SOBOLEV EMBEDDING INTO BMO AND WEAK-L^∞ FOR 1-DIMENSIONAL LOG-CONCAVE PROBABILITY MEASURE

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Abstract. We characterize rearrangement invariant spaces X with respect to a one dimensional probability log-concave measure μ such that the Sobolev embedding

$$\|u\|_{BMO(\mathbb{R},\mu)} \leq C \left\|u'\right\|_X + \|u\|_{L^1(\mathbb{R},\mu)}$$

holds, for any function $u \in L^1(\mathbb{R},\mu)$, whose real-valued weakly derivative u' belongs to X, where $BMO(\mathbb{R},\mu)$ is the space of functions with bounded mean oscillation with respect to μ. We investigate the embedding in weak-$L^\infty(\mathbb{R},\mu)$, too.

1. Introduction

Let μ be an absolutely continuous symmetric log-concave probability measure on the line, *i.e.* $d\mu = Ze^{-\Phi(x)}\,dx$, where $\Phi(x)$ is a even convex function and Z is the normalization constant in order to have $\mu(\mathbb{R}) = 1$. Let X be a rearrangement-invariant (*r.i.*) space[1] on $(\mathbb{R},\mu)$ (see Section 2 below). The Sobolev space $V_X^1(\mathbb{R},\mu) := V_X^1$ is the space of functions $u \in L^1(\mathbb{R},\mu)$ of those real-valued weakly differentiable functions on $\mathbb{R}$ which first derivative belong to X.

Poincaré type inequalities of the form

$$\left\| u - \int_{\mathbb{R}} u d\mu \right\|_Y \leq C \left\|u'\right\|_X, \qquad u \in V_X^1 \tag{1.1}$$

where X, Y are $r.i$ spaces on $(\mathbb{R},\mu)$ was studied either for the Gauss measure or for more general probability measures (see [7], [12] and [13]). These inequalities are strictly related to the isoperimetric function $I(t)$ of the measure μ (see Section 2 below for its definition). It was proved in [13, Theorem 6] that if X and Y are rearrangement invariant spaces on $(\mathbb{R},\mu)$ then the inequality (1.1) holds if and only if the isoperimetric Hardy operator Q_I defined on measurable functions on $(0,1)$ by

$$Q_I u(t) = \int_t^{1/2} \frac{u(s)}{I(s)}\,ds,$$

is bounded from X to Y.

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[1]Loosely speaking, in an *r.i.* space the norm of a function depends only on the measure of its level sets.

An elementary consequence of (2.2) in Section 2 is that

$$\int_0^{1/2} \frac{ds}{I(s)} = +\infty,$$

which implies that there is not any $r.i$ space X such Q_I is bounded from X to L^∞, thus the Sobolev-Poincaré

$$\left\| u - \int_{\mathbb{R}} ud\mu \right\|_{L^\infty} \leq C \left\| u' \right\|_X , \tag{1.2}$$

never holds, in other words there is not any summability property of u' which guarantees $u \in L^\infty$, in particular $u' \in L^\infty$ does not imply that $u \in L^\infty$.

An appropriate substitute for L^∞ when L^∞ does not work is the space $BMO(\mathbb{R}, \mu)$ of functions having bounded mean oscillation, defined as the functions $u \in L^1(\mathbb{R}, \mu)$ such that

$$\|u\|_{*,(\mathbb{R},\mu)} = \sup_{J \text{ interval } \subset \mathbb{R}} \frac{1}{\mu(J)} \int_J \left| u - \frac{1}{\mu(J)} \int_J ud\mu \right| d\mu < +\infty. \tag{1.3}$$

It is clear that the functional in (1.3) is not a norm since it vanishes in constant functions however it is easy to verify that $BMO(\mathbb{R}, \mu)$ is a Banach space under the norm[2],

$$\|u\|_{BMO(\mathbb{R},\mu)} = \|u\|_{*,(\mathbb{R},\mu)} + \|u\|_{L^1(\mathbb{R},\mu)} .$$

Closely related to $BMO(\mathbb{R}, \mu)$ is the Bennett-DeVore-Sharpley space weak-$L^\infty(\mathbb{R}, \mu)$[3], defined by (cf. [2]),

$$L^{\infty,\infty}(\mathbb{R}, \mu) = \left\{ u : \|u\|_{L^{\infty,\infty}} = \sup_{0<t<1} \left(u_\mu^{**}(t) - u_\mu^*(t) \right) < +\infty \right\}$$

where $u_\mu^{**}(t) = \frac{1}{t} \int_0^t u_\mu^*(s)ds$, and u_μ^* is the non increasing rearrangement of u with respect to the measure μ (see section 2 below).

Notice that $L^{\infty,\infty}$ is not a linear space and $\|\cdot\|_{L^{\infty,\infty}}$ is not a norm. The relation between the space weak-$L^\infty(\mathbb{R}, \mu)$ and $BMO(\mathbb{R}, \mu)$ when μ is a not doubling absolutely continuos measure was obtained in [1] (see [2] and [3] for the Lebesgue measure), where it was shown that there is a constant $c > 0$ such that

$$\|u\|_{L^{\infty,\infty}} = \sup_{0<t<1} \left(u_\mu^{**}(t) - u_\mu^*(t) \right) \leq c \|u\|_{BMO(\mathbb{R},\mu).} \tag{1.4}$$

The main objective of this paper is to establish criteria that ensure a function belongs to $BMO(\mathbb{R}, \mu)$ or to weak-$L^\infty(\mathbb{R}, \mu)$ in terms of the summability properties of its derivative (see [6] for Lebesgue measure). More precisely, we characterize all $r.i$ spaces X on $(\mathbb{R}, \mu)$ such that the corresponding Sobolev space $V_X^1(\mathbb{R}, \mu)$ is embedded into $BMO(\mathbb{R}, \mu)$ or in weak-$L^\infty(\mathbb{R}, \mu)$. Furthermore, we characterize the largest $r.i$ space X such that $V_X^1(\mathbb{R}, \mu)$ is continuously embedded into $BMO(\mathbb{R}, \mu)$.

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[2]This definition of $BMO(\mathbb{R}, \mu)$ is similar to the classical one, but here the measure μ is not doubling. It has shown in [11] that some of the properties that BMO satisfies when the measure is doubling are satisfied also if is non doubling, for example, the John-Nirenberg inequality holds, $BMO(\mathbb{R}, \mu)$ is the dual of $H_{at}^1(\mathbb{R}, \mu)$ and the operators which are bounded from $H_{at}^1(\mathbb{R}, \mu)$ into $L^1(\mathbb{R}, \mu)$ and from L^∞ into $BMO(\mathbb{R}, \mu)$ are bounded on $L^p(\mathbb{R}, \mu)$ for $1 < p < \infty$.

[3]For $1 < p < \infty$, the weak-L^p spaces satisfy (see [3])

$$\sup_{0<t<1} t^{1/p}(f_\mu^{**}(t) - f_\mu^*(t)) \leq \sup_{0<t<1} t^{1/p} f_\mu^{**}(t) = \|f\|_{L^{p,\infty}} ,$$

thus the space $L^{\infty,\infty}$ is indeed the limit of the spaces weak-L^p as $p \to \infty$.

2. Preliminaries

Let μ be an absolutely continuous symmetric log-concave probability measure on the line, *i.e.* $d\mu = Ze^{-\Phi(x)}\,dx = f(x)dx$, where $\Phi(x)$ is a even convex function and Z is the normalization constant in order to have $\mu(\mathbb{R}) = 1$. We denote by

$$F\,(x) = \int_{-\infty}^{x} f(x)dx \qquad \forall x \in \mathbb{R}\cup\{-\infty, +\infty\} \tag{2.1}$$

its distribution function.

Let J be an interval. If $A \subset J$ is a Borel set, then the *relative perimeter* of A *w.r.t.* (with respect to) μ is defined by

$$P_\mu(A, J) = \liminf_{h\to 0} \frac{\mu(A_h) - \mu(A)}{h},$$

where $A_h = \{x \in J : d(x, A) < h\}$.

The isopemetric profile of μ is defined by,

$$I(t) := \inf\{P_\mu(A, \mathbb{R}) : \mu(A) = t\}, \quad (0 \le t \le 1)$$

Since μ is a log-concave symmetric measure *(see* [5]*)*, we have that

$$I(t) = f(F^{-1}(t)),$$

in particular I is a continuous concave function symmetric with respect to $1/2$, such that $I(0) = I(1) = 0$, moreover, for $0 < t < 1$

$$F^{-1}(t) = \int_{\frac{1}{2}}^{t} \frac{1}{I(s)} ds. \tag{2.2}$$

In an analogue way the *relative isoperimetric profile* is defined by

$$h_J(t) := \inf\{P_\mu(A, J) : A \subset J,\ \mu(A) = t\}, \quad (0 \le t \le \mu(J))\,.$$

We briefly recall the basic definitions and conventions we use about rearrangement of functions and rearrangement invariant spaces. We refer the reader to [3] and [16] for a complete treatment.

Let Ω be a measurable subset of $\mathbb{R}$ and u be a real-valued measurable function on Ω. The *non-increasing rearrangement* of u *w.r.t.* μ is given by

$$u_\mu^*(t) = \sup\{s \ge 0 : \mu_{|u|}(s) > t\} \quad 0 < t < \mu(\Omega)$$

and the *signed non-increasing rearrangement* of u *w.r.t.* μ is given by

$$u_\mu^\circ(t) = \sup\{s \in \mathbb{R} : \mu_u(s) > t\} \quad 0 < t < \mu(\Omega),$$

where $\mu_u(s) = \mu\{x \in \Omega : u(x) > s\}$.

We say that a Banach function space $X = X(\mathbb{R}, \mu)$ on $(\mathbb{R}, \mu)$ is a rearrangement-invariant (*r.i.*) space, if $v \in X$ implies $u \in X$ for all $\mu-$measurable functions such that $u_\mu^* = v_\mu^*$ and, moreover, $\|u\|_X = \|v\|_X$. The functional $\|\cdot\|_X$ will be called a rearrangement invariant norm. Typical examples of *r.i.* spaces are the L^p-spaces, Orlicz spaces, Lorentz spaces, Marcinkiewicz spaces, etc.

Since $\mu(\mathbb{R}) = 1$, for any *r.i.* space X we have

$$L^\infty(\mathbb{R}) \subseteq X \subseteq L^1(\mathbb{R}, \mu), \tag{2.3}$$

with continuous embedding.

For rearrangement invariant norms $\|.\|_X$, we can compare the size of elements by comparing their averages, as expressed by a majoration principle, sometimes referred to as the Calderón-Hardy Lemma:

$$\text{Suppose that } \int_0^t u_\mu^*(s)ds \leq \int_0^t v_\mu^*(s)ds, \text{ holds for all } 0<t<1 \Rightarrow \|u\|_X \leq \|v\|_X . \tag{2.4}$$

The associate space X' of X is the *r.i.* space of all measurable functions u in for which the *r.i.* norm given by

$$\|v\|_{X'} = \sup_{u\neq 0} \frac{\int_{\mathbb{R}} |u(x)v(x)|\, d\mu}{\|u\|_X} = \sup_{u\neq 0} \frac{\int_0^1 u_\mu^*(s)v_\mu^*(s)ds}{\|u\|_X}. \tag{2.5}$$

In particular, the following *generalized Hölder's inequality* holds

$$\int_{\mathbb{R}} |u(x)v(x)|\, d\mu \leq \|u\|_X \|v\|_{X'} . \tag{2.6}$$

Given a function $u \in X$ and an interval J, from Hölder inequality, it easily follows that

$$\frac{1}{2}\left\| u\chi_J - \frac{1}{\mu(J)}\int_J u\ d\mu \right\|_X \leq \inf_{\alpha\in\mathbb{R}} \|(u-\alpha)\,\chi_J\|_X \leq \left\| u\chi_J - \frac{1}{\mu(J)}\int_J u\ d\mu \right\|_X . \tag{2.7}$$

Given a *r.i.* space $X = X(\mathbb{R},\mu)$, there exists a *unique* *r.i.* space $\bar{X} = \bar{X}(0,1)$ on $((0,1),m)$, (m denotes the Lebesgue measure on the interval $(0,1)$) such that

$$\|u\|_X = \|u_\mu^*\|_{\bar{X}}. \tag{2.8}$$

$\bar{X}$ is called the *representation space* of $X(\mathbb{R},\mu)$ (a characterization of the norm $\|\cdot\|_{\bar{X}}$ is available in [3]). Moreover, we have $\|u\|_X = \|u_\mu^\circ\|_{\bar{X}}$.

In the framework of our paper we will use the Lorentz type space $\Lambda(\mu)$ associated to isoperimetric profile I defined by

$$\Lambda(\mu) := \left\{ u : \ \|u\|_{\Lambda(\mathbb{R},\mu)} := \int_0^1 u_\mu^*(t)\frac{I(t)}{t}dt < +\infty \right\}, \tag{2.9}$$

which is a r.i. space (see [8]) since by the concavity of I the function $\frac{I(t)}{t}$ is decreasing. Notice that

$$\int_0^{1/2} u_\mu^*(t)\frac{I(t)}{t}dt \leq \|u\|_{\Lambda(\mathbb{R},\mu)} \leq 2\int_0^{1/2} u_\mu^*(t)\frac{I(t)}{t}dt. \tag{2.10}$$

We shall finish this section with a Lemma in which we collect some results that relate isoperimetry, derivative and rearrangements.

Lemma 1. *Let μ be an absolutely continuous symmetric log-concave probability measure on the line. Let $J \subseteq \mathbb{R}$ be an interval. Assume that the following relative isopemetric inequality holds:*

$$P_\mu(A,J) \geq \tilde{h}(\mu(A)); \quad (A \text{ Borel set contained in } J),$$

where $\tilde{h} : [0,\mu(J)] \to [0,\infty)$ is a continuos concave function symmetric around $\mu(J)/2$, such that $\tilde{h}(0) = \tilde{h}(\mu(J)) = 0$. Let $u \in V_{L^1}^1(\mathbb{R},\mu)$ and let $v = u\chi_J$. Then

(1) *The rearrangements v_μ^* and v_μ° are locally absolutely continuous functions.*

(2) *Denote by* $[\cdot]^*$ *the rearrangement with respect to Lebesgue measure on* $[0,\mu(J)]$, *then*

$$\int_0^s \left[\tilde{h}(\bullet)\left(-v_\mu^*\right)'(\bullet)\right]^*(r)dr \leq \int_0^s (v')_\mu^*(r)dr \quad 0<s<\mu(J), \tag{2.11}$$

and the same inequality holds for v_μ°.

Notice that the inequality (2.11), by (2.4), implies that for any r.i *space* X,

$$\left\|\tilde{h}(\bullet)\left(-v_\mu^*\right)'(\bullet)\right\|_{\bar{X}} \leq \|v'\|_X \ \text{ and } \ \left\|\tilde{h}(\bullet)\left(-v_\mu^\circ\right)'(\bullet)\right\|_{\bar{X}} \leq \|v'\|_X .$$

Proof. For v_μ^* this result was proved in [7] and [12] for Gauss measure and in [13] in the general context of metric spaces. It is easy to see that the same proof works for v_μ°. □

3. Main result

In this section we characterize the embedding of $V_X^1(\mathbb{R},\mu)$ into $BMO(\mathbb{R},\mu)$ or in weak-$L^\infty(\mathbb{R},\mu)$.

To this end we will need the following technical result (whose proof will be given in the Appendix). If μ a log-concave symmetric probability measure and I is its isoperimetric profile, then there exist a constant $C>0$ such that

$$I(t) \leq \int_0^t \frac{I(s)}{s}ds \leq CI(t) \ \text{ for } 0\leq t\leq \frac{1}{2}. \tag{3.1}$$

Theorem 1. *Let* X *be a* r.i. *space. The following statements are equivalent:*

i) $D=\sup_{0<s<\frac{1}{2}} \frac{1}{s}\left\|\frac{r}{I(r)}\chi_{(0,s)}(r)\right\|_{\overline{X}'}<+\infty$;

ii) there exists a constant $C_1>0$ *such that*

$$\|u\|_{*,(\mathbb{R},\mu)} \leq C_1 \|u'\|_X$$

for all $u\in V_X^1(\mathbb{R},\mu)$;

iii) there exists a constant $C_2>0$ *such that*

$$\|u\|_{L^{\infty,\infty}(\mathbb{R},\mu)} \leq C_2\left(\|u'\|_X+\|u\|_{L^1(\mathbb{R},\mu)}\right)$$

for all $u\in V_X^1(\mathbb{R},\mu)$.

In order to prove Theorem 1 we follow arguments contained in [6]. We shall need three preliminary results.

Lemma 2. *For every* $u\in V_{L^1}^1(\mathbb{R},\mu)$ *the following inequality holds*

$$\left\|u-\int_{\mathbb{R}} u d\mu\right\|_{\Lambda(\mu)} \leq 4C\|u'\|_{L^1(\mathbb{R},\mu)}, \tag{3.2}$$

where C *is the same constant that in (3.1) and* $\Lambda(\mu)$ *is the Lorentz space associated to isoperimetric profile of* μ.

Proof. Let $u\in V_{L^1}^1(\mathbb{R},\mu)$, we can assume that $\int_{\mathbb{R}} u=0$ (otherwise we consider $u-\int_{\mathbb{R}} ud\mu$). By Lemma 1, u_μ^* is a locally absolutely continuos function, therefore we can write

$$u_\mu^*(t)=\int_t^{1/2}\left(-u_\mu^*\right)'(s)ds+u_\mu^*(1/2). \tag{3.3}$$

Using (3.3) and (2.10) we have that

$$\begin{aligned}
\|u\|_{\Lambda(\mu)} &\leq 2\left(\int_0^{1/2}\left(\int_t^{1/2}\left(-u_\mu^*\right)'(\tau)\right)\frac{I(t)}{t}dt + u_\mu^*(1/2)\int_0^{1/2}\frac{I(t)}{t}dt\right) \\
&\leq 2\left(\int_0^{1/2}\left(-u_\mu^*\right)'(t)\left(\int_t^{1/2}\frac{I(\tau)}{\tau}d\tau\right)dt + u_\mu^*(1/2)\int_0^{1/2}\frac{I(t)}{t}dt\right) \\
&\leq 2C\left(\int_0^{1/2}\left(-u_\mu^*\right)'(t)I(t)dt + I\left(\frac{1}{2}\right)u_\mu^*(1/2)\right) \quad \text{(by (3.1))} \\
&\leq 2C\left(\int_0^{1/2}\left(-u_\mu^*\right)'(t)I(t)dt + 2I\left(\frac{1}{2}\right)\int_0^{1/2}u_\mu^*(t)dt\right).
\end{aligned}$$

Using again Lemma 1, we get that

$$\begin{aligned}
\int_0^{1/2}\left(-u_\mu^*\right)'(t)I(t)dt &= \int_0^{1/2}\left[I(\bullet)\left(-u_\mu^*\right)'(\bullet)\right]^*(r)dr \\
&\leq \int_0^{1/2}\left(u'(\bullet)\right)_\mu^*(r)dr \\
&\leq \|u'\|_{L^1(\mathbb{R},\mu)},
\end{aligned}$$

and, by the Poincaré inequality (see Proposition 1.4 of [4]), we have that

$$2I\left(\frac{1}{2}\right)\int_0^{1/2}u_\mu^*(t)\,dt \leq 2I\left(\frac{1}{2}\right)\|u\|_{L^1(\mathbb{R},\mu)} \leq \|u'\|_{L^1(\mathbb{R},\mu)}.$$

Therefore

$$\|u\|_{\Lambda(\mu)} \leq 4C\,\|u'\|_{L^1(\mathbb{R},\mu)}.$$

□

Remark 1. In the case of Gauss measure inequality (3.2) is well-known (see *e.g.* [17] and [9]).

Lemma 3. *Let $J \subset \mathbb{R}$ be an interval. Then for all Borel set $A \subset J$ the following relative isoperimetric inequality holds:*

$$P_\mu(A,J) \geq \widetilde{h}_J(\mu(A)), \tag{3.4}$$

where

$$\widetilde{h}_J(s) = \frac{1}{16C}\min\{I(s), I(\mu(J)-s)\},$$

here I is the siopemetric profile of μ and C is the same constant that appears in (3.1).

Proof. Let $J=(a,b)$ with $-\infty < a < b < +\infty$, and let u be a Lipschitz function on $\mathbb{R}$. Let us define

$$U(x) = \begin{cases} u(a) & x \in (-\infty, a), \\ u(x) & x \in (a,b), \\ u(b) & x \in (b,\infty). \end{cases}$$

Obviously $U \in V^1_{L^1}(\mathbb{R},\mu)$ and

$$U'(x) = u'(x) \text{ on } J \text{ and } U'(x) = 0 \text{ outside } J.$$

Lemma 2 applied to the function U, gives

$$\left\| U - \int_{\mathbb{R}} U \right\|_{\Lambda(\mu)} \leq 4C \left\| U' \right\|_{L^1(\mathbb{R},\mu)} = 4C \left\| u' \chi_J \right\|_{L^1(\mathbb{R},\mu)} = 4C \left\| u' \right\|_{L^1(J,\mu)} . \tag{3.5}$$

Obviously,

$$\begin{aligned} \left\| U - \int_{\mathbb{R}} U \right\|_{\Lambda(\mu)} &\geq \inf_{\alpha \in \mathbb{R}} \left\| U - \alpha \right\|_{\Lambda(\mu)} \\ &\geq \inf_{\alpha \in \mathbb{R}} \left\| (U - \alpha) \chi_J \right\|_{\Lambda(\mu)} \\ &= \inf_{\alpha \in \mathbb{R}} \left\| (u - \alpha) \chi_J \right\|_{\Lambda(\mu)} \\ &\geq \frac{1}{2} \left\| u \chi_J - \frac{1}{\mu(J)} \int_J u d\mu \right\|_{\Lambda(\mu)} \qquad \text{(by (2.7))}. \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6) we obtain

$$\left\| u \chi_J - \frac{1}{\mu(J)} \int_J u d\mu \right\|_{\Lambda(\mu)} \leq 8C \left\| u' \right\|_{L^1(J,\mu)} .$$

Suppose that $A \subset J$ is a Borel set. We may assume, without loss, that $P_\mu(A; J) < \infty$. By [4, Lemma 3.7] we can select a sequence $\{u_n\}_{n \in N}$ of Lipschitz functions such that $u_n \to \chi_A$ pointwise, and

$$P_\mu(A; J) \geq \lim \sup_{n \to \infty} \left\| (u_n)' \right\|_{L^1(J,\mu)} .$$

Consequently, by Fatou lemma

$$\begin{aligned} 8CP_\mu(A; J) &\geq 8C \lim \sup_{n \to \infty} \left\| (u_n)' \right\|_{L^1(J,\mu)} \geq \lim \sup_{n \to \infty} \left\| u_n \chi_J - \frac{1}{\mu(J)} \int_J u_n d\mu \right\|_{\Lambda(\mu)} \\ &\geq \left\| \chi_A - \frac{1}{\mu(J)} \int_A d\mu \right\|_{\Lambda(\mu)} \\ &= \left(1 - \frac{\mu(A)}{\mu(J)} \right) \int_0^{\mu(A)} \frac{I(s)}{s} ds + \frac{\mu(A)}{\mu(J)} \int_{\mu(A)}^{\mu(J)} \frac{I(s)}{s} ds. \end{aligned}$$

If $\mu(A) \leq \frac{\mu(J)}{2}$, then

$$\begin{aligned} \left\| \chi_A - \frac{1}{\mu(J)} \int_A d\mu \right\|_{\Lambda(\mu)} &= \int_0^{\mu(J)} \left(\chi_A - \frac{\mu(A)}{\mu(J)} \right)_\mu^* (s) \frac{I(s)}{s} ds \\ &= \left(1 - \frac{\mu(A)}{\mu(J)} \right) \int_0^{\mu(A)} \frac{I(s)}{s} ds + \frac{\mu(A)}{\mu(J)} \int_{\mu(A)}^{\mu(J)} \frac{I(s)}{s} ds \\ &\geq \left(1 - \frac{\mu(A)}{\mu(J)} \right) I(\mu(A)) \geq \frac{1}{2} I(\mu(A)). \end{aligned}$$

In case that $\mu(A) \geq \frac{\mu(J)}{2}$, then, since the measure μ has continuos density, the sets A and $J \backslash A$ has the same perimeter; since $\mu(J \backslash A) \leq \frac{\mu(J)}{2}$ we get

$$8CP_\mu(A; J) = 8CP_\mu(J \backslash A; J) \geq \frac{1}{2} I(\mu(J) - \mu(A)).$$

$\square$

Lemma 4. *Let $u \in V^1_{L^1}(\mathbb{R},\mu)$ and J an interval. Then;*

$$\int_J \left| u\chi_J - (u\chi_J)^\circ_\mu\left(\frac{\mu(J)}{2}\right)\right| d\mu = \int_0^{\mu(J)} -\left((u\chi_J)^\circ_\mu\right)'(s)\min\{s,\mu(J)-s\}ds. \tag{3.7}$$

Proof. By absolutely continuity of $(u\chi_J)^\circ_\mu$ (Lemma 1) and Fubini's theorem, we have that

$$\int_J \left| u\chi_J - (u\chi_J)^\circ_\mu\left(\frac{\mu(J)}{2}\right)\right| d\mu = \int_0^{\mu(J)} \left|\left(u\chi_J - (u\chi_J)^\circ_\mu\left(\frac{\mu(J)}{2}\right)\right)^\circ_\mu\right| ds$$

$$= \int_0^{\frac{\mu(J)}{2}} \left[(u\chi_J)^\circ_\mu(s) - (u\chi_J)^\circ_\mu\left(\frac{\mu(J)}{2}\right)\right] ds + \int_{\frac{\mu(J)}{2}}^{\mu(J)} \left[(u\chi_J)^\circ_\mu(s) - (u\chi_J)^\circ_\mu\left(\frac{\mu(J)}{2}\right)\right] ds$$

$$= \int_0^{\frac{\mu(J)}{2}}\int_s^{\frac{\mu(J)}{2}} -\left((u\chi_J)^\circ_\mu\right)'(\tau)d\tau ds + \int_{\frac{\mu(J)}{2}}^{\mu(J)}\int_{\frac{\mu(J)}{2}}^{s} -\left((u\chi_J)^\circ_\mu\right)'(\tau)d\tau ds$$

$$= \int_0^{\mu(J)} \min\{s,\mu(J)-s\}\left(-(u\chi_J)^\circ_\mu\right)'(s)ds.$$

□

Proof of Theorem 1.
i)$\Longrightarrow$ *ii)*
Given $u \in V^1_X(\mathbb{R},\mu)$ and an interval J with $\mu(J) = a$, we denote $v = u\chi_J$. Then,

$$\begin{aligned}
V(a) &:= \frac{1}{2}\int_0^a \left| v^\circ_\mu(s) - \frac{1}{a}\int_0^a v^\circ_\mu(t)dt\right| ds \le \inf_\alpha \int_0^a |v-\alpha|^\circ_\mu(s)ds \text{ (by (2.7))}\\
&\le \int_J \left| v - v^\circ_\mu\left(\frac{a}{2}\right)\right| d\mu = \frac{1}{a}\int_0^a -\left(v^\circ_\mu\right)'(s)\min\{s,a-s\}ds \quad \text{(by (3.7))}\\
&\le \left\|\widetilde{h}_I(r)\left(-\left(v^\circ_\mu\right)'\right)\chi_{(0,a)}(r)\right\|_{\overline{X}} \left\|\frac{\min\{r,a-r\}}{\widetilde{h}_I(r)}\chi_{(0,a)}(r)\right\|_{\overline{X}'} \quad \text{(Hölder's inequality)}\\
&\le \|u'\|_X \left\|\frac{\min\{r,a-r\}}{\widetilde{h}_I(r)}\chi_{(0,a)}(r)\right\|_{\overline{X}'} \quad \text{(by Lemma 1).}
\end{aligned}$$

Using the symmetry of $\widetilde{h}_I$ around the point $a/2$ and the definition of $\widetilde{h}_I$, it follows that

$$\begin{aligned}
\left\|\frac{\min\{r,a-r\}}{\widetilde{h}_I(r)}\chi_{(0,a)}(r)\right\|_{\overline{X}'} &= 2\left\|\frac{r}{\widetilde{h}_I(r)}\chi_{(0,\frac{a}{2})}(r)\right\|_{\overline{X}'}\\
&\le 32C\left\|\frac{r}{I(r)}\chi_{(0,\frac{a}{2})}(r)\right\|_{\overline{X}'}.
\end{aligned}$$

Therefore

$$V(a) \le 32C\,\|u'\|_X \left\|\frac{r}{I(r)}\chi_{(0,\frac{a}{2})}(r)\right\|_{\overline{X}'}.$$

By the definition of the rearrangements

$$\frac{1}{a}\int_J \left| v - \frac{1}{a}\int_J v d\mu \right| d\mu = \frac{1}{a}\int_0^a \left(v - \frac{1}{a}\int_J v d\mu \right)_\mu^* ds$$
$$= \frac{1}{a}\int_0^a \left| v_\mu^\circ(s) - \frac{1}{a}\int_0^\alpha v_\mu^\circ(t)dt \right| ds$$
$$= \frac{2}{a}V(a),$$

thus

$$\frac{1}{a}\int_J \left| v - \frac{1}{a}\int_J v d\mu \right| d\mu \leq 64C \left\| u' \right\|_X \sup_{0<s<\frac{1}{2}} \frac{1}{s} \left\| \frac{r}{I(r)} \chi_{(0,s)}(r) \right\|_{\overline{X}'},$$

taking the supremum we get

$$\left\| u \right\|_{*,(\mathbb{R},\mu)} \leq 64C \left\| u' \right\|_X \sup_{0<s<\frac{1}{2}} \frac{1}{s} \left\| \frac{r}{I(r)} \chi_{(0,s)}(r) \right\|_{\overline{X}'}.$$

ii)⇒iii)

As was pointed in the introduction (inequality (1.4)) there is a positive constant c such that

$$\left\| u \right\|_{L^{\infty,\infty}} = \sup_{0<t<1} \left(u_\mu^{**}(t) - u_\mu^*(t) \right) \leq c \left\| u \right\|_{BMO(\mathbb{R},\mu)}$$
$$= c\left(\left\| u \right\|_{*,(\mathbb{R},\mu)} + \left\| u \right\|_{L^1(\mathbb{R},\mu)} \right).$$

By hypothesis *ii)* it follows that

$$\left\| u \right\|_{L^{\infty,\infty}} \leq C_2 \left(\left\| u' \right\|_{(\mathbb{R},\mu)} + \left\| u \right\|_{L^1(\mathbb{R},\mu)} \right)$$

for some positive constant $C_2 > 0$.

iii)⇒ i)

Given a positive measurable function $g \in \bar{X}$, with *suppg* $\subset (0, 1/2)$ we consider

$$G(t) = \int_t^1 g(s)\frac{ds}{I(s)}, \qquad t \in (0,1)$$

and define

$$u(x) = G(F(x)) \qquad x \in \mathbb{R},$$

where F is the distribution function of μ defined by (2.1). Then

$$|u'(x)| = \left| -g(F(x))\frac{F'(x)}{I(F(x))} \right| = g(F(x)).$$

Moreover (see [13, Secction 5]),

$$(u')_\mu^*(t) = g_\mu^*(t), \tag{3.8}$$

and

$$u_\mu^*(t) = \int_t^1 g(s)\frac{ds}{I(s)}. \tag{3.9}$$

From (3.8) we get that $u \in V_X^1(\mathbb{R},\mu)$, and from (3.9) and Fubini's theorem, we obtain

$$\frac{1}{s}\int_0^s \left[u_\mu^*(r) - u_\mu^*(s) \right] dr = \frac{1}{s}\int_0^s \frac{\tau}{I(\tau)} g(\tau) d\tau.$$

By hypothesis *iii*) it follows

$$\frac{1}{s}\int_0^s \frac{\tau}{I(\tau)} g(\tau)d\tau \le C_2\left(\left\|u'\right\|_X + \left\|u\right\|_{L^1(\mathbb{R},\mu)}\right)$$

Obviously, $u_\mu^\circ(1/2) = 0$ (since $suppg \subset (0,1/2)$), then by Lemma 4 (with $J = \mathbb{R}$) we obtain

$$\begin{aligned}
\left\|u\right\|_{L^1(\mathbb{R},\mu)} &= \int_{\mathbb{R}} |u|\, d\mu = \int_0^{1/2} -\left(u_\mu^\circ\right)'(s)sds = \int_0^{1/2} -\left(u_\mu^\circ\right)'(s)I(s)\frac{s}{I(s)} \\
&\le \frac{2}{I(1/2)}\int_0^{1/2} -\left(u_\mu^\circ\right)'(s)I(s)ds \quad \text{(by Lemma1)} \\
&\le \frac{2}{I(1/2)}\left\|u'\right\|_{L^1(\mathbb{R},\mu)} \\
&\le \frac{2}{I(1/2)}\left\|u'\right\|_X,
\end{aligned}$$

i.e. there is a constant c such that

$$\frac{1}{s}\int_0^s \frac{\tau}{I(\tau)} g(\tau)d\tau \le c\left\|u'\right\|_X = c\left\|g\right\|_{\bar{X}} \quad \text{(by (3.8))}.$$

Therefore

$$\sup_{g\in\bar{X}, suppg\subset(0,1/2)} \frac{\frac{1}{s}\int_0^s \frac{\tau}{I(\tau)}g(\tau)d\tau}{\left\|g\right\|_{\bar{X}}} \le c.$$

By (2.5) the left-hand side equals to $\frac{1}{s}\left\|\frac{r}{I(r)}\chi_{(0,s)}(r)\right\|_{\overline{X}'}$ for $0 < s < 1/2$ and then

$$\sup_{0<s<1/2} \frac{1}{s}\left\|\frac{r}{I(r)}\chi_{(0,s)}(r)\right\|_{\overline{X}'} \le c.$$

□

In the next proposition we are able to identify (depending on μ) the largest *r.i.* space such that embedding of $V_X^1(\mathbb{R},\mu)$ into $BMO(\mathbb{R},\mu)$ holds.

Proposition 1. *The space*

$$M(\mu) = \left\{u : \sup_{t\in(0,1/2)} \frac{1}{I(t)}\int_0^t u_\mu^*(s)ds < +\infty\right\}$$

is the largest r.i. space such that ii) or iii) of Theorem 1 holds, i.e. for any X r.i. space such that ii) or iii) of Theorem 1 holds, then we have

$$X \subseteq M(\mu).$$

Proof. We have to check that if $X = M(\mu)$ then D is finite. By (3.1) it follows that $(M(\mu))' = \Lambda(\mu)$ (see *e.g.* [15]). Then,using the monotonicity of function $\frac{r}{I(r)}$ and (3.1), we have

$$\sup_{0<s<\frac{1}{2}} \frac{1}{s}\left\|\frac{r}{I(r)}\chi_{(0,s)}(r)\right\|_{\Lambda(\mu)} \le \frac{1}{2I(s)}\left\|\chi_{(0,s)}(r)\right\|_{\Lambda(\mu)} = \frac{1}{2I(s)}\int_0^s \frac{I(t)}{t}dt \le \frac{1}{2}C.$$

Let us assume that $X \nsubseteq M(\mu)$. Then there exists a function $g \in X$ with $\|g\|_X = 1$ such that $g \notin M(\mu)$ and there exists a sequence $\{t_k\}_{k\in\mathbb{N}} \subset \left(0, \frac{1}{2}\right)$ such that $\frac{1}{I(t_k)} \int_0^t g_\mu^*(t_k) ds \to +\infty$. Moreover by (2.5) and monotonicity of function $\frac{r}{I(r)}$

$$\begin{aligned} \sup_{0<s<\frac{1}{2}} \frac{1}{s} \left\| \frac{r}{I(r)} \chi_{(0,s)}(r) \right\|_{\overline{X}'} &\geq \sup_{k\in\mathbb{N}} \frac{1}{t_k} \int_0^{1/2} g_\mu^*(s) \left(\frac{r}{I(r)} \chi_{(0,s)}(r) \right)_\mu^* (s) ds \\ &\geq \sup_{k\in\mathbb{N}} \frac{1}{t_k} \int_0^{\frac{t_k}{2}} g_\mu^*(s) \frac{t_k - s}{I(t_k - s)} ds \\ &\geq \sup_{k\in\mathbb{N}} \frac{1}{2I(\frac{t_k}{2})} \int_0^{\frac{t_k}{2}} g_\mu^*(s) ds \to +\infty. \end{aligned}$$

This gives a contradiction and the assert follows. □

Remark 2. We point out that if a measure μ is non doubling, Calderón-Zygmund operators may be bounded on $L^2(\mu)$ but not from L^∞ into $BMO(\mathbb{R}, \mu)$. To overcome this difficulty some new BMO-type spaces have been introduced (see [19], [14] and references quoted there in, and in the case of the gaussian measure see also [10]). This new BMO-type spaces are imbedded into $BMO(\mathbb{R}, \mu)$, therefore our Theorem 1 also works.

We finish the paper with some examples of the largest *r.i.* space

$$M(\mu) = \left\{ u : \sup_{t\in(0,1/2)} \frac{1}{I(t)} \int_0^t u_\mu^*(s) ds < +\infty \right\}$$

for which embedding of $V_X^1(\mathbb{R}, \mu)$ into $BMO(\mathbb{R}, \mu)$ holds. We have to consider the asymptotic behavior of the isoperimetric profile for different measure (see *e.g.* [18]).

- *Gauss measure* $d\mu = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{|x|^2}{2}\right) dx$.
  We know that $\lim_{t\to 0} \frac{I(t)}{t\left(2\log\frac{1}{t}\right)^{\frac{1}{2}}} = 1$, then $M(\mu)$ is the Zygmund space (see *e.g.* [3])
  $$M(\mu) = L^\infty (\log L)^{-\frac{1}{2}} (\mathbb{R}, \mu).$$
- *Exponetial-type measure* $d\mu = \frac{1}{2p^{\frac{1}{p}}\Gamma\left(1+\frac{1}{p}\right)} \exp\left(-\frac{|x|^p}{p}\right) dx$ *for* $p > 1$.
  For this kind of measure, $\lim_{t\to 0} \frac{I(t)}{t\left(p\log\frac{1}{t}\right)^{1-\frac{1}{p}}} = 1$ and $M(\mu)$ is the Zygmund space
  $$M(\mu) = L^\infty (\log L)^{\frac{1}{p}-1} (\mathbb{R}, \mu).$$
- *Two-sided exponential* $d\mu = \frac{1}{2} \exp(-|x|) dx$.
  It is well-known that $I(t) = \min\{t, 1-t\}$, then $M(\mu)$ is the Lebesgue space
  $$M(\mu) = L^\infty(\mathbb{R}).$$
- $d\mu = \frac{1}{Z} \exp\left(-|x|^p \log\left(\exp\left(\frac{2\alpha}{2-p}\right) + |x|^\alpha\right)\right) dx$, $\alpha \geq 0$ and $p \in [1, 2]$.
  In this case $M(\mu)$ is the generalized Lorentz-Zygmund space
  $$M(\mu) = \left\{ u : \sup_{t\in(0,1)} \frac{1}{t\left(1-\log t\right)^{1-\frac{1}{p}} \left(1+\log\left(1-\log(t)\right)\right)^{\frac{\alpha}{p}}} \int_0^t u_\mu^*(s) ds < +\infty \right\},$$
  since $\lim_{t\to 0} \frac{I(t)}{t\left(\log\frac{1}{t}\right)^{1-\frac{1}{p}} \left(\log\log\left(e+\frac{1}{t}\right)\right)^{\frac{\alpha}{p}}} = \ell \in (0, +\infty)$.

- $d\mu = \frac{1}{Z_\Phi} \exp\left(-\Phi(|x|)\right) dx$ *with* $\Phi(t)$ *is convex and* $\sqrt{\Phi(t)}$ *concave on* $(0, +\infty)$. For these measure it is possible to prove that $\lim_{t\to 0} \frac{I(t)}{t\Phi'\left(\Phi^{-1}\left(\log\frac{1}{t}\right)\right)} = \ell \in (0, +\infty)$.

4. Appendix: proof of inequality (3.1)

We organize the proof in 3 steps.

Step 1. For every $0 < \alpha < 1$, we claim that $\liminf_{t\to 0} \ t^{-\alpha} I(t) = 0$.

Indeed, if there exists $0 < \alpha < 1$ such that $\liminf_{t\to 0} \ t^{-\alpha} I(t) = \ell > 0$, then

$$\sup_{0<t<1/2} \frac{t^\alpha}{I(t)} = c > 0,$$

thus

$$\int_0^{\frac{1}{2}} \frac{dt}{I(t)} \le c \int_0^{\frac{1}{2}} \frac{dt}{t^\alpha} < +\infty,$$

but by (2.2) we known $\int_0^{\frac{1}{2}} \frac{dt}{I(t)} = +\infty$.

Step 2. For every $0 < \alpha < 1$, the function $t^{-\alpha} I(t)$ is increasing near zero.

We have to prove the derivative of this function is non negative near zero. Since

$$\left(t^{-\alpha} I(t)\right)' = -\alpha t^{-\alpha-1} I(t) + t^{-\alpha} I'(t) \ge 0 \Leftrightarrow \frac{I'(t)}{I(t)} \ge \frac{\alpha}{t},$$

integrating the last inequality between r and $1/2$, and after a straightforward computation we obtain

$$I\left(\frac{1}{2}\right) \ge \frac{I(r)}{2^\alpha r^\alpha},$$

which is positive if r is small enough by Step 1.

Step 3. Proof of (3.1).

Fix $0 < \alpha < 1$, by the previous step, we know that there exists $\varepsilon > 0$ such that $t^{-\alpha} I(t)$ is increasing in $(0, \varepsilon)$. For $0 < t \le \varepsilon$,

$$\int_0^t \frac{I(s)}{s} ds = \int_0^t \frac{I(s)}{s^\alpha} s^{\alpha-1} ds \le t^{-\alpha} I(t) \int_0^t s^{\alpha-1} ds \le \frac{I(t)}{\alpha}.$$

Otherwise when $\varepsilon \le t \le 1/2$, we have

$$\begin{aligned}
\int_0^t \frac{I(s)}{s} ds &= \int_0^\varepsilon \frac{I(s)}{s} ds + \int_\varepsilon^t \frac{I(s)}{s} ds \\
&= \varepsilon^{-\alpha} I(\varepsilon) \int_0^\varepsilon t^{\alpha-1} dt + \frac{I(\varepsilon)}{\varepsilon}(t - \varepsilon) \\
&\le C I(t),
\end{aligned}$$

for some positive constant C, since I is an increasing function.

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References

[1] D. Aalto, Weak L^∞ and BMO in metric spaces. Boll. Unione Mat. Ital. (9) 5 (2012), n. 2, 369–385.

[2] C. Bennett, R. De Vore and R. Sharpley, *Weak-L^∞ and BMO*, Annals of Math. **113** (1981), 601–611.

[3] C. Bennett, R.C. Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics, 129, Academic Press, Orlando (1988).

[4] S. Bobkov, C. Houdré, Some connections between isoperimetric and Sobolev-type inequalities. Mem. Amer. Math. Soc. 129 (1997), n. 616, viii+111 pp.

[5] S. Bobkov, Extremal properties of half-spaces for log-concave distributions. Ann. Probab. 24 (1996), n. 1, 35–48.

[6] A. Cianchi, L. Pick, *Sobolev embeddings into BMO, VMO and L^∞*, Ark. Mat. 36 (1998), n. 2, 317–340.

[7] Cianchi, Andrea; Pick, Luboš, Optimal Gaussian Sobolev embeddings, J. Funct. Anal. 256 (2009), n. 11, 3588–3642.

[8] G.G. Lorentz, On the theory of spaces Λ, Pacific J. Math. 49 (1951), 411-429.

[9] F. Feo, M.R. Posteraro, *Logarithmic Sobolev trace inequalities*, Asian J. Math. 17 (2013), n. 3, 569–582.

[10] G. Mauceri, S. Meda, *BMO and H^1 for the Ornstein-Uhlenbeck operator*, J. Funct. Anal. 252 (2007), n. 1, 278–313.

[11] Mateu, J.; Mattila, P.; Nicolau, A.; Orobitg, J. BMO for nondoubling measures. Duke Math. J. 102 (2000), n. 3, 533–565.

[12] J. Martín and M. Milman, Isoperimetry and symmetrization for logarithmic Sobolev inequalities, J. Funct. Anal. 256 (2009) 149–178.

[13] J. Martin and M. Milman, *Pointwise symmetrization inequalities for Sobolev functions and applications*, Adv. Math **225** (2010), 121-199.

[14] F. Nazarov,S. Treil,A. Volberg, The Tb-theorem on non-homogeneous spaces. Acta Math. 190 (2003), no. 2, 151–239.

[15] A. Gogatishivili, L. Pick, Embeddings and duality theorems for weak classical Lorentz spaces, Canad. Math. Bull. 49 (2006), n. 1, 82–95.

[16] S. G. Krein, Yu I. Petunin, E. M. Semenov, Interpolation of linear operators, *Transl. Math. Monogr. Amer. Math, Soc.* **54,** Providence**,** (1982).

[17] E. Pelliccia, G. Talenti, A proof of a logarithmic Sobolev inequality, Calc. Var. Partial Differential Equations 1 (1993), no. 3, 237–242.

[18] C, Roberto, *Isoperimetry for product probability measures*, Markov Processes and Related Fields, 16, n. 4, 617-634 (2010).

[19] X. Tolsa, BMO, H 1 , and Calderón-Zygmund operators for non doubling measures. Math. Ann. 319 (2001), n. 1, 89–149.

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