Sharing the surplus in games with externalities within and across issues[*]

Effrosyni Diamantoudi[†]   Inés Macho-Stadler[‡]   David Pérez-Castrillo[§]

Licun Xue[¶]

February 23, 2015

Abstract

We consider issue-externality games in which agents can cooperate on multiple issues and externalities are present both within and across issues, that is, the amount a coalition receives in one issue depends on how the players are organized on all the issues. Examples of such games are several firms competing in multiple markets, and countries negotiating both a trade agreement (through, e.g., WTO) and an environmental agreement (e.g., Kyoto Protocol). We propose a way to extend (Shapley) values for partition function games to issue-externality games. We

[†]CIREQ and Department of Economics, Concordia University, 1455 boul. de Maisonneuve Ouest, Montréal, Québec, Canada H3G 1M8; Email: ediamant@alcor.concordia.ca.

[‡]Universitat Autonoma de Barcelona and Barcelona GSE, Dept. Economía e Hist. Económica, Edificio B, 08193 Bellaterra - Barcelona, Spain; Email: ines.macho@uab.es.

[§]Corresponding author. Universitat Autonoma de Barcelona and Barcelona GSE, Dept. Economía e Hist. Económica, Edificio B, 08193 Bellaterra - Barcelona, Spain; Email: david.perez@uab.es. Ph: +34 935811405. Fax: +34 935812012.

[¶]CIREQ and Department of Economics, McGill University, 855 Sherbrooke Street West, Montréal, Québec, Canada H3A 2T7; Email: licun.xue@mcgill.ca.

1

characterize our proposal through axioms that extend the Shapley axioms to our more general environment. The solution concept that we propose can be applied to many interesting games, including intertemporal situations where players meet sequentially.

JEL Classification numbers: C71, D62.

Keywords: externalities, cooperative game theory, Shapley value, linked issues.

1   Introduction

A central question in game theory is how to share the joint surplus among players when they cooperate. For *games in characteristic form* where the worth of a coalition depends only on the composition of this coalition, Shapley (1953) uses an axiomatic approach to characterize the unique value or payoff allocation that satisfies the properties (axioms) of efficiency, linearity, anonymity, and dummy player. This value can also be seen as an operator that assigns an expected marginal contribution to each player in a game with respect to a uniform distribution over the set of all permutations on the set of players. Alternatively, the Shapley value can be obtained as the sum of the dividends that accrue for a player from the various coalitions in which he could participate (Harsanyi, 1959) and through the potential approach proposed by Hart and Mas-Colell (1989).

Even though the Shapley value possesses many desirable properties and has inspired a host of studies, it cannot be applied to situations where externalities are present. In many economic situations, the worth of a coalition of players depends not only on the members of that coalition but also on how the rest of the players are organized. For example, in the context of international trade, the welfare of a trade union depends on whether the outside countries form other trade unions; in an oligopolistic market the profits of a cartel depend not only on the composition of this cartel but also on the organization of other firms in the market. As a natural extension of the games in characteristic form, Thrall and Lucas (1963) introduced *games in partition function form* in which the worth of a coalition is determined by the partition of the remaining players. Using the axiomatic approach, a number of authors have proposed extensions of the Shapley value for games in partition

2

function form. Contributions in this line of research include the works of Myerson (1977), Bolger (1989), Feldman (1996), Albizuri, Arin and Rubio (2005), Macho-Stadler, Pérez-Castrillo and Wettstein (2007), Pham Do and Norde (2007), McQuillin (2009), and Dutta, Ehlers and Kar (2010).[1]

The worth of a coalition in a partition function game depends only on the organization of all players in *this* game. Thus, if different games correspond to different issues under consideration, then it is taken for granted that all issues are independent and thus can be analyzed separately. However, there are interesting economic situations with multiple *linked* issues in that the amount a coalition receives in one issue depends on how the players are organized on all the issues. Put differently, in these environments, there are not only multiple issues but also *externalities across these issues*. Consider, for instance, several firms competing in multiple markets. Cooperation in one market can have an impact on the profits obtained in the other markets either through the cost functions or through the demand functions (due to product complementarity/substitutability). Alternatively, consider countries negotiating both a trade agreement (through, e.g., WTO) and an environmental agreement (e.g., Kyoto Protocol). These two issues, namely trade and environment, are linked through production. For example, the accelerated growth triggered by trade liberalization supported by the WTO is likely to raise CO2 emissions, making it more difficult for the participants in the environmental agreement to comply with their obligations under the Kyoto Protocol.

In situations such as described above, one can no longer consider each issue in isolation in order to determine the value or the payoff allocation. The alternative approach of simply "adding up" the two issues and then computing the value of each player also seems erroneous as it imposes that players be organized in the same way or form the same coalitions on different issues. In this paper we introduce a value that takes into account the externalities that the formation of coalitions on one issue may create on the worth of all the coalitions on the other issues.

We take the axiomatic approach and propose an extension of the Shapley value to games where there are *externalities both within and across issues*. First, we present a

---

[1] Macho-Stadler, Pérez-Castrillo and Wettstein (2006) provided mechanisms that implement a family of extensions of the Shapley value for games in partition function form.

definition of *issue-externality games*, as a natural extension of the partition form games to environments with linked issues.[2] We consider scenarios where forming the grand coalition on the set of all issues jointly is the efficient outcome and the worth must be allocated to all the players.[3] This leads us to include, for convenience, the usual efficiency axiom in the definition of value.[4] Our value concept builds on a reference value for partition function games. That is, we extend some reference value that has been proposed to deal with only within-issue externalities to environments where externalities across issues are also present. We show that the classic axioms of linearity, player anonymity, and dummy player can be easily extended from partition function games to issue-externality games. Also the "strong dummy property", which captures the idea that when dummy players are added to or excluded from a game the remaining players should receive the same payoffs, extends to issue-externality games. In addition, we show that when the above axioms hold for the reference value, our extension of the Shapley value satisfies the additional properties of issue symmetry and dummy issue (which mirror, with respect to issues, the axioms of dummy player and player anonymity), as well as two axioms that capture the way inter-issue externalities are considered: issue-externality anonymity and issue-externality symmetry.

Our main result is that the afore-mentioned axioms characterize our proposed value: If a value for issue-externality games satisfies the axioms of linearity, player anonymity, strong dummy player, issue symmetry, dummy issue, issue-externality anonymity, and

---

[2]Nax (2014) considers a similar class of games that he calls multiple membership games. His approach is, however, different from ours since he focuses on extending the core allocation proposed by Bloch and de Clippel (2010) for combined games, to games with externalities across issues (multiple membership games).

[3]The efficiency axiom on issue-externality games does not require that the formation of the grand coalition on a particular issue maximizes the total value in that issue. It may be the case that forming the grand coalition on an issue is efficient because it maximizes the joint value of all the issues although it does not maximize the value on that issue.

[4]See Maskin (2003) and de Clippel and Serrano (2008) for a discussion of the possible consequences of including externalities on the efficiency of the outcome. Maskin (2003) suggests that in situations in which coalitions generate significant positive externalities, we should not expect that the grand coalition will form. This might be a reason why the Shapley value and the core have not been used in settings with externalities.

issue-externality symmetry, then it can be obtained as an extension (using our procedure) of a value for partition function games that satisfies the axioms of linearity, player anonymity and strong dummy player.

The solution concept that we propose can be applied to many interesting games, including intertemporal situations where players meet sequentially.[5] In these games, each "date" at which players make decisions can be associated to an "issue." Therefore, a multi-stage game corresponds to a game with several issues. When the payoffs that the players may obtain at a stage depend on the decisions taken at previous periods, the game is an issue-externality game. Additionally, our approach provides values even for situations where the set of active players in a certain period may change over time as a function of the coalitions formed by the active players at earlier periods.

This paper is organized as follows. Section 2 presents "issue-externality games" to capture externalities within and across issues. Section 3 introduces our proposed value concept. Section 4 presents the axioms. Section 5 establishes the relationship between the axioms satisfied by the value for partition function games and those satisfied by the proposed value for issue-externality games. The latter section also states our main characterization result and it applies it to one particular value for partition function games. Section 6 illustrates our value concept through two examples and Section 7 concludes. The proofs of all the results are delineated in the Appendix.

2   The Model

In this section, we formulate "issue-externality games" with transferable payoffs that generalize partition function games. We denote by $N = \{1, ..., n\}$ the *set of players*. A *coalition S* is a subset of players, that is, $S \subseteq N$. We denote by P a *partition* (coalition structure) of the set of players N and, for technical convenience, we follow the convention that the empty set $\emptyset$ is in P for every partition P. The *set of all partitions* of N is denoted by $\mathcal{P}$.

In our environment, players can cooperate on several issues. We denote by A the

---

[5] Beja and Gilboa (1990) propose a class of "two-stage games" and characterize all the semivalues in this class of games.

finite set of issues the players are concerned with. Players can form different coalitions and partitions on different issues. Hence, to represent the way in which the players are organized, we need to specify a partition of N for each issue. Let $P^A = (P^a)_{a \in A}$ denote a vector of $|A|$ partitions of the set N, indexed by issues in A and $\mathcal{P}^A$ denote the set of vectors of $|A|$ partitions of N.[6]

An *embedded coalition* is a triplet $(S; a; P^A)$, where S is a coalition, a is an issue, and P^A is a vector of $|A|$ partitions of N such that $S \in P^a$, where P^a is the component (partition) in P^A that corresponds to issue a. An embedded coalition, hence, specifies a coalition S formed on an issue a together with the structures of coalitions formed by all the players on all issues P^A such that coalition S is an element of the partition on issue a. $ECL(N, A)$ is the set of all embedded coalitions for a given set of players N and a set of issues A.

We represent the worth that a group of players can achieve through a real-valued function $v : ECL(N, A) \to \mathbb{R}$ that associates a real number to each embedded coalition. Hence, $v(S; a; P^A)$, with $a \in A$, $S \in P^a$ and $P^A \in \mathcal{P}^A$, is the total utility available for division among members of coalition S in issue a when the players are organized on the issues in A according to the partition vector P^A. We assume that the value function satisfies $v(\emptyset; a; P^A) = 0$ for all $a \in A$ and $P^A \in \mathcal{P}^A$. The *game* (N, A, v) is called an *issue-externality game*. We denote $\mathcal{G}$ as the set of such games.

Example 1 presents a game with externalities across issues with two players, $N = \{1, 2\}$, and two issues, $A = \{a, b\}$.

---

[6] $|\Omega|$ denotes the cardinality of any set Ω.

$\downarrow a$ $\quad$ $b \rightarrow$	$\{1\}, \{2\}$	$\{1,2\}$
$\{1\}, \{2\}$	$\quad\quad 2,2$ $3,1$	$\quad\quad 5$ $2,4$
$\{1,2\}$	$\quad\quad 1,4$ $\boxed{5}$	$\quad\quad 6$ 7

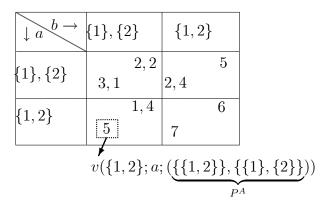$$v(\{1,2\}; a; (\underbrace{\{\{1,2\}\}, \{\{1\}, \{2\}\}}_{P^A}))$$

Table 1: Example 1

We say that the game (N, A, v) has *no externalities within issues* if the worth of a coalition S on any issue a is independent of the way the rest of the players are organized on that issue. Otherwise, the game has externalities within issues. Formally,[7]

Definition 1 *The game (N, A, v) has externalities within issue $a \in A$ if for some $P^{A \backslash a} \in \mathcal{P}^{A \backslash a}$, $P^a, O^a \in \mathcal{P}$, and $S \in P^a \cap O^a$, we have $v\left(S; a; \left(P^a, P^{A \backslash a}\right)\right) \neq v\left(S; a; \left(O^a, P^{A \backslash a}\right)\right).$*

When a game has externalities within issues, the worth of a coalition on issue a depends on the organization of the other players on this issue. In a multi-issue environment, the worth of a coalition S formed on a particular issue a may depend not only on the way the rest of the players are organized on issue a but also on the organization of all players on the other issues. When this happens, we say that the game exhibits externalities across issues. More formally:

Definition 2 *The game (N, A, v) has externalities across issues if for some $a \in A$, $P^a \in \mathcal{P}$, $S \in P^a$, and $P^{A \backslash a}, O^{A \backslash a} \in \mathcal{P}^{A \backslash a}$, we have $v(S; a; (P^a, P^{A \backslash a}) \neq v(S; a; (P^a, O^{A \backslash a}).$*

Example 1 represents a game with externalities across issues. For instance, the stand-alone coalition of player 1 in issue a obtains a payoff of 3 if the partition in issue b is $\{\{1\}, \{2\}\}$ while it obtains a payoff of 2 if the grand coalition forms in issue b.

Issues are said to be linked if there are externalities across them. Linked issues cannot be analyzed separately and must be included in the same game.

---

[7]For notational simplicity, we use $A \backslash a$, $P^{A \backslash a}$ and $\mathcal{P}^{A \backslash a}$ instead of $A \backslash \{a\}$, $P^{A \backslash \{a\}}$ and $\mathcal{P}^{A \backslash \{a\}}$, and similarly for other sets throughout this paper.

The objective of this paper is to propose a way to share the surplus generated when players cooperate in an issue-externality game. We formalize the proposed division through a value. A *value* Φ specifies the payoff to players in N for any game (N, A, v), that is, a value Φ is a function from the set of games $\mathcal{G}$ to $\mathbb{R}^{|N|}$ such that $\sum_{i \in N} \Phi_i (N, A, v) = \sum_{a \in A} v(N; a; N^{|A|})$. Note that we incorporate the *efficiency axiom* into the definition of the value. We have in mind those economic environments where efficiency requires that all players cooperate on all the issues, that is, $\sum_{a \in A} v(N; a; N^{|A|}) \geq \sum_{a \in A} \sum_{S \in P^a} v(S; a; P^A)$ for every vector of partitions P^A.

3   A Value for Games with Externalities within and across Issues

The class of issue-externality games $\mathcal{G}$ that we consider is quite large, encompassing partition function games as a special class. Recall that a *partition function game* is a pair (N, u), where u is a function that associates a real number with each pair (S, P), with $S \in P$, $P \in \mathcal{P}$. That is, $u : ECL(N) \to \mathbb{R}$ where $ECL(N) \equiv \{(S, P) \mid S \in P, P \in \mathcal{P}\}$. Thus, a partition function game represents a situation where players are concerned with a single issue, although the representation abstracts away from the identity of the issue.[8] A natural generalization of this property to issue-externality games is that a solution concept does not depend on how issues are labelled or identified, which we shall refer to as "issue symmetry," reminiscent of player anonymity. Once this axiom is invoked, we can represent partition function games as special cases of issue-externality games. More precisely, let PFG be the set of partition function games and denote by α a particular issue. Then PFG can be viewed as a collection of issue-externality games with a single issue, that is, $A = \{\alpha\}$, by defining $v(S; \alpha; P) \equiv u(S, P)$ for every $(S; \alpha; P) \in ECL(N, \alpha)$. Therefore, the value Φ defined for $\mathcal{G}$ also constitutes a value for PFG. Given that PFG encompasses the class of characteristic function games as a special case, Φ defined for $\mathcal{G}$ immediately provides a value for games in characteristic function form.

---

[8]Consequently, all solution concepts, including values, for this game depend only on the information embedded in (N, u), not on the identity of the issue under consideration.

The Shapley value is one of the most important value solutions for games in characteristic form. One natural way to define a value concept for PFG is to extend the Shapley value to PFG. There have been several such extensions in the literature. In the same vein, we propose value concepts for $\mathcal{G}$ by extending values defined for PFG to our broader class of games $\mathcal{G}$.

We consider a particular value ϕ^* defined for PFG. We build a value concept for $\mathcal{G}$ that treats externalities across issues (i.e., inter-issue externalities) in a "similar" way as ϕ^* treats externalities within issues (i.e., intra-issue externalities). To this end, we view the contribution of each player $i \in N$ as the sum of the contributions of $|A|$ "delegates" of player i, one delegate per issue. That is, we disentangle the $|A|$ contributions of player i as if they would come from $|A|$ players. Then, we define a game with a single issue and $|A||N|$ "delegates." Finally, we apply the value ϕ^* to this new game.

Formally, we denote by $N(a)$ the *replica* of the set N pertaining to issue a. A typical player, coalition, and partition with respect to issue a shall be identified as $i(a)$, $S(a)$, and $P(a)$, respectively. Also, we use $N(A)$ to denote the union of all replicas of N, that is, $N(A) = \underset{a \in A}{\cup} N(a)$; hence, $N(A)$ has $|A||N|$ players. For example, if $N = \{1, 2, 3\}$ and $A = \{a, b\}$, then $N(a) = \{1(a), 2(a), 3(a)\}$, $N(b) = \{1(b), 2(b), 3(b)\}$, and $N(A) = \{1(a), 2(a), 3(a), 1(b), 2(b), 3(b)\}$. For a coalition $T \subseteq N(A)$, we denote $T(a) \equiv T \cap N(a)$. Similarly, the partition obtained by the intersection of $N(a)$ with the elements of a partition Q of $N(A)$ is denoted by $Q(a)$, that is, $Q(a) = \{T(a) \mid T \in Q\}$. $Q(a)$ is the partition of $N(a)$ as induced by Q. In our previous example, if $T = \{2(a), 2(b), 3(b)\}$, then $T(b) = \{2(b), 3(b)\}$ and if $Q = \{\{2(a), 2(b), 3(b)\}, \{1(a), 3(a), 1(b)\}\}$, then $Q(b) = \{\{2(b), 3(b)\}, \{1(b)\}\}$. Finally, for $T \subseteq N(A)$, $\widetilde{T}(a) \equiv \{i \in N \mid i(a) \in T(a)\}$ is the set of players whose $a-$replicas are in T, and for each $a \in A$, $\widetilde{Q}(a) \equiv \{\widetilde{T}(a) \mid T \in Q\}$ is the partition of N on issue a as induced by Q, for every partition Q of $N(A)$.

Definition 3 *Given a game (N, A, v), we define the partition function game $(N(A), \hat{v})$ as follows:*

$$\hat{v}(T, Q) \equiv \sum_{a \in A} v\left(\widetilde{T}(a); a; \left(\widetilde{Q}(b)\right)_{b \in A}\right) \tag{1}$$

for any $(T, Q) \in ECL(N(A))$, that is, for any partition Q of $N(A)$ and any coalition $T \in Q$.

We can think of "$\hat{\ }$" as an operator that transforms a function from $ECL(N,A)$ to $\mathbb{R}$ to a function from $ECL(N(A))$ to $\mathbb{R}$. Such a transformation turns a game with multiple linked issues to a game with a single issue where the value of any coalition $T \subseteq N(A)$ can depend on the organization Q of all the agents.

Once (N, a, v) is transformed to $(N(A), \hat{v})$, we can apply the value ϕ^* to this game and $\phi_k^*(N(A), \hat{v})$ is the payoff for any player $k \in N(A)$. Notice that

$$\sum_{k \in N(A)} \phi_k^*(N(A), \hat{v}) = \hat{v}(N(A), \{N(A)\}) = \sum_{a \in A} v(\widetilde{N}(a); a; (\widetilde{N}(b))_{b \in A}) = \sum_{a \in A} v(N; a; N^{|A|}).$$

(2)

Then, we consider the sharing rule Φ^* for (N, A, v) obtained by summing, for every player $i \in N$, the payoff that all his replicas (delegates) $i(a) \in N(a)$ obtain. That is,

Definition 4 *Given a value ϕ^* for PFG, we define the value Φ^* for the class of games $\mathcal{G}$ as:*

$$\Phi_i^*(N, A, v) \equiv \sum_{a \in A} \phi_{i(a)}^*(N(A), \hat{v})$$

for any game $(N, A, v) \in \mathcal{G}$.

It is immediate from (2) that the value Φ^* is efficient as long as ϕ^* is efficient. We will consider values ϕ^* for *PFG* that extend the original Shapley value and we will examine the properties or axioms that characterize the definition of Φ^* as given above. In the next section, we propose a list of reasonable axioms to impose on a value.

4 Axioms

We start the section with the axioms underlying the construction of the Shapley value for games in characteristic form. We adapt these axioms to the class of issue-externality games $\mathcal{G}$. We first define the operations of *addition* and *multiplication by a scalar*, and the notions of *permutation of games* and *dummy player*.

Definition 5 *The addition of two games (N, A, v) and (N, A, v') is defined as the game $(N, A, v + v')$ where $(v + v')(S; a; P^A) = v(S; a; P^A) + v'(S; a; P^A)$ for all $(S; a; P^A) \in ECL(N, A)$. Similarly, given a game (N, A, v) and a scalar $\lambda \in \mathbb{R}$, the game $(N, A, \lambda v)$ is defined by $(\lambda v)(S; a; P^A) = \lambda v(S; a; P^A)$ for all $(S; a; P^A) \in ECL(N, A)$.*

Let $\sigma_N : N \to N$ be a permutation of N. For $S \subseteq N$, let $\sigma_N(S) = \{\sigma(i) \mid i \in S\}$ and for partition P^a on issue $a \in A$, let $\sigma_N(P^a) = \{\sigma(S) \mid S \in P^a)$. Furthermore, for any payoff vector $x \in \Re^{|N|}$, $\sigma_N x$ is the payoff vector such that $(\sigma_N x)_i = x_{\sigma_N^{-1}(i)}$ for all $i \in N$.

Definition 6 *For any permutation σ_N of N, the σ_N-permutation of the game (N, A, v), denoted by $(N, A, \sigma_N v)$, is defined by $(\sigma_N v)(S; a; P^A) = v(\sigma_N^{-1}(S); a; \sigma_N^{-1}(P^A))$ for all $(S; a; P^A) \in ECL(N, A)$, where $\sigma_N^{-1}(P^A) = (\sigma_N^{-1}(P^a))_{a \in A}$.*

Definition 7 *Player $j \in N$ is a dummy player in the game (N, A, v) if for any $(S; a; P^A) \in ECL(N, A)$ it is the case that $v(S; a; P^A) = v(S'; a; O^A)$ for any embedded coalition $(S'; a; O^A)$ that can be deduced from $(S; a; P^A)$ by solely changing the affiliation of player j in some issues.*

Hence, a dummy player j has no effect in the game: in any issue a (i) he alone receives zero for any organization of the other players; (ii) he has no effect on the worth of any coalition S; (iii) if player j is not a member of S, changing the organization of players outside S in issue a by moving player j around will not affect the worth of S, and (iv) changing the affiliation of player j in any issue other than a does not change the worth of any coalition formed on issue a.

We adapt the three original Shapley (1953) value axioms to our environment:

1. *Linearity*: A value Φ satisfies the linearity axiom if:

    1.1. $\Phi(N, A, v + v') = \Phi(N, A, v) + \Phi(N, A, v')$ for any two games (N, A, v) and (N, A, v') in $\mathcal{G}$.

    1.2. $\Phi(N, A, \lambda v) = \lambda \Phi(N, A, v)$ for any $\lambda \in \mathbb{R}$ and for any game (N, A, v) in $\mathcal{G}$.[9]

One of the implications of linearity is that when a group of players face two issue-externality games with the same set of issues, each player's payoff does not depend on whether they consider the two games separately or they simply analyze a "combined"

---

[9]In games without any type of externalities, additivity (part 1.1), dummy and anonymity axioms imply the property on the multiplication for a scalar (part 1.2). As shown in Macho-Stadler, Pérez-Castrillo and Wettstein (2007), in games with externalities within an issue there are values that are additive but not linear, that is, they satisfy part 1.1 (and the other basic axioms) but not part 1.2.

game. Following Myerson (1991, p. 437-438), an alternative interpretation is that players are uncertain of the issue-externality games they are going to play and the linearity axiom asserts that players' expected payoffs are the same whether they analyze the game before or after the uncertainty is resolved.

2. *Player anonymity*: A value Φ satisfies the player anonymity axiom if for any game (N, A, v) in $\mathcal{G}$ and for any permutation $\sigma_N : N \to N$, $\Phi(N, A, \sigma_N v) = \sigma_N \Phi(N, A, v)$.

This axiom can be replaced by the following stronger version. Let M be such that $|M| = |N|$ and let σ_{NM} be a bijection from the set N to the set M. $(M, A, \sigma_{NM} v)$ is a game defined by $\sigma_{NM} v(S; a; P^A) = v(\sigma_{NM}^{-1}(S); a; \sigma_{NM}^{-1}(P^A))$ for all $(S; a; P^A) \in ECL(M, A)$.

2'. *Player anonymity*: A value Φ satisfies the player anonymity axiom if for any game (N, A, v) in $\mathcal{G}$ and for any bijection $\sigma_{NM} : N \to M$, $\Phi(M, A, \sigma_{NM} v) = \sigma_{NM} \Phi(N, A, v)$.

3. *Dummy player*: A value Φ satisfies the dummy player axiom if, for any game (N, A, v) in $\mathcal{G}$, $\Phi_j(N, A, v) = 0$ if player j is a dummy player in the game (N, A, v).

Next, we consider an axiom that reflects ideas akin to player anonymity but with respect to the issue: the name of the issue should not influence the payoffs players obtain in a game. We shall refer to this axiom as *issue symmetry*.[10]

4 *Issue symmetry*: A value Φ satisfies the issue symmetry axiom if, for any two games $(N, \{a\}, v)$ and $(N, \{b\}, v')$, if $v'(S; b; P) = v(S; a; P)$ for any $P \in \mathcal{P}$ and $S \in P$, then $\Phi(N, \{a\}, v) = \Phi(N, \{b\}, v')$.

Thus, issue symmetry states that in a game with a single issue, renaming the issue alone does not change the value, that is, Φ depends on the game $(N, \{a\}, v)$ through v.[11] Once the axiom of issue symmetry is invoked, *PFG* can be viewed as a special class of

---

[10]For games in characteristic form, symmetry and anonymity are synonymous. In our environment we shall use anonymity to refer to properties for players and symmetry for issues.

[11]In fact, this axiom can be replaced by a stronger version. Let A and B be two sets of issues such that $|A| = |B|$ and let μ_{AB} be a bijection from the set A to the set B. Then the $\mu_{AB}-renaming$ of

single-issue games in $\mathcal{G}$; moreover, by restricting axioms 1-3 to these single-issue games, we recover these axioms for PFG as well as characteristic function games, which constitute a special case of PFG.

Axioms 1-3 characterize a unique value in characteristic function form games (Shapley, 1953). Let (N, w) be a game in characteristic function form, where $w : 2^N \to \mathbb{R}$ is the characteristic function. The Shapley value φ is then given by

$$\varphi_i(w) = \sum_{S \subseteq N} \beta_i(S) w(S) = \sum_{\substack{S \subseteq N \\ S \ni i}} \beta_i(S) MC_i(S) \text{ for all } i \in N, \tag{3}$$

where $MC_i(S)$ is the marginal contribution of player $i \in S$ to coalition S, that is, $MC_i(S) \equiv w(S) - w(S \backslash \{i\})$ and

$$\beta_i(S) = \begin{cases} \frac{(|S|-1)!(n-|S|)!}{n!} & \text{for all } S \subseteq N \text{ such that } i \in S \\ -\frac{|S|!(n-|S|-1)!}{n!} & \text{for all } S \subseteq N \text{ such that } i \in N \backslash S. \end{cases}$$

The three basic Shapley value axioms are compatible with many values defined for PFG and they leave even more leeway regarding values for issue-externality games. We now discuss some other axioms that allow us to give more structure to values in this large class of games.

First, we introduce a stronger dummy axiom that is implied by the previous three axioms in characteristic function games. Hence, it is satisfied by the Shapley value defined for this class of games but is a more demanding property than the dummy axiom when we enlarge the domain of games under consideration.

3' *Strong dummy player*: A value Φ satisfies the strong dummy player axiom if, for any game (N, A, v), $\Phi_i(N \backslash j, A, v_{-j}) = \Phi_i(N, A, v)$ for all $i \in N \backslash j$ if j is a dummy player in game (N, A, v), where $v_{-j}(S; a; P^A) \equiv v(S \cup j; a; P^A_{+j})$ for all $(S; a; P^A) \in ECL(N \backslash j, A)$, and P^A_{+j}, with $S \cup j \in P^a_{+j}$, is similar to P^A except that player j is affiliated with some coalition in P^b for any issue b.

_____

issues in game (N, A, v) denoted by $(N, \mu_{AB}A, \mu_{AB}v) = (N, B, \mu_{AB}v)$ is defined by $(\mu_{AB}v)(S; b; P^B) = v(S; \mu_{AB}^{-1}(b); \mu_{AB}^{-1} P^B)$ for all $(S; b; P^B) \in ECL(N, B)$, where $\mu_{AB}^{-1} P^B$ applies the bijection μ_{AB}^{-1} to the components of the vector of partitions P^B. A value Φ satisfies the (stronger version of) issue symmetry axiom if $\Phi(N, \mu_{AB}A, \mu_{AB}v) = \Phi(N, A, v)$ for all $\mu_{AB}-$renaming of issues in (N, A, v).

The strong dummy axiom states that when a dummy player is added or removed from a game, the payoffs of the remaining players do not change. This property is not satisfied by all the proposals for games with externalities within issues. The values proposed by Myerson (1977), Feldman (1996), Macho-Stadler *et al.* (2007), Pham Do and Norde (2007), de Clippel and Serrano (2008), and McQuillin (2009) satisfy the strong dummy player axiom;[12] in contrast, those of Bolger (1989) and Albizuri *et al.* (2005) do not.[13]

We now consider an axiom that reflects ideas akin to dummy player axioms but with respect to issues. The elimination of an issue that generates neither worth nor externalities should not change players' payoffs. We shall refer to this axiom as *dummy issue*. To formulate this axiom, we need to define the notion of dummy issue. Issue $d \in A$ is a *dummy issue* in the game (N, A, v) if $v(S; d; P^A) = 0$ for all $P^A \in \mathcal{P}^A$ and all $S \in P^a$ and $v(S; a; (P^d, P^{A \setminus d})) = v(S; a; (O^d, P^{A \setminus d}))$ for all $a \neq d$, $P^d, O^d \in \mathcal{P}$, $P^{A \setminus d} \in \mathcal{P}^{A \setminus d}$, and $S \in P^a$. Hence, no coalition can obtain any worth in a dummy issue, and the organization of the players in a dummy issue has no effect on the worth of any coalition in any other issue.

5 *Dummy issue*: A value Φ satisfies the dummy issue axiom if for any game (N, A, v) in $\mathcal{G}$, $\Phi(N, A \setminus d, v_{-d}) = \Phi(N, A, v)$ where d is a dummy issue in (N, A, v) and $v_{-d}(S; a; P^{A \setminus d}) \equiv v(S; a; (P^d, P^{A \setminus d}))$ for any $(S; a; P^{A \setminus d}) \in ECL(N, A \setminus d)$ and any partition P^d of N.

Finally, we introduce two axioms that capture how cross-issue externalities are dealt with. The first is an axiom of anonymity on externalities across issues; it ensures that externalities across issues are treated in such a way that a player's payoff does not depend on the identities of the players exerting the *cross-issue externalities*; rather, it depends only on the extent of these externalities.

---

[12]Among the values based on the "average approach" defined in Macho-Stadler *et al.* (2007), some satisfy the strong dummy player axiom while others do not. To illustrate this, note that all the values just mentioned but Myerson's are in the family of the average approach. To show that there are some values that do not satisfy the axiom let us define the "value alternate", which consists of applying a value in the class of average values (for example, the value proposed by Macho-Stadler *et al.*, 2007) to games with an odd number of players and another one (for example the one by de Clippel and Serrano, 2008) to games with an even number of players.

[13]These two values are not in the family of values that satisfy the average approach.

6 *Issue-externality anonymity*: A value Φ satisfies the issue-externality anonymity axiom if for any game (N, A, v) in $\mathcal{G}$ and any $i \in N$, it is the case that $\Phi_i(N, A, v_{\sigma_N}) = \Phi_i(N, A, v)$ for all permutations σ_N that satisfy $\sigma_N(i) = i$, where $v_{\sigma_N}(S; a; P^A) \equiv v(S; a; (P^a, O^{A \setminus a}))$ for all $(S; a; P^A) \in ECL(N, A)$, and for all $b \in A \setminus a$ either $O^b = \sigma_N P^b$ or $O^b = P^b$.

In the presence of cross-issue externality, a coalition's worth in issue a also depends on P^b, where $b \in A \setminus a$. When the names or roles of two players in P^b for some $b \in A \setminus a$ are interchanged, how does this affect the payoff of a player i (whose name stays the same across different issues) through the channel of cross-issue externalities? The issue-externality anonymity stipulates that player i's payoff should not change, i.e., player i's payoff does not depend on the identities of the players who exert cross-issue externalities. The issue-externality anonymity axiom differs from player anonymity axiom as the latter axiom stipulates that players' payoffs do not depend on the specific names they have in the entire game.

The second axiom pertaining to cross-issue externalities is an axiom of symmetry among issues where externalities are created. A player's payoff should not depend on the name of the issue from which externalities originate. More precisely, consider a set of players M whose only role in the game is to induce externalities on others through their organization on one of the issues. Our issue-externality symmetry axiom then says that players' payoffs depend only on the extent of these externalities not on the issue from which players in M exert their externalities. To formulate this axiom, we first define the concept of "externality players on a single issue."

Definition 8 *Let $a \in A$. $M \subset N$ is a set of a-externality players if $v(S; b; P^A) = v(T; b; Q^A)$ for all $(S; b; P^A), (T; b; Q^A) \in ECL(N, A)$ such that[14]*

(i) $Q^c \cap (N \setminus M) = P^c \cap (N \setminus M)$ for all $c \in A$,

(ii) $Q^a \cap M = P^a \cap M$, and

(iii) $S \setminus M = T \setminus M$.

---

[14] For $P \in \mathcal{P}$ and $S \subseteq N$, $P \cap S$ is the partition on the set S obtained from P by removing the players in $N \setminus S$.

Thus, players in M affect any coalition's worth *only* through their organization on issue a, and the externalities M generates do not interact with those by $N \backslash M$. Moreover, no player in M can add to the worth of any coalition. The next axiom says that if we "transfer" the externalities exerted by M from issue a to another issue b, players' payoffs should not change. To state the axiom, we use the following definition:

Definition 9 *Given a game* (N, A, v) *and a set* M *of a-externality players, the game* $v_{M,ab}$ *is the transformation of game v by moving the externalities induced by M from issue a to issue b, that is,* $v_{M,ab}(S; c; P^A) \equiv v(T; c; O^A)$ *for all* $(S; c; P^A) \in ECL(N, A)$, *where* $O^{A \backslash \{a\}} = P^{A \backslash \{a\}}$, $O^a \cap (N \backslash M) = P^a \cap (N \backslash M)$, $O^a \cap M = P^b \cap M$, *and* $S \backslash M = T \backslash M$.[15]

   7 *Issue-externality symmetry*: A value Φ satisfies the issue-externality symmetry axiom if for any game (N, A, v) in $\mathcal{G}$, $\Phi_i(N, A, v_{M,ab}) = \Phi_i(N, A, v)$ for all $i \in N$, for any $a, b \in A$, and for any set M of $a-$externality players.

5   Characterization of the Value

In Section 3 we defined a value Φ^* for the class of games $\mathcal{G}$ by extending a reference value ϕ^* for PFG. We now relate the properties of these two values. Note that axioms for PFG can be obtained by restricting axioms defined in Section 3 to single-issue games in $\mathcal{G}$. First we show that the value Φ^* satisfies a series of properties related to those satisfied by the reference value ϕ^*. Proposition 1 states that the classic axioms of linearity, player anonymity, dummy player, and strong dummy player can be extended from ϕ^* to Φ^*.

Proposition 1 *(i) If ϕ^* satisfies the linearity axiom in PFG, then Φ^* satisfies the linearity axiom in $\mathcal{G}$.*
(ii) If ϕ^* satisfies the player anonymity axiom in PFG, then Φ^* satisfies the player anonymity axiom in $\mathcal{G}$.
(iii) If ϕ^* satisfies the dummy player axiom in PFG, then Φ^* satisfies the dummy player

---

[15]Note that for each $c \neq a$, if $S \in P^c$, then $S \in O^c$. However, it is possible that $S \in P^a$ and $S \notin O^a$. In the original game, M exerts externalities through $O^a \cap M$ while in the transformed game, M exerts externalities through $P^b \cap M$.

axiom in $\mathcal{G}$.

(iv) If ϕ^* satisfies the strong dummy player in PFG, then Φ^* satisfies the strong dummy player axiom in $\mathcal{G}$.

Proposition 2 shows that when the reference value ϕ^* satisfies the strong dummy player axiom, then the properties of dummy issue and issue symmetry, which extend to issues the ideas of dummy player and player symmetry, are satisfied by the value Φ^*.

Proposition 2 *(i) Φ^* satisfies the issue symmetry axiom in $\mathcal{G}$.*
(ii) If ϕ^* satisfies the strong dummy player axiom in PFG, then Φ^* satisfies the dummy issue axiom in $\mathcal{G}$.

Finally, Proposition 3 states that the two axioms that capture the way inter-issue externalities are considered are also satisfied given the construction of the value Φ^*, as long as the reference value ϕ^* satisfies the classic axioms of linearity and player anonymity.

Proposition 3 *(i) If ϕ^* satisfies linearity and player anonymity in PFG, then Φ^* satisfies the issue-externality anonymity axiom in $\mathcal{G}$.*
(ii) If ϕ^* satisfies player anonymity in PFG, then Φ^* satisfies the issue-externality symmetry axiom in $\mathcal{G}$.

Propositions 1 to 3 show that if we construct a value Φ^* for the class of issue-externality games by the procedure proposed in Definition 3, starting with a value ϕ^* for *PFG* that satisfies the axioms of linearity, player anonymity, and strong dummy player, then the seven axioms that we formulated in Section 4 hold for the value Φ^*. Our main result shows that the converse is also true. That is, if a value Φ for $\mathcal{G}$ satisfies the seven axioms, then it can be constructed through the proposed procedure, using a reference value ϕ for *PFG* that satisfies the axioms of linearity, player anonymity, and strong dummy player.

Theorem 1 *A value Φ in $\mathcal{G}$ satisfies the axioms of linearity, player anonymity, strong dummy player, issue symmetry, dummy issue, issue-externality anonymity, and issue-*

externality symmetry[16]*, if and only if there exists a value ϕ in PFG that satisfies linearity, player anonymity, and strong dummy player such that*

$$\Phi_i(N, A, v) = \sum_{a \in A} \phi_{i(a)}(N(A), \hat{v})$$

for any game (N, A, v) and any player $i \in N$, where

$$\hat{v}(T, Q) \equiv \sum_{a \in A} v(\widetilde{T}(a); a; (\widetilde{Q}(b))_{b \in A})$$

for any partition Q of $N(A)$ and any coalition $T \in Q$.

We need only to prove the necessity part of Theorem 1. The detailed proof is relegated the appendix and its structure is outlined as follows: We first use additivity to "decompose" an issue-externalty game (N, A, v) into a collection of games $\{(N, A, v_a)\}_{a \in A}$, where in each (N, A, v_a) any coalition can only obtain non-zero worth in issue a. For each such game (N, A, v_a), we make use of strong dummy player axiom by adding replicas of each player, one per issue, to the game. We then appeal to issue-externality anonymity to "encode" the cross-issue externalities in any issue $b \in A$ so that they are exerted by the $b-$replicas of the players. Next, by using issue-externality symmetry we encode all the externalities to issue a, knowing that $b-$replicas of the players are the set of $b-$externality players. After eliminating the dummy issues, we end up with a partition function game. The structure of the proof makes it easy to see the role each axiom plays.

Another property that our value concept satisfies is *independence*. To formulate this axiom, we first define the union of two issue-externality games. The union of (N, A, v)

---

[16]Note that issue-externality anonymity, issue-externality symmetry, and dummy issues are the key new axioms that are specific to our issue-externality games and they become superfluous in games with a single issue (corresponding to partition function games). As we shall see, these three axioms enable us to "transform" an issue-externality game to a partition-function game, based on which our value concept is defined. To see, for example, that issue-externality symmetry axiom is independent from the rest of axioms, consider a simple example with three-player (1,2, and 3) and two issues (a and b) where $\{23\}$ is the set of $b-$externality players: When 2 and 3 belong to the same coalition in P^b, $v(S; a; P^A) = 1$ if $1 \in S$ and 0 if $1 \notin S$; $v(S; b; P^A) = 0$ for all P^A. Without issue-externality symmetry, the game cannot be reduced to a partition-function game and there are multitude of values compatible with the rest of the axioms. With issue-externality symmetry, the value can be constructed from an auxiliary partition function game.

and (N, B, w) such that $A \cap B = \varnothing$ is defined as a game $(N, A \cup B, v \cup w)$ where $(v \cup w)(S; c; P^{A \cup B}) = v(S; c; P^A)$ if $c \in A$ and $(v \cup w)(S; c; P^{A \cup B}) = w(S; c; P^B)$ if $c \in B$. The axiom of independence states that players' payoffs are the same whether we analyze two games separately or the union of the two games.

8 *Independence*: $\Phi(N, A \cup B, v \cup w) = \Phi(N, A, v) + \Phi(N, B, w)$ for all games (N, A, v) and (N, B, w) such that $A \cap B = \varnothing$.

We notice that the property of independence is an axiom related to linearity, as it stipulates how the value should treat combinations of games with the same set of players.

Proposition 4 *If a value Φ satisfies linearity and dummy issue, then it satisfies the independence axiom.*

It is easy to verify that the independence axiom implies the dummy issue axiom. Therefore,

Corollary 1 *Under the linearity axiom, a value Φ satisfies the dummy issue axiom if and only if it satisfies the independence axiom.*

However, the axiom of independence does not imply the axiom of linearity, even if we assume efficiency, player anonymity, and dummy player in addition to dummy issue. Independence relates the values of two games with different sets of issues through an "expanded" game that contains both sets of issues. On the other hand, linearity relates the payoffs in two games with the same set of issues through a "combined" game where the worth of each embedded coalition in the two games is added. The dummy issue axiom makes it possible to use the property of linearity to delineate independence; hence, Proposition 4 holds. Nevertheless, linearity is a strong property for the class of games with fixed set of issues that is not implied by the other axioms, even if we include independence. The Shapley value for games with one issue is not characterized if we substitute linearity by independence.

To illustrate the above discussion, consider any value ϕ for PFG satisfying efficiency, player anonymity and strong dummy player but not linearity.[17] Define now

$$\Phi_i(N, A, v) \equiv \sum_{a \in A} \phi_{i(a)}(N(A), \hat{v})$$

as in Definition 4. The value for issue-externality games Φ satisfies efficiency, player anonymity, strong dummy player, dummy issue and independence. However, it does not satisfy the linearity axiom.

Theorem 1 characterizes a method to construct a value for the class of issue-externality games based on any value for PFG that satisfies linearity, player anonymity, and strong dummy player. One consequence of the characterization result is that the seven axioms mentioned in the Theorem cannot determine a unique value but a class of values. The reason is that the axioms do not allow selecting a unique way to deal with intra-issue externalities because they do not state properties indicating how intra-issue externalities should be rewarded or punished. However, uniqueness of a value is important for applications. We now show that, form Theorem 1, it is possible to characterize a particular value for issue-externality games once we extend to this class of games the properties underlying the selected value for PFG. Indeed, once we introduce axioms stating how to treat intra-issue externalities, Theorem 1 can be invoked to characterize a unique value for the class of issue-externality games.

As an illustration, we present the previous procedure for the value identified in Macho-Stadler *et al.* (2007):

$$\phi_i^{MPW}(M, u) = \sum_{(S,P) \in ECL(M)} \frac{\Pi_{T \in P \backslash S}(|T| - 1)!}{(|M| - |S|)!} \beta_i(S) u(S, P) \text{ for all } i \in M.$$

---

[17]For example, denote ϕ_1 the value proposed by Macho-Stadler et al. (2007) and ϕ_2 the one proposed by de Clippel and Serrano (2008). Both satisfy efficiency, linearity, player anonymity, and strong dummy player. Consider the value ϕ defined as follows:

$$\phi(N, v) = \phi_1(N, v) \text{ if } v(N) \leq 5$$
$$\phi(N, v) = \phi_2(N, v) \text{ if } v(N) > 5.$$

It is immediate that the value ϕ satisfies efficiency, player anonymity, and strong dummy player, but it does not satisfy linearity.

The value ϕ^{MPW} is characterized in the class of *PFG* by the axioms of linearity, strong player anonymity,[18] dummy player, and similar influence. We now extend in a straightforward way the axioms of strong player anonymity and similar influence to issue-externality games. Both axioms refer to intra-issue externalities.

The first axiom strengthens player anonymity by imposing, in addition to symmetric treatment of individual players, the symmetric treatment of "externalities" generated by players. It requires that the payoff of a player should not change after permutations in the set of players $N\backslash S$ in issue a, for any embedded coalition (S, a, P). Formally, given an embedded coalition $\left(S, a, P^A\right)$, we denote by $\sigma_{S,a,P^a} P^a$ a new partition such that $S \in \sigma_{S,a,P^a} P^a$, and the other coalitions in issue a result from a permutation of the set $N\backslash S$ applied to $P^a\backslash S$. That is, after the permutation $\sigma_{S,a,P^a} P^a$, the partitions for the issues different from a remain unchanged and, in issue a, only the players outside S are reorganized in sets whose size distribution is the same as in $P^a\backslash S$. Given the permutation $\sigma_{S,a,P^a} P^a$, the permutation of the game v denoted by $\sigma_{S,a,P^a} v$ is defined by $\sigma_{S,a,P^a} v\left(S, a, P^A\right) \equiv v\left(S, a, \left(P^{A\backslash a}, \sigma_{S,a,P^a} P^a\right)\right)$, $\sigma_{S,a,P^a} v\left(S, a, \left(P^{A\backslash a}, \sigma_{S,a,P^a} P^a\right)\right) \equiv v\left(S, a, P^A\right)$, and $\sigma_{S,a,P^a} v\left(R, b, Q^A\right) \equiv v\left(R, b, Q^A\right)$ for all $v\left(R, b, Q^A\right) \in ECL(N, A, v)\backslash \left\{\left(S, a, P^A\right), \left(P^{A\backslash a}, \sigma_{S,a,P^a} P^a\right)\right\}$.

9 *Strong player anonymity*: A value Φ satisfies the strong player anonymity axiom if it satisfies player anonymity and for any game (N, A, v) in $\mathcal{G}$, $\Phi\left(N, A, \sigma_{S,a,P} v\right) = \Phi\left(N, A, v\right)$ for any $\left(S, a, P^A\right) \in ECL(N, A, v)$.[19]

Finally, the similar influence axiom states that if the only difference between two games is that a pair of players generates an externality within an issue in the first game when they are together whereas they generate a similar externality in the same issue in the second game when they are separated, then these two players should obtain the

---

[18]In Macho-Stadler *et al.* (2007), the "strong player anonymity" axiom was called the "strong symmetry" axiom.

[19]In Macho-Stadler *et al.* (2007), it is proven that the "strong player anonymity" axiom, together with linearity and dummy player, leads to a natural method of constructing a solution, that is called the average approach: each coalition is associated a worth that is some average of what the coalition can obtain in the different scenarios, and then it allocates to each player her Shapley value in this average game.

same payoffs in both games. Formally, we say that a pair of different players i and j has *similar influence* in games (N, A, v) and (N, A, v') if $v(T, b, Q^A) = v'(T, b, Q^A)$ for all $(T, b, Q^A) \in ECL \backslash \{(S, a, P^A), (S, a, P^{A\prime})\}$, $v(S, a, P^A) = v'(S, a, P^{A\prime})$, and $v(S, a, P^{A\prime}) = v'(S, a, P^A)$, where the only difference between the vectors of partitions P^A and $P^{A\prime}$ is that $\{i\}, \{j\} \in P^a \backslash S$ while $\{i, j\} \in P^{a\prime} \backslash S$.

10. *Similar influence*: A value Φ satisfies the similar influence axiom if for any two games (N, A, v) and (N, A, v') and for any pair of players $\{i, j\}$ that has similar influence in those games, we have $\Phi_i(N, A, v) = \Phi_i(N, A, v')$ and $\Phi_j(N, A, v) = \Phi_j(N, A, v')$.

Proposition 5 characterizes the value for the class of games with externalities within and across issues that satisfies the strong player anonymity and similar influence axioms. The proof of the proposition follows from our Theorem 1 and Theorem 2 in Macho-Stadler *et al.* (2007). We also use Proposition 2 in the latter, where it is shown that the strong dummy player axiom is implied by dummy player together with the other axioms.

Proposition 5 *A value Φ in $\mathcal{G}$ satisfies the axioms of linearity, strong player anonymity, dummy player, similar influence, issue symmetry, dummy issue, issue-externality anonymity, and issue-externality symmetry, if and only if*

$$\Phi_i(N, A, v) = \Phi_i^{MPW}(N, A, v) \equiv \sum_{a \in A} \phi_{i(a)}^{MPW}(N(A), \hat{v})$$

for any game (N, A, v) and any player $i \in N$.

6 Two Examples

Example 1. Consider a duopoly competing in two markets, a and b (see Bulow *et al.*, 1985, and Nax, 2014). Suppose that the two firms have the option to merge their operations in one or both markets. The firms' profits depend on the market structures in both markets according to the payoffs given in Figure 1 (see Section 2). Then, we cannot analyze the two markets separately because they are linked; that is, there are externalities across the two markets. It is also inappropriate to add up the worth in the two markets. We use the $\phi^{MPW}(M, u)$ and Φ^{MPW}.

In our example, $M = N(A) = \{1(a), 2(a), 1(b), 2(b)\}$ and $\hat{v}$ is determined by equation (1). A straightforward computation yields

$$\phi_{1(a)}^{MPW} = 3.25, \phi_{2(a)}^{MPW} = 3.25, \phi_{1(b)}^{MPW} = 2.50, \phi_{2(b)}^{MPW} = 4.00,$$

implying that in this game, the two firms share total profits from merging in both markets as follows:

$$\Phi_1^{MPW} = 5.75, \Phi_2^{MPW} = 7.25.$$

Example 2. The class of issue-externality games $\mathcal{G}$, and the value that we propose, can accommodate situations where players meet sequentially. For example, players can meet and form coalitions at date $t = 1$ (issue a), meet again at date $t = 2$ (issue b), and the worth of the coalitions at $t = 2$ depends on the partition formed at $t = 1$. We can even consider situations where new players are active or not at $t = 2$ (that is, in issue b) depending on the coalitions formed at $t = 1$ (that is, in issue a). For example, at $t = 1$ players 1 and 2 may form a coalition or not. If they form a coalition, then player 3 participates at $t = 2$; if players 1 and 2 do not form a coalition, then players 1 and 2 are the only ones creating worth in issue b. This situation can be formalized as a game with three players and two issues where player 3 does not influence payoffs in issue a and, if players 1 and 2 do not form a coalition at $t = 1$, player 3 also does not generate any worth in issue b.

The payoffs in Figure 2 may represent such a situation. Firms 1 and 2 are initially active in the market. Firm 3 only exists if firms 1 and 2 form a coalition in issue a (that is, at $t = 1$). Therefore, the worth of any (embedded) coalition in issue a does not change if player 3 is added to or removed from it. The same holds for issue b if players 1 and 2 belong to different coalitions in issue a. On the other hand, if firms 1 and 2 are together in issue a, then firm 3 is an active player in issue b and can influence the worth obtained when he forms coalitions with either of the two players, or with both of them.

$t = 2 \rightarrow$ $\downarrow t = 1$	$\{1\}, \{2\}, \{3\}$	$\{1\}, \{2,3\}$	$\{1,3\}, \{2\}$	$\{1,2\}, \{3\}$	$\{1,2,3\}$
$\{1\}, \{2\}, \{3\}$	5, 5, 0 5, 5, 0	5, 5 5, 5, 0	5, 5 5, 5, 0	8, 0 5, 5, 0	8 5, 5, 0
$\{1\}, \{2,3\}$	5, 5, 0 5, 5	5, 5 5, 5	5, 5 5, 5	8, 0 5, 5	8 5, 5
$\{1,3\}, \{2\}$	5, 5, 0 5, 5	5, 5 5, 5	5, 5 5, 5	8, 0 5, 5	8 5, 5
$\{1,2\}, \{3\}$	4, 4, 4 12, 0	2, 12 12, 0	12, 2 12, 0	10, 6 12, 0	25 12, 0
$\{1,2,3\}$	4, 4, 4 12	2, 12 12	12, 2 12	10, 6 12	25 12

Table 2: Example 2

In Example 2, the value generated by the grand coalition is 37 and, according to the proposal ϕ^{MPW}, must be shared as

$$\phi_{1(a)}^{MPW} = \phi_{2(a)}^{MPW} = 9.521, \ \phi_{3(a)}^{MPW} = 0$$
$$\phi_{1(b)}^{MPW} = \phi_{2(b)}^{MPW} = 6.333, \ \phi_{3(b)}^{MPW} = 5.292,$$

which implies the following total payoffs:

$$\Phi_1^{MPW} = \Phi_2^{MPW} = 15.854, \ \Phi_3^{MPW} = 5,292.$$

Our proposal allows us to compute the payoff allocation from players' contributions in the different issues. The "delegates" of firms 1 and 2 in issue a obtain a total of $\phi_{1(a)}^{MPW} + \phi_{2(a)}^{MPW} = 19.042$, which is higher than the worth of 12 that they generate in that issue. Therefore, our value allocates a total worth of 7.042 to the externality that the firms' behavior in issue a generates on the value created in issue b.

Example 2 illustrates how to apply our values to the class of "two-stage games" proposed by Beja and Gilboa (1990). In these games, agents form a coalition in the first stage, which entitles its members to play a prespecified cooperative game at the second stage. We can think of the first stage as issue a ($t = 1$) and the second stage as issue b

$(t = 2)$, with the property that worth is only obtained in issue b. Beja and Gilboa (1990) characterize all the semivalues in this class of game, where semivalues satisfy linearity, player anonymity, dummy player, and monotonicity. Our approach provides more structure to the values by introducing axioms on the way externalities should be treated within and across issues (in this case, across issues, between the coalitions formed at $t = 1$ and the game played at $t = 2$); in particular, this allows us to identify the payoff that each player obtains due to his participation in each stage.

For example, Beja and Gilboa (1990) consider the following majority game. There are three players with "relative weights" or "vote counts" of $(2, 2, 3)$. If a coalition of at least two players is formed at stage 1, then the players in that coalition play a majority game to share a worth of 1. Therefore, if the coalition $\{1, 2\}$ is formed, then the two players together get 1 and each obtains a payoff of 0.5 if they do not form a coalition at $t = 2$; if the grand coalition forms at stage 1 then at stage 2 any coalition of two player obtains 1; however, player 3 ends up with a payoff of 1 in the majority game at stage 2 if either coalition $\{1, 3\}$ or $\{2, 3\}$ is formed at $t = 1$. According to the proposal ϕ^{MPW}, the worth of 1 must be shared as

$$
\begin{aligned}
\phi_{1(a)}^{MPW} &= \phi_{2(a)}^{MPW} = \phi_{3(a)}^{MPW} = 0.07777 \\
\phi_{1(b)}^{MPW} &= \phi_{2(b)}^{MPW} = 0.17222, \ \phi_{3(b)}^{MPW} = 0.4222,
\end{aligned}
$$

which implies players' payoffs of $\Phi_1^{MPW} = \Phi_2^{MPW} = 0.25$ and $\Phi_3^{MPW} = 0.5$. The contribution of the three players to build a winning coalition in stage 1 is the same, hence they receive the same payoff 0.07777 for this contribution. However, player 3 has more power in stage 2, which is acknowledged with a payoff of 0.4222 compared with 0.17222 each for players 1 and 2.

Finally, Example 2 also suggests that issue-externality games can accommodate situations with several linked issues where different players are "active" in each issue: the set of players is $N = N_a \cup N_b$, players in N_a take a relevant decision on issue a while N_b is the relevant set in issue b, with $N_a \cap N_b = \varnothing$. Such a situation arises when different generations of players or different sets of countries deciding on different issues with externalities within and across them.

7   Conclusion

This paper considers situations where players interact in several issues and the issues are linked because the worth of a coalition in one issue depends on the organization of the players in the other issues. We have proposed a way to extend values that have been put forward to deal with externalities within issues to games where there are externalities both within and across issues. We have shown that any value for this class of games satisfies the axioms of linearity, player anonymity, strong dummy player, issue symmetry, dummy issue, issue-externality anonymity, and issue-externality symmetry, if and only if the value can be obtained as an extension of a value for partition function games that satisfy the axioms of linearity, player anonymity and strong dummy player.

8   Appendix

Proof of Proposition 1.   (i) Consider two games (N, A, v) and (N, A, v'). Since ϕ^* satisfies linearity, we have

$$\phi_k^*(N(A), \hat{v} + \widehat{v'}) = \phi_k^*(N(A), \hat{v}) + \phi_k^*(N(A), \widehat{v'}) \text{ for every } k \in N(A).$$

Also, following (1), it is easy to check that $\widehat{v + v'} = \hat{v} + \widehat{v'}$. Hence,

$$\Phi_i^*(N, A, v + v') = \sum_{a \in A} \phi_{i(a)}^*(N(A), \widehat{v + v'}) = \sum_{a \in A} \phi_{i(a)}^*(N(A), \hat{v} + \widehat{v'}) =$$

$$\sum_{a \in A} \phi_{i(a)}^*(N(A), \hat{v}) + \sum_{a \in A} \phi_{i(a)}^*(N(A), \widehat{v'}) = \Phi_i^*(N, A, v) + \Phi_i^*(N, A, v')$$

for all $i \in N$, and Φ^* satisfies part 1.1 of the linearity axiom. Similarly, for the multiplication by an scalar λ it is the case that $\lambda \phi_k^*(N(A), \hat{v}) = \phi_k^*(N(A), \lambda \hat{v})$ for every $k \in N(A)$ and $\widehat{\lambda v} = \lambda \hat{v}$. Hence,

$$\Phi_i^*(N, A, \lambda v) = \sum_{a \in A} \phi_{i(a)}^*(N(A), \widehat{\lambda v}) = \sum_{a \in A} \phi_{i(a)}^*(N(A), \lambda \hat{v}) =$$

$$\sum_{a \in A} \lambda \phi_{i(a)}^*(N(A), \hat{v}) = \lambda \Phi_i^*(N, A, v)$$

for all $i \in N$, and Φ^* satisfies part 1.2 of the linearity axiom.

(ii) The player anonymity axiom of ϕ^* implies that $\phi^*_{\sigma k}(N(A), \sigma \hat{v}) = \phi^*_k(N(A), \hat{v})$ for any $k \in N(A)$ and for any permutation σ on the set $N(A)$. Take now a permutation σ_N on the set N and denote $\sigma_{N(A)}$ the permutation on the set $N(A)$ that associates player $i(a)$ to $(\sigma_N(i))(a)$, for every $i \in N$, $a \in A$. Consider the game (N, A, v). Then,

$$\widehat{\sigma_N v}(T, Q) = \sum_{a \in A} \sigma_N v\left(\widetilde{T}(a); a; (\widetilde{Q}(b))_{b \in A}\right) = \sum_{a \in A} v(\sigma_N \widetilde{T}(a); a; \left(\sigma_N \widetilde{Q}(b)\right)_{b \in A}) =$$

$$\sum_{a \in A} v((\widetilde{\sigma_{N(A)} T})(a); a; \left(\left(\widetilde{\sigma_{N(A)} Q}\right)(b)\right)_{b \in A}) = \hat{v}(\sigma_{N(A)} T, \sigma_{N(A)} Q) = \left(\sigma_{N(A)} \hat{v}\right)(T, Q).$$

Consequently,

$$\Phi^*_i(N, A, \sigma_N v) = \sum_{a \in A} \phi^*_{i(a)}(N(A), \widehat{\sigma_N v}) = \sum_{a \in A} \phi^*_{i(a)}(N(A), \sigma_{N(A)} \hat{v}) =$$

$$\sum_{a \in A} \phi^*_{\sigma_{N(A)}(i(a))}(N(A), \hat{v}) = \sum_{a \in A} \phi^*_{(\sigma_N(i))(a)}(N(A), \hat{v}) = \Phi^*_{\sigma_N(i)}(N, A, v)$$

for each $i \in N$. Hence, Φ^* satisfies the player anonymity axiom.

(iii) We first prove that if $j \in N$ is a dummy player in the game (N, A, v), then all the replicas $j(a)$, for all $a \in A$, are dummy players in $(N(A), \hat{v})$. Consider any $(T, Q) \in ECL(N(A))$ and any (T', Q') obtained from (T, Q) by changing the affiliation of player $j(a)$. For any such (T', Q'), it is always the case that $Q'(b) = Q(b)$ for any $b \neq a$, since we are changing the affiliation of a player that belongs to $N(a)$. There are two possibilities: a) It can be the case that $Q'(a) = Q(a)$. Then,

$$\hat{v}(T', Q') = \sum_{b \in A} v(\widetilde{T'}(b); b; (\widetilde{Q'}(c))_{c \in A}) = \sum_{b \in A} v(\widetilde{T}(b); b; (\widetilde{Q}(c))_{c \in A}) = \hat{v}(T, Q).$$

b) Or it can be the case that $Q'(a) \neq Q(a)$ when $j(a)$ changes affiliation. In this case,

$$v(\widetilde{T'}(b); b; (\widetilde{Q'}(c))_{c \in A}) = v(\widetilde{T}(b); b; (\widetilde{Q}(c))_{c \in A})$$

for any embedded coalition $(\widetilde{T}(b); b; (\widetilde{Q}(c))_{c \in A})$ and for all $b \in A$ because $(\widetilde{T'}(b); b; (\widetilde{Q'}(c))_{c \in A})$ can be deduced from $(\widetilde{T}(b); b; (\widetilde{Q}(c))_{c \in A})$ by changing the affiliation of the dummy player j within issue a in (N, A, v). Hence again, $\hat{v}(T', Q') = \hat{v}(T, Q)$.

This ends the proof that all the replicas $j(a)$, for all $a \in A$, are dummy players in $(N(A), \hat{v})$.

If ϕ^* satisfies the dummy player axiom, then $\phi^*_{j(a)}(N(A), \hat{v}) = 0$ for all $a \in A$ since $j(a)$ is a dummy player in $(N(A), \hat{v})$. Therefore,

$$\Phi^*_j(N, A, v) = \sum_{a \in A} \phi^*_{j(a)}(N(A), \hat{v}) = 0$$

and Φ^* satisfies the dummy player axiom.

(iv) Consider a dummy player $j \in N$ in the game (N, A, v) and a particular issue $a \in A$. First, since ϕ^* satisfies the strong dummy player property and $j(a)$ is a dummy player in $(N(A), \hat{v})$,

$$\phi^*_k(N(A) \backslash j(a), \hat{v}_{-j(a)}) = \phi^*_k(N(A), \hat{v})$$

for all $k \in N(A) \backslash j(a)$. Second, player $j(b)$, for $b \neq a$, is also a dummy player in the game $(N(A) \backslash j(a), \hat{v}_{-j(a)})$. (If we have two dummy players in any *PFG*, the second dummy player is still dummy in the game where we have eliminated the first one.) Applying this procedure sequentially to all the issues in A, and denoting $j(A) = \underset{a \in A}{\cup} j(a)$, we have that

$$\phi^*_k(N(A) \backslash j(A), \hat{v}) = \phi^*_k(N(A) \backslash j(A), \hat{v}_{-j(A)})$$

for all $k \in N(A) \backslash j(A)$. Therefore,

$$\Phi^*_i(N \backslash j, A, v_{-j}) = \sum_{a \in A} \phi^*_{i(a)}((N \backslash j)(A), \widehat{v_{-j}}) = \sum_{a \in A} \phi^*_{i(a)}(N(A) \backslash j(A), \hat{v}_{-j(A)}) =$$

$$\sum_{a \in A} \phi^*_{i(a)}(N(A), \hat{v}) = \Phi^*_i(N, A, v)$$

for all $i \in N \backslash j$ and Φ^* satisfies the strong dummy player axiom. ∎

Proof of Proposition 2. (i) This property is trivially satisfied.

(ii) If d is a dummy issue in the game (N, A, v), then all the replicas $i(d)$, for any $i \in N$, are dummy players in $(N(A), \hat{v})$, because, by the definition of dummy issue, $\hat{v}(T, Q) = \hat{v}(T', Q')$ for all (T', Q') obtained from (T, Q) by changing the affiliation of player $i(d)$, for any $i \in N$.

Given that ϕ^* satisfies the strong dummy player axiom, then if the n dummy players $i(d)$ are dropped off $N(A)$

$$\phi^*_k(N(A) \backslash (i(d))_{i \in N}, \hat{v}_{-(i(d))_{i \in N}}) = \phi^*_k(N(A), \hat{v}),$$

which implies that for all $i \in N$

$$\Phi_i^*(N, A\backslash d, v_{-d}) = \sum_{a \in A} \phi_{i(a)}^*(N(A\backslash d), \widehat{v_{-d}} = \sum_{a \in A} \phi_{i(a)}^*(N(A)\backslash\{i(d)\}_{i \in N}, \hat{v}_{-(i(d))_{i \in N}}) =$$

$$\sum_{a \in A} \phi_{i(a)}^*(N(A), \hat{v}) = \Phi_i^*(N, A, v)$$

and Φ^* satisfies the dummy issue axiom. $\blacksquare$

Proof of Proposition 3. (i) Consider the game (N, A, v) and, for any $a \in A$, define (N, A, v_a) as

$$v_a(S; a; P^A) \equiv v(S; a; P^A) \text{ for all } (S; a; P^A) \in ECL(N, A)$$
$$v_a(S; b; P^A) \equiv 0 \text{ for all } b \in A\backslash a, \ (S; b; P^A) \in ECL(N, A).$$

It is immediate that $v = \sum_{a \in A} v_a$. The linearity of ϕ^* implies the linearity of Φ^* (Proposition 1); hence

$$\Phi^*(N, A, v) = \sum_{a \in A} \Phi^*(N, A, v_a).$$

Similarly, consider the game (N, A, v_{σ_N}), where σ_N is a permutation of the set of players N. Remember that the function v_{σ_N} is defined as $v_{\sigma_N}(S; a; P^A) \equiv v\left(S; a; \left(P^a, O^{A\backslash a}\right)\right)$ for all $(S; a; P^A) \in ECL(N, A)$, where $O^b = \sigma_N P^b$ or $O^b = P^b$ for all $b \in A\backslash a$. Let $B \subset A\backslash a$ be the subset of issues where σ_N applies, i.e., $O^b = \sigma_N P^b$ for all $b \in B$ and $O^b = P^b$ for all $b \in A\backslash B\backslash a$. For any particular $a \in A$, we define $(N, A, (v_{\sigma_N})_a)$ as

$$(v_{\sigma_N})_a (S; a; P^A) \equiv v_{\sigma_N}(S; a; P^A) \text{ for all } (S; a; P^A) \in ECL(N, A)$$
$$(v_{\sigma_N})_a (S; b; P^A) \equiv 0 \text{ for all } b \in A\backslash a, \ (S; b; P^A) \in ECL(N, A).$$

Given that $v_{\sigma_N} = \sum_{a \in A} (v_{\sigma_N})_a$, the linearity of Φ^* implies

$$\Phi^*(N, A, v_{\sigma_N}) = \sum_{a \in A} \Phi^*(N, A, (v_{\sigma_N})_a).$$

We now prove that $\Phi_i^*(N, A, (v_{\sigma_N})_a) = \Phi_i^*(N, A, v_a)$ for all $i \in N$ for whom $\sigma_N(i) = i$, which will prove part (i) of the proposition.

For any $i \in N$,

$$\Phi_i^*(N, A, v_a) = \sum_{b \in A} \phi_{i(b)}^*(N(A), \widehat{v_a})$$

29

where

$$\widehat{v_a}(T, Q) = \sum_{b \in A} v_a(\widetilde{T}(b); b; (\widetilde{Q}(c))_{c \in A}) = v_a(\widetilde{T}(a); a; (\widetilde{Q}(c))_{c \in A})$$

for any $(T, Q) \in ECL(N(A))$, since the other terms in the sum are zero by construction of the function v_a. Also,

$$\Phi_i^*(N, A, (v_{\sigma_N})_a) = \sum_{b \in A} \phi_{i(b)}^*(N(A), \widehat{(v_{\sigma_N})_a})$$

where

$$\widehat{(v_{\sigma_N})_a}(T, Q) = \sum_{b \in A} (v_{\sigma_N})_a (\widetilde{T}(b); b; (\widetilde{Q}(c))_{c \in A}) = (v_{\sigma_N})_a (\widetilde{T}(a); a; (\widetilde{Q}(c))_{c \in A})$$

for any $(T, Q) \in ECL(N(A))$. We notice that, by definition of v_{σ_N}, $(v_{\sigma_N})_a (S; a; P^A) = v(S; a; (P^a, O^{A \setminus a}))$ for all $(S; a; P^A) \in ECL(N, A)$, where $O^b = \sigma_N P^b$ or $O^b = P^b$ for all $b \in A \setminus a$ (and $(v_{\sigma_N})_a (S; b; P^A) = 0$ for all $b \in A \setminus a$, $(S; b; P^A) \in ECL(N, A)$). Since $(v_{\sigma_N})_a$ only permutes the roles of the players involved in a subset of issues $B \subset A \setminus a$, $\widehat{(v_{\sigma_N})_a}$ only permutes the roles of the players in each $N(b)$, for all $b \in B$. In fact, $\widehat{(v_{\sigma_N})_a} = \sigma_{N(A)} \widehat{v_a}$, where the permutation $\sigma_{N(A)}$ is as follows:

$$\sigma_{N(A)}(i(c)) = i(c) \text{ for all } i \in N \text{ and all } c \in A \setminus B.$$
$$\sigma_{N(A)}(i(b)) = (\sigma_N (i)) (b) \text{ for all } i \in N \text{ and all } b \in B.$$

Given that ϕ^* satisfies player anonymity,

$$\phi_{i(c)}^*(N(A), \widehat{(v_{\sigma_N})_a}) = \phi_{i(c)}^*(N(A), \sigma_{N(A)} \widehat{v_a}) = \phi_{i(c)}^*(N(A), \widehat{v_a})$$

for all $i \in N$ and $c \in A \setminus B$ and

$$\phi_{i(b)}^*(N(A), \widehat{(v_{\sigma_N})_a}) = \phi_{(\sigma_N(i))(b)}^*(N(A), \sigma_{N(A)} \widehat{v_a}) = \phi_{(\sigma_N(i))(b)}^*(N(A), \widehat{v_a})$$

for all $i \in N$ and all $b \in B$. In particular, $\phi_{i(b)}^*(N(A), \widehat{(v_{\sigma_N})_a}) = \phi_{i(b)}^*(N(A), \widehat{v_a})$ for all $i \in N$ for whom $\sigma_N(i) = i$. This implies that $\Phi_i^*(N, A, (v_{\sigma_N})_a) = \Phi_i^*(N, A, v_a)$ for any $i \in N$ for whom $\sigma_N(i) = i$, and the result holds.

(ii) Consider the game (N, A, v), a set M of $a-$externality players and $b \neq a$. We will show that if ϕ^* satisfies linearity and player anonymity in *PFG*, then $\Phi_i^* (N, A, v_{M,ab}) =$

$\Phi_i^*(N, A, v)$ for all $i \in N$. Notice that $\Phi_i^*(N, A, v) = \sum_{c \in A} \phi_{i(c)}^*(N(A), \widehat{v})$, where $\widehat{v}(T, Q) = \sum_{c \in A} v(\widetilde{T}(c); c; (\widetilde{Q}(d))_{d \in A})$ and $\Phi_i^*(N, A, v_{M,ab}) = \sum_{c \in A} \phi_{i(c)}^*(N(A), \widehat{v_{M,ab}})$, where $\widehat{v_{M,ab}}(T, Q) = \sum_{c \in A} v_{M,ab}(\widetilde{T}(c); c; (\widetilde{Q}(d))_{d \in A})$. We consider the following permutation $\sigma_{N(A)}$ on the set $N(A)$: $\sigma_{N(A)}(i(a)) = i(b)$ and $\sigma_{N(A)}(i(b)) = i(a)$ for all $i \in M$ and $\sigma_{N(A)}(k) = k$ otherwise. Applying the permutation $\sigma_{N(A)}$ to the value function $\widehat{v}$ has the same effect as going from v to $v_{M,ab}$: it moves the roles of players in M from issue a to issue b. Hence, $\sigma_{N(A)}\widehat{v} = \widehat{v_{M,ab}}$.

Given that the value ϕ^* satisfies anonymity, it is the case that

$$\phi_{i(c)}^*(N(A), \widehat{v_{M,ab}}) = \phi_{i(c)}^*(N(A), \sigma_{N(A)}\widehat{v}) = \phi_{\sigma_{N(A)}(i(c))}^*(N(A), \widehat{v}).$$

Given that $\sigma_{N(A)}$ only permutes replicas of the same players (those in M), it is the case that

$$\sum_{c \in A} \phi_{i(c)}^*(N(A), \widehat{v_{M,ab}}) = \sum_{c \in A} \phi_{\sigma_{N(A)}(i(c))}^*(N(A), \widehat{v}) = \sum_{c \in A} \phi_{i(c)}^*(N(A), \widehat{v})$$

(since $\phi_{\sigma_{N(A)}(i(a))}^*(N(A), \widehat{v}) + \phi_{\sigma_{N(A)}(i(b))}^*(N(A), \widehat{v}) = \phi_{i(b)}^*(N(A), \widehat{v}) + \phi_{i(a)}^*(N(A), \widehat{v})$ for $i \in M$). Therefore, $\Phi_i^*(N, A, v_{M,ab}) = \Phi_i^*(N, A, v)$ as we wanted to prove. $\blacksquare$

Proof of Theorem 1. The sufficiency part of the Theorem is a corollary of Propositions 1, 2, and 3. We prove the necessity part through a series of steps. Take any game (N, A, v) in $\mathcal{G}$.

Step 1.- For any $a \in A$, we define the following game (N, A, v_a):

$$v_a(S; a; P^A) \equiv v(S; a; P^A) \text{ for all } (S; a; P^A) \in ECL(N, A)$$
$$v_a(S; b; P^A) \equiv 0 \text{ for all } b \in A\backslash a, (S; b; P^A) \in ECL(N, A).$$

That is, the worth of a coalition on issue a in the game v_a is the same as that in v; however, the worth of a coalition on any other issue is zero in game v_a. Note that the organization of the players on issues other than a influences the worth of coalitions in issue a in the game v_a in the same way as it does in v.

It is immediate that

$$v = \sum_{a \in A} v_a.$$

Therefore, if Φ satisfies the axiom of *linearity* then,

$$\Phi(N, A, v) = \sum_{a \in A} \Phi(N, A, v_a).$$

Step 2.- For each (N, A, v_a), we now define a related game $(N(A), A, w_a)$, which is similar to (N, A, v_a) except that we add $(|A| - 1)n$ dummy players. More precisely, for each $b \in A \setminus a$, let $N(b) = \{i(b) \mid i \in N\}$ be the $b-$replica of N and for convenience, let $N(a) \equiv N$ (i.e., $N(a)$ is the original set of players). Then the set of players in the new game is $N(A) = \cup_{b \in A} N(b)$ with $N(A) \setminus N(a)$ being dummy players. Therefore, for every $a \in A$, $(N(A), A, w_a)$ is defined as follows:[20]

$$w_a(T; a; Q^A) \equiv v_a(\widetilde{T}(a); a; (\widetilde{Q^b}(a))_{b \in A})$$

for all $(T; a; Q^A) \in ECL(N(A), A)$ (i.e., for all vector Q^A of $|A|$ partitions of $N(A)$ and any $T \in Q^a$), and

$$w_a(T; b; Q^A) \equiv v_a(\widetilde{T}(a); b; (\widetilde{Q^b}(a))_{b \in A}) = 0$$

for all $b \in A \setminus a$ and all $(T; b; Q^A) \in ECL(N(A), A)$.

Given that Φ satisfies the axioms of *strong dummy player* and *player anonymity* (2'), we have

$$\begin{aligned}
\Phi_i(N(A), A, w_a) &= \Phi_i(N, A, v_a) \text{ for all } i \in N(a) = N \\
\Phi_i(N(A), A, w_a) &= 0 \text{ for all } i \in N(A) \setminus N(a).
\end{aligned}$$

Step 3.- Next, for each $a \in A$, we define another game $(N(A), A, z_a)$ that is related to $(N(A), A, w_a)$ in the following sense. First, as in $(N(A), A, w_a)$, a coalition of players obtains worth only on issue a. Second, only players in $N(a)$ create worth. Third, the inter-issue externalities in $(N(A), A, z_a)$ are "similar" to those in $(N(A), A, w_a)$; however, there is one important difference: in game $(N(A), A, z_a)$, the externalities originating from each issue $b \in A \setminus a$ are exerted by players in $N(b)$, rather than by players in $N(a)$ as in game $(N(A), A, w_a)$. That is, the game $(N(A), A, z_a)$ is defined as follows:

$$z_a(T; a; Q^A) \equiv w_a(T; a; R^A)$$

---

[20] As previously done, we denote $T(b) = T \cap N(b)$ for any coalition T of $N(A)$ and $Q(b) = \{T \cap N(b) \mid T \in Q\}$ for any partition Q of $N(A)$, for any $b \in A$.

for all $(T; a; Q^A) \in ECL(N(A), A)$, where R^A is a vector of $|A|$ partitions of $N(A)$ such that $R^a = Q^a$ and for every $b \in A\backslash a$, R^b is obtained from Q^b by exchanging the memberships of $i(a)$ and $i(b)$ for each $i \in N$,[21] and

$$z_a(T; b; Q^A) \equiv 0$$

for all $b \in A\backslash a$ and all $(T; b; Q^A) \in ECL(N(A), A)$.

Note that $z_a(T; a; Q^A) = v_a\left(\widetilde{T}(a); a; \left(\widetilde{Q^b}(b)\right)_{b \in A}\right)$ for all $(T; a; Q^A) \in ECL(N(A), A)$.[22]

We claim that, by *issue-externality anonymity* axiom,

$$\sum_{b \in A} \Phi_{i(b)}(N(A), A, z_a) = \Phi_i(N, A, w_a) \text{ for all } i \in N = N(a). \tag{4}$$

We prove this claim by decomposing the change from $(N(A), A, w_a)$ to $(N(A), A, z_a)$ in $|N|(|A| - 1)$ stages. In each stage, we switch the membership of some $i(a) \in N(a)$ with that of $i(b) \in N(b)$ in the partition P^b on some issue $b \in A\backslash a$. In doing so, $i(b)$ takes the role of $i(a)$ in generating externalities from issue b. Note that the identities of the players who create worth (always on issue a) remain the same. Then, by the issue-externality anonymity axiom, the value of every player different from $i(a)$ and $i(b)$ should not change; hence, the sum of the values for players $i(a)$ and $i(b)$ should not change either. Repeating this argument cross issues implies that after $|A| - 1$ stages of switching the membership of $i(a) \in N(a)$ with $i(b) \in N(b)$ for every issue $b \in A\backslash a$, the sum of the values for all replicas of player i remain unchanged while the value of each of the remaining players stays the same throughout these stages. By repeating the above stages for all $i(a) \in N(a)$, we complete our transformation from $(N(A), A, w_a)$ to $(N(A), A, z_a)$ and obtain equation (4).

Step 4.- For each $(N(A), A, z_a)$, we now define a related game $(N(A), A, r_a)$ such that all externalities are generated from issue a. Recall that in $(N(A), A, z_a)$, for any $(T; a; Q^A) \in ECL(N(A), A)$, the worth of T depends only on $(Q^b(b))_{b \in A}$; moreover, only a coalition of players in $N(a)$ can create worth and it does so only on issue a. In fact, for each $b \in A\backslash a$, $N(b)$ is a set of b−externality players in $(N(A), A, z_a)$. We define the game $(N(A), A, r_a)$ by encoding the externalities exerted by $N(b)$ for all $b \in A\backslash b$ in z_a:

---

[21] Thus, $R^b(a) = Q^b(b)$ for all $b \in A\backslash a$.

[22] Recall that Q^b is a partition of $N(A)$ on issue b and $Q^b(b)$ is the partition of $N(b)$ induced by Q^b; $\widetilde{Q^b}(b)$ is obtained from $Q^b(b)$ by replacing each $i(b) \in N(b)$ with i.

$$r_a(T; a; Q^A) \equiv z_a(T; a; R^A)$$

for all $(T; a; Q^A) \in ECL(N(A), A)$ where R^A is a vector of $|A|$ partitions of $N(A)$ such that $R^a = Q^a$ and for every $b \in A \backslash a$, R^b is such that $R^b \cap N(b) = Q^a \cap N(b)$. Thus, r_a can be obtained from z^a from $(|A| - 1)$ steps of transformation, each involving moving the externalities induced by $N(b)$, for a particular $b \in A \backslash a$, from issue b to issue a.

Note that $r_a(T; a; Q^A) = v_a \left(\widetilde{T}(a); a; \left(\widetilde{Q^a}(b) \right)_{b \in A} \right)$ for all $(T; a; Q^A) \in ECL(N(A), A)$. By the *issue-externality symmetry* axiom,

$$\Phi_k(N(A), A, r_a) = \Phi_k(N, A, w_a) \text{ for all } k \in N(A).$$

We also note that all issues in $A \backslash a$ are dummy issues in $(N(A), A, r_a)$.

Step 5.- Finally, we define game $(N(A), a, s_a)$ by eliminating the set of *dummy issues* $A \backslash a$ in $(N(A), A, r_a)$, that is,

$$s_a(T; a; Q) \equiv r_a(T; a; Q^A)$$

for any $(T; a; Q) \in ECL(N(A), a)$ and any vector Q^A of $|A|$ partitions of $N(A)$ that satisfies $Q^a = Q$. By the dummy issue axiom, we have

$$\Phi_k(N(A), a, s_a) = \Phi_k(N(A), A, r_a) \text{ for all } k \in N(A).$$

Note that $(N(A), a, s_a)$ is a game with a single issue (a in this case). Therefore, we can consider $(N(A), a, s_a)$ as a *PFG*, that we denote $(N(A), \widetilde{s_a})$. Moreover, when it is applied to games with only one issue, the issue symmetry axiom implies that the value Φ depends only on the function that gives the worth of each embedded coalition, not on the identity of the issue itself. Thus, Φ also defines a value for *PFG*. Let ϕ be this value. Hence,

$$\phi_k(N(A), \widetilde{s_a}) = \Phi_k(N(A), a, s_a) \text{ for all } k \in N(A).$$

Therefore, Steps 1-5 allow us to obtain the following series of equalities for every $i \in N$:

$$\Phi_i(N, A, v) = \sum_{a \in A} \Phi_i(N, A, v_a) = \sum_{a \in A} \Phi_{i(a)}(N(A), A, w_a) =$$

$$\sum_{a \in A} \sum_{b \in A} \Phi_{i(b)}(N(A), A, z_a) = \sum_{a \in A} \sum_{b \in A} \Phi_{i(b)}(N(A), A, r_a) = \sum_{a \in A} \sum_{b \in A} \Phi_{i(b)}(N(A), a, s_a) =$$

$$\sum_{a \in A} \sum_{b \in A} \phi_{i(b)}(N(A), \widetilde{s_a}) = \sum_{b \in A} \sum_{a \in A} \phi_{i(b)}(N(A), \widetilde{s_a}).$$

We now prove that $\hat{v} = \sum_{a \in A} \widetilde{s}_a$. Consider any partition Q of $N(A)$ and any coalition $T \in Q$. By construction,

$$\widetilde{s}_a(T; Q) = s_a(T; a; Q) = r_a(T; a; Q^A),$$

where Q^A is any vector of $|A|$ partitions of $N(A)$ that satisfies $Q^a = Q$. Also,

$$r_a(T; a; Q^A) = v_a\left(\widetilde{T}(a); a; \left(\widetilde{Q^a}(b)\right)_{b \in A}\right) = v_a(\widetilde{T}(a); a; \left(\widetilde{Q}(b)\right)_{b \in A}) = v(\widetilde{T}(a); a; \left(\widetilde{Q}(b)\right)_{b \in A}).$$

Hence,

$$\sum_{a \in A} \widetilde{s}_a(T; Q) = \sum_{a \in A} v(\widetilde{T}(a); a; \left(\widetilde{Q}(b)\right)_{b \in A}) = \hat{v}(T, Q).$$

Finally, linearity of Φ implies that the value ϕ is also linear and $\phi_k(N(A), \hat{v}) = \sum_{a \in A} \phi_k(N(A), \widetilde{s}_a)$ for all $k \in N(A)$. Therefore,

$$\Phi_i(N, A, v) = \sum_{b \in A} \phi_{i(b)}(N(A), \hat{v})$$

which completes the proof of Theorem 1. ∎

Proof of Proposition 4. Take two games (N, A, v) and (N, B, w), with $A \cap B = \varnothing$, and consider a value Φ that satisfies the dummy issue axiom. We add to the first game $|B|$ dummy issues, obtaining the game $(N, A \cup B, v')$ where v' is a characteristic function such that

$$\begin{aligned}
v'(S; a; P^{A \cup B}) &= v(S; a; P^A) \text{ for all } a \in A, S \in P^a, P^a \in P^A \\
v'(S; b; P^{A \cup B}) &= 0 \text{ for all } b \in B, S \in P^b, \text{ and } P^b \in P^B.
\end{aligned}$$

By the dummy issue property, Φ assigns the same payoff in both games to any player $i \in N$, i.e.,

$$\Phi_i(N, A, v) = \Phi_i(N, A \cup B, v').$$

Similarly, if we add to the game (N, B, w) a set of $|A|$ dummy issues, we obtain the game $(N, A \cup B, w')$ where w' is a characteristic function such that

$$\begin{aligned}
w'(S; a; P^{A \cup B}) &= 0 \text{ for all } a \in A, S \in P^a, P^a \in P^A \\
w'(S; b; P^{A \cup B}) &= w(S; b; P^A) \text{ for all } b \in B, S \in P^b, \text{ and } P^b \in P^B.
\end{aligned}$$

Again, by the dummy issue axiom, we have

$$\Phi_i(N, B, w) = \Phi_i(N, A \cup B, w'), \text{ for all } i \in N.$$

Since Φ satisfies *linearity*,

$$\Phi_i(N, A \cup B, v') + \Phi_i(N, A \cup B, w') = \Phi_i(N, A \cup B, v' + w').$$

Finally, we notice that the game $(N, A \cup B, v' + w')$ is equivalent to $(N, A \cup B, v' \cup w')$; hence,

$$\Phi_i(N, A, v) + \Phi_i(N, B, w) = \Phi_i(N, A \cup B, v' \cup w')$$

and the independence axiom is satisfied. ■

References

[1] Albizuri, M.J., Arin, J., Rubio, J.: An axiom system for a value for games in partition function form. Int. Game Theory Rev. 7 , 63–73 (2005).

[2] Beja, A., Gilboa, I.: Values for two-stage games: Another view of the Shapley axioms. Int. J. Game Theory 19, 17-31 (1990).

[3] Bloch, F., de Clippel, G.: Core of combined games. J. Econ. Theory 145, 2424-2434 (2010).

[4] Bolger, E.M.: A set of axioms for a value for partition function games. Int. J. Game Theory 18, 37-44 (1989).

[5] Bulow, J.I., Geanakoplos, J.D., Klemperer, P.D.: Multimarket oligopoly: Strategic substitutes and complements. J. Polit. Econ. 93, 488-511 (1985).

[6] de Clippel, G., Serrano, R.: Marginal contributions and externalities in the value. Econometrica 76, 1413–1436 (2008).

[7] Dutta, B., Ehlers, L., Kar, A.: Externalities, potential, value and consistency. J. Econ. Theory 134, 2380-2411 (2010).

[8] Feldman, B.E.: Bargaining, coalition formation, and value. Ph.D. thesis, State University of New York at Stony Brook (1996).

[9] Harsanyi, J.C.: A bargaining model for the cooperative n-person game. In Contributions to the Theory of Games IV, ed by A.W. Tucker, R.D. Luce, 325-355, Princeton University Press, Princeton (1959).

[10] Hart, S., Mas-Colell, A.: Potential, value and consistency. Econometrica 57, 589-614 (1989).

[11] Macho-Stadler, I., Pérez-Castrillo, D., Wettstein, D.: Efficient bidding with externalities. Games Econ. Behav. 57, 304-320 (2006).

[12] Macho-Stadler, I., Pérez-Castrillo, D., Wettstein, D.: Sharing the surplus: An extension of the Shapley value for environments with externalities. J. Econ. Theory 135, 339-356 (2007).

[13] Maskin, E.: Bargaining, coalitions and externalities, Presidential Address to the Econometric Society, Institute for Advanced Study, Princeton (2003).

[14] McQuillin, B.: The extended and generalized Shapley value: simultaneous consideration of coalitional externalities and coalitional structure. J. Econ. Theory 144, 696–721 (2009).

[15] Myerson, R.B.: Game theory: Analysis of conflict. Harvard University Press (1991).

[16] Myerson, R.B.: Values of games in partition function form. Int. J. Game Theory 6, 23-31 (1977).

[17] Nax, H.H.: A note on the core of TU-cooperative games with multiple membership externalities. Games 5, 191-203 (2014).

[18] Pham Do, K.H., Norde, H.: The Shapley value for partition function form games. Int. Game Theory Rev. 9, 353–360 (2007).

[19] Shapley, L.S.: A value for n-person games. In: H.W. Kuhn, A.W. Tucker (Eds.) Contributions to the Theory of Games II, 307-317 (1953).

[20] Thrall, R.M., Lucas, W.F.: n-Person games in partition function form. Naval Research Logist Quarterly 10, 281–298 (1963).