and $i \in \{0, 1, \ldots, u\}$ there exists a family of embedded antipodal distance-regular coset graphs $\Gamma^{(i)*}$ with 2^{m+i+1} vertices and diameter 4. Specifically:*

- *The graph $\Gamma^{(0)*}$ is the complete bipartite graph $K_{n+1,n+1}$.*
- *$\Gamma^{(i)*}$ has intersection array*

$$(2^m, 2^m - 1, 2^m - 2^{m-i}, 1; 1, 2^{m-i}, 2^m - 1, 2^m).$$

- *$\Gamma^{(i)*}$ is a subgraph of $\Gamma^{(i+1)*}$ for all $i = 0, 1, \ldots, u - 1$.*
- *$\Gamma^{(i)*}$ covers $\Gamma^{(j)*}$, where $j = 0, 1, \ldots, i - 1$ with size of the fibre $r_{i,j} = 2^{i-j}$. In particular, $\Gamma^{(i)*}$ is 2^i-fold antipodal cover graph of $K_{n+1,n+1}$.*
- *The graphs $\Gamma^{(i)*}$ are distance-transitive for $i = 0, 1, u$ when $m \geq 8$ and $i = 0, 1, 2, 3$ when $m = 6$.*

From [14] we know that $\Gamma^{(i)}$ (respectively $\Gamma^{(i)*}$) is distance-transitive if and only if $C^{(i)}$ (respectively $C^{(i)*}$) is completely transitive, for $i = 0, \ldots, u$. Moreover, by Theorem 3.11, we deduce that if $\Gamma^{(i)}$ is distance-transitive, so is $\Gamma^{(i)*}$.

Note that $\Gamma^{(i)}$ and $\Gamma^{(i)*}$ are connected graphs, for all $i = 0, \ldots, u$.

Corollary 4.6. *Let $i \in \{1, \ldots, u\}$, where $2u = m$. If i does not divide m and $2^i - 1$ does not divide u, then $\Gamma^{(i)}$ and $\Gamma^{(i)*}$ are not distance-transitive graphs. Hence, $C^{(i)}$ and $C^{(i)*}$ are not completely transitive codes.*

Proof. In [11], all distance-transitive r-fold antipodal cover graphs Γ of $K_{k,k}$ are classified. In our case, $r = 2^i$ and $k = 2^m$. From the main theorem in [11], if Γ is a connected graph then one of the following must hold: 2^m is a power of 2^i, or $2^m = 2^i$, or $2^i - 1$ divides $2m$.

The second possibility has no sense in our situation, since $i \leq u = m/2$. Hence, we obtain that m is a multiple of i or $2^i - 1$ divides $2m = 4u$. This proves the statement. $\square$

Note that, for $i \in \{0, 1, u\}$ the codes $C^{(i)}$ and $C^{(i)*}$ are completely transitive and the corresponding graphs $\Gamma^{(i)}$ and $\Gamma^{(i)*}$ are distance-transitive. Also, for $m = 6$, all codes $C^{(i)}$ and $C^{(i)*}$ are completely transitive and the graphs $\Gamma^{(i)}$ and $\Gamma^{(i)*}$ are distance-transitive. All graphs $\Gamma^{(i)*}$ for $i = 1, \ldots, u$ seem to be isomorphic to the graphs constructed in [11] when the conditions of Corollary 4.6 are satisfied. Therefore, we conjecture that when one of such divisibility conditions is satisfied, then $\Gamma^{(i)*}$ is a distance-transitive graph and, hence, $C^{(i)*}$ is a completely transitive code. Moreover, in such cases, we conjecture that $C^{(i)}$ is also completely transitive and

$\Gamma^{(i)}$ is also distance-transitive. However, the question about complete transitivity of codes $C^{(i)}$ and $C^{(i)*}$ for $i \neq 0, 1, u$ is open and needs more attention.

The first graphs $\Gamma^{(1)}$ and $\Gamma^{(1)*}$ are well known distance-transitive graphs (see [3, 4] and references there).

Graphs $\Gamma^{(u)}$ and $\Gamma^{(u)*}$ are also known. The corresponding codes $C^{(u)}$ and $C^{(u)*}$ have been constructed by Kasami [12] and have been presented in a very symmetric form by Calderbank and Goethals [6]. They proved that these codes form an association scheme [7], which immediately implies the existence of the corresponding distance-regular graphs $\Gamma^{(u)}$ and $\Gamma^{(u)*}$ (Ch. 11 in [5]).

All graphs $\Gamma^{(i)}$ for $i = 0, 1, \ldots, u$ have been constructed by Godsil and Hensel using the Quotient Construction [10]. But it was not mentioned in all references above that some of these graphs are distance-transitive. Besides, except for the graphs $\Gamma^{(u)}$, it was not stated that these graphs can be constructed as coset graphs.

References

[1] L.A. Bassalygo, G.V. Zaitsev, V.A. Zinoviev, *Uniformly packed codes*, Problems Inform. Transmiss. **10-1** (1974), 9-14.

[2] L.A. Bassalygo, V.A. Zinoviev, *A note on uniformly packed codes*, Problems Inform. Transmiss. **13-3** (1977), 22-25.

[3] J. Borges, J. Rifa, V.A. Zinoviev, *New families of completely regular codes and their corresponding distance regular coset graphs*, Designs, Codes and Cryptography **70** (2014), 139-148.

[4] J. Borges, J. Rifa, V.A. Zinoviev, *Families of completely transitive codes and distance transitive graphs*, Discrete Mathematics **324** (2014), 68-71.

[5] A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer, Berlin, 1989.

[6] A.M. Calderbank, J. M. Goethals, *Three-weights codes and association schemes*, Philips J. Res. **39** (1984), 143-152.

[7] P. Delsarte, An algebraic approach to the association schemes of coding theory, Philips Research Reports Supplements, **10** (1973).

[8] A. Gardiner, *Antipodal Covering Graphs*, J. Combin. Theory Ser. B **16** (1974), 255-273.

[9] M. Giudici, C. E. Praeger, *Completely Transitive Codes in Hamming Graphs*, Europ. J. Combinatorics **20** (1999), 647-662.

[10] C.D. Godsil, A.D. Hensel, *Distance regular covers of the complete graph*, J. Comb. Theory, Ser. B **56** (1992), 205-238.

[11] A.A. Ivanov, R.A. Lieber, T. Penttila, C.E. Praeger, *Antipodal Distance-transitive Covers of Complete Bipartite Graphs*, Europ. J. Combinatorics **18** (1997), 11-13.

[12] T. Kasami, *The weight enumerators for several classes of subcodes of the 2nd order binary Reed-Muller codes*, Information and Control **18** (1971), 369-394.

[13] A. Neumaier, *Completely regular codes*, Discrete Maths. **106/107** (1992), 335-360.

[14] J. Rifà, J. Pujol, *Completely transitive codes and distance transitive graphs*, Proc, 9th International Conference, AAECC-9, LNCS, Springer-Verlag **539** (1991), 360-367.

[15] J. Rifà, V.A. Zinoviev, *On lifting perfect codes*, IEEE Trans. on Inform. Theory **57-9** (2011), 5918-5925.

[16] P. Solé, *Completely Regular Codes and Completely Transitive Codes*, Discrete Maths. **81** (1990), 193-201.