THE INVERSION HEIGHT OF THE FREE FIELD IS INFINITE

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Abstract. Let X be a finite set with at least two elements, and let k be any commutative field. We prove that the inversion height of the embedding $k\langle X\rangle \hookrightarrow D$, where D denotes the universal (skew) field of fractions of the free algebra $k\langle X\rangle$, is infinite. Therefore, if H denotes the free group on X, the inversion height of the embedding of the group algebra kH into the Malcev-Neumann series ring is also infinite. This answers in the affirmative a question posed by Neumann in 1949 [32, p. 215].

We also give an infinite family of examples of non-isomorphic fields of fractions of $k\langle X\rangle$ with infinite inversion height.

We show that the universal field of fractions of a crossed product of a field by the universal enveloping algebra of a free Lie algebra is a field of fractions constructed by Cohn (and later by Lichtman). This extends a result by A. Lichtman.

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1. Introduction

Let X be a set with $|X| \geq 2$. Let H be the free group on X, and let k be a commutative field. Choose a total order on H such that $(H, <)$ is an ordered group. Consider the Malcev-Neumann series ring $k((H, <))$ associated with the group ring kH. B.H. Neumann conjectured in [32, p. 215] that

(N) the inversion height of the embedding $kH \hookrightarrow k((H, <))$ is infinite. Equivalently, in the (skew) subfield $E = E(X)$ of $k((H, <))$ generated by kH there exist elements which need an arbitrarily large number of nested inversions to be constructed as rational expressions from elements of kH.

The field $E = E(X)$ can be characterized by its categorical properties. It was proved by Lewin [22] that it is the universal field of fractions of kH and, hence, it is also the universal field of fractions of $k\langle X\rangle$, the free algebra on X; because of that E is usually named *the free field on* X. We recall that $k\langle X\rangle$ can also be regarded as the enveloping algebra of the free Lie algebra on X.

The interest in conjecture (N) was renewed in [14] where the theory of quasideterminants was developed. C. Reutenauer brilliantly proved in [34, Theorem 2.1] (see also [5, Chapter 4]) that the conjecture holds when X is infinite. As suggested in [34, Section 5.2], it was expected that (N) should hold in general because a free algebra R over a set of at least two elements contains many subalgebras S that are isomorphic to a free algebra over an infinite (countable) set. The difficulty in settling the question with this approach was being able to choose a subalgebra S such that the universal field of fractions of S can be viewed inside that of $R = k\langle X\rangle$ and that, in addition, the inversion height is preserved through the embedding. In this paper, we overcome this problem considering the more flexible structure of crossed product. Viewing R as a crossed product of the subalgebra S with *something else* we can produce, via Reuteneauer's result, elements in E of arbitrary inversion height. Hence we give the final step to solve conjecture (N).

Crossed products can be considered in the group context, in the context of Lie algebras or, unifying both settings, for Hopf algebras. They have proved to be specially suitable for induction-type arguments and in the construction of quantum deformations of classical algebraic objects.

Throughout the paper, we give several constructions of elements in the free field E of arbitrary inversion height, keeping in parallel the point of view of crossed products of Lie algebras and that of group crossed products. In Section 4, we give the most elementary constructions to produce elements of arbitrarily large inversion height inside the free field. We make a variation of an embedding of free algebras due to Cohn [10] that allows us to regard the free algebra as an Ore differential extension of a free algebra on infinitely many variables. Then we are able to give an elementary solution to conjecture (N) in Theorem 4.5. We remark that such extensions are the easiest examples of crossed products of Lie algebras.

On the group side, if H is a free group, any onto group homomorphism φ from H to an infinite cyclic group allows us to view kH as a skew Laurent polynomial ring with coefficients in the group algebra over the free group $\operatorname{Ker}\varphi$, again this is the easiest example of crossed product of groups. Such a description of the group algebra allows us to give in Theorem 4.6 another elementary solution to conjecture (N).

In Section 5, we make essential use of the theory of crossed products of groups to produce infinitely many non-isomorphic embeddings of the free algebra into division rings of infinite inversion height. Hence, the property of having infinite inversion height does not characterize the universal field of fractions.

In Sections 6 and 7, we develop some specific theory of crossed products for Lie algebras, and we give a construction of a field of fractions, as a subfield of a power series ring, for the crossed product of a field by a residually nilpotent Lie algebra with a Q-basis. In the case of a free Lie algebra H or, more generally, when the crossed product is a fir, this gives a construction of the universal field of fractions. In Section 8, we use this theory to produce further examples of elements with arbitrarily large inversion height in the free field. A different line of applications of this construction is given in Example 7.17 to the enveloping algebra of the free Poisson field, cf. [28].

In the case of an ordered group, the Malcev-Neumann series ring gives a very neat way to embed a crossed product of an arbitrary field by the group into a field. As mentioned before, when the group is free, this yields an embedding of the universal field of fractions of the crossed product in such power series rings. This was proved by Lewin in [22] using a deep result of Hughes on the uniqueness of certain fields of fractions [18].

On the Lie algebra side, a well known result of Cohn implies that any crossed product of a field by the universal enveloping algebra of a Lie algebra can be embedded into a field, cf. Proposition 6.6, which we call the canonical field of fractions. But an analog of the Malcev-Neumann series ring construction, possibly containing the canonical field of fractions, is missing in the setting of *ordered Lie algebras*. Our main results in Section 6 aim to fill this gap in the case of crossed products of residually nilpotent Lie algebras with a Q-basis. In our constructions, we follow and extend results and ideas due to Lichtman [25, 26].

As we have already mentioned, all our results on inversion height are based on results of Reutenauer's. It seems an interesting and challenging question to extend Reutenauer's results from commutative fields to arbitrary (skew) fields. We note that our approach of passing from the case of countably infinitely many variables to the finite one does not use any commutativity and it works for general crossed products.

2. Preliminaries

We begin this section by fixing some notions that will be used throughout the paper.

All rings are assumed to be associative and with 1. A morphism of rings $\alpha\colon R \to S$ always preserves 1's, i.e. α sends 1_R to 1_S.

By an *embedding* $\iota\colon R \hookrightarrow E$ we mean an injective morphism of rings where we identify R with its image in E.

A *domain* is a nonzero ring R such that the product of any two nonzero elements is nonzero.

Following [11], a *field* E is a nonzero ring such that every nonzero element has an inverse, i.e. if $x \in E \setminus \{0\}$ there exists $x^{-1} \in E$ such that $xx^{-1} = x^{-1}x = 1$.

Note that domains and fields are not assumed to be commutative. In the literature, our concept of field is also known as division ring or skew field.

2.1. Skew polynomial rings and skew Laurent series.

The material of this subsection is quite standard and can be found in the literature. We follow the notation of [15, Chapter 1].

Let S be a ring and $\alpha\colon S \to S$ an injective endomorphism of rings.

A (left) *α-derivation* is an additive map $\delta\colon S \to S$ such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$.

We denote by $S[x;\alpha,\delta]$ the *skew polynomial ring*. It is a ring extension of S which is a free left S-module with basis $\{1, x, \dots, x^n, \dots\}$, thus the elements can be uniquely written as

$$a_0 + a_1x + \dots + a_nx^n \text{ with } a_i \in S,\ n \in \mathbb{N},\ a_n \neq 0;$$

the product is determined by the rule $xa = \alpha(a)x + \delta(a)$ for all $a \in S$. When $\delta = 0$, we write $S[x;\alpha]$ instead of $S[x;\alpha,0]$, and when α is the identity on S, we write $S[x;\delta]$ instead of $S[x;\alpha,\delta]$.

If S is a domain, the ring $S[x;\alpha,\delta]$ is also a domain. If S is a left Ore domain, then $S[x;\alpha,\delta]$ is a left Ore domain. If S is a field, we denote its left Ore field of fractions by $S(x;\alpha,\delta)$ (respectively $S(x;\alpha)$, $S(x;\delta)$).

When $\delta = 0$, we can consider the skew series ring $S[[x;\alpha]]$ which consists of all infinite series

$$a_0 + a_1x + \dots + a_nx^n + \cdots,\ a_n \in S \text{ for all } n \in \mathbb{N},$$

with component-wise addition and multiplication based on the commutation rule

$$xa = \alpha(a)x, \text{ for all } a \in S.$$

The set $\{1, x, \dots, x^n, \dots\}$ is a left Ore set in $S[[x;\alpha]]$, and we denote its Ore localization by $S((x;\alpha))$. The elements of $S((x;\alpha))$ are of the form

$$x^{-r}\sum_{n=0}^{\infty} a_n x^n \text{ with } r\in\mathbb{N},\ a_n\in S \text{ for all } n.$$

If S is a field, $S((x;\alpha))$ is a field that contains $S(x;\alpha)$. If α is bijective, the elements of $S((x;\alpha))$ can be written as $\sum_{n\geq -r} a_n x^n$ with $r\in\mathbb{N}$ and $a_n\in S$ for all n.

When $\delta=0$ and α is bijective, the subring of $S((x;\alpha))$ consisting of the polynomials of the form

$$a_{-m}x^{-m}+a_{-m+1}x^{-m+1}+\cdots+a_0+a_1x+\cdots+a_nx^n, \quad \text{with } a_i\in S,\ m,n\in\mathbb{N},$$

is called the *skew Laurent polynomial ring* and denoted by $S[x,x^{-1};\alpha]$. If S is a left Ore domain, $S[x,x^{-1};\alpha]$ is also a left Ore domain. If S is a field, the left Ore field of fractions is $S(x;\alpha)$.

When $\delta\neq 0$ and α is injective, a similar ring of series can be constructed (to understand its definition, notice that the relation $xa=\alpha(a)x+\delta(a)$ implies that $ax^{-1}=x^{-1}\alpha(a)+x^{-1}\delta(a)x^{-1}$). Introduce a new variable $y(=x^{-1})$, and consider the ring of series

$$a_0+ya_1+\cdots+y^na_n+\cdots \text{ with } a_n\in S \text{ for all } n\in\mathbb{N},$$

(coefficients on the right) with component-wise addition and multiplication based on the commutation rule

$$ay=y\alpha(a)+y^2\alpha(\delta(a))+\cdots+y^n\alpha(\delta^{n-1}(a))+\cdots=\sum_{n\geq 1}y^n\alpha(\delta^{n-1}(a)), \quad (2.1)$$

for each $a\in S$. This ring of series will be denoted by $S[[y;\alpha,\delta]]$. The set $\{1,y,\dots,y^n,\dots\}$ is a right Ore set and we denote by $S((y;\alpha,\delta))$ its Ore localization. So the elements of $S((y;\alpha,\delta))$ are of the form

$$\Big(\sum_{n=0}^{\infty}y^na_n\Big)y^{-r} \text{ with } r\in\mathbb{N},\ a_n\in S \text{ for all } n\in\mathbb{N}.$$

The idea for the construction of the ring $S((y;\alpha,\delta))$ goes back to Schur, such rings are also called *formal pseudo-differential operator rings*. We note that if S is a field, then $S((y;\alpha,\delta))$ is a field.

From 2.1, it is easy to see that the assignment $x\mapsto y^{-1}$ induces an injective morphism of rings $S[x;\alpha,\delta]\to S((y;\alpha,\delta))$ which is the identity on S. The universal property of the Ore localization, allows us to extend this embedding to an embedding of fields $S(x;\alpha,\delta)\to S((y;\alpha,\delta))$.

Finally, we observe that if α is an automorphism then the elements of $S((y;\alpha,\delta))$ can be written, in a unique way, in the form

$$\sum_{n\geq l}^{\infty}a_ny^n \text{ with } l\in\mathbb{Z},\ a_n\in S \text{ for all } n\geq l.$$

2.2. **Crossed products and Malcev-Neumann series.** Let R be a ring, and let G be a group. We define a *crossed product* RG (of R by G) as an associative ring which contains R constructed in the following way. It is a free left R-module with basis $\overline{G}$, a copy (as a set) of G. The elements in RG are uniquely written as $\sum_{x\in G} a_x\bar{x}$ where only a finite number of $a_x \in R$ are nonzero. Multiplication is determined by the two rules below:

Twisting. For $x, y \in G$

$$\bar{x}\bar{y} = \tau(x,y)\overline{xy}$$

where $\tau\colon G\times G \longrightarrow R^\times$ is a set map and $R^\times$ denotes the group of units of R.

Action. For $x\in G$ and $r\in R$

$$\bar{x}r = {}^{\sigma(x)}r\bar{x}$$

where $\sigma\colon G\to \operatorname{Aut}(R)$ is a set map, $\operatorname{Aut}(R)$ denotes the group of automorphisms of R and ${}^{\sigma(x)}r$ denotes the image of r by $\sigma(x)$. Hence if $\sum_{x\in G} a_x\bar{x}, \sum_{x\in G} b_x\bar{x} \in RG$, their product is

$$\sum_{x\in G}\Big(\sum_{yz=x} a_y{}^{\sigma(y)}b_z\tau(y,z)\Big)\bar{x}. \tag{2.2}$$

In Lemma 2.1, we shall recall the conditions on σ and τ that make this multiplication associative.

If H is a subgroup of G, then $RH = \{\eta\in RG \mid \operatorname{supp}\eta\subseteq H\}$ is the naturally embedded sub-crossed product.

If $d\colon G\to R^\times$ assigns to each element $x\in G$ a unit d_x, then $\widetilde{G} = \{\tilde{x} = d_x\bar{x} \mid x\in G\}$ is another R-basis for RG in terms of which RG still has the form of a crossed product. Therefore, after a change of basis if necessary, we will always suppose that $1_{RG} = \bar{1}$.

Lemma 2.1. ([33, Lemma 1.1]) *Let R be a ring and let G be a group. Let $\tau\colon G\times G\to R^\times$ and $\sigma\colon G\to \operatorname{Aut}(R)$ be set maps. Then τ and σ are the action and the twisting, respectively, of a crossed product RG if and only if for any x, y, $z\in G$ the following two conditions are satisfied*

(i) $\tau(x,y)\tau(xy,z) = {}^{\sigma(x)}\tau(y,z)\tau(x,yz)$.

(ii) $\sigma(x)\sigma(y) = \eta(x,y)\sigma(xy)$ *where $\eta(x,y)\in\operatorname{Aut}(R)$ is defined by*

$$\eta(x,y)(r) = \tau(x,y)r\tau(x,y)^{-1},$$

for any $r\in R$.

Here $1_{RG}=\bar{1}$ if and only if $\tau(1,x)=\tau(x,1)=1$ for any $x\in G$ (whence, by (ii), $\sigma(1)=\operatorname{Id}_R$).

Remark 2.2. Let R be a subring of a ring S, and let G be a group. Assume that a crossed product RG, with action σ and twisting τ, is given. We will be interested in extending σ to S in order to have a crossed product SG that has RG as a subring. The first requirement to do this is that, for any $x\in G$, the automorphism $\sigma(x)$ extends to an automorphism of S. Once this is done, it is enough to check condition (ii) in Lemma 2.1; a way to guarantee the required equality is to assume that the embedding $R \hookrightarrow S$ is a ring epimorphism, so that two automorphisms of S that coincide over R must coincide over S.

Since we are interested in extensions of crossed product structures, their description via the twisting and the action is quite convenient for us. A more conceptual way to describe a crossed product of a ring R by a group G is as a G-graded ring $T = \bigoplus_{x\in G} T_x$ such that $T_1 = R$ and, for each $x \in G$, the homogeneous component T_x contains a unit $\bar{x}$, see [33, Exercise 2, p. 18].

A crucial property of crossed products is the following. If N is a normal subgroup of G then $RG = RN\frac{G}{N}$, where the latter is some crossed product of the group G/N over the ring RN.

If R is any ring and C denotes an infinite cyclic group, then any crossed product $RC \cong R[x, x^{-1}; \alpha]$ for a suitable ring automorphism $\alpha\colon R \to R$ given by conjugation by x.

We refer the reader to [33] for further details on crossed products. If k is a commutative field and R is a k-algebra, then the construction of RG is a particular case of a Hopf algebra crossed product, see [31, Chapter 7].

We say that a group G is an *orderable group* if there exists a total order $<$ on G which is compatible with the product defined on G, that is, $x < y$ implies that $zx < zy$ and $xz < yz$ for all $x, y, z \in G$. In this event $(G, <)$ is an *ordered group.*

Given a ring R, an ordered group $(G, <)$ and a crossed product group ring RG, the *Malcev-Neumann series ring* $R((G, <))$ consists of the formal sums

$$f = \sum_{x\in G} a_x \bar{x},$$

such that $\operatorname{supp} f = \{x \in G \mid a_x \neq 0\}$ is a well-ordered subset of G. The sum is defined component-wise and the product is defined as in (2.2).

It was proved independently by A.I. Malcev [29] and B.H. Neumann [32] that if R is a field then $R((G, <))$ is also a field. Let $f = \sum_{x\in G} a_x\bar{x}$ be a nonzero series in $R((G, <))$. Set $x_0 = \min\{x \in G \mid x \in \operatorname{supp} f\}$ and $g = a_{x_0}\bar{x}_0 - f$. Observe that $\operatorname{supp} g(a_{x_0}\bar{x}_0)^{-1} \subseteq \{x \in G \mid x > 1\}$. As in [21, Corollary 14.23], it can be proved that $\sum_{m\geq 0}(g(a_{x_0}\bar{x}_0)^{-1})^m$ is a well-defined element in $R((G, <))$, that is, for each $x \in G$, the set $L_x = \{m \geq 0 \mid x \in \operatorname{supp}(g(a_{x_0}\bar{x}_0)^{-1})^m\}$ is finite and, moreover, $\{x \in G \mid L_x \neq \emptyset\}$ is a well-ordered subset of G. Then

$$f^{-1} = (a_{x_0}\bar{x}_0)^{-1} \sum_{m\geq 0} \left(g(a_{x_0}\bar{x}_0)^{-1}\right)^m \tag{2.3}$$

2.3. **Universal fields, matrix localization and the free field.** See [11, Chapter 4] for the missing details. Let R be a ring. An *epic R-field* is a morphism of rings $\iota\colon R \to E$ with E a field which is rationally generated by the image of ι. If ι is injective, it is called a *field of fractions* of R.

A morphism of rings is said to be *local* if it takes non-units to non-units. If $A \to F$ is a local morphism of rings with F a field, then A is a local ring. A *local morphism of R-rings* between two epic R-fields $\iota_1\colon R \to F_1$ and $\iota_2\colon R \to F_2$ is a local morphism of rings $g\colon A \to F_2$ such that $\iota_1(R) \subseteq A$ and $g\iota_1 = \iota_2$. We denote by $\mathcal{L}(F_1, F_2)$ the set of all local morphism of R-rings between $\iota_1\colon R \to F_1$ and $\iota_2\colon R \to F_2$.

We give an example, suggested to us by the referee, which shows that there might be many elements in $\mathcal{L}(F_1, F_2)$.

Example 2.3. Let k be a commutative field. We consider two epic R-fields of $R = k[x, y]$. The first one is the field of fractions of R, $\iota_1\colon R \to k(x, y) = F_1$. The second is defined as follows, let $\iota_2\colon R \to k = F_2$ be the morphism of rings that sends x and y to zero. Clearly, ι_2 factorizes in a unique way through $A = k[x, y]_{(x,y)}$, the localization of R at (the complement of) the prime ideal (x, y), and this gives an element $f \in \mathcal{L}(F_1, F_2)$.

For any $n \geq 1$, consider $k[x, y, z]/(y^n z - x) \cong k[x, y, \frac{x}{y^n}] = S_n$. Given any $a \in k^\times = k \setminus \{0\}$, there is a morphism of k-algebras $h_{(n,a)}\colon S_n \to k$ such that $h_{(n,a)}(x) = h_{(n,a)}(y) = 0$ and $h_{(n,a)}(\frac{x}{y^n}) = a$. Define $A_{(n,a)}$ to be the localization of S_n at the (complement of) prime ideal $\ker h_{(n,a)}$. Then $A_{(n,a)}$ is a local ring that has a unique local onto morphism $f_{(n,a)}\colon A_{(n,a)} \to k$ extending $h_{(n,a)}$. Since $A_{(n,a)}$ embeds in $k(x, y)$, we conclude that $f_{(n,a)} \in \mathcal{L}(F_1, F_2)$. Finally, observe that f and $\{f_{(n,a)}\}_{(n,a)\in\mathbb{N}\times k^\times}$ are different elements of $\mathcal{L}(F_1, F_2)$.

Consider two epic R-fields $\iota_1\colon R \to F_1$ and $\iota_2\colon R \to F_2$. Let $g_1\colon A_1 \to F_2$, $g_2\colon A_2 \to F_2$ be elements of $\mathcal{L}(F_1, F_2)$. The relation $g_1 \sim g_2$ if and only if there exists $g\colon A \to F_2 \in \mathcal{L}(F_1, F_2)$ with $A \subseteq A_1 \cap A_2$ and $g_i(a) = g(a)$ for any $a \in A$ and $i = 1, 2$, is an equivalence relation. By definition, a *specialization* from $\iota_1\colon R \to F_1$ to $\iota_2\colon R \to F_2$ is an element of the quotient set $\mathcal{L}(F_1, F_2)/\sim$. We now show that there exists at most one specialization between two epic R-fields. Suppose that $\mathcal{L}(F_1, F_2) = \{g_i\colon A_i \to F_2\}_{i\in I}$ is not empty. Then $A = \{a \in \bigcap_{i\in I} A_i \mid g_i(a) = g_j(a) \text{ for any } i, j \in I\}$ is a subring of $\bigcap_{i\in I} A_i$ that contains $\iota_1(A)$. Moreover, the map $g\colon A \to F_2$ defined as the restriction of some (and hence all) g_i to A is a local morphism. Therefore $g \in \mathcal{L}(F_1, F_2)$. Therefore $g_i \sim g_j$ for $i, j \in I$.

If $R \to E$ is an epic R-field the unique endomorphism of $R \to E$ in the category of epic R-fields is the class of the identity map. Note that this class consists of the identity map alone. Hence, two epic R-fields $\iota_1\colon R \to F_1$ and $\iota_2\colon R \to F_2$ are isomorphic in the category of epic R-fields if and only if there exists an isomorphism $g\colon F_1 \to F_2$ such that $g\iota_1 = \iota_2$.

Epic R-fields (objects) together with specializations (morphisms) form a category. The composition of specializations is as follows. Let $\iota_i\colon R \to F_i$, $i = 1, 2, 3$, be three epic R-fields such that there exist specializations s_i from $\iota_i\colon R \to F_i$ to $\iota_{i+1}\colon R \to F_{i+1}$, $i = 1, 2$. Thus there exist $g_i \in \mathcal{L}(F_i, F_{i+1})$, $g_i\colon A_i \to F_{i+1}$, $i = 1, 2$. Consider the local subring $A_1' = g_1^{-1}(A_2)$ of F_1, and note that it contains $\iota_1(R)$. Then $g_2 g_{1|A_1'}$ is an element of $\mathcal{L}(F_1, F_3)$. Its class in $\mathcal{L}(F_1, F_3)/\sim$ is the composition $s_2 s_1$ of the specializations s_1 and s_2.

If there exists an initial object in this category, it is called a *universal R-field*. Thus, if a universal R-field exists, it is unique up to isomorphism. As an example, note that if R is a commutative domain and F is its field of fractions, then the natural embedding $R \hookrightarrow F$ is a universal R-field. Indeed, if $\iota\colon R \to E$ is another R-field and $\mathfrak{p}$ is the kernel of ι, then ι can be extended to $R_{\mathfrak{p}} \subseteq F$. Thus we obtain the local morphism of R-rings $R_{\mathfrak{p}} \to E$.

Let R be a ring and let $\varphi\colon R \to E$ be an epic R-field. The set $\mathcal{P}_E$ of all square matrices with entries in R whose image via φ is not invertible in E form a *prime matrix ideal of* R. Let us call $\mathcal{P}_E$ the associated prime matrix ideal to the epic R-field $\varphi\colon R \to E$. The universal localization of R at the set of all square matrices with entries in R whose image via φ is invertible is a local ring, denoted by $R_{\mathcal{P}_E}$, such that the canonical map $R_{\mathcal{P}_E} \to E$ induces an R-isomorphism between the residue field of $R_{\mathcal{P}_E}$ and E.

The correspondence between isomorphism classes of epic R-fields and prime matrix ideals of a ring R is in fact bijective. If $\mathcal{P}$ is a prime matrix ideal of R, then $R_{\mathcal{P}}$ is a local ring, its residue field E is an epic R-field and $\mathcal{P}_E = \mathcal{P}$.

The next result explains the interplay between specializations, prime matrix ideals and universal localization.

Theorem 2.4. *Let R be a ring, and let $R \to F_1$ and $R \to F_2$ be epic R-fields with associated prime matrix ideals $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Then the following statements are equivalent:*

(i) *There exists a specialization $F_1 \to F_2$.*
(ii) $\mathcal{P}_1 \subseteq \mathcal{P}_2$.
(iii) *The canonical localization homomorphism $R \to R_{\mathcal{P}_1}$ factors through the canonical localization homomorphism $R \to R_{\mathcal{P}_2}$.*

In particular, if $\mathcal{P}_2$ is a minimal prime matrix ideal and the above equivalent conditions hold, then:

(a) $\mathcal{P}_1 = \mathcal{P}_2$ *and*
(b) *Let $\rho\colon A_1 \to F_2$ be a local morphism of R-rings with $A_1 \subseteq F_1$. Since $\mathcal{P}_2 \subseteq \mathcal{P}_1$, there exists a specialization $F_2 \to F_1$. The composition of the specializations $F_1 \to F_2$ and $F_2 \to F_1$ gives an endomorphism of $R \to F_1$ in the category of epic R-fields. Hence $A_1 = F_1$.*

Note also that (iii) in the theorem above implies that if F_2 is given by universal localization of R (at a prime matrix ideal of R), then F_2 is isomorphic to F_1. Therefore, by Theorem 2.4 (ii), $\mathcal{P}_2$ is a minimal prime matrix ideal.

All prime matrix ideals contain the set of non-full matrices. The set $\mathcal{P}$ of non-full matrices is a prime matrix ideal, hence the least prime matrix ideal, if and only if R is a Sylvester domain; in this case, $R_{\mathcal{P}}$ is a field and, hence, it is a universal field of fractions. A free algebra (or more generally a semifir) is a Sylvester domain. The universal field of fractions of a free algebra is usually called a *free field*.

It is useful to keep the following important result in mind.

Remark 2.5. Let G be a free group on a nonempty set X, k a field and kG a crossed product. Lewin proved that the universal field of fractions of kG (and of

$k\langle X\rangle$) is the field of fractions of kG inside $k((G,<))$ for any group ordering $<$ on G, see [22] and the remark in [23, Section 2]. An easier proof of this fact was given by C. Reutenauer [35] (or see [36]). Observe that if N is a subgroup of G (or $Y \subseteq X$), then the universal field of fractions of kN (respectively $k\langle Y\rangle$) is the field of fractions of kN ($k\langle Y\rangle$) inside $k((G,<))$.

3. Inversion height

Suppose that $\iota\colon R \hookrightarrow E$ is an embedding of a domain R into a field E.

Set $E_\iota(-1) = \emptyset$, $E_\iota(0) = R$, and we define inductively for $n \geq 0$:

$$E_\iota(n+1) = \begin{array}{c}\text{subring of } E\\ \text{generated by}\end{array} \left\{r, s^{-1} \mid r, s \in E_\iota(n),\ s \neq 0\right\}.$$

Then $E_\iota = \bigcup_{n=0}^{\infty} E_\iota(n)$ is the *field of fractions of* R *inside* E. That is, E_ι is the field rationally generated by R inside E or, equivalently, the intersection of all subfields of E that contain R.

We define $\mathrm{h}(\iota)$, the *inversion height of* R (inside E), as ∞ if there is no $n \in \mathbb{N}$ such that $E_\iota(n)$ is a field. Otherwise,

$$\mathrm{h}(\iota) = \min\{n \mid E_\iota(n) \text{ is a field}\}.$$

Notice that if $\mathrm{h}(\iota) = n$, then $E_\iota(m) = E_\iota(n)$ for all $m \geq n$.

Given an integer $n \geq 0$, we say that an element $f \in E_\iota$ has *inversion height* n if $f \in E_\iota(n) \setminus E_\iota(n-1)$, and we write $\mathrm{h}(\iota, f) = n$. In other words, $\mathrm{h}(\iota, f)$ says how many nested inversions are needed to express an element of E_ι in terms of elements of R, and $\mathrm{h}(\iota)$ is the supremum of all $\mathrm{h}(\iota, f)$ with $f \in E_\iota$.

We now give some easy remarks that will be used throughout.

Remarks 3.1. Let $\iota\colon R \hookrightarrow E$ be an embedding of a domain R in a field E.

(a) If $\kappa\colon E \hookrightarrow L$ is an embedding in a field L, then $\kappa\iota$ is an embedding such that $E_\iota(n) = L_{\kappa\iota}(n)$ for all $n \geq -1$. Therefore $E_\iota = L_{\kappa\iota}$, $\mathrm{h}(\iota) = \mathrm{h}(\kappa\iota)$, and $\mathrm{h}(\iota, f) = \mathrm{h}(\kappa\iota, f)$ for all $f \in L_{\kappa\iota}$.

(b) On the other hand, if S is a subring of R and we consider the embedding $\varepsilon = \iota_{|S}\colon S \hookrightarrow E$, then $E_\varepsilon(n) \subseteq E_\iota(n)$, and thus $\mathrm{h}(\iota, f) \leq \mathrm{h}(\varepsilon, f)$ for all $f \in E_\varepsilon$.

One of the problems when dealing with inversion height is the fact that we cannot be more accurate in Remarks 3.1(b). That is, we may know $\mathrm{h}(\varepsilon, f)$ for some f or even $\mathrm{h}(\varepsilon)$, but usually it is not useful if we want to compute $\mathrm{h}(\iota, f)$ or $\mathrm{h}(\iota)$. Our key results on inversion height (Propositions 3.4 and 3.5) state that $\mathrm{h}(\varepsilon, f) = \mathrm{h}(\iota, f)$ in certain important cases.

Lemma 3.2. *Let k be a commutative field, and let R be a k-algebra with a fixed embedding $\iota\colon R \hookrightarrow E$ into a field E. If $f \in E_\iota$ satisfies $h(\iota, f) \leq m$, then there exists a finitely generated k-subalgebra S of R such that $f \in E_\varepsilon$ and $h(\varepsilon, f) \leq m$ where $\varepsilon = \iota_{|S}\colon S \to E$.*

Proof. The proof is by induction on m. For $m = 0$ the claim is clear. Suppose that the claim is true for $m-1 \geq 0$. Since $f \in E_\iota(m)$, $f = \sum_{j=1}^{r} f_{1j} \cdots f_{l_j j}$ where, for each i, j, either $f_{ij} \in E_\iota(m-1)$ or f_{ij} is the inverse of some nonzero element in $E_\iota(m-1)$. The induction hypothesis implies that there exist $S_{1j}, \dots, S_{l_j j}$ finitely generated k-subalgebras of R such that $f_{ij} \in E_{\varepsilon_{ij}}$, where $\varepsilon_{ij} = \iota_{|S_{ij}} \colon S_{ij} \to E$, and $h(\varepsilon_{ij}, f_{ij}) \leq m$. Let S be the smallest subalgebra of R containing S_{ij} for all i, j, and let $\varepsilon = \iota_{|S} \colon S \to E$. Then $f \in E_\varepsilon$, and $h(\varepsilon, f) \leq m$ because $E_{\varepsilon_{ij}}(m) \subseteq E_\varepsilon(m)$. This proves the result. □

Lemma 3.3. *Let S be a domain with a fixed embedding $\varepsilon \colon S \hookrightarrow F$ into a field F. Let $\alpha \colon F \to F$ be a morphism of rings and $\delta \colon F \to F$ be an α-derivation.*

(i) *If $\alpha(S) \subseteq S$ and $\delta(S) \subseteq S$, then*

$$\alpha(F_\varepsilon(n)) \subseteq F_\varepsilon(n) \quad \text{and} \quad \delta(F_\varepsilon(n)) \subseteq F_\varepsilon(n)$$

for all $n \geq 0$. Hence, $F_\varepsilon(n)((y;\alpha,\delta)) \hookrightarrow F_\varepsilon((y;\alpha,\delta))$ and $F_\varepsilon(n)((x;\alpha)) \hookrightarrow F_\varepsilon((x;\alpha))$.

(ii) *If $\alpha(S) = S$, then α induces an automorphism of $F_\varepsilon(n)$ for each $n \geq 0$, and thus it induces an automorphism on F_ε.*

Proof. (i) The hypothesis ensures that $\alpha(F_\varepsilon(0)) \subseteq F_\varepsilon(0)$ and $\delta(F_\varepsilon(0)) \subseteq F_\varepsilon(0)$. Since for each $f \in F \setminus \{0\}$, $\alpha(f^{-1}) = \alpha(f)^{-1}$ and $\delta(f^{-1}) = -\alpha(f)^{-1}\delta(f)f^{-1}$ cf. Lemma 4.3, using the definition of $F_\varepsilon(n)$, it is easy to prove the first claim inductively.

The second claim follows from the first and the commutativity of the following diagram

$$\begin{array}{ccc}
F_\varepsilon(n)[[y;\alpha,\delta]] & \xhookrightarrow{\ \eta\ } & F_\varepsilon[[y;\alpha,\delta]] \\
\big\downarrow & & \big\downarrow \\
F_\varepsilon(n)((y;\alpha,\delta)) & \xhookrightarrow{\ \nu\ } & F_\varepsilon((y;\alpha,\delta))
\end{array}$$

where the vertical arrows are given by the right Ore localization at the powers of y, η is induced from $F_\varepsilon(n) \hookrightarrow F_\varepsilon$, and ν is given by the universal property of Ore localization. Similarly for $F_\varepsilon(n)((x;\alpha))$.

(ii) Assume that $\alpha \colon S \to S$ is an automorphism. We prove, by induction on n, that $\alpha \colon F_\varepsilon(n) \to F_\varepsilon(n)$ is an isomorphism for each $n \geq 0$. Our hypothesis ensures the case $n = 0$. Assume that $n > 0$ and $\alpha \colon F_\varepsilon(n-1) \to F_\varepsilon(n-1)$ is onto, hence an automorphism. As for any $r \in F_\varepsilon(n-1) \setminus \{0\}$, $\alpha(r^{-1}) = \alpha(r)^{-1} \in F_\varepsilon(n)$ and $F_\varepsilon(n-1) = \alpha(F_\varepsilon(n-1))$, we deduce that all the ring generators of $F_\varepsilon(n)$ are in $\alpha(F_\varepsilon(n))$, which implies that $\alpha \colon F_\varepsilon(n) \to F_\varepsilon(n)$ is onto. □

Proposition 3.4. *Let S be a domain, let $\alpha \colon S \to S$ be an injective ring endomorphism, and let $\delta \colon S \to S$ be an α-derivation. Set $R = S[x;\alpha,\delta]$. Suppose that $\varepsilon \colon S \hookrightarrow F$ is a field of fractions of S, that α and δ extend to F and that*

$$\alpha\big(F_\varepsilon(n) \setminus F_\varepsilon(n-1)\big) \subseteq F_\varepsilon(n) \setminus F_\varepsilon(n-1), \tag{3.1}$$

for each integer $n \geq 0$. Let $E = F(x;\alpha,\delta)$, and let $\iota\colon R \hookrightarrow E$ be the natural embedding of R in E. Consider the field of skew Laurent series $F((y;\alpha,\delta))$. Then

(i) *For each $n \geq 0$, $E_\iota(n) \subseteq F_\varepsilon(n)((y;\alpha,\delta))$.*
(ii) *Let $f \in F$. If $\mathrm{h}(\varepsilon, f) = n$, then $\mathrm{h}(\iota, f) = n$.*
(iii) *$\mathrm{h}(\iota) \geq \mathrm{h}(\varepsilon)$.*

Proof. To simplify the notation, let $\mathcal{L}_n = F_\varepsilon(n)((y;\alpha,\delta))$ for each $n \geq 0$. By Lemma 3.3(i), $\mathcal{L}_n$ is a subring of $F((y;\alpha,\delta))$.

(i) We proceed by induction on n. For $n = 0$, observe that $E_\iota(0) = S[x;\alpha,\delta]$ which we regard as embedded in $S((y;\alpha,\delta)) = \mathcal{L}_0$, cf. §2.1.

Suppose that the result holds for $n \geq 0$. Let $f \in E_\iota(n) \setminus \{0\}$. As an element in $\mathcal{L}_n$, $f = (\sum\limits_{m\geq 0} y^m a_m) y^{-r}$ with $a_m \in F_\varepsilon(n)$. Suppose that m_0 is the first natural number such that $a_{m_0} \neq 0$. Then f can be written as $y^{m_0}\big(1 - \sum\limits_{m\geq 1} y^m b_m\big) a_{m_0} y^{-r}$ where $b_m = -a_{m+m_0} a_{m_0}^{-1} \in F_\varepsilon(n+1)$. Hence

$$f^{-1} = y^r a_{m_0}^{-1} \left(\sum_{s\geq 0} \left(\sum_{m\geq 1} y^m b_m \right)^s \right) y^{-m_0}. \tag{3.2}$$

Consider the series $h = \sum_{s\geq 0} \left(\sum_{m\geq 1} y^m b_m\right)^s$. It can be uniquely expressed as $h = \sum_{n\geq 0} y^n c_n$ with $c_n \in F$ for each n. If $s > n$, the terms of $\left(\sum_{m\geq 1} y^m b_m\right)^s$ do not contribute to the computation of c_n. Thus c_n is obtained from $\sum_{0\leq s\leq n} \left(\sum_{m\geq 1} y^m b_m\right)^s$. Now Lemma 3.3(i) implies that $c_n \in F_\varepsilon(n+1)$ because $b_m \in F_\varepsilon(n+1)$ for each m. Hence $\sum\limits_{s\geq 0} \big(\sum\limits_{m\geq 1} y^m b_m\big)^s \in F_\varepsilon(n+1)[[y;\alpha,\delta]] \subseteq \mathcal{L}_{n+1}$. As $\mathcal{L}_{n+1}$ is a ring, it follows that $f^{-1} = y^r a_{m_0}^{-1} h y^{-m_0} \in \mathcal{L}_{n+1}$.

We have shown that the generators of $E_\iota(n+1)$ are contained in the ring $\mathcal{L}_{n+1}$, therefore $E_\iota(n+1) \subseteq \mathcal{L}_{n+1}$, as desired.

(ii) If S is a field, the result is clear. So suppose that S is not a field and let $f \in F$ with $f \in F_\varepsilon(n+1) \setminus F_\varepsilon(n)$ for some $n \geq 0$. If $f \in E_\iota(n)$ then, by (i), $f \in \mathcal{L}_n$ so that there exist $a_m \in F_\varepsilon(n)$ and $r \geq 0$ such that $f = \big(\sum_{m\geq 0} y^m a_m\big) y^{-r}$. By (2.1), fy^r is a series of the form $y^r \alpha^r(f) + \sum_{m\geq 1} y^{r+m} b_m$, but since $fy^r = \sum_{m\geq 0} y^m a_m$ with all $a_m \in F_\varepsilon(n)$, we get a contradiction because $a_r = \alpha^r(f) \in F_\varepsilon(n+1) \setminus F_\varepsilon(n)$ by the hypothesis (3.1).

(iii) follows from (ii). □

Note that if α is an automorphism of S, then (3.1) in Proposition 3.4 holds.

Proposition 3.5. *Let S be a domain, $\alpha\colon S \to S$ be an automorphism and $R = S[x, x^{-1};\alpha]$. Suppose that $\varepsilon\colon S \hookrightarrow F$ is a field of fractions of S and that α extends to F. Let $E = F(x;\alpha)$ and $\iota\colon R \hookrightarrow E$ be the natural embedding of R in E. Consider the field of skew Laurent series $F((x;\alpha))$. Then*

(i) *For each $n \geq 0$, $E_\iota(n) \subseteq F_\varepsilon(n)((x;\alpha))$.*

(ii) *Let* $f \in F$. *If* $\mathrm{h}(\varepsilon, f) = n$, *then* $\mathrm{h}(\iota, f) = n$.
(iii) $\mathrm{h}(\iota) \geq \mathrm{h}(\varepsilon)$.

Proof. Consider $\mathcal{L}_n = F_\varepsilon(n)((x;\alpha))$ as a subring of $F((x;\alpha))$. Then proceed as in the proof of Proposition 3.4 □

4. Two solutions

We shall use the following notation. Let A be an $n \times n$ matrix with entries over a ring. Let $i, j, p, q \in \{1, \dots, n\}$. By A^{ij} we denote the matrix obtained from A by deleting the i-th row and the j-th column. By r_p^j we mean the row vector obtained from the p-th row of A deleting the j-th entry. And by s_q^i we denote the column vector obtained from the q-th column of A by deleting the i-th entry.

Let k be a commutative field and X be a set. Let $A = (x_{ij})$ be an $n \times n$ matrix with entries over the free k-algebra $k\langle X\rangle$. We say that A is a *generic matrix* (over $k\langle X\rangle$) if the x_{ij}'s are distinct variables in X. If $\iota\colon k\langle X\rangle \hookrightarrow E$ is the universal field of fractions of $k\langle X\rangle$, then such a generic matrix is invertible over E. Moreover the (j,i)-th entry of $A^{-1} \in M_n(E)$ is the inverse of

$$|A|_{ij} = x_{ij} - r_i^j (A^{ij})^{-1} s_j^i.$$

The element $|A|_{ij}$ is known as the (i,j)-th *quasideterminant* of A [14].

Theorem 4.1. (C. Reutenauer [34, Theorem 2.1]) *Let k be a commutative field and let X be a finite set of cardinality at least n^2, where $1 \leq n < \infty$. Let $\iota\colon k\langle X\rangle \hookrightarrow E$ be the embedding of the free algebra $k\langle X\rangle$ in its universal field of fractions E. Let A be an $n \times n$ generic matrix. If f is an entry of $A^{-1} \in M_n(E)$, then* $\mathrm{h}(\iota, f) = n$.

To adapt this result to our purposes, we note the following Corollary.

Corollary 4.2. *Let k be a commutative field, let Z be an infinite set and let N be the free group on Z. Let $\varepsilon'\colon kN \hookrightarrow F$ be the universal field of fractions of the group algebra kN, and $\varepsilon = \varepsilon'_{|k\langle Z\rangle}\colon k\langle Z\rangle \hookrightarrow F$. Then* $\mathrm{h}(\varepsilon') = \mathrm{h}(\varepsilon) = \infty$. *Indeed, if A_n is an $n \times n$ generic matrix and f is an entry of $A_n^{-1} \in M_n(F)$, then* $\mathrm{h}(\varepsilon, f) = n$ *and* $\mathrm{h}(\varepsilon', f) = n - 1$.

Proof. First of all notice that since $k\langle Z\rangle \subseteq kN$,

$$E_\varepsilon(m) \subseteq E_{\varepsilon'}(m) \subseteq E_\varepsilon(m+1) \subseteq E_{\varepsilon'}(m+1) \tag{4.1}$$

for each integer $m \geq 0$. Thus if $\mathrm{h}(\varepsilon) = \infty$, then $\mathrm{h}(\varepsilon') = \infty$.

Let A_n be an $n \times n$ generic matrix. Recall that if Y is a subset of Z and $\eta = \varepsilon_{|k\langle Y\rangle}\colon k\langle Y\rangle \hookrightarrow F$, then F_η is the universal field of fractions of $k\langle Y\rangle$, cf. Remark 2.5. Thus if Y is any finite subset of Z that contains the entries of A_n and f is an entry of A_n^{-1}, then $\mathrm{h}(\eta, f) = n$ by Theorem 4.1. Now Lemma 3.2 implies that $\mathrm{h}(\varepsilon, f) = n$, and (4.1) that $\mathrm{h}(\varepsilon', f) \geq n - 1$.

Since Z is an infinite set, there exist $n \times n$ generic matrices A_n for each natural number $n \geq 1$ and therefore $\mathrm{h}(\iota)$ is not finite by the foregoing.

We prove that $\mathrm{h}(\varepsilon', f) \leq n - 1$ by induction on $n \geq 1$. If $n = 1$, the result follows because $f \in Z$ and therefore $f^{-1} \in N$. Suppose the claim holds for $n \geq 1$.

Consider an $(n+1)\times(n+1)$ generic matrix $A_{n+1}=(x_{ij})$. Then f is the (j,i)-th entry of A_{n+1}^{-1}. Thus $f=\left(x_{ij}-r_i^j(A_{n+1}^{ij})^{-1}s_j^i\right)^{-1}$ for some i,j. Since A_{n+1}^{ij} is an $n\times n$ generic matrix, the induction hypothesis implies that if g is any entry of $(A_{n+1}^{ij})^{-1}$ then $\mathrm{h}(\varepsilon',g)\leq n-1$. Therefore $\mathrm{h}(\varepsilon',f)\leq n$. □

4.1. **First solution.** If x, y are two elements of a ring, we denote by $[x,y]$ the element $[x,y]=xy-yx$.

We are interested in extending derivations to certain localizations of R. We recall the following easy and well known formula which implies that such extensions, if they exist, are unique.

Lemma 4.3. *Let R be a ring, and let $\delta\colon R\to R$ be a derivation. If $r\in R$ is invertible, then $\delta(r^{-1})=-r^{-1}\delta(r)r^{-1}$. Hence, if $R\to D$ is a field of fractions of R and δ, $\delta'\in \mathrm{Der}\,(D)$ are such that $\delta(r)=\delta'(r)$, for any $r\in R$, then $\delta=\delta'$.*

In the next lemma, we show that derivations can be extended to matrix localizations.

The result, at least for fields of fractions of Sylvester domains, is well known and the proof for the general case follows the same pattern. However we include it for completeness' sake.

Recall that if R is a ring, $\delta\colon R\to R$ is a derivation if and only if the map $R\to M_2(R)$ given by $r\mapsto\begin{pmatrix} r & \delta(r)\\ 0 & r\end{pmatrix}$, for any $r\in R$, is a ring homomorphism.

For the proof of the next result it is useful to keep in mind the following explicit description of an isomorphism between $M_{2n}(S)$ and $M_n(M_2(S))$ for any natural number n and any given ring S. The elements of $M_n(M_2(S))$ are matrices of the form

$$A=\begin{pmatrix} A_{11} & \cdots & A_{1n}\\ \vdots & \ddots & \vdots\\ A_{n1} & \cdots & A_{nn}\end{pmatrix}$$

where $A_{ij}=\left(\begin{smallmatrix} a_{ij} & b_{ij}\\ c_{ij} & d_{ij}\end{smallmatrix}\right)\in M_2(S)$ for each $i,j\in\{1,\dots,n\}$. The map $\rho_n\colon M_n(M_2(S))\to M_{2n}(S)$ defined by

$$\begin{pmatrix} A_{11} & \cdots & A_{1n}\\ \vdots & \ddots & \vdots\\ A_{n1} & \cdots & A_{nn}\end{pmatrix}\mapsto\begin{pmatrix} a_{11} & \cdots & a_{1n} & b_{11} & \dots & b_{1n}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ a_{n1} & \cdots & a_{nn} & b_{n1} & \dots & b_{nn}\\ c_{11} & \cdots & c_{1n} & d_{11} & \dots & d_{1n}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ c_{n1} & \cdots & c_{nn} & d_{n1} & \dots & d_{nn}\end{pmatrix}$$

is an isomorphism of rings.

Lemma 4.4. *Let R be a ring, Φ a set of square matrices over R, and let $R\to R_\Phi$, $a\mapsto\hat{a}$, be the matrix localization of R at Φ. Then any derivation $\delta\colon R\to R$, $a\mapsto a^\delta$, extends to a unique derivation of R_Φ.*

In particular, if $R \hookrightarrow D$ is the universal field of fractions of a Sylvester domain R, then any derivation on R can be uniquely extended to D.

Proof. For each matrix $A = (a_{ij}) \in M_n(R)$, denote by $\hat{A}$ the matrix $(\hat{a}_{ij}) \in M_n(R_\Phi)$ and denote by A^δ the matrix $(a_{ij}^\delta) \in M_n(R)$.

For each natural number n, consider the map $\psi_n\colon M_n(R) \to M_{2n}(R_\Phi)$ given by $A \mapsto \left(\begin{smallmatrix} \hat{A} & \widehat{A^\delta} \\ 0 & \hat{A} \end{smallmatrix}\right)$. Since δ is a derivation, ψ_n is a morphism of rings. For each $n \times n$ matrix $A \in \Phi$, the matrix $\psi_n A$ is invertible in $M_{2n}(R_\Phi)$. Indeed, since $\hat{A}$ is invertible in $M_n(R_\Phi)$ by definition, the matrix

$$\begin{pmatrix} \hat{A}^{-1} & -\hat{A}^{-1}\widehat{A^\delta}\hat{A}^{-1} \\ 0 & \hat{A}^{-1} \end{pmatrix} \tag{4.2}$$

is the inverse of $\psi_n A$. Thus the image of any $n \times n$ matrix in Φ by the morphism $\rho_n^{-1}\psi_n\colon M_n(R) \to M_n(M_2(R_\Phi))$ is invertible. Note that if $A = (a_{ij})$, then

$$\rho_n^{-1}\psi_n A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \tag{4.3}$$

where $A_{ij} = \left(\begin{smallmatrix} \hat{a}_{ij} & \widehat{a_{ij}^\delta} \\ 0 & \hat{a}_{ij} \end{smallmatrix}\right)$. Hence, the morphism $R \to M_2(R_\Phi)$, $a \mapsto \left(\begin{smallmatrix} \hat{a} & \widehat{a^\delta} \\ 0 & \hat{a} \end{smallmatrix}\right)$ is Φ-inverting, and there exists a unique morphism $\theta\colon R_\Phi \to M_2(R_\Phi)$ making the diagram

$$\begin{array}{ccc} R & \longrightarrow & M_2(R_\Phi) \\ & \searrow \quad \nearrow_{\theta} & \\ & R_\Phi & \end{array}$$

commutative. Hence, if $a \in R$, $\theta(\hat{a}) = \left(\begin{smallmatrix} \hat{a} & \widehat{a^\delta} \\ 0 & \hat{a} \end{smallmatrix}\right)$. Now, if $x \in R_\Phi$ is an entry of $\hat{A}^{-1}$ for some $A \in \Phi$, ρ_n^{-1} applied to the matrix in (4.2) shows that $\theta(x) = \left(\begin{smallmatrix} x & x^\Delta \\ 0 & x \end{smallmatrix}\right)$ for some $x^\Delta \in R_\Phi$. Therefore, if x is any element in R_Φ, $\theta(x)$ is of the form $\left(\begin{smallmatrix} x & x^\Delta \\ 0 & x \end{smallmatrix}\right)$ for some $x^\Delta \in R_\Phi$ because we have proved that this happens for a set of generators of R_Φ (the image of R and the entries of the inverses of the matrices in Φ) and θ is a morphism of rings. Hence $\Delta\colon R_\Phi \to R_\Phi$, $x \mapsto x^\Delta$, is a derivation extending δ, as desired.

The last part follows from the first one because if R is a Sylvester domain, then its universal field of fractions is of the form R_Φ where Φ is the set of all full matrices over R, cf. § 2.3. □

The next result is based on the ideas of [10], where a particular kind of embedding of a free algebra of infinite countable rank into free algebra of rank two is given.

Theorem 4.5. *Let k be a commutative field and $k\langle x, y_1, \dots, y_n\rangle$ be the free algebra with $n \geq 1$. Let $\iota\colon k\langle x, y_1, \dots, y_n\rangle \hookrightarrow E$ be the universal field of fractions of $k\langle x, y_1, \dots, y_n\rangle$. Then $\mathrm{h}(\iota) = \infty$. Moreover, if*

$$A_m = \begin{pmatrix} w_0 & w_m & \cdots & w_{m^2-m} \\ w_1 & w_{m+1} & \cdots & w_{m^2-m+1} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m-1} & w_{2m-1} & \cdots & w_{m^2-1} \end{pmatrix},$$

where

$$w_0 = y_1, \quad w_{i+1} = [x, w_i] \ \text{ for } i \geq 0,$$

and f is an entry of $A_m^{-1} \in M_m(E)$, then $\mathrm{h}(\iota, f) = m$.

Proof. Set $Z = \{z_0, z_1, \dots, z_m, \dots\}$, $S = k\langle Z\rangle$, $R = k\langle x, y_1, \dots, y_n\rangle$, and let $\varepsilon\colon S \hookrightarrow F$ be the universal field of fractions of S.

Proceeding as in [10, Lemma 2.1] or using Lemma 4.4, it can be shown that there exists a derivation $\delta\colon S \to S$ such that $\delta(z_i) = z_{i+n}$, for each $i \in \mathbb{N}$, and that it can be extended to a unique derivation of F.

Express each integer $i \geq 0$ (uniquely) as $i = r_i n + j_i$ with $0 \leq j_i \leq n-1$. As in [10, Theorem 2.2], one can prove that there is an embedding $\beta_0\colon S \to R$ defined by

$$\beta_0(z_i) = \begin{cases} y_{i+1} & \text{for } 0 \leq i \leq n-1 \\ [x, \dots [x, [x, y_{j_i+1}]] \cdots] \text{ with } r_i \text{ factors } x & \text{for } n-1 < i. \end{cases}$$

which is honest (and 1-inert). Thus β_0 can be extended to a morphism of rings $\beta_0\colon F \hookrightarrow E$. Again as in [10], identifying S and F with their images via β_0, we get that $R = S[x; \delta]$ and that $E = F(x; \delta)$. Since $\mathrm{h}(\varepsilon) = \infty$ by Corollary 4.2, we also have $\mathrm{h}(\iota) = \infty$ by Proposition 3.4(iii).

Now let f be an entry of the inverse of A_m. Note that the matrix A_m is (the image of) a generic matrix over S. Thus Corollary 4.2 says that $\mathrm{h}(\varepsilon, f) = m$. Therefore $\mathrm{h}(\iota, f) = m$ by Proposition 3.4(ii). □

4.2. **Second solution.**

Theorem 4.6. *Let k be a commutative field, $X = \{x, y_1, \dots, y_n\}$ be a finite set with $n \geq 1$, and H be the free group on X. Let kH denote the group algebra. Let $\iota'\colon kH \hookrightarrow E$ be the universal field of fractions of kH, and $\iota = \iota'_{|k\langle X\rangle}\colon k\langle X\rangle \hookrightarrow E$. Then $\mathrm{h}(\iota) = \mathrm{h}(\iota') = \infty$. Moreover, if*

$$A_m = \begin{pmatrix} z_0 & z_m & \cdots & z_{m^2-m} \\ z_1 & z_{m+1} & \cdots & z_{m^2-m+1} \\ \vdots & \vdots & \cdots & \vdots \\ z_{m-1} & z_{2m-1} & \cdots & z_{m^2-1} \end{pmatrix} \quad \text{where } z_i = x^i y_1 x^{-i},$$

and f is an entry of $A_m^{-1} \in M_m(E)$, then $\mathrm{h}(\iota, f) = m$ and $\mathrm{h}(\iota', f) = m-1$.

Proof. Fix an order on H such that $(H,<)$ is an ordered group. We identify E with the field of fractions of the group algebra kH inside $L = k((H,<))$.

Let $C = \langle c\rangle$ be the infinite cyclic group. Consider the morphism of groups $\varphi\colon H \to C$ given by $x \mapsto c$ and $y_j \mapsto 1$ for $1 \le j \le n$. Let $N = \ker\varphi$. Thus H is the extension of N by the infinite cyclic group generated by x. It is well known that N is a free group with basis the infinite set $Z = \{x^i y_j x^{-i} \mid 1 \le j \le n,\ i \in \mathbb{Z}\}$, see for example [20, Section 36]. We denote by kN the group algebra on N.

Let $\varepsilon' = \iota'_{|kN}\colon kN \hookrightarrow F$ and $\varepsilon = \iota'_{|k\langle Z\rangle}\colon k\langle Z\rangle \hookrightarrow F$ be the universal field of fractions of kN and $k\langle Z\rangle$ respectively, where we identify F with $E_\varepsilon = E_{\varepsilon'}$, the subfield rationally generated by kN inside E.

Let $\alpha\colon E \to E$ be the automorphism of E given by $f \mapsto xfx^{-1}$ for all $f \in E$. Notice that α restricts to an automorphism of kN and also to an automorphism of $k\langle Z\rangle$. Then α can be extended to an automorphism of F by Lemma 3.3(ii). Notice also that kH is a crossed product of the ring kN by the cyclic group generated by x, hence $kH = (kN)[x,x^{-1};\alpha]$, cf. §2.1. Let $\iota_Z = \iota'_{|k\langle Z\rangle[x,x^{-1};\alpha]}\colon k\langle Z\rangle[x,x^{-1};\alpha] \to E$.

Observe that F is contained in $k((N,<)) \subseteq L$. If $n_1x^{r_1} = n_2x^{r_2}$, for $n_1, n_2 \in N$ and $r_1, r_2 \in \mathbb{Z}$, then $r_1 = \varphi(n_1x^{r_1}) = \varphi(n_2x^{r_2}) = r_2$ and, hence, also $n_1 = n_2$; this allows to prove that the powers of x are $k((N,<))$-linearly independent. In particular, the powers of x are F-linearly independent. Therefore there is an embedding $\Upsilon\colon F[x,x^{-1};\alpha] \hookrightarrow E$ and, by the universal property of the Ore localization, $E = F(x;\alpha)$.

Note that the entries of A_m belong to Z. Let $f \in F$, be one of the entries of A_m^{-1}. By Corollary 4.2, $\mathrm{h}(\varepsilon', f) = m-1$. Now, if we define $S = kN$ and $R = kH = (kN)[x,x^{-1};\alpha]$, Proposition 3.5(ii) implies that $h(\iota', f) = m-1$.

Similarly, by Corollary 4.2, $\mathrm{h}(\varepsilon, f) = m$. Now, if we set $S = k\langle Z\rangle$ and $R = k\langle Z\rangle[x,x^{-1};\alpha]$, Proposition 3.5(ii) implies that $h(\iota_Z, f) = m$.

Since $k\langle X\rangle \subseteq k\langle Z\rangle[x,x^{-1};\alpha] \subseteq kH$ we obtain that

$$E_\iota(m-1) \subseteq E_{\iota_Z}(m-1) \subseteq E_\iota(m) \subseteq E_{\iota_Z}(m),$$

$$E_\iota(m-1) \subseteq E_{\iota'}(m-1) \subseteq E_\iota(m) \subseteq E_{\iota'}(m).$$

The first expression says that $m \le \mathrm{h}(\iota, f)$, and the second one $\mathrm{h}(\iota, f) \le m$. Therefore $\mathrm{h}(\iota, f) = m$.

Since m is any natural number ≥ 1, we obtain that there exist elements $f \in E$ with any prescribed inversion height $m \ge 1$. Therefore $\mathrm{h}(\iota) = \mathrm{h}(\iota') = \infty$. □

5. Other embeddings of infinite inversion height

Let S be a ring, G a group and SG a crossed product (determined by an action σ and a twisting τ as in section 2.2). Let $\varepsilon\colon S \hookrightarrow F$ be an epimorphism of rings such that the automorphism $\sigma(x) \in \mathrm{Aut}(S)$ can be extended to a unique automorphism of F for every $x \in G$. Following Remark 2.2 or as in [37, Lemma 4], it is easy to prove that there exists a crossed product FG with an embedding $\kappa\colon SG \to FG$ with $\kappa_{|S} = \varepsilon$ and $\kappa(\bar{x}) = \bar{x}$ for every $x \in G$.

If $\varepsilon\colon S \hookrightarrow F$ is a field of fractions, then $S \hookrightarrow F_\varepsilon(n)$ and $F_\varepsilon(n) \hookrightarrow F$ are ring epimorphisms for each n. Suppose now that we are in the situation of the foregoing paragraph. By Lemma 3.3(ii), $\sigma(x)$ can be extended to $F_\varepsilon(n)$ for each $x \in G$ and $n \geq 0$. Thus we obtain the embeddings

$$SG \hookrightarrow F_\varepsilon(n)G \hookrightarrow FG$$

for each $n \geq 0$. If, moreover, $(G,<)$ is an ordered group, we get the embeddings of Malcev-Neumann series rings

$$S((G,<)) \hookrightarrow F_\varepsilon(n)((G,<)) \hookrightarrow F((G,<))$$

for each $n \geq 0$.

The next result is a general version for Malcev-Neumann series of Proposition 3.4.

Theorem 5.1. *Let S be a domain with a field of fractions $\varepsilon\colon S \hookrightarrow F$. Let $(G,<)$ be an ordered group. Consider a crossed product SG which can be extended to a crossed product FG. Let $E = F((G,<))$ be the associated Malcev-Neumann series ring, and let $\iota\colon SG \hookrightarrow E$ be the natural embedding. Then*

(i) *$E_\iota(n) \subseteq F_\varepsilon(n)((G,<))$ for each integer $n \geq 0$.*
(ii) *Let $f \in F$. If $\mathrm{h}(\varepsilon,f) = n$, then $\mathrm{h}(\iota,f) = n$.*
(iii) *$\mathrm{h}(\iota) \geq \mathrm{h}(\varepsilon)$.*

Proof. Throughout the proof let σ and τ denote the action and the twisting, respectively, of the crossed product FG. To simplify the notation, for any $n \geq 0$, we will denote $F_\varepsilon(n)((G,<))$ simply by $\mathcal{L}_n$.

We prove (i) by induction on n. For $n = 0$ the result is clear because $E_\iota(0) = SG \subseteq \mathcal{L}_0$. So suppose that (i) holds for $n \geq 0$, we must prove it for $n+1$.

By the definition of $E_\iota(n+1)$ and because $\mathcal{L}_{n+1}$ is a ring, it suffices to prove that if $f \in E_\iota(n) \setminus \{0\}$ then $f^{-1} \in \mathcal{L}_{n+1}$. By the induction hypothesis, $f \in \mathcal{L}_n$. Suppose that $f = \sum_{x\in G} a_x\bar{x}$ with $a_x \in F_\varepsilon(n)$. Let $x_0 = \min\{x \in G \mid x \in \operatorname{supp} f\}$. By equation (2.3),

$$f^{-1} = (a_{x_0}\bar{x}_0)^{-1} \sum_{m\geq 0} \big(g(a_{x_0}\bar{x}_0)^{-1}\big)^m,$$

where $g = a_{x_0}\bar{x}_0 - f \in \mathcal{L}_n$. Note that $(a_{x_0}\bar{x}_0)^{-1} = \bar{x}_0^{-1}a_{x_0}^{-1} = \tau(x_0^{-1},x_0)^{-1}\overline{x_0^{-1}}a_{x_0}^{-1} = \tau(x_0^{-1},x_0)^{-1\sigma(x_0^{-1})}a_{x_0}^{-1}\overline{x_0^{-1}}$. Since $\tau(x_0^{-1},x_0)^{-1\sigma(x_0^{-1})}a_{x_0}^{-1} \in F_\varepsilon(n+1)$, $g(a_{x_0}\bar{x}_0)^{-1}$ and, hence, $(g(a_{x_0}\bar{x}_0)^{-1})^m$ are in $\mathcal{L}_{n+1}$ for any $m \geq 0$. As explained at the end of Section 2.2, the series $\sum_{m\geq 0}(g(a_{x_0}\bar{x}_0)^{-1})^m$ is well defined in E. Since, for each $x \in G$, the coefficient of $\bar{x}$ in $\sum_{m\geq 0}(g(a_{x_0}\bar{x}_0)^{-1})^m$ is an element of $F_\varepsilon(n+1)$, then $\sum_{m\geq 0}(g(a_{x_0}\bar{x}_0)^{-1})^m \in \mathcal{L}_{n+1}$. Therefore $f^{-1} \in \mathcal{L}_{n+1}$ as we wanted to see.

(ii) If S is a field, the result is clear. So suppose that S is not a field. Let $f \in F_\varepsilon(n+1)\setminus F_\varepsilon(n)$ for some integer $n \geq 0$. Since $S \subseteq SG$, clearly $f \in F_\varepsilon(n+1) \subseteq$

$E_\iota(n+1)$. Suppose that $\mathrm{h}(\iota,f)\leq n$, that is, $f\in E_\iota(n)$. By (i), $f\in\mathcal{L}_n$, hence $f=\sum_{x\in G}a_x\bar{x}$ with $a_x\in F_\varepsilon(n)$. Since two series $\sum_{x\in G}b_x\bar{x}$, $\sum_{x\in G}c_x\bar{x}\in E$, where $b_x,c_x\in F$ for each $x\in G$, are equal if and only if $b_x=c_x$ for each $x\in G$, we deduce that $f=a_1\in F_\varepsilon(n)$, a contradiction. Hence $\mathrm{h}(\iota,f)=n+1$.

Statement (iii) follows from (ii). □

If G is a group and $x,y\in G$, by (x,y) we denote the commutator $(x,y)=x^{-1}y^{-1}xy$.

It is well known that a torsion-free nilpotent group is orderable. Also, the free product of orderable groups is orderable. Hence, if we are given a set of torsion-free nilpotent groups $\{G_i\}_{i\in I}$, the free product $*_{i\in I}G_i$ is an orderable group.

Corollary 5.2. *Let k be a commutative field, I be a set of cardinality at least two and $\{G_i\}_{i\in I}$ be a set of nontrivial torsion-free nilpotent groups. Set $G=\underset{i\in I}{*}G_i$, and let $(G,<)$ be a structure of ordered group on G. Let kG be the group ring and $\iota\colon kG\hookrightarrow E=k((G,<))$ be the natural embedding in its Malcev-Neumann series ring. Then $\mathrm{h}(\iota)=\infty$. Indeed, let $x\in G_i\setminus\{1\}$ and $y\in G_j\setminus\{1\}$ with $i\neq j$. If f is any entry of the inverse of the $n\times n$ matrix*

$$A_n=\begin{pmatrix}(x,y)&(x,y^2)&\cdots&(x,y^n)\\(x^2,y)&(x^2,y^2)&\cdots&(x^2,y^n)\\\vdots&\vdots&\cdots&\vdots\\(x^n,y)&(x^n,y^2)&\cdots&(x^n,y^n)\end{pmatrix},\tag{5.1}$$

then $\mathrm{h}(\iota,f)=n-1$.

In particular, if X is a set of cardinality at least two and G is the free group on X, then the universal fields of fractions $\iota_1\colon kG\hookrightarrow F$ and $\iota_2\colon k\langle X\rangle\hookrightarrow F$ are of infinite inversion height. Indeed, let $x,y\in X$ be different elements, if f is any entry of the inverse of the $n\times n$ matrix (5.1), then $\mathrm{h}(\iota_1,f)=n-1$.

Proof. Consider $\bigoplus_{i\in I}G_i$, the subgroup of the cartesian product $\prod\limits_{i\in I}G_i$ consisting of all $(x_i)_{i\in I}\in\prod\limits_{i\in I}G_i$ such that $x_i=1$ for almost all $i\in I$.

For each $i\in I$, let $\pi_i\colon G_i\hookrightarrow\bigoplus_{i\in I}G_i$ be the canonical inclusion and let $\pi\colon *_{i\in I}G_i\to\bigoplus_{i\in I}G_i$ be the unique morphism of groups such that $\pi_{|G_i}=\pi_i$. Set $N=\ker\pi$, then N is a free group. Indeed, the same arguments of [17, Appendix] prove that if we fix a total order $\prec$ on I, then N is the free group on the nontrivial elements of the set of commutators

$$\Big\{(x_{i_s}\cdots x_{i_{r+1}},x_{i_r}\cdots x_{i_2}x_{i_1})\mid x_{i_j}\in G_{i_j},\ i_s\succ\cdots\succ i_2\succ i_1\in I\Big\}.$$

Since the cardinality of I is at least two, and each G_i is an infinite group for each i, N is not finitely generated.

Therefore G is the extension of the free group N by the group $G/N\cong\bigoplus_{i\in I}G_i$. Thus the group ring kG can be regarded as the crossed product $(kN)(G/N)$. Recall that since G/N is locally nilpotent, any crossed product $F\frac{G}{N}$, with F a field, is an Ore domain.

If $\varepsilon = \iota_{|kN} : kN \hookrightarrow E_\varepsilon$, then ε is the universal field of fractions of kN, cf. Remark 2.5. Since any automorphism of kN can be extended to E_ε by [12, Corollary 7.5.16], the crossed product $kN\frac{G}{N}$ extends to $E_\varepsilon\frac{G}{N}$, see Remark 2.2. Another way of proving this extension can be found in [37, Proposition 2.5(1)].

If for each $\alpha \in \frac{G}{N}$, we pick a coset representative $x_\alpha \in G$, then the set $\{x_\alpha\}_{\alpha\in\frac{G}{N}}$ is linearly independent over E_ε (in fact over $k((N,<))$). Thus $E_\varepsilon\frac{G}{N} \hookrightarrow E$. Hence $kG \hookrightarrow E_\varepsilon\frac{G}{N} \hookrightarrow E_\iota$. The crossed product $E_\varepsilon\frac{G}{N}$ is an Ore domain and then, by the universal property of the Ore localization, $E_\varepsilon\frac{G}{N} \hookrightarrow E_\iota$ is the Ore field of fractions of $E_\varepsilon\frac{G}{N}$.

The group $\frac{G}{N}$ is orderable, so that fixing a group ordering $\prec$ on G/N, we can form the Malcev-Neumann power series ring $E_\varepsilon((\frac{G}{N},\prec))$ which is a field containing $E_\varepsilon\frac{G}{N}$. Again, by the universal property of the Ore localization, the field of fractions of $E_\varepsilon\frac{G}{N}$ inside $E_\varepsilon((\frac{G}{N},<))$ is isomorphic to E_ι. Now we can use Theorem 5.1(iii) to deduce that $\mathrm{h}(\iota) \geq \mathrm{h}(\varepsilon)$; since $\mathrm{h}(\varepsilon) = \infty$ by Corollary 4.2, then $\mathrm{h}(\iota) = \infty$. Also, by Theorem 5.1(ii) and Corollary 4.2, $\mathrm{h}(\iota, f) = n - 1$.

Now we prove the second part of the Corollary as a consequence of the foregoing. Let X be a set of cardinality at least 2 and G be the free group on X. Consider the group ring kG, the free algebra $k\langle X\rangle$ and the universal fields of fractions $\iota_1 : kG \hookrightarrow F$ and $\iota_2 : k\langle X\rangle \hookrightarrow F$. First recall that we can identify F with the field of fractions of kG (or $k\langle X\rangle$) inside the Malcev-Neumann power series ring $k((G,<))$ for any structure of ordered group $(G,<)$ on G, cf. Section 2.3. Next note that $G = \mathop{*}_{x\in X} G_x$ where G_x is the infinite cyclic group for each $x \in X$. Hence, applying the foregoing, we obtain that $\mathrm{h}(\iota_1) = \infty$ and $\mathrm{h}(\iota_1, f) = n-1$ where f is any entry of the inverse of the matrix (5.1). To show that $\mathrm{h}(\iota_2) = \infty$, observe that $F_{\iota_1}(n) \subseteq F_{\iota_2}(n+1) \subseteq F_{\iota_1}(n+1)$. □

Proposition 5.3. *Let k be a commutative field. For each finite set X with $|X| \geq 2$, there exist infinitely many non-isomorphic fields of fractions $\iota : k\langle X\rangle \to D$ such that $\mathrm{h}(\iota) = \infty$. Moreover, if we denote by H the free group on X and by kH the group algebra on H, there exist infinitely many non-isomorphic fields of fractions $\varepsilon : kH \to D$ such that $\mathrm{h}(\varepsilon) = \infty$.*

Proof. Step 1: *We define a suitable poly-orderable group Γ_r for each integer $r \geq 1$.*

We follow the notation in [13, Chapter 1]. Fix $r \geq 1$. Let Y be the connected graph with vertex set $VY = \mathbb{Z}$, edge set $EY = \{e_i \mid i \in \mathbb{Z}\}$ and incidence functions $\bar{\iota}(e_i) = i$ and $\bar{\tau}(e_i) = i+1$, i.e.

$$\cdots \xrightarrow{e_{i-1}} \overset{i}{\bullet} \xrightarrow{e_i} \overset{i+1}{\bullet} \xrightarrow{e_{i+1}} \cdots$$

Let $(G(\ ),Y)$ be the graph of groups

$$\cdots \longrightarrow \overset{G(i)}{\bullet} \xrightarrow{G(e_i)} \overset{G(i+1)}{\bullet} \xrightarrow{G(e_{i+1})} \cdots$$

where $G(i)$ is the free abelian group on $\{T_i, T_{i+1}, \dots, T_{i+r}\}$ and $G(e_i)$ the free abelian group on $\{T_{i+1}, \dots, T_{i+r}\}$ for each $i \in \mathbb{Z}$. Let N_r be the fundamental group of $(G(\ ), Y)$, i.e. $N_r = \pi(G(\ ), Y, Y_0)$ with $Y_0 = Y$. Then, by definition,

$$N_r = \left\langle T_i,\ i \in \mathbb{Z} \,\middle|\, \begin{smallmatrix} T_iT_{i+1}=T_{i+1}T_i \\ T_iT_{i+2}=T_{i+2}T_i \\ \cdots \\ T_iT_{i+r}=T_{i+r}T_i \end{smallmatrix} \right\rangle. \tag{5.2}$$

Also N_r can be viewed as

$$\cdots G(i-1) *_{G(e_{i-1})} G(i) *_{G(e_i)} G(i+1) *_{G(e_{i+1})} \cdots . \tag{5.3}$$

Consider the morphism of groups $\theta\colon N_r \to \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}$ defined by $\theta(T_i) = f_i$ where f_i is the sequence $(x_n)_{n\in\mathbb{Z}}$ with $x_i = 1$ and $x_n = 0$ for $n \neq i$. It is easy to deduce from (5.2) that θ is well defined. Let $L_r = \ker\theta$. Observe that $\theta_{|G(i)}$ is injective for each $i \in \mathbb{Z}$. Hence L_r is a free group by [13, Proposition 7.10]. Moreover, L_r is not commutative because for example $T_0T_{r+1}T_0^{-1}T_{r+1}^{-1}$ and $T_{2r+2}T_{3r+3}T_{2r+2}^{-1}T_{3r+3}^{-1}$ belong to L_r, but they do not commute as can be deduced from (5.3). In a similar way, it can be shown that L_r is not finitely generated.

Define now $\Gamma_r = N_r \rtimes C$, where $C = \langle S\rangle$ is the infinite cyclic group, and C acts on N_r as $T_i \mapsto T_{i+1}$, i.e. $ST_iS^{-1} = T_{i+1}$. Hence Γ_r has the subnormal series

$$1 \lhd L_r \lhd N_r \lhd \Gamma_r,$$

with $\Gamma_r/N_r = C$ infinite cyclic, $N_r/L_r \cong \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}$ a torsion-free abelian group and L_r a non commutative free group. Hence all factors are orderable groups.

Step 2: *We prove that there exists a field of fractions of the group algebra* $k\Gamma_r$, $\delta_r\colon k\Gamma_r \hookrightarrow E^r$ *say, with* $\mathrm{h}(\delta_r) = \infty$ *for each integer* $r \geq 1$.

Let $r \geq 1$. Consider $\beta_r\colon kL_r \hookrightarrow Q_r$ the universal field of fractions of the free group algebra kL_r. Consider kN_r as a crossed product $(kL_r)\frac{N_r}{L_r}$. Any automorphism of kL_r can be extended to an automorphism of Q_r by [12, Corollary 7.5.16]. Hence we can consider a crossed product $Q_r\frac{N_r}{L_r}$ that contains $(kL_r)\frac{N_r}{L_r}$ in the natural way, see Remark 2.2. Since N_r/L_r is a torsion-free abelian group, $Q_r\frac{N_r}{L_r}$ is an Ore domain. Let $\gamma_r\colon kN_r \hookrightarrow Q_r\frac{N_r}{L_r} \hookrightarrow D_r$ where D_r is the Ore field of fractions of $Q_r\frac{N_r}{L_r}$. The group ring $k\Gamma_r$ can be regarded as a skew Laurent polynomial ring $(kN_r)[S, S^{-1}; \alpha]$ where α is given by left conjugation by S, cf. §2.2. Observe that conjugation by S induces an automorphism on L_r, thus on kL_r and on Q_r. Therefore it can be extended to an automorphism of $Q_r\frac{N_r}{L_r}$. Since $Q_r\frac{N_r}{L_r}$ is an Ore domain and conjugation by S gives an automorphism of $Q_r\frac{N_r}{L_r}$, it can be extended to an automorphism α of D_r. Hence there are k-algebra embeddings

$$k\Gamma_r = (kN_r)[S, S^{-1}; \alpha] \hookrightarrow Q_r\frac{N_r}{L_r}[S, S^{-1}; \alpha] \hookrightarrow D_r[S, S^{-1}; \alpha].$$

Let E^r be the Ore field of fractions of $D_r[S, S^{-1}; \alpha]$, and $\delta_r\colon k\Gamma_r \hookrightarrow E^r$ be the natural embedding. Observe that it is a field of fractions of $k\Gamma_r$.

Since $Q_r\frac{N_r}{L_r}$ and $D_r[S,S^{-1};\alpha]$ are Ore domains, we can think of E^r and D_r as embedded in $D_r((S;\alpha))$ and $Q_r((\frac{N_r}{L_r},<))$ for a certain group ordering $<$ on N_r/L_r, respectively. Now by Proposition 3.5(iii), $\mathrm{h}(\delta_r)\geq \mathrm{h}(\gamma_r)$. By Theorem 5.1(iii), $\mathrm{h}(\gamma_r)\geq \mathrm{h}(\beta_r)$. By Corollary 4.2, $\mathrm{h}(\beta_r)=\infty$. Therefore $\mathrm{h}(\delta_r)=\infty$.

Step 3: *We show that, for each pair of integers $1\leq r\leq s$, the free algebra $k\langle X_0,X_1,\dots,X_r\rangle$ embeds in $k\Gamma_s$ via $X_i\mapsto T_0^iS$.*

We shall prove that the above substitution induces an embedding of the free monoid on $X_0,X_1,\dots,X_r$ into Γ_s. To this aim it suffices to show that in a given product $p=(T_0^{i_0}S)\cdots(T_0^{i_m}S)$ the parameters m and $i_0,\dots,i_m$ are uniquely determined. Since $\Gamma_s=N_s\rtimes C$, then $p=aS^m$ for a suitable $a\in N_s$ by the definition of the action of C on N_s. Since this expression of p is unique, m and a are uniquely determined. Moreover, the coordinates at $j=0,\dots,m$ of $\theta(a)$ are $i_0,\dots,i_m$, respectively, so that they are uniquely determined. This finishes the proof of the claim.

Step 3′: *Let H_r be the free group on $\{X_0,X_1,\dots,X_r\}$. Then, for each pair of integers $1\leq r\leq s$, the free group algebra kH_r embeds in E^s via $X_i\mapsto 1+T_0^iS$.*

Recall from the proof of Step 2 that E^s is the Ore field of fractions of $D_r[S,S^{-1};\alpha]$. There is a natural valuation $\upsilon\colon D_r[S,S^{-1};\alpha]\to\mathbb{Z}\cup\{\infty\}$. Indeed, let $f\in D_r[S,S^{-1};\alpha]$, thus $f=\sum_{i\in\mathbb{Z}}f_iS^i$, where $f_i\in D_r$ and almost all $f_i=0$. Then υ is defined by $\upsilon(f)=\min\{i\in\mathbb{Z}\mid f_i\neq 0\}$ and $\upsilon(0)=\infty$. By the definition of υ, $\upsilon(T_0^iS)=1$. Since the free k-algebra $k\langle X_0,X_1,\dots,X_r\rangle$ embeds in E^s via $X_i\mapsto T_0^iS$, by Step 3, [24, Corollary 1, p.524] implies that the k-algebra generated by $\{1+T_0^iS,\ (1+T_0^iS)^{-1}\mid i=0,1,\dots,r\}$ is the free group k-algebra on the set $\{1+T_0^iS\mid i=0,1,\dots,r\}$, as desired. We note that in order to apply [24, Corollary 1, p.524], it is enough that k is a central subfield of E^s and not the whole center of E^s (as stated in Lichtman's result) by [27, Lemma 1].

Step 4: *Let us prove that, for each pair of integers $1\leq r\leq s$, there is a field of fractions $\iota_{rs}\colon k\langle X_0,X_1,\dots,X_r\rangle\hookrightarrow E^s$, defined by $\iota_{rs}(X_i)=T_0^iS$, of infinite inversion height.*

Let ι_{rs} be the embedding of Step 3 composed with δ_s. Note that $S=\iota_{rs}(X_0)$ and $T_0=\iota_{rs}(X_1)\iota_{rs}(X_0)^{-1}$. Since Γ_s is generated by S and T_0, $k\Gamma_s\subseteq E^s_{\iota_{rs}}(1)$. Therefore E^s is generated, as a field, by the image of ι_{rs} and $\mathrm{h}(\iota_{rs})\geq\mathrm{h}(\delta_s)=\infty$.

Step 4′: *Let us prove that, for each pair of integers $1\leq r\leq s$, there is a field of fractions $\varepsilon_{rs}\colon kH_r\hookrightarrow E^s$, defined by $\varepsilon_{rs}(X_i)=1+T_0^iS$, of infinite inversion height.*

Since both $E_{\iota_{rs}}(0)$, $E_{\varepsilon_{rs}}(0)$ contain the field k, $E_{\iota_{rs}}(0)\subseteq E_{\varepsilon_{rs}}(0)$. Hence, by induction, $E_{\iota_{rs}}(n)\subseteq E_{\varepsilon_{rs}}(n)$ for any natural number $n\geq 0$. On the other hand, the fact that $E_{\varepsilon_{rs}}(0)\subseteq E_{\iota_{rs}}(1)$ implies that $E_{\varepsilon_{rs}}(n)\subseteq E_{\iota_{rs}}(n+1)$ for all $n\geq 0$. Therefore E^s is generated, as a field, by the image of ε_{rs} and $\mathrm{h}(\varepsilon_{rs})=\infty$ because $\mathrm{h}(\iota_{rs})=\infty$.

Step 5: *Let $1 \leq r \leq s < s'$ be integers. Then the two fields of fractions*

$$\iota_{rs}\colon k\langle X_0, X_1, \dots, X_r\rangle \hookrightarrow E^s \quad \textit{and} \quad \iota_{rs'}\colon k\langle X_0, X_1, \dots, X_r\rangle \hookrightarrow E^{s'}$$

are not isomorphic.

First of all observe that there does not exist an isomorphism of groups $\Gamma_s \to \Gamma_{s'}$ such that $S \mapsto S$ and $T_0 \mapsto T_0$.

If there is an isomorphism of rings $\eta_{ss'}\colon E^s \to E^{s'}$ such that $\iota_{rs'} = \eta_{ss'}\iota_{rs}$, then

$$\eta_{ss'}(S) = \eta_{ss'}(\iota_{rs}(X_0)) = \iota_{rs'}(X_0) = S,$$

$$\eta_{ss'}(T_0) = \eta_{ss'}(\iota_{rs}(X_1)\iota_{rs}(X_0)^{-1}) = \iota_{rs'}(X_1)\iota_{rs'}(X_0)^{-1} = T_0.$$

Hence the restriction of $\eta_{ss'|\Gamma_s}\colon \Gamma_s \to \Gamma_{s'}$ gives an isomorphism of groups sending $S \mapsto S$ and $T_0 \mapsto T_0$, a contradiction.

Step 5′: *Let $1 \leq r \leq s < s'$ be integers. Then the two fields of fractions*

$$\varepsilon_{rs}\colon kH_r \hookrightarrow E^s \quad \textit{and} \quad \varepsilon_{rs'}\colon kH_r \hookrightarrow E^{s'}$$

are not isomorphic.

If there is an isomorphism of rings $\eta_{ss'}\colon E^s \to E^{s'}$ such that $\varepsilon_{rs'} = \eta_{ss'}\varepsilon_{rs}$, then the restriction $\eta_{ss'|\Gamma_s}\colon \Gamma_s \to \Gamma_{s'}$ gives an isomorphism of groups sending $S \mapsto S$ and $T_0 \mapsto T_0$ as in Step 5, a contradiction. □

Corollary 5.4. *Let k be a commutative field and $Z = \{z_0, z_1, \dots\}$ be an infinite countable set. Then the free algebra $k\langle Z\rangle$ has infinitely many non-isomorphic fields of fractions $\iota\colon k\langle Z\rangle \to D$ such that $\mathrm{h}(\iota) = \infty$. Moreover, if we denote by H_Z the free group on Z, then the free group algebra kH_Z has infinitely many non-isomorphic fields of fractions $\varepsilon\colon kH_Z \to D$ such that $\mathrm{h}(\varepsilon) = \infty$.*

Proof. The free k-algebra case follows from [16, Proposition 2.3] and Proposition 5.3.

For the free group k-algebra, we need to take a closer look at the embedding of $k\langle Z\rangle$ given in [16, Proposition 2.3]. Let $k\langle x, y\rangle$ be the free algebra on two generators. For each $s \geq 2$, consider the embedding given in Step 4 of Proposition 5.3, $\iota_{2s}\colon k\langle x, y\rangle \hookrightarrow E^s$, $x \mapsto T_0S$, $y \mapsto S$. Now consider the embedding $\alpha\colon k\langle Z\rangle \hookrightarrow k\langle x, y\rangle$, $z_i \mapsto xy^i$ given in [16, Proposition 2.3], and consider the composition $\iota_{2s}\alpha$. Observe that $\iota_{2s}\alpha(z_i) = T_0S^{i+1}$. Since $\upsilon(T_0S^{i+1}) = i+1$, where υ is the valuation defined in Step 3', one can apply [24, Corollary 1, p.524] to obtain the embedding $kH_Z \to E^s$, $z_i \mapsto 1 + T_0S^{i+1}$. It can be proved as in Step 5' of Proposition 5.3 that these fields of fractions of kH_Z are not isomorphic if $s \neq s'$. □

6. Crossed products of a ring by a universal enveloping algebra

Throughout this section, k will denote a commutative field. If L is a Lie k-algebra, we will denote by $U(L)$ its *universal enveloping algebra.*

Let L be a Lie k-algebra. Let S be a k-algebra with a k-linear embedding ${}^{-}\colon L \to S$, $x \mapsto \bar{x}$. Let $\mathcal{C}$ be a k-linear independent subset of L. Suppose that we have defined a total order $<$ in $\mathcal{C}$. The set of *standard monomials* in $\mathcal{C}$ is the subset of S consisting of the monomials of the form $\bar{x}_1\bar{x}_2\cdots\bar{x}_m$ with $m \geq 0$, $x_i \in \mathcal{C}$ and $x_1 \leq x_2 \leq \cdots \leq x_m$ where we understand that the identity element in S is the standard monomial corresponding to $m = 0$. We will say that a standard monomial that is the product of m elements of $\mathcal{C}$ has degree m. If ${}^{-}\colon L \to U(L)$ is the natural map and $\mathcal{B} = \{x_i \mid i \in I\}$ is a totally ordered basis of L, the Poincaré-Birkhoff-Witt (PBW) Theorem states that the standard monomials in $\mathcal{B}$ form a k-basis of $U(L)$.

Let L be a Lie k algebra with a basis with a totally ordered basis $\mathcal{B} = \{x_i \mid i \in I\}$. Suppose that R is a k-algebra, and let $\operatorname{Der}_k(R)$ denote the set of k-linear derivations of R. A k-algebra S containing R is called a *crossed product* of R by $U(L)$ (and written $R{*}U(L)$) provided that there is a k-linear embedding ${}^{-}\colon L \to S$, $x \mapsto \bar{x}$, such that:

(i) S is a free left R-module with basis the standard monomials in $\mathcal{B}$.

(ii) There exist a k-linear map (called an *action*) $\delta\colon L \to \operatorname{Der}_k(R)$, $x \mapsto \delta_x$, and a k-bilinear antisymmetric map (called a *twisting*) $t\colon L \times L \to R$, $(x, y) \mapsto t(x, y)$ such that the following two conditions hold:

$$\bar{x}a = a\bar{x} + \delta_x(a) \quad \text{for all } x \in L \text{ and } a \in R, \tag{6.1}$$

$$\bar{x}\bar{y} - \bar{y}\bar{x} = \overline{[x, y]} + t(x, y) \quad \text{for all } x, y \in L. \tag{6.2}$$

Crossed products for Lie Algebras were introduced in [30, 1.7.12] and in [7].

Note that (i) in the above definition is equivalent to say that S has the left R-module structure as $R \otimes_k U(L)$. It is not hard to show that condition (i) is independent of the choice of the basis $\mathcal{B}$ of L and of the total order $<$ of $\mathcal{B}$.

Remark 6.1. Given any k-algebra R and Lie k-algebra L, at least one crossed product of R by $U(L)$ exists, the k-algebra $R \otimes_k U(L)$. In this example, $\delta\colon R \to \operatorname{Der}_k(R)$ sends every element of R to the zero map and $t\colon L \times L \to R$ is such that $t(x, y) = 0$ for all $x, y \in L$. More generally, if $\delta\colon R \to \operatorname{Der}_k(R)$ is any morphism of Lie k-algebras and, again, $t\colon L \times L \to R$ is such that $t(x, y) = 0$ for all $x, y \in L$, there exists a crossed product $R{*}U(L)$ with action δ and twisting t by Lemma 6.3.

One of the most important properties of crossed products is the following result which is [3, Lemma 1.1]. We will need to know how the identification (6.3) is made, thus we sketch its proof.

Lemma 6.2. *Suppose R is a k-algebra, L a Lie k-algebra, $R * U(L)$ is a crossed product and H is an ideal of L. Then there exists a crossed product decomposition*

$$R * U(L) = (R * U(H)) * U(L/H). \tag{6.3}$$

Proof. Set $T = R * U(H)$. Let W be a subspace of L with $L = H \oplus W$ and let $\sigma\colon L/H \to W$ be a k-vector space isomorphism. Let $\mathcal{D}$ be an ordered basis for L/H and let $\mathcal{C}$ be one for H. Then $\mathcal{B} = \mathcal{C} \cup \{\sigma(d) \mid d \in \mathcal{D}\}$ is a basis for L. We order it by extending the orders in $\mathcal{C}$ and $\mathcal{D}$ to $\mathcal{B}$ and determining that the elements of $\mathcal{C}$ come first. Then $R * U(L)$ has the additive structure of $T * U(L/H)$ by the PBW-theorem. Let $\tilde{}$ denote the composition of σ followed by $\bar{}$. Then, for each $x \in L/H$ and $t \in T$, we have that $\zeta_x(t) = t\tilde{x} - \tilde{x}t \in T$. Thus we get a k-linear map $\zeta\colon L/H \to \operatorname{Der}_k(T)$, $x \mapsto \zeta_x$. Also it is not very difficult to prove that, for each $x, y \in L/H$, $s(x,y) = \tilde{x}\tilde{y} - \tilde{y}\tilde{x} - \widetilde{[x,y]} \in T$. We thus define the k-linear map $s\colon L/H \times L/H \to T$, $(x,y) \mapsto s(x,y)$ and see that ζ and s satisfy (6.1) and (6.2). ☐

For a given embedding of rings $R \hookrightarrow D$, we will be interested in extending a crossed product $R * U(L)$ to a crossed product $D * U(L)$. In order to do that we need to be precise on the conditions that δ and t must satisfy. This is explained in the next lemma which can be viewed as a corollary of [31, Theorem 7.1.10]

Lemma 6.3. *Let R be a k-algebra, and let L be a Lie k-algebra. Suppose that there exist a k-linear map $\delta\colon L \to \operatorname{Der}_k(R)$, $x \mapsto \delta_x$, and a k-bilinear antisymmetric map $t\colon L \times L \to R$, $(x,y) \mapsto t(x,y)$. They define a crossed product $R * U(L)$ if and only if δ and t satisfy the following relations:*

(i) $\delta_x(t(y,z)) + \delta_y(t(z,x)) + \delta_z(t(x,y)) + t(x,[y,z]) + t(y,[z,x]) + t(z,[x,y]) = 0.$
(ii) $[\delta_x, \delta_y] = \delta_{[x,y]} + \partial_{t(x,y)}$ *where $\partial_{t(x,y)}$ denotes the k-derivation of R defined by $a \mapsto [t(x,y), a] = t(x,y)a - at(x,y)$ for all $a \in R$.*

Moreover, $R * U(L)$ can be constructed as the k-coproduct of R with $T(L)$, the k-tensor algebra over L, modulo the two-sided ideal $\mathcal{I}$ generated by the set

$$\{xa - ax - \delta_x(a), \quad xy - yx - [x,y] - t(x,y) \mid \text{ for any } x, y \in L \text{ and } a \in R\},$$

and it is free as a right and as a left R-module. More precisely, if $\mathcal{B} = \{e_j \mid j \in J\}$ is a fixed ordered basis for L, then the set $\mathcal{G}$ of standard monomials on $\mathcal{B}$ is a basis of $R*U(L)$ as a right and as a left R-module; and if, for any $m \geq 0$, $\mathcal{G}_m \subseteq \mathcal{G}$ denotes the set of standard monomials of degree at most m, then $\sum_{x \in \mathcal{G}_m} xR = \sum_{x \in \mathcal{G}_m} Rx$.

Remarks 6.4.

(a) Let $R \hookrightarrow D$ be an extension of k-algebras, and let L be a Lie k-algebra such that there exists a crossed product $R * U(L)$. To extend the crossed product structure to a crossed product $D * U(L)$ in such a way that there is a ring inclusion $R * U(L) \hookrightarrow D * U(L)$ extending $R \hookrightarrow D$ and such that $\overline{x} \mapsto \overline{x}$, for any $x \in L$, one has:
  (1) to extend the action δ_R to a k-linear map $\delta_D\colon L \to \operatorname{Der}_k(D)$ in such a way that, for any $r \in R$, $\delta_D(x)(r) = \delta_R(x)(r)$;
  (2) to make sure that condition (ii) in Lemma 6.3 is satisfied.

  Notice that the twisting must be the same for both crossed products, so that it is not necessary to verify condition (i) in Lemma 6.3.

(b) Usually, we will be working with ring embeddings $R \hookrightarrow D$ such that the derivations over R extend in a unique way to D (as in Lemma 4.4), so that conditions (1) and (2) above will be automatically satisfied.

(c) We will also find contexts in which there is a k-algebra embedding $R*U(L) \to E$ such that $D \subseteq E$, and then we will need, not only the existence of the k-algebra $D*U(L)$, but also that $D*U(L)$ is a subalgebra of E. To be able to do that we will need

   (3) to make sure that the standard monomials are left D-independent in E.

The existence of a PBW-basis for $R*U(L)$, asserted in Lemma 6.3, gives a structure of filtered ring to $R*U(L)$ by setting, for any $m \geq 0$, $\mathcal{F}_m$ to be the R-subbimodule of $R*U(L)$ generated by the monomials of degree at most m. By the definition of crossed product and Lemma 6.3, the associated graded ring is a polynomial ring over R in the commutative variables given by the basis of the Lie algebra L. For further quoting we summarize this in the next Lemma.

Lemma 6.5. *Let R be a k-algebra, and let L be a Lie k-algebra. Suppose that there exists a crossed product $R*U(L)$. Fix $\mathcal{B}$ to be a basis of L, then $\operatorname{gr}(R*U(L)) \cong R[\mathcal{B}]$, that is, a polynomial algebra over R in the commuting variables $\mathcal{B}$.*

In the foregoing lemma, if R is a field, then $\operatorname{gr}(R*U(L))$ is an Ore domain, which implies that $R*U(L)$ embeds in a field with some good properties. This is expressed more precisely in the next proposition.

Proposition 6.6. *Let L be a Lie k-algebra and K be a field with k as a central subfield. For each crossed product $K*U(L)$, there is a canonically constructed field of fractions*

$$K*U(L) \hookrightarrow \mathfrak{D}(K*U(L)).$$

Suppose that N is a subalgebra of L. The following properties are satisfied:

(i) *The following diagram is commutative*

$$\begin{array}{ccc} K*U(N) & \hookrightarrow & \mathfrak{D}(K*U(N)) \\ \downarrow & & \downarrow \\ K*U(L) & \hookrightarrow & \mathfrak{D}(K*U(L)) \end{array}$$

(ii) *If $\mathcal{B}_N$ is a basis of N and $\mathcal{C}$ is a set of elements of $L \setminus N$ such that $\mathcal{B}_N \cup \mathcal{C}$ is a basis of L, then the standard monomials in $\mathcal{C}$ are linearly independent over $\mathfrak{D}(K*U(N))$.*

(iii) *If N is an ideal of L, then the subring of $\mathfrak{D}(K*U(L))$ generated by $K*U(L)$ and $\mathfrak{D}(K*U(N))$ is a crossed product $\mathfrak{D}(K*U(N))*U(L/N)$ extending $(K*U(N))*U(L/N)$ in the natural way.*

Proof. By Lemma 6.5, $\operatorname{gr}(K*U(L))$ is an Ore domain. Now [11, Theorem 2.6.5] or [25] imply the existence of the construction $K*U(L) \hookrightarrow \mathfrak{D}(K*U(L))$.

Conditions (i) and (ii) can be proved in exactly the same way as for the embedding $U(L) \hookrightarrow \mathfrak{D}(U(L))$, see [26, Proposition 2]. Condition (iii) follows as in [26, Section 2.3]. □

Now we turn our attention to crossed products where the underlying Lie algebra is free.

Lemma 6.7. *Let R be a k-algebra. Let H be the free Lie k-algebra on a set X. If $R * U(H)$ is a crossed product then, for each $x \in X$, there exists a k-derivation $\partial_x\colon R \to R$ such that $R * U(H) \cong \coprod_{x\in X} R[x;\partial_x]$, the ring coproduct over R.*

In particular, if $R = K$ is a field then $K * U(H)$ is a fir.

Proof. Consider the Lie k-algebra structure of $R * U(H)$ where the Lie product is given by $[a,b] = ab - ba$ for all $a, b \in R * U(H)$. Consider the morphism of Lie k-algebras $\tilde{}\colon H \to R * U(H)$ which sends each $x \in X$ to $\bar{x}$. Thus $\tilde{z}\tilde{w} - \tilde{w}\tilde{z} = \widetilde{[z,w]}$ for all $z, w \in H$.

By induction on the length of the Lie words on X and then extending by linearity to H, it is not difficult to show that for each $z \in H$,

$$\tilde{z} = \bar{z} + b_z \text{ for some } b_z \in R. \tag{6.4}$$

Hence $\tilde{}\colon H \to R * U(H)$ is a k-linear embedding.

Also from (6.4), it follows that

$$\tilde{z}a = a\tilde{z} + \partial_z(a) \quad \text{for each } a \in R \text{ and } z \in H, \tag{6.5}$$

where $\partial_z \in \mathrm{Der}_k(R)$ and is given by $a \mapsto \delta_z(a) + [b_z, a]$. Thus we have just proved that $R * U(H)$ can be regarded as a crossed product with trivial twisting.

It is known that $U(H)$ is the free k-algebra on the set X. Let $\mathcal{B}$ be a basis of H, $\mathcal{S}$ be the set of standard monomials in $\mathcal{B}$ (inside $U(H)$) and M be the free monoid on X. Then M and $\mathcal{S}$ are two basis of $U(H)$. Consider now $\widetilde{M}$, the monoid generated by $\widetilde{X} = \{\tilde{x} \mid x \in X\}$ inside $R * U(H)$. Let $\widetilde{\mathcal{S}}$ be the set of the standard monomials in $\mathcal{B}$ (inside $R * U(H)$), and note that it is a basis of $R * U(H)$. Since the twisting is trivial, the change of basis from one of the basis, $\mathcal{S}$ or M, of $U(H)$ to the other is the same as the change of basis from one of the sets, $\widetilde{\mathcal{S}}$ or $\widetilde{M}$, to the other. Therefore $\widetilde{M}$ is the free monoid on $\widetilde{X}$ and it is a basis of $R * U(H)$ as a left R-module.

From (6.5), we deduce that, for each $x \in X$, there exists a morphism of R-rings

$$\varphi_x\colon R[x;\partial_x] \to R * U(H)$$

which sends $x \mapsto \tilde{x}$. Consider now the unique morphism of R-rings

$$\varphi\colon \coprod_{x\in X} R[x;\partial_x] \to R * U(H)$$

extending all φ_x. Proceeding as in [4, Section 4] it is possible to prove that the free monoid on X is a right and left R-basis of $\coprod_{x\in X} R[x;\partial_x]$. Thus φ is an isomorphism.

The statement when R is a field follows from [9, §6]. □

Hence for a free Lie algebra H and a crossed product $K*U(H)$, Lemma 6.7 implies the existence of the universal field of fractions of $K*U(H)$ and Proposition 6.6 the existence of $K*U(H)\hookrightarrow \mathfrak{D}(K*U(H))$. We will show in the next section that both fields of fractions are in fact the same. The next result, which is a generalization of [26, Lemma 3.1] to crossed products, will be useful in proving this assertion.

Proposition 6.8. *Let K be a field with k as a central subfield. Let H be a Lie k-algebra and let N be an ideal of H. Consider a crossed product $K*U(H)$. Suppose that the following two conditions are satisfied:*

(1) *$K*U(H)$ has a universal field of fractions $K*U(H)\hookrightarrow E$.*
(2) *$R=K*U(N)$ has a prime matrix ideal $\mathcal{P}$ whose localization $R_{\mathcal{P}}$ is a field of fractions of R.*

Then $K*U(H)=R*U(H/N)$ (cf. Lemma 6.2) can be extended to a crossed product structure $R_{\mathcal{P}}*U(H/N)$, the embedding $K*U(H)\hookrightarrow E$ can be extended to $R_{\mathcal{P}}*U(H/N)\hookrightarrow E$ and this embedding is the universal field of fractions of $R_{\mathcal{P}}*U(H/N)$.

Proof. First note that since $R_{\mathcal{P}}$ is a field of fractions, $\mathcal{P}$ is a minimal prime matrix ideal (cf. §2.3).

We view $K*U(H)$ as $R*U(H/N)$. By Lemma 4.4, for each $x\in H/N$, the k-derivation δ_x of R can be extended to $R_{\mathcal{P}}$. We denote this extension again by δ_x.

By Lemma 4.3 the extension of the derivations from R to $R_{\mathcal{P}}$ is unique and, hence, by Remarks 6.4(b) there is a crossed product $R_{\mathcal{P}}*U(H/N)$ extending the original one.

Let $\mathcal{B}$ be a basis of H/N. By Proposition 6.6, $R_{\mathcal{P}}*U(H/N)$ has a field of fractions $R_{\mathcal{P}}*U(H/N)\hookrightarrow D=\mathfrak{D}(R_{\mathcal{P}}*U(H/N))$. Clearly the restriction $K*U(H)\hookrightarrow D$ is a field of fractions of $K*U(H)$. Thus there exists a $K*U(H)$-specialization ρ from E to D. By Theorem 2.4(b), ρ gives by restriction an isomorphism between the subfield E_N of E generated by R and $R_{\mathcal{P}}$. Moreover, the standard monomials on $\mathcal{B}$ are linearly independent over E_N in E because, by Proposition 6.6, their images via ρ are linearly independent over $R_{\mathcal{P}}$ in D. Thus the subring of E generated by E_N and $\{\bar{x}\mid x\in H/N\}$ is a crossed product isomorphic to $R_{\mathcal{P}}*U(H/N)$, because $\rho^{-1}\circ\delta$ and $\rho^{-1}\circ\tau$ induce an action and twisting, respectively, for this subring where δ and τ are the action and the twisting of $R_{\mathcal{P}}*U(H/N)$. Thus $R_{\mathcal{P}}*U(H/N)\hookrightarrow E$ is a field of fractions of $R_{\mathcal{P}}*U(H/N)$. To prove that it is the universal field of fractions, observe that any $(R_{\mathcal{P}}*U(H/N))$-field is a $(K*U(H))$-field that contains the field $R_{\mathcal{P}}$. Hence there exists a $(K*U(H))$-specialization from E whose domain contains $R_{\mathcal{P}}$ by Theorem 2.4(b), and thus, arguing as above, it also contains $R_{\mathcal{P}}*U(H/N)$. Hence, such a specialization is also an $(R_{\mathcal{P}}*U(H/N))$-specialization. □

The next corollary is relatively easy but it gives an idea of how *weak* the structure of crossed product is.

Corollary 6.9. *For each field K with k as a central subfield and each Lie k-algebra L, there exists a field D that contains K and a crossed product $D * U(L)$ that has a universal field of fractions.*

Proof. Let X be a set of generators of L. Let H be the free Lie k-algebra on X. Consider the morphism of Lie algebras $H \to L$ that is the identity on X, and let N be the kernel of this morphism. Note that $L \cong H/N$.

Consider a crossed product $K * U(H)$ (this is always possible by Remark 6.1). Set $R = K * U(N)$. Then $K * U(H) = R * U(L)$. By Lemma 6.7, $K * U(H)$ has a universal field of fractions E. Since N is also a free Lie k-algebra, R is a fir by Lemma 6.7. Thus R has a universal field of fractions and it is of the form $R_{\mathcal{P}}$ where $\mathcal{P}$ is the prime matrix ideal consisting of the non-full matrices over R. By Proposition 6.8, there is a crossed product $R_{\mathcal{P}} * U(L)$ and $R_{\mathcal{P}} * U(L) \hookrightarrow E$ is its universal field of fractions. □

Let G be a group, and fix an isomorphism $G \cong H/N$ where H is a free group and N is a normal subgroup (hence, it is a free group) of H. Consider a group ordering on H. For any field K consider the group algebra KH. It was proved in [37, Proposition 2.5], that the crossed product structure $KH = (KN)(H/N)$ can be extended to $K((N))(H/N)$ and this is a subring of the Malcev-Neumann series field $K((H))$. This result combined with the fact that the universal field of fractions of KH can be regarded as a subring of $K((H))$, allows us to prove a result analogous to Corollary 6.9 for the case of groups. That is, for any field K and any group G there is a field D containing K and a crossed product DG that has a universal field of fractions.

7. A field of fractions of a crossed product of a residually nilpotent Lie algebra.

Throughout this section, k will denote a commutative field.

In this section we present a ring of series introduced by A. I. Lichtman in [26]. This ring of series $K((H))$ is constructed from a crossed product $K * U(H)$ of a field K by $U(H)$ where H is a residually nilpotent Lie algebra satisfying the Q-condition (see Section 7.2). It will play the role of the Malcev-Neumann series ring $K((G,<))$ constructed from a crossed product KG of a field K by an ordered group $(G,<)$.

We will give a detailed exposition of the construction of the ring of series for several reasons. First, we wish to clarify and generalize in some aspects the construction given in [26]. Secondly, in Theorem 7.16 we prove that, for a free Lie algebra H, this power series ring contains the universal field of fractions of $K * U(H)$, this is an extension of [26, Theorem 1]. Having in mind the analogy between free groups and free Lie algebras, this result can be viewed as a counterpart of Lewin's Theorem [22]. Moreover, as an application, we will produce further examples of elements with arbitrary inversion height inside the free field.

The construction is divided into two parts. In Section 7.1 we shall construct a ring of series for a crossed product of a ring R by $U(L)$ where L is a nilpotent Lie algebra. In Section 7.2, we give the general construction using the preceding case. The main idea is presented in Construction 7.2, which is a generalization of [26, Section 4]. We begin with a lemma on the extension of derivations.

Lemma 7.1. *Let S be a k-algebra and $\{I_n\}_{n\geq 0}$ a descending chain of two-sided ideals of S with $I_0 = R$. Let $\delta\colon S \to S$ be a k-derivation such that $\delta(I_n) \subseteq I_n$ for all $n \geq 0$, so that, for any $n \geq 1$, δ induces a derivation $\delta_n\colon S/I_n \to S/I_n$. Consider the completion $\widehat{S}$ of S with respect to the topology induced by $\{I_n\}_{n\geq 0}$. For any $n \geq 1$, let $\pi_n\colon \hat{S} \to S/I_n$ denote the canonical projection. Then δ can be extended to a derivation $\delta\colon \widehat{S} \to \widehat{S}$ which is uniquely determined by the property $\pi_n \circ \delta = \delta_n \circ \pi_n$ for any $n \geq 1$.*

Proof. For all $n \geq 1$, the diagram

$$\begin{array}{ccc} \frac{S}{I_{n+1}} & \xrightarrow{\delta_{n+1}} & \frac{S}{I_{n+1}} \\ \downarrow & & \downarrow \\ \frac{S}{I_n} & \xrightarrow{\delta_n} & \frac{S}{I_n} \end{array}$$

where the vertical arrows are the canonical projections, is commutative. Hence, by the universal property of the completion $\hat{S} = \varprojlim_{n\geq 1} S/I_n$, the maps $\delta_n \circ \pi_n$ induce a unique k-linear map $\delta\colon \hat{S} \to \hat{S}$ that will be a derivation because δ_n is one for any $n \geq 1$. Notice that, by the definition of this extension, $\pi_n \circ \delta = \delta_n \circ \pi_n$ for any $n \geq 1$. □

We remark that, in the statement of Lemma 7.1, the k-algebra S may not embed in $\widehat{S}$.

Construction 7.2. Let L be a Lie k-algebra, R a k-algebra and $R * U(L)$ a crossed product.

Suppose that the *center* of L, $\mathcal{Z}(L) = \{x \in L \mid [L, x] = 0\}$, is not zero. Fix a nonzero element $u \in \mathcal{Z}(L)$. The k-subspace $N = ku$ is an ideal of L and $[L, N] = 0$. Note that $R * U(N)$, the k-subalgebra of $R * U(L)$ generated by R and $\bar{u}$, is a skew polynomial ring $R[\bar{u}; \delta_u]$.

Let $x \in L$, then

$$\bar{x}\bar{u} - \bar{u}\bar{x} = \overline{[x, u]} + t(x, u) = t(x, u) \in R \tag{7.1}$$

Thus the restriction of the inner derivation of $R * U(L)$ determined by $\bar{x}$ induces a k-derivation $R[\bar{u}; \delta_u] \to R[\bar{u}; \delta_u]$, $f \mapsto \bar{x}f - f\bar{x}$. Notice that it extends $\delta_x\colon R \to R$, thus we will denote the extension again by $\delta_x\colon R[\bar{u}; \delta_u] \to R[\bar{u}; \delta_u]$.

Introduce the new variable $z=(\bar u)^{-1}$, and let $R((z;\delta_u))$ be the power series ring described in Section 2.1. Observe that if we want to extend δ_x to $R((z;\delta_u))$, we have to define $\delta_x(z)=-z\delta_x(\bar u)z$.

Lemma 7.3. *Following the notation of Construction 7.2, for each $x\in L$, the derivation $\delta_x\colon R[\bar u;\delta_u]\to R[\bar u;\delta_u]$, $f\mapsto \bar x f-f\bar x$, can be extended to a derivation*

$$R((z;\delta_u))\to R((z;\delta_u)),\quad \sum_n a_nz^n\mapsto\sum_n\delta_x(a_nz^n)=\sum_n\left(\delta_x(a_n)z^n+a_n\delta_x(z^n)\right),$$

where $\delta_x(z)$ is defined as

$$\delta_x(z)=-z\delta_x(\bar u)z=\sum_{i=1}^{\infty}(-1)^i\delta_u^{i-1}(\delta_x(\bar u))z^{i+1}. \tag{7.2}$$

Proof. Define $S=R\langle z\ ;\ za=az-z\delta_u(a)z,\ a\in R\rangle$. In other words, the k-algebra S is isomorphic to the coproduct $R\coprod_k k[z]$ modulo the two-sided ideal generated by $\{za=az-z\delta_u(a)z,\ a\in R\}$. Let $\varepsilon_1\colon k[z]\to S$ and $\varepsilon_2\colon R\to S$ be the natural homomorphisms of k-algebras. By the universal property of the coproduct, there is a k-algebra homomorphism $\varphi\colon S\to R[[z;\delta_u]]$ such that, for any $n\geq 1$, $\varphi(\varepsilon_1(z^n))=z^n$ and, for any $a\in R$, $\varphi(\varepsilon_2(a))=a$. Therefore ε_1 and ε_2 are injective homomorphisms. To simplify the notation, we just identify R and $k[z]$ with their image in S without making any reference to the embeddings ε_1 and ε_2.

The proof of the lemma is divided into three parts. First we extend the derivation $\delta_x\colon R\to R$ to S. Secondly, we prove that the completion of S with respect to the topology induced by $\{Sz^n\}_{n\geq 0}$ is $R[[z;\delta_u]]$. Finally, we use Lemma 7.1 and the Ore property to obtain our result.

Step 1: *We claim that the k-derivation $\delta_x\colon R\to R$ can be extended to S by defining*

$$\delta_x(z)=-z\delta_x(\bar u)z.$$

To prove the claim we must show that there is a morphism of k-algebras $\Phi\colon S\to\mathbb{T}_2(S)$, $f\mapsto\left(\begin{smallmatrix} f & \delta_x(f)\\ 0 & f\end{smallmatrix}\right)$, where $\mathbb{T}_2(S)$ is the ring of 2×2 upper triangular matrices over S.

There is a morphism of k-algebras $\Phi_1\colon k[z]\to\mathbb{T}_2(S)$ given by $\Phi_1(p(z))=\left(\begin{smallmatrix} p(z) & \delta_x(p(z))\\ 0 & p(z)\end{smallmatrix}\right)$ for any $p(z)\in k[z]$. There is also a morphism of k-algebras $\Phi_2\colon R\to\mathbb{T}_2(S)$ given by $\Phi_2(a)=\left(\begin{smallmatrix} a & \delta_x(a)\\ 0 & a\end{smallmatrix}\right)$ for any $a\in R$. By the universal property of the coproduct, there is a unique k-algebra homomorphism $\Phi_3\colon R\coprod_k k[z]\to\mathbb{T}_2(S)$ such that $\Phi_3(z)=\Phi_1(z)$ and such that, for any $a\in R$, $\Phi_3(a)=\Phi_2(a)$.

We show that, for any $a\in R$, $za-az+z\delta_u(a)z\in\operatorname{Ker}\Phi_3$. This is equivalent to the matrix equality

$$\begin{pmatrix} z & \delta_x(z)\\ 0 & z\end{pmatrix}\begin{pmatrix} a & \delta_x(a)\\ 0 & a\end{pmatrix}=$$

$$=\begin{pmatrix} a & \delta_x(a)\\ 0 & a\end{pmatrix}\begin{pmatrix} z & \delta_x(z)\\ 0 & z\end{pmatrix}-\begin{pmatrix} z & \delta_x(z)\\ 0 & z\end{pmatrix}\begin{pmatrix} \delta_u(a) & \delta_x(\delta_u(a))\\ 0 & \delta_u(a)\end{pmatrix}\begin{pmatrix} z & \delta_x(z)\\ 0 & z\end{pmatrix},$$

which yields

$$\begin{pmatrix} za & z\delta_x(a)+\delta_x(z)a \\ 0 & za \end{pmatrix} =$$

$$= \begin{pmatrix} az & a\delta_x(z)+\delta_x(a)z \\ 0 & az \end{pmatrix} - \begin{pmatrix} z\delta_u(a)z & z\delta_u(a)\delta_x(z)+z\delta_x(\delta_u(a))z+\delta_x(z)\delta_u(a)z \\ 0 & z\delta_u(a)z \end{pmatrix}.$$

Hence $za - az + z\delta_u(a)z \in \operatorname{Ker}\Phi_3$ if and only if the equality

$$z\delta_x(a)+\delta_x(z)a \overset{(*)}{=} a\delta_x(z)+\delta_x(a)z+z\delta_u(a)\delta_x(z)+z\delta_x(\delta_u(a))z+\delta_x(z)\delta_u(a)z$$

holds.

After substituting $-z\delta_x(\bar{u})z$ for $\delta_x(z)$, the right hand side of the equality $(*)$ is equal to

$$-az\delta_x(\bar{u})z+\delta_x(a)z+z\delta_u(a)z\delta_x(\bar{u})z-z\delta_x(\delta_u(a))z+z\delta_x(\bar{u})z\delta_u(a)z.$$

Now, the left hand side of $(*)$ is

$$\begin{aligned} z\delta_x(a)+\delta_x(z)a &= \delta_x(a)z-z\delta_u(\delta_x(a))z-z\delta_x(\bar{u})za \\ &= \delta_x(a)z-z\delta_u(\delta_x(a))z-z\delta_x(\bar{u})az+z\delta_x(\bar{u})z\delta_u(a)z \\ &= \delta_x(a)z-z\delta_u(\delta_x(a))z-z[\delta_x(\bar{u}),a]z-za\delta_x(\bar{u})z+z\delta_x(\bar{u})z\delta_u(a)z \\ &= \delta_x(a)z-z\delta_u(\delta_x(a))z-z[\delta_x(\bar{u}),a]z-az\delta_x(\bar{u})z+z\delta_u(a)z\delta_x(\bar{u})z+z\delta_x(\bar{u})z\delta_u(a)z. \end{aligned}$$

After eliminating equal terms on both sides of $(*)$, we see that it holds if and only if

$$-z\delta_u(\delta_x(a))z-z[\delta_x(\bar{u}),a]z=-z\delta_x(\delta_u(a))z.$$

Equivalently,

$$-z([\delta_x,\delta_u](a)-[\delta_x(\bar{u}),a])z=0. \tag{7.3}$$

By Lemma 6.3(ii) and because $[x,u]=0$, $[\delta_x,\delta_u](a)=[t(x,u),a]$ for all $a\in R$. Therefore, applying (7.1) we can conclude that

$$[\delta_x,\delta_u](a)=[\delta_x(\bar{u}),a], \quad \text{for all } a\in R.$$

This verifies the equality (7.3), and we can deduce that Φ_3 induces the map $\Phi\colon S\to \mathbb{T}_2(S)$ which must be a morphism of k-algebras. This shows the existence of the claimed extension of the derivation $\delta_x\colon S\to S$.

Step 2: *Let us prove that the completion of S with respect to the topology induced by $\{Sz^n\}_{n\geq 0}$ is isomorphic to $R[[z;\delta_u]]$.*

Let T_1 be a subset of $R\setminus\{1\}$ such that $T_1\cup\{1\}$ is k-basis of R, and let $T_2=\{z^n\}_{n\geq 1}$ (thus $T_2\cup\{1\}$ is a k-basis of $k[z]$). By [4, Proposition 4.1], or by [11, comment after Lemma 5.2.2], $R\coprod_k k[z]$ has a k-basis consisting of all monomials $t_1t_2\cdots t_n$, where $t_i\in T_1\cup T_2$ but no two neighboring factors belong to the same T_i. Since, by the definition of S, $zR\subseteq Sz$, the ideal Sz^n is two-sided for any $n\geq 1$. Moreover, if we fix $s\in S$ and $n\geq 0$, then $s=a_0+a_1z+\cdots+a_nz^n+r_n$ where $a_i\in R$ for $i\in\{0,\dots,n\}$ and $r_n\in Sz^{n+1}$. Consider the k-algebra homomorphism

$\varphi\colon S \to R[[z;\delta_u]]$ defined at the beginning of the proof. Let $\pi_n\colon R[[z;\delta_u]] \to R[[z;\delta_u]]/R[[z;\delta_u]]z^n$ denote the canonical projection. Since $\pi_n \circ \varphi(s) = a_0 + a_1 z + \cdots + a_n z^n$, the powers of z are right and left R-independent in S and the elements $a_0, \dots, a_n$ are uniquely determined. This implies that $\pi_n \circ \varphi$ induces an isomorphism $S/Sz^n \cong R[[z;\delta_u]]/R[[z;\delta_u]]z^n$. Since $\varprojlim R[[z;\delta_u]]/R[[z;\delta_u]]z^n \cong R[[z;\delta_u]]$, we conclude that the completion of S with respect to the topology induced by the two-sided ideals $\{Sz^n\}_{n\geq 0}$ is isomorphic to $R[[z;\delta_u]]$ and that the isomorphism $\hat{\varphi}\colon \hat{S} \to R[[z;\delta_u]]$ is defined by $\hat{\varphi}(s) = \sum_{i=0}^{\infty} a_i z^i$. Finally, the natural morphism $S \to \hat{S} \xrightarrow{\hat{\varphi}} R[[z;\delta_u]]$ sends $z \mapsto z$, $a \mapsto a$ and

$$za \mapsto za = \sum_{i\geq 1} (-1)^{i-1} \delta_u^{i-1}(a) z^i \tag{7.4}$$

for all $a \in R$.

Step 3: *Let us extend $\delta_x\colon R((z;\delta_u)) \to R((z;\delta_u))$.*

By Lemma 7.1, the derivation $\delta_x\colon S \to S$ extends to $\delta_x\colon R[[z;\delta_u]] \to R[[z;\delta_u]]$ as

$$\delta_x\left(\sum_{i\geq 0} a_i z^i\right) = \sum_{i\geq 0} \delta_x(a_i z^i).$$

Since $R((z;\delta_u))$ is the left Ore localization of $R[[z;\delta_u]]$ at the set $\{1, z, \dots, z^n, \dots\}$, δ_x also extends to a derivation of $R((z;\delta_u))$ in a unique way (cf. Lemma 4.4).

Since $\bar{u} = z^{-1}$, the equality $za = az - z\delta_u(a)z$ implies that $\bar{u}a = a\bar{u} + \delta_u(a)$ for each $a \in R$; hence $R[\bar{u};\delta_u] \hookrightarrow R((z;\delta_u))$. Also, as $\delta_x(z^{-1}) = -z^{-1}\delta_x(z)z^{-1}$, $\delta_x(z^{-1}) = \delta_x(\bar{u})$.

To prove that the derivation δ_x has the properties claimed in the statement, it only remains to compute $\delta_x(z)$. By definition, $\delta_x(z) = -z\delta_x(\bar{u})z$. By (7.1), $\delta_x(\bar{u}) \in R$. Then, by (7.4),

$$\delta_x(z) = -z\delta_x(\bar{u})z = \sum_{i\geq 1} (-1)^i \delta_u^{i-1}(\delta_x(\bar{u})) z^{i+1},$$

as desired. □

By Lemma 6.2 and following the notation of Construction 7.2, $R * U(L) = (R * U(N)) * U\left(\frac{L}{N}\right) = R[\bar{u};\delta_u] * U\left(\frac{L}{N}\right)$. Now we are ready to extend this crossed product structure to $R((z;\delta_u))$.

Corollary 7.4. *Following the notation of Construction 7.2, there exists a crossed product structure $R((z;\delta_u)) * U\left(\frac{L}{N}\right)$ such that*

$$R * U(L) = R[\bar{u};\delta_u] * U\left(\frac{L}{N}\right) \hookrightarrow R((z;\delta_u)) * U\left(\frac{L}{N}\right).$$

Proof. By the proof of Lemma 6.2, we know that, for each $w \in L/N$, there exists $x \in L$ such that the k-derivation $\delta_w\colon R[\bar{u};\delta_u] \to R[\bar{u};\delta_u]$ given by the definition of $R[\bar{u};\delta_u] * U(L/N)$ coincides with $\delta_x\colon R[\bar{u};\delta_u] \to R[\bar{u};\delta_u]$. We extend it to a

k-derivation of $R((z;\delta_u))$ as in Lemma 7.3, we denote the extension also by δ_w. This gives a map $\xi\colon L/N \to \mathrm{Der}_k(R((z;\delta_u)))$ which is k-linear by the proof of Lemma 6.2. By Remarks 6.4(a), to obtain our result it only remains to prove that condition (ii) in Lemma 6.3 is satisfied.

By (7.1) and Lemma 7.3,

$$\delta_w(f) = \sum_{i\geq l} a_i' z^i \quad \text{for any } f = \sum_{i\geq l} a_i z^i \in R((z;\delta_u)). \tag{7.5}$$

That is, if the coefficients of degree smaller than l of f are zero, then the coefficients of degree smaller than l of $\delta_w(f)$ are zero.

Let $f = \sum_{i\geq l} a_i z^i \in R[[z;\delta_u]]$ and $w_1, w_2 \in L/N$, we want to prove that

$$[\delta_{w_1}, \delta_{w_2}](f) = \delta_{[w_1,w_2]}(f) - [f, t(w_1,w_2)].$$

Fix $p > l$ and set $f = f_1 + f_2$, where $f_1 = \sum_{i\geq l}^{p} a_i z^i$ and $f_2 = \sum_{i\geq p+1} a_i z^i$. The derivations $[\delta_{w_1}, \delta_{w_2}]$ and $\delta_{[w_1,w_2]} - \partial_{t(w_1,w_2)}$ coincide on $R[\bar{u};\delta_u]$, hence on z by Lemma 4.3, and therefore on f_1. Thus

$$[\delta_{w_1}, \delta_{w_2}](f) = [\delta_{w_1}, \delta_{w_2}](f_1) + [\delta_{w_1}, \delta_{w_2}](f_2) = \delta_{[w_1,w_2]}(f_1) - \partial_{t(w_1,w_2)}(f_1) + \sum_{i\geq p+1} c_i z^i.$$

Since p was arbitrary, both derivations coincide on f and we obtain our result. □

7.1. **The case of hypercentral Lie algebras.** A Lie k-algebra L is *hypercentral* if there exist an ordinal ν and a chain of ideals $\{L_\mu\}_{\mu\leq\nu}$ of L that satisfy the following conditions:

(i) $L_0 = 0$, $L_\nu = L$.
(ii) $L_\mu \subset L_{\mu+1}$ for all $0 \leq \mu < \nu$.
(iii) $L_{\mu'} = \bigcup_{\mu<\mu'} L_\mu$ for all limit ordinals $\mu' \leq \nu$.
(iv) $[L, L_{\mu+1}] \subseteq L_\mu$ for all $\mu < \nu$, or equivalently, $L_{\mu+1}/L_\mu$ is contained in the center of L/L_μ.

We will say that $\{L_\mu\}_{\mu\leq\nu}$ is a *hypercentral series* of L.

Many times hypercentral algebras are defined in terms of the upper central series [1, p. 8], we see the equivalence of both definitions in the next remark.

Remark 7.5. For any Lie k-algebra L, let $\mathcal{Z}(L)$ denote the center of L. The upper central series is an ascending chain of ideals of L defined by transfinite induction as

(1) $\mathcal{Z}_0(L) = 0$;
(2) if μ is an ordinal and $\mathcal{Z}_\mu(L)$ is already defined, then $\mathcal{Z}_{\mu+1}(L)/\mathcal{Z}_\mu(L) = \mathcal{Z}\left(L/\mathcal{Z}_\mu(L)\right)$;
(3) if μ is a limit ordinal, then $\mathcal{Z}_\mu(L) = \bigcup_{\tau<\mu} \mathcal{Z}_\tau(L)$.

In general, for any Lie k-algebra L there exists an ordinal ν such that $\mathcal{Z}_\nu(L) = \mathcal{Z}_{\nu+1}(L)$. It is easy to prove, by transfinite induction, that a hypercentral Lie k-algebra L with hypercentral series $\{L_\mu\}_{\mu\leq\nu}$ satisfies that, for any $\mu \leq \nu$, $L_\mu \subseteq \mathcal{Z}_\mu(L)$. As a consequence, a Lie k-algebra is hypercentral if and only if there exists and ordinal ν such that $\mathcal{Z}_\nu(L) = L$.

For our purposes, the most important example of hypercentral Lie algebra is that of a nilpotent Lie algebra. Indeed, if L is a nilpotent Lie k-algebra, it is enough to choose $L_0 = 0$, $L_1 = \mathcal{Z}(L)$, and for $i \geq 1$, $L_{i+1}/L_i = \mathcal{Z}(L/L_i)$. It is not difficult to prove that any hypercentral Lie algebra is locally nilpotent.

Fix a hypercentral Lie k-algebra L together with a hypercentral series $\{L_\mu\}_{\mu\leq\nu}$ of L. For each $0 \leq \mu < \nu$, we pick in $L_{\mu+1}$ a set of elements $\mathcal{B}_\mu$ which gives a basis of $L_{\mu+1}/L_\mu$, and we endow $\mathcal{B}_\mu$ with a well-ordered set structure. Set $\mathcal{B} = \bigcup_{\mu<\nu} \mathcal{B}_\mu$. Observe that $\mathcal{B}$ is a basis of L. Then we order $\mathcal{B}$ extending the ordering in each $\mathcal{B}_\mu$ in the following way: given $u_1 \in \mathcal{B}_{\mu_1}$ and $u_2 \in \mathcal{B}_{\mu_2}$ we set

$$u_1 < u_2 \quad \text{iff} \quad \begin{cases} \mu_1 < \mu_2, \text{ or} \\ \mu_1 = \mu_2 \text{ and } u_1 \text{ is smaller than } u_2 \text{ in } \mathcal{B}_{\mu_1}. \end{cases} \tag{7.6}$$

Then $(\mathcal{B}, <)$ is a well-ordered set. Thus, we can suppose that there exists an ordinal ε such that $\mathcal{B} = \{u_\gamma\}_{0\leq\gamma<\varepsilon}$ and $u_{\gamma_1} \leq u_{\gamma_2}$ if and only if $\gamma_1 \leq \gamma_2$.

For each $0 \leq \beta \leq \varepsilon$, define N_β to be the k-subspace of L generated by $\{u_\gamma \mid \gamma < \beta\}$. By convention, $N_0 = 0$. Observe that N_β is an ideal of L, hence a Lie subalgebra of L, and that

$$[L, u_\beta] \subset N_\beta.$$

Let R be a k-algebra. For a fixed crossed product $R*U(L)$, we shall construct, by transfinite induction, a ring of series $R((N_\beta))$ and a crossed product $R((N_\beta)) * U(L/N_\beta)$, for each $\beta \leq \varepsilon$, such that the following properties are satisfied for $\gamma < \beta \leq \varepsilon$

(a) $R((N_\gamma)) \hookrightarrow R((N_\beta))$,
(b) $R * U(L) \hookrightarrow R((N_\gamma)) * U(L/N_\gamma) \hookrightarrow R((N_\beta)) * U(L/N_\beta)$ extending the embedding of (a) in the natural way.

We define $R((N_0)) = R$. Let $0 < \beta$ be an ordinal and suppose that we have defined $R((N_\gamma))$ for all $\gamma < \beta$ so that conditions (a) and (b) are satisfied. Suppose first that β is not a limit ordinal, thus $\beta = \gamma + 1$ for some ordinal γ. Set $R_\gamma = R((N_\gamma))$ and $T_\gamma = R_\gamma[\bar{u}_\gamma; \delta_{u_\gamma}]$. Introduce a new variable $z_\gamma = (\bar{u}_\gamma)^{-1}$. Define $R((N_\beta)) = R_\gamma((z_\gamma; \delta_{u_\gamma}))$ as in Construction 7.2. By Corollary 7.4, there exists a crossed product structure $R((N_\beta)) * U(L/N_\beta)$ such that

$$R * U(L) \hookrightarrow R((N_\gamma)) * U(L/N_\gamma) \hookrightarrow R((N_\beta)) * U(L/N_\beta).$$

Thus conditions (a) and (b) are satisfied.

Suppose now that β is a limit ordinal. Define $R((N_\beta)) = \bigcup_{\gamma<\beta} R((N_\gamma))$. Set

$$M = \varinjlim_{\gamma<\beta} R((N_\gamma)) * U(L/N_\gamma)$$

where the maps in the direct limit are the ones given by condition (b).

The embeddings $R((N_\gamma)) \hookrightarrow M$, for any $\gamma < \beta$, induce a k-algebras embedding $R((N_\beta)) \hookrightarrow M$. We want to prove that M has a natural crossed product structure of $R((N_\beta))$ by $U(L/N_\beta)$. To this aim, we show that M satisfies conditions (i) and (ii) in the definition of a crossed product. For that it is helpful to have in mind the proof of Lemma 6.2.

Consider $\mathcal{S} = \{u_\alpha\}_{\beta\leq\alpha<\varepsilon} \subseteq L$. Let $\mathcal{M}$ be the set of standard monomials on $\mathcal{S}$. Abusing notation, we may suppose that $\mathcal{M} \subseteq R((N_\gamma)) * U(L/N_\gamma)$ for all $\gamma < \beta$, and the embedding $R((N_{\gamma_1})) * U(L/N_{\gamma_1}) \hookrightarrow R((N_{\gamma_2})) * U(L/N_{\gamma_2})$ can be viewed as the identity on $\mathcal{M}$ for $\gamma_1 < \gamma_2 < \beta$.

Let $f \in M$. There exists $\gamma_0 < \beta$ such that $f \in R((N_{\gamma_0})) * U(L/N_{\gamma_0})$. So that $f = \sum_{i=1}^n f'_{m'_i} m'_i$ where, for $i = 1, \dots, n$, $f'_{m'_i} \in R((N_{\gamma_0}))$ and m'_i is a standard monomial on $\{u_\alpha\}_{\gamma_0\leq\alpha<\varepsilon}$. There exists $\gamma_0 \leq \gamma < \beta$ such that, for each $i = 1, \dots, n$, $m'_i = n_i m_i$ where $n_i \in R * U(N_\gamma)$ and $m_i \in \mathcal{M}$. Hence $f = \sum_{i=1}^n f_{m_i} m_i$, with $f_{m_i} \in R((N_\gamma))$ and $m_i \in \mathcal{M}$ for any $i = 1, \dots, n$. This shows that M is generated by $\mathcal{M}$ as a left $R((N_\beta))$-module. Now we want to prove that the elements of $\mathcal{M}$ are left linearly independent over $R((N_\beta))$.

Let $m_1, \dots, m_n \in \mathcal{M}$ and let $f_{m_1}, \dots, f_{m_n} \in R((N_\beta))$. Then there exists $\gamma < \beta$ such that $f_{m_1}, \dots, f_{m_n} \in R((N_\gamma))$, and $\sum_{i=1}^n f_{m_i} m_i = 0$ implies that $f_{m_1} = \dots = f_{m_n} = 0$. This shows that M is a free left $R((N_\beta))$-module with basis $\mathcal{M}$.

For each $\gamma < \beta$, we identify the subspace of L generated by $\mathcal{S}$ with a subspace of L/N_γ in the natural way. Let $x \in L$ be any k-linear combination of elements in $\mathcal{S}$. For each $\gamma < \beta$, the crossed product structure $R((N_\gamma)) * U(L/N_\gamma)$ defines a derivation $\delta_{x,\gamma}\colon R((N_\gamma)) \to R((N_\gamma))$. Moreover, if $\gamma_1 < \gamma_2 < \beta$, since $R((N_{\gamma_1})) \subseteq R((N_{\gamma_2}))$ and $R((N_{\gamma_1}) * U(L/N_{\gamma_1})) \hookrightarrow R((N_{\gamma_2})) * U(L/N_{\gamma_2})$ we have that δ_{x,γ_1} equals δ_{x,γ_2} on $R((N_{\gamma_1}))$. Define $\delta_x\colon R((N_\beta)) \to R((N_\beta))$ in the natural way, that is, for each $f \in R((N_\beta))$ there exists $\gamma < \beta$ such that $f \in R((N_\gamma))$, then we set $\delta_x(f) = \delta_{x,\gamma}(f)$. Then $\delta\colon L/N_\beta \to \mathrm{Der}_k(R((N_\beta)))$, $x \mapsto \delta_x$, is defined, where we are identifying the subspace of L generated by $\mathcal{S}$ and L/N_β in the natural way. Let $x, y \in L/N_\beta$ and $f \in R((N_\beta))$. The equality $\bar{x}f = f\bar{x} + \delta_x(f)$ holds because $f \in R((N_\gamma))$ for some $\gamma < \beta$. Let $t\colon L \times L \to R((N_\beta))$ be given by the crossed product structure of $(R * U(N_\beta)) * U(L/N_\beta) \cong R * U(L)$. Hence, in particular $t(x, y) \in R * U(N_\beta)$. Then $\bar{x}\bar{y} - \bar{y}\bar{x} = \overline{[x, y]} + t(x, y)$. Thus conditions (6.1) and (6.2) are satisfied. Therefore $M = R((N_\beta)) * U(L/N_\beta)$ and $R * U(L) \hookrightarrow R((N_\gamma)) * U(L/N_\gamma) \hookrightarrow R((N_\beta)) * U(L/N_\beta)$ for $\gamma < \beta$.

This completes the transfinite induction, we then define $R((L)) = R((N_\varepsilon))$.

Remarks 7.6. (a) Notice that the whole construction of the ring $R((L))$ depends on the crossed product $R * U(L)$ we start with. It also depends on the order $<$ in (7.6) of the basis $\{u_\gamma\}_{0\leq\gamma<\varepsilon}$ of L obtained from the hypercentral series $\{L_\mu\}_{\mu\leq\nu}$ of L. The same hypercentral series $\{L_\mu\}_{\mu\leq\nu}$ can give rise to different rings of series $R((L))$ because $R((L))$ depends on the basis $\mathcal{B}_\mu$ and the different well-ordered set structures that each $\mathcal{B}_\mu$ can be given. Also, different hypercentral series can give rise to the same ring of series $R((L))$ if we choose the same basis $\{u_\gamma\}_{0\leq\gamma<\varepsilon}$ and the same order $<$ obtained as in (7.6).

(b) By construction, if $\beta < \varepsilon$, then

$$R((L)) = R((N_\beta))((L/N_\beta)) \tag{7.7}$$

where we are identifying the ordered set $\{u_\alpha\}_{\beta\leq\alpha<\varepsilon}$ with an ordered basis of the hypercentral Lie algebra L/N_β in the natural way.

(c) If R is a domain then $R((L))$ is also a domain.

(d) If L' is a subalgebra of L with a basis $\mathcal{B}'\subseteq\mathcal{B}$, where we understand that the order of $\mathcal{B}'$ is inherited from that of $\mathcal{B}$, then $R((L'))\hookrightarrow R((L))$ in the natural way. Indeed, if we define $N'_\beta=\{u_\gamma \mid u_\gamma\in\mathcal{B}',\ \gamma<\beta\}$, then $R((N'_\beta))\subseteq R((N_\beta))$ in the natural way for each $0\leq\beta<\varepsilon$.

Now we want to introduce the so called *least coefficient map* $\ell\colon R((L))\to R$. Let $f\in R((L))$. By the definition of $R((L))$, there exists $0\leq\beta\leq\varepsilon$ such that $f\in R((N_\beta))$. We define the least coefficient map by induction on β. If $\beta=0$, i.e. $f\in R$, we define $\ell(f)=f$. Let now β be any ordinal, and assume recursively that ℓ has been defined on $\bigcup_{\gamma<\beta}R((N_\gamma))$. If β is a limit ordinal, the function ℓ has already been defined on all $R((N_\gamma))$. Thus suppose that β is not a limit ordinal. There exists an ordinal γ such that $\beta=\gamma+1$. By construction, $R((N_\beta))=R((N_\gamma))((z_\gamma;\delta_{u_\gamma}))$. Hence f is a series in z_γ with coefficients in $R((N_\gamma))$. Let $f_1\in R((N_\gamma))$ be the coefficient of the least element in $\operatorname{supp} f$ as a series in z_γ. We define $\ell(f)=\ell(f_1)$. We say that $\ell(f)$ is the *least coefficient* of f.

We collect some properties of the least coefficient map in the following Lemma.

Lemma 7.7. *Let L be a hypercentral Lie k-algebra and $\mathcal{B}$ a basis of L ordered as in* (7.6)*. Let $\ell\colon R((L))\to R$ be the least coefficient map. Let $f,g\in R((L))$. The following hold true:*

(i) *$\ell(f)=f$ if, and only if, $f\in R$.*

(ii) *$\ell(f)=0$ if, and only if, $f=0$.*

(iii) *Let L' be a subalgebra of L with a basis $\mathcal{B}'\subseteq\mathcal{B}$, where we understand that the order of $\mathcal{B}'$ is inherited from that of $\mathcal{B}$. If $f\in R((L'))$, then the least coefficient of f viewed as an element of $R((L'))$ coincides with $\ell(f)$.*

(iv) *If R is a domain, then $\ell(fg)=\ell(f)\ell(g)$.*

(v) *If $\ell(f)$ is invertible in R, then f is invertible in $R((L))$. If R is a domain, the converse is true.*

Proof. (i) and (ii) follow easily from the construction.

(iii) follows by construction, defining N'_β as in Remarks 7.6(d) and by induction on β.

We prove (iv) by induction on the least ordinal β such that $f,g\in R((N_\beta))$. Observe that β is not a limit ordinal. If $\beta=0$, the result is clear by (i). Suppose that $\beta>0$ and the result is true for $\gamma<\beta$. As $\beta=\gamma+1$, $f,g\in R((N_\beta))=R((N_\gamma))((z_\gamma;\delta_{u_\gamma}))$. Let $f_1,g_1\in R((N_\gamma))$ be the coefficient of the least element in $\operatorname{supp} f$, $\operatorname{supp} g$, respectively, as a series in z_γ. Then $f_1g_1=(fg)_1\in R((N_\gamma))$ because of the way series in one indeterminate are multiplied and the fact that $R((N_\gamma))$ is a domain (since R is). Now observe that, by the definition of the least coefficient map, $\ell(f_1)=\ell(f)$, $\ell(g_1)=\ell(g)$ and $\ell(fg)=\ell((fg)_1)$. Thus applying the induction hypothesis

$$\ell(fg)=\ell((fg)_1)=\ell(f_1g_1)=\ell(f_1)\ell(g_1)=\ell(f)\ell(g).$$

Again, we prove (v) by induction on the least ordinal β such that $f \in R((N_\beta))$. Notice that β is not a limit ordinal. If $\beta = 0$, the result is clear. Suppose that $\beta > 0$ and the result is true for $\gamma < \beta$. As $\beta = \gamma + 1$, $f \in R((N_\beta)) = R((N_\gamma))((z_\gamma; \delta_{u_\gamma}))$. Let $f_1 \in R((N_\gamma))$ be the coefficient of the least element in $\operatorname{supp} f$ as a series in z_γ. Since $\ell(f) = \ell(f_1)$, the induction hypothesis implies that f_1 is invertible in $R((N_\gamma))$. Therefore f is invertible.

Suppose now that R is a domain and that f is invertible. Applying (i) and (iv), we get $\ell(f^{-1})\ell(f) = \ell(f^{-1}f) = 1 = \ell(1) = \ell(ff^{-1}) = \ell(f)\ell(f^{-1})$. $\square$

As a first outcome, we obtain a slight generalization of [26, Section 5].

Corollary 7.8. *Let L be a hypercentral Lie k-algebra. Let K be a field with k as a central subfield. Any crossed product $K * U(L)$ is an Ore domain and $K((L))$ is a field that contains the Ore field of fractions of $K * U(L)$.*

Proof. Any hypercentral Lie k-algebra is locally nilpotent. Thus $K*U(L)$ is locally an iterated skew polynomial ring $K[x_1;\delta_1]\cdots[x_n;\delta_n]$, which is an Ore domain. We have already seen that $K*U(L) \hookrightarrow K((L))$. Now $K((L))$ is a field by Lemma 7.7(v). By the universal property of the Ore localization, the Ore field of fractions of $K * U(L)$ is contained in $K((L))$. $\square$

7.2. **The residually nilpotent case.** Let H be a Lie k-algebra. We say that H is *residually nilpotent* if H has a descending sequence of ideals

$$H = H_1 \supseteq H_2 \supseteq \cdots \supseteq H_i \supseteq H_{i+1} \supseteq \cdots \tag{7.8}$$

with $[H, H_i] \subseteq H_{i+1}$ for all i, and such that $\bigcap_{i\geq 1}^{\infty} H_i = 0$. In this event, we call $\{H_i\}_{i\geq 1}$ an *RN-series* of H. The RN-series $\{H_i\}_{i\geq 1}$ satisfies the *Q-condition* if, for each i, there exists a set of elements $\mathcal{C}_i$ of H_i which gives a basis of H_i/H_{i+1} such that $\mathcal{C} = \bigcup_{i=1}^{\infty} \mathcal{C}_i$ is a basis of H. We also say that $\mathcal{C}$ is a *Q-basis* of H.

Given a Q-basis $\mathcal{C}$, a *canonical ordering* of $\mathcal{C}$ is an ordering $<$ of $\mathcal{C}$ obtained as follows. First we give an (arbitrary) well ordered set structure to $\mathcal{C}_i$ for each $i \geq 1$. Then we order $\mathcal{C}$ extending the order in each $\mathcal{C}_i$ in the following way: given $u_1 \in \mathcal{C}_{i_1}$ and $u_2 \in \mathcal{C}_{i_2}$ we set

$$u_1 < u_2 \quad \text{iff} \quad \begin{cases} i_1 > i_2, \text{ or} \\ i_1 = i_2 \text{ and } u_1 \text{ is smaller than } u_2 \text{ in } \mathcal{C}_{i_1}. \end{cases} \tag{7.9}$$

Notice that there may exist infinitely many canonical orderings of $\mathcal{C}$.

We remark that $(\mathcal{C}, <)$ need not be a well-ordered set, but $\bigcup_{i=1}^{m} \mathcal{C}_i$ can be viewed as a well ordered basis of H/H_{m+1} for any m under the obvious identification.

Not all residually nilpotent Lie k-algebras have Q-bases. Important examples of residually nilpotent Lie k-algebras with Q-bases are the following.

Examples 7.9. (1) Suppose that H is a nilpotent Lie k-algebra. Let $H = H_1 \supseteq \cdots \supseteq H_{n+1} = 0$ be an RN-series. If $\mathcal{C}_i$ is a set of elements of H_i which gives a basis of H_i/H_{i+1}, then clearly $\mathcal{C} = \bigcup_{i=1}^{n} \mathcal{C}_i$ is a Q-basis of H.

(2) Suppose that H is a *positively graded* Lie k-algebra, that is, there exists a sequence $\{N_i\}_{i\geq 1}$ of subspaces of H such that $H = \bigoplus_{i=1}^{\infty} N_i$ and $[N_i, N_j] \subseteq N_{i+j}$ for all $i, j \geq 1$. If we now define $H_i = \bigoplus_{j\geq i} N_j$, and $\mathcal{C}_i$ as any basis of N_i for each $i \geq 1$, then it is easy to see that $\{H_i\}_{i=1}^{\infty}$ is an RN-series and $\mathcal{C} = \bigcup_{i=1}^{\infty} \mathcal{C}_i$ a Q-basis of H.

Examples of these algebras are the Lie algebras arising from torsion-free nilpotent and residually torsion-free nilpotent groups using the lower central series (of the groups), and the graded Lie algebras that appear in [38, Examples A,B,C,D].

Fix a residually nilpotent Lie k-algebra H with an RN-series $\{H_i\}_{i=1}^{\infty}$ that has a Q-basis $\mathcal{C} = \bigcup_{i=1}^{\infty} \mathcal{C}_i$ and a canonical ordering of $\mathcal{C}$.

Note that for each $n > m \geq 1$, H/H_m and H_m/H_n are nilpotent and hence hypercentral Lie k-algebras. Moreover

$$H_m/H_m = 0 < H_{m-1}/H_m < \cdots < H_2/H_m < H/H_m$$

is a chain of ideals of H/H_m with $[H/H_m, H_p/H_m] \subseteq H_{p+1}/H_m$, and

$$H_n/H_n = 0 < H_{n-1}/H_n < \cdots < H_m/H_n$$

is a chain of ideals of H_m/H_n with $[H_m/H_n, H_{m+p}/H_n] \subseteq H_{m+p+1}/H_n$.

For $1 \leq i \leq m-1$, let $\mathcal{B}_{m,i}$ be the basis of $\frac{H_i/H_m}{H_{i+1}/H_m} \cong H_i/H_{i+1}$ obtained via the natural identification with $\mathcal{C}_i$. Let $\mathcal{B}_m = \bigcup_{i=1}^{m-1} \mathcal{B}_{m,i}$ be the basis of H/H_m with the well order inherited from $\bigcup_{i=1}^{m-1} \mathcal{C}_i$.

Let R be a k-algebra and consider a crossed product $R * U(H)$.

For each $m \geq 1$, set $R_m = R * U(H_m)$. Then, with each basis $\mathcal{B}_m$ fixed, we can construct the embedding $R_m * U(H/H_m) \hookrightarrow R_m((H/H_m))$. If $n > m$, since $R_m = R_n * U(H_m/H_n)$ and $R_n((H/H_n)) = R_n((H_m/H_n))((H/H_m))$, we obtain the commutativity of the following diagram

$$\begin{array}{ccc} R_m * U(H/H_m) & \hookrightarrow & R_m((H/H_m)) = R_n * U(H_m/H_n)((H/H_m)) \\ \| & & \downarrow \\ R_n * U(H/H_n) & \hookrightarrow & R_n((H/H_n)) = R_n((H_m/H_n))((H/H_m)) \end{array} \tag{7.10}$$

This allows us to define

$$R((H)) = \varinjlim_{m} R_m((H/H_m)).$$

For each $n > m \geq 1$, let $\ell_m \colon R_m((H/H_m)) \to R_m$ be the least coefficient map, and let $t_m \colon R_m \to R_{m+1}$ be the least coefficient map of $R_{m+1}((H_m/H_{m+1}))$ restricted to R_m (or equivalently, the restriction of ℓ_{m+1} to R_m by Lemma 7.7(iii)).

The commutativity of the diagram

$$\begin{array}{ccc} R_m((H/H_m)) & \xrightarrow{\ell_m} & R_m \\ \big\downarrow & & \big\downarrow{\scriptstyle t_m} \\ R_{m+1}((H/H_{m+1})) & \xrightarrow{\ell_{m+1}} & R_{m+1} \end{array} \tag{7.11}$$

follows from (7.10).

Note that, because of Lemma 7.7(i), each t_m is the identity on $R_{m+1} \subseteq R_m$, and hence on R.

We claim that if $f \in R * U(L)$, there exists $m \geq 1$ such that $\ell_m(f) \in R$. Indeed, we may express $f = \sum_{i=1}^{n} a_i m_i$ where each $a_i \in R$ and each m_i is a standard monomial in the set $\mathcal{C}$. Thus there exists $m \geq 1$ such that f is an R-linear combination of the standard monomials in $\bigcup_{i=1}^{m-1} \mathcal{C}_i$. Now, by definition of $\ell_m \colon R_m((H/H_m)) \to R_m$, it follows that $\ell_m(f) \in R$, and the claim is proved.

Let now $f \in R((H))$. There exists $m \geq 1$ such that $f \in R_m((H/H_m))$. By the preceding paragraph and the commutativity of (7.11), there exists m_0 such that $\ell_{m_0}(f) \in R$. The commutativity of (7.11) and the fact that ℓ_l is the identity on R for each l implies that $\ell_n(f) = \ell_{m_0}(f)$ for all $n \geq m_0$. Thus we have a well defined map $\ell \colon R((H)) \to R$ where for each $f \in R((H))$, $\ell(f) = \ell_{m_0}(f)$ where m_0 is any natural number such that $\ell_{m_0}(f) \in R$. The map $\ell \colon R((H)) \to R$ is called the *least coefficient map* of $R((H))$, and $\ell(f)$ the *least coefficient* of $f \in R((H))$.

Lemma 7.10. *Let H be a residually nilpotent Lie k-algebra with an RN-series $\{H_i\}_{i=1}^{\infty}$ that has a Q-basis $\mathcal{C} = \bigcup_{i=1}^{\infty} \mathcal{C}_i$. For each canonical ordering of $\mathcal{C}$, the least coefficient map $\ell \colon R((H)) \to R$ satisfies the properties (i)-(v) in Lemma 7.7.*

Proof. (i) and (ii) are clear from the construction.

(iii) Define $H'_m = H_m \cap H'$ and $R'_m = R * U(H'_m)$. Then

$$R((H')) = \varinjlim_{m} R'_m((H'/H'_m))$$

and the result follows from Lemma 7.7(iii).

(iv) Let $f, g \in R((H))$. There exists $m \geq 1$ such that $f, g \in R_m((H/H_m))$ and $\ell_m(f), \ell_m(g) \in R$. Now apply Lemma 7.7(iv).

(v) if $\ell(f)$ is invertible, then $\ell_m(f)$ is invertible for some m such that $f \in R_m((H/H_m))$. By Lemma 7.7(v), f is invertible in $R_m((H/H_m))$, and therefore in $R((H))$. If R is a domain, then so is R_m for all m. Now apply Lemma 7.7(v). □

From all this, we obtain the extension of [26, Theorem 2] to crossed products $K * U(H)$. More precisely, Lemma 7.10(v) yields the following result.

Corollary 7.11. *Let H be a residually nilpotent Lie k-algebra with an RN-series $\{H_i\}_{i=1}^{\infty}$ that has a Q-basis $\mathcal{C} = \bigcup_{i=1}^{\infty} \mathcal{C}_i$. Let K be a field with k as a central subfield. For any crossed product $K * U(H)$ and any canonical ordering of $\mathcal{C}$, then there exists a ring of series $K((H))$ which is a field that contains $K * U(H)$.*

The subfield of $K((H))$ generated by $K * U(H)$ will be denoted by $K(H)$.

7.3. **Main results.** The next result gives a condition that ensures when two fields of fractions of a crossed product are isomorphic. It is the generalization of [26, Section 6, Corollary] to crossed products. Although weaker, it should be regarded as a similar result to [18, Theorem].

Theorem 7.12. *Let H be a residually nilpotent Lie k-algebra with an RN-series $\{H_i\}_{i=1}^{\infty}$ that has a Q-basis $\mathcal{C} = \bigcup_{i=1}^{\infty} \mathcal{C}_i$. Let K be a field with k as a central subfield. Consider a crossed product $K * U(H)$ and suppose that it has a field of fractions $K * U(H) \hookrightarrow D$. For each $m \geq 1$, denote by D_m the subfield of D generated by $K * U(H_m)$. Assume that, for each $m \geq 1$, the standard monomials in $\bigcup_{i=1}^{m-1} \mathcal{C}_i$ are linearly independent over D_m. Then $K(H)$ and D are isomorphic fields of fractions of $K * U(H)$.*

Proof. For each $m \geq 1$, set $R_m = K * U(H_m)$ and consider $K * U(H)$ as $R_m * U(H/H_m)$. Fix $x \in H/H_m$, the derivation δ_x of R_m extends to D as the inner derivation $\delta_x(d) = \bar{x}d - d\bar{x}$; since $\delta_x(R_m) \subseteq R_m$ and D_m is a field of fractions of R_m, we deduce from Lemma 4.3 that $\delta_x(D_m) \subseteq D_m$. Therefore, the subring of D generated by D_m and $\bigcup_{i=1}^{m-1} \mathcal{C}_i$ is a crossed product $D_m * U(H/H_m)$ because conditions (i) and (ii) in the definition of crossed product are easily verified.

For each $m \geq 1$, we can consider $D_m * U(H/H_m) \hookrightarrow D_m((H/H_m))$ because H/H_m is a nilpotent Lie k-algebra. Since $R_m * U(H/H_m) \hookrightarrow D_m * U(H/H_m) \hookrightarrow D$ and D is a field of fractions of $R_m * U(H/H_m) = K * U(H)$, and since $D_m * U(H/H_m)$ is an Ore domain, we get that $D_m * U(H/H_m) \hookrightarrow D$ is the Ore field of fractions of $D_m * U(H/H_m)$ and, moreover, $D \hookrightarrow D_m((H/H_m))$ by Corollary 7.8. Note also that $R_m((H/H_m)) \hookrightarrow D_m((H/H_m))$. Hence we have $K * U(H) \hookrightarrow R_m((H/H_m)) \hookrightarrow D_m((H/H_m))$ and, by the universal property of the direct limit, $K * U(H) \hookrightarrow K((H)) \hookrightarrow \varinjlim D_m((H/H_m))$. This implies that the field of fractions of $K * U(H)$ in $K((H))$, which is $K(H)$, and in $\varinjlim D_m((H/H_m))$ must coincide. Now note that D is the field of fractions of $K * U(H)$ in $\varinjlim D_m((H/H_m))$ because D is the field of fractions of $K * U(H)$ in $D_m((H/H_m))$ for each m. Therefore $K(H) = D$, as desired. □

By [18], it is known that if G is an orderable group, K a field and KG a crossed product, then the field of fractions $K(G)$ inside the Malcev-Neumann series ring $K((G))$ does not depend on the ordering of G. The following theorem should be seen as an analogous result.

Theorem 7.13. *Let H be a residually nilpotent Lie k-algebra with an RN-series $\{H_i\}_{i=1}^{\infty}$ that has a Q-basis $\mathcal{C} = \bigcup_{i=1}^{\infty} \mathcal{C}_i$. Let K be a field with k as a central subfield. Consider a crossed product $K * U(H)$. Then the field $K(H)$ does not depend on the RN-series with a Q-basis chosen, nor on the Q-basis $\mathcal{C}$ chosen nor on the canonical ordering of $\mathcal{C}_i$ chosen. In fact $K * U(H) \hookrightarrow K(H)$ and $K * U(H) \hookrightarrow \mathfrak{D}(K * U(H))$ (cf. Proposition 6.6) are isomorphic fields of fractions.*

Proof. First note that the construction of $\mathfrak{D}(K * U(H))$ does not depend on the RN-series with a Q-basis chosen, nor on the Q-basis $\mathcal{C}$ nor on the canonical ordering of $\mathcal{C}$, see [11, Theorem 2.6.5] or [25].

Let $\{H_i\}_{i=1}^{\infty}$ be an RN-series with a Q-basis $\mathcal{C} = \bigcup_{i=1}^{\infty} \mathcal{C}_i$ and choose a canonical ordering of $\mathcal{C}$.

For each $m \geq 1$, $\bigcup_{i=1}^{m-1} \mathcal{C}_i$ is a set of elements in H which give a basis of H/H_m. By Proposition 6.6(ii), the standard monomials in $\bigcup_{i=1}^{m-1} \mathcal{C}_i$ are linearly independent over $\mathfrak{D}(K * U(H_m))$. Hence $K * U(H) \hookrightarrow K(H)$ and $K * U(H) \hookrightarrow \mathfrak{D}(K * U(H))$ are isomorphic fields of fractions of $K * U(H)$ by Theorem 7.12. □

The next result should be regarded as a weaker version of [19, Theorem], along the lines of [37, Proposition 2.5(3)(ii)].

Corollary 7.14. *Let H be a Lie k-algebra. Let K be a field containing k as a central subfield. Consider a crossed product $K * U(H)$.*

Suppose that N is an ideal of H such that both N and H/N are residually nilpotent and they both have RN-series with Q-basis. Then the natural embedding $K * U(H) \hookrightarrow K(N)(\frac{H}{N})$ gives a field of fractions of $K * U(H)$ isomorphic to $K * U(H) \hookrightarrow \mathfrak{D}(K * U(H))$.

Moreover, if H is residually nilpotent with an RN-series that has a Q-basis, then $K * U(H) \hookrightarrow K(H)$ and $K * U(H) \hookrightarrow K(N)(\frac{H}{N})$ are isomorphic fields of fractions.

Proof. By Proposition 6.6(iii), we have $\mathfrak{D}(K * U(N)) * U(H/N) \subseteq \mathfrak{D}(K * U(H))$. Now Theorem 7.12 and again Proposition 6.6, imply that $\mathfrak{D}\big(\mathfrak{D}(K * U(N)) * U(H/N)\big) \cong \mathfrak{D}(K * U(H))$. By Theorem 7.13, $K(N)$ and $\mathfrak{D}(K * U(N))$ are isomorphic fields of fractions of $K * U(N)$. Hence $\mathfrak{D}(K(N) * U(\frac{H}{N})) \cong \mathfrak{D}(K * U(H))$. Again by Theorem 7.13, $K(N)(\frac{H}{N}) \cong \mathfrak{D}(K * U(H))$ as fields of fractions of $K * U(H)$.

If $K(H)$ exists, then Theorem 7.13 implies that $K(H) \cong \mathfrak{D}(K * U(H)) \cong K(N)(\frac{H}{N})$. □

We showed in Lemma 6.7 that $K * U(H)$, the crossed product of a field K by $U(H)$ where H is a free Lie k-algebra, is a fir. Thus it has a universal field of fractions. We are going to prove that $K * U(H) \hookrightarrow K(H)$ and $K * U(H) \hookrightarrow \mathfrak{D}(K * U(H))$ are both the universal field of fractions. This result was already known for $U(H)$ [26, Theorem 1], where the proof relies on the existence of some specialization (see [26, Lemma 3.1]). The techniques for the construction of such specialization do not work for crossed products. In our proof, the role of [26, Lemma 3.1] is played by Proposition 6.8.

Remark 7.15. Let H be a free Lie k-algebra. Then H is graded. Indeed $H = \bigoplus_{i \geq 1} N_i$ where each N_i is the subspace generated by the Lie monomials of degree i. Then $H_i = \bigoplus_{j \geq i} N_j$ is the i-th term of the lower central series of H. Let $\mathcal{C}_i$ be a basis of N_i for $i \geq 1$. Therefore we are in the situation of Examples 7.9 and we can deduce that $\bigcup_{i \geq 1} \mathcal{C}_i$ is a Q-basis of the residually nilpotent algebra H.

Theorem 7.16. *Let H be a free Lie k-algebra, K a field with k as a central subfield and consider a crossed product $K * U(H)$. Then $K * U(H) \hookrightarrow K(H)$ and $K*U(H) \hookrightarrow \mathfrak{D}(K*U(H))$ coincide with the universal field of fractions of $K*U(H)$.*

Proof. Denote by $K * U(H) \hookrightarrow E$ the universal field of fractions of $K * U(H)$. We follow the notation of Remark 7.15.

It is known that any subalgebra of a free Lie algebra is a free Lie algebra. Thus, for each $m \geq 1$, $K * U(H_m)$ is a fir and therefore it has a universal field of fractions $K*U(H_m) \hookrightarrow R_{\mathcal{P}_m}$ which, by Lemma 6.7, is a universal localization at the prime matrix ideal $\mathcal{P}_m$. Now, by Proposition 6.8, $K*U(H) \hookrightarrow R_{\mathcal{P}_m} * U(H/H_m) \hookrightarrow E$. Hence the conditions of Theorem 7.12 are satisfied. Thus we can deduce that $K * U(H) \hookrightarrow E$ and $K * U(H) \hookrightarrow K(H)$ are isomorphic fields of fractions. By Theorem 7.13, $K * U(H) \hookrightarrow \mathfrak{D}(K * U(H))$ is also isomorphic to the universal field of fractions of $K * U(H)$. □

For the missing details and definitions in the next example, the reader is referred to [28] and the references therein.

Example 7.17. Let $Q = P(x_1, \dots, x_n)$ be the free Poison field over k in the variables $x_1, \dots, x_n$ and let Q^e be its universal enveloping algebra.

In [28, Theorem 1], it is proved that Q^e satisfies the weak algorithm for a certain filtration of Q^e. Thus Q^e is a free ideal ring and, therefore, it has a universal field of fractions. Although not stated explicitly, it is also proved in [28, Proposition 1, Corollary 1] that Q^e is in fact a crossed product $K * U(H)$ of a commutative field K over $U(H)$, the universal enveloping algebra of the free Lie algebra H on $x_1, \dots, x_n$. Indeed, by [28, Proposition 1], the morphism given in [8, Theorem 5] is in fact an isomorphism by a basis argument, and thus Q^e is a crossed product as stated. Then, by Theorem 7.16, $Q^e \hookrightarrow \mathfrak{D}(Q^e)$ and $Q^e \hookrightarrow K(H)$ are the universal field of fractions of Q^e. We stress that it cannot be deduced from the results in [26] that these embeddings are the universal field of fractions of Q^e.

We remark in passing that if R is an ordered k-algebra with positive cone $P(R)$ (for unexplained terminology see for example [11, Section 9.6]), and H is a residually nilpotent Lie k-algebra with a Q-basis, then $R((H))$ is an ordered ring for any crossed product $R * U(H)$. In particular, if $R = K$ is a field, $K((H))$, $K(H)$ and $\mathfrak{D}(K * U(H))$ are ordered fields. Indeed, if $\ell \colon R((H)) \to R$ is the least coefficient map, then $\mathfrak{P} = \{f \in R((H)) \mid \ell(f) \in P(R)\}$ is a positive cone for $R((H))$. Clearly, $\mathfrak{P} \cap -\mathfrak{P} = \emptyset$ and $\mathfrak{P} \cup -\mathfrak{P} = R((H)) \setminus \{0\}$. Moreover $\mathfrak{P} \cdot \mathfrak{P} \subseteq \mathfrak{P}$ by Lemma 7.10(iv), and it is not difficult to prove that $\mathfrak{P} + \mathfrak{P} \subseteq \mathfrak{P}$.

8. Inversion height: using crossed products of Lie algebras.

Let R be a k-algebra with a field of fractions $\varepsilon \colon R \hookrightarrow D$. Let H be a residually nilpotent Lie k-algebra with an RN-series $\{H_i\}_{i=1}^{\infty}$ that has a Q-basis $\mathcal{C} = \bigcup_{i=1}^{\infty} \mathcal{C}_i$. Consider a crossed product $R * U(H)$ and suppose that it can be extended to a crossed product structure $D * U(H)$. Then, by Remarks 6.4(b) and Lemma 4.3, we can consider the crossed product $D_\varepsilon(n) * U(H)$ for each $n \geq 0$. Moreover,

$$R * U(H) \hookrightarrow D_\varepsilon(n) * U(H) \hookrightarrow D_\varepsilon(n+1) * U(H) \hookrightarrow D * U(H).$$

Consider the embedding

$$\iota\colon R * U(H) \hookrightarrow \mathcal{L}_n = D_\varepsilon(n)((H)) \hookrightarrow \mathcal{L}_{n+1} = D_\varepsilon(n+1)((H)) \hookrightarrow D((H)) = E.$$

Note that if $f \in D_\varepsilon(n)((H))$, then the least coefficient map $\ell\colon D((H)) \to D$ is such that $\ell(f) \in D_\varepsilon(n)$.

With this notation, we can prove an analogous result to Theorem 5.1.

Theorem 8.1. *In the situation described above, the following hold true*

(i) $E_\iota(n) \subseteq \mathcal{L}_n$ *for each integer* $n \geq 0$.
(ii) *Let* $f \in D$. *If* $\mathrm{h}(\varepsilon, f) = n$, *then* $\mathrm{h}(\iota, f) = n$.
(iii) $\mathrm{h}(\iota) \geq \mathrm{h}(\varepsilon)$.

Proof. (i) We proceed by induction on n. For $n = 0$, the result holds since $R * U(H) \hookrightarrow R((H))$. Suppose that the result holds for $n \geq 0$. Let $0 \neq f \in E_\iota(n) \subseteq D_\varepsilon(n)((H))$. Consider the least coefficient map $\ell\colon D_\varepsilon(n)((H)) \to D_\varepsilon(n)$.

For each $m \geq 1$, set $R_m = D_\varepsilon(n) * U(H_m)$, $S_m = D_\varepsilon(n+1) * U(H_m)$ and consider $\ell_m\colon R_m((H/H_m)) \to R_m$.

Fix an m such that $\ell(f) = \ell_m(f) \in D_\varepsilon(n)$. We will prove that $f^{-1} \in S_m((H/H_m))$ by transfinite induction. Let β be the least ordinal such that $f \in R_m((N_\beta))$. If $\beta = 0$, then $f \in D_\varepsilon(n)$ and clearly $f^{-1} \in D_\varepsilon(n+1) \subseteq S_m((H/H_m))$. If $\beta \neq 0$, there exists an ordinal number γ such that $\beta = \gamma + 1$ and $R_m((N_\beta)) = R_m((N_\gamma))((z_\gamma; \delta_{u_\gamma}))$. Express f as a series in z_γ,

$$f = f_1 z_\gamma^{l_0} + \sum_{l > l_0} c_l z_\gamma^l$$

where $f_1, c_l \in R_m((N_\gamma))$ for all c_l and $\ell_m(f) = \ell_m(f_1)$. Define $g = f_1 z_\gamma^{l_0} - f$. Then

$$f^{-1} = (f_1 z_\gamma^{l_0})^{-1} \sum_{q \geq 0} (g (f_1 z_\gamma^{l_0})^{-1})^q.$$

By the transfinite induction hypothesis, $f_1^{-1} \in S_m((H/H_m))$. Now it can be proved that $(g(f_1 z_\gamma^{l_0})^{-1})^q \in S_m((N_\gamma))((z_\gamma; \delta_{u_\gamma}))$ for each q. Therefore $f^{-1} \in S_m((H/H_m)) \subseteq \mathcal{L}_{n+1}$.

(ii) Let $f \in D_\varepsilon(n+1) \setminus D_\varepsilon(n)$. Since $D_\varepsilon(n+1) \subseteq D$, $\ell(f) = f$. Suppose that $f \in D_\varepsilon(n)((H))$. Then $f = \ell(f) \in D_\varepsilon(n)$, a contradiction.

(iii) Follows from (ii). □

Let I be a set of cardinality at least two and let $\{H_i\}_{i \in I}$ be a set of Lie k-algebras. Let H be the free product of these algebras, that is, $H = \coprod_{i \in I} H_i$. Consider the direct sum $\bigoplus_{i \in I} H_i$. For each $i \in I$, let $\pi_i\colon H_i \hookrightarrow \bigoplus_{i \in I} H_i$ be the canonical inclusion. Let $\pi\colon \coprod_{i \in I} H_i \to \bigoplus_{i \in I} H_i$ be the unique morphism of Lie k-algebras such that $\pi_{|_{H_i}} = \pi_i$. The subalgebra $\ker \pi$ is called the *cartesian subalgebra* of the free product H.

By $[x, y]_n$, we denote the product $[[\dots[[x, y], y], \dots], y]$ with n factors y.

Corollary 8.2. *Let I be a set of cardinality at least two and let $\{H_i\}_{i\in I}$ be a set of nilpotent Lie k-algebras. Set $H=\coprod_{i\in I} H_i$. Let $U(H)$ be the universal enveloping algebra of H and consider the embedding $\iota\colon U(H)\hookrightarrow \mathfrak{D}(K*U(H))$. Then $\mathrm{h}(\iota)=\infty$. Indeed, let $x\in H_i\setminus\{0\}$ and $y\in H_j\setminus\{0\}$ with $i\neq j$. If f is any entry of the inverse of the $n\times n$ matrix*

$$A_n=\begin{pmatrix} [[x,y],x] & [[x,y],x]_2 & \dots & [[x,y],x]_n \\ [[x,y]_2,x] & [[x,y]_2,x]_2 & \dots & [[x,y]_2,x]_n \\ \dots & \dots & \ddots & \dots \\ [[x,y]_n,x] & [[x,y]_n,x]_2 & \dots & [[x,y]_n,x]_n \end{pmatrix} \tag{8.1}$$

then $\mathrm{h}(\iota,f)=n$.

In particular, if X is a set of cardinality at least two and $k\langle X\rangle$ is the free k-algebra on X, then the universal field of fractions $\iota\colon k\langle X\rangle\hookrightarrow F$ is of infinite inversion height. Indeed, let $x,y\in X$ be different elements. If f is any entry of the inverse of the $n\times n$ matrix (8.1)*, then $\mathrm{h}(\iota,f)=n$.*

Proof. Let N be the cartesian subalgebra of H. By [6, Theorem 4.10.5], N is a free Lie k-algebra on an infinite set Y, and thus $U(N)$ is a free k-algebra on Y. Moreover, it is not difficult to see that $H/N\cong\bigoplus_{i\in I} H_i$ is a residually nilpotent Lie k-algebra with an RN-series that has a Q-basis. By Corollary 7.14, $U(H)\hookrightarrow\mathfrak{D}(K*U(H))$ can be viewed as $U(H)\hookrightarrow k(N)(\frac{H}{N})\hookrightarrow k(N)((\frac{H}{N}))$. By Theorem 4.1 and Theorem 7.16, $\varepsilon\colon U(N)\hookrightarrow K(N)$ is of infinite inversion height. By Theorem 8.1, $h(\iota)=\infty$.

Moreover, using [6, Section 4.10], Y can be chosen to contain the elements $[[x,y]_i,x]_j$. By Theorem 4.1, for each entry f of A_n^{-1}, $\mathrm{h}(\varepsilon,f)=n$. Applying Theorem 8.1, we obtain that $\mathrm{h}(\iota,f)=n$.

When H is the free Lie algebra on a set X, put $I=X$. Then H is the free product of the abelian (and hence nilpotent) Lie k-algebras generated by each $x\in X$. Now apply the foregoing, and note that $\mathfrak{D}(U(H))$ is the universal field of fractions of $U(H)$, by Theorem 7.16. □

We remark that the statement of Corollary 8.2 works for any set $\{H_i\}_{i\in I}$ of residually nilpotent Lie k-algebras with a Q-basis because they induce a natural RN-series with a Q-basis in $\bigoplus_{i\in I} H_i$. Also, it is known that the free product of residually nilpotent Lie algebras is a residually nilpotent Lie algebra, see for example [6, p.175]. On the other hand, assuming that the H_i's satisfy the Q-condition, we do not know whether there exists an RN-series of the free product $\coprod_{i\in I} H_i$ with a Q-basis.

Note that by choosing different elements (or changing the basis) of N, other elements of prescribed inversion height n can be found.

Another way of obtaining the second part of Corollary 8.2 is the following. By [2], if $N\neq H$ is an ideal of the free (non commutative) Lie algebra H, then N is a non-finitely generated free Lie algebra. Thus, choosing N such that H/N is nilpotent, we get elements of inversion height n for any n^2 different free generators of N.

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