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ON THE LIMIT CYCLES BIFURCATING FROM AN ELLIPSE OF A QUADRATIC CENTER

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ABSTRACT. Consider the class of all quadratic centers whose period annulus has a periodic solution whose phase curve is an ellipse E. The period annulus of any of such quadratic centers has cyclicity at least one, and this one is due to a family of algebraic limit cycles (formed by ellipses) bifurcating from the ellipse E.

1. INTRODUCTION, BRIEF SURVEY OF LITERATURE AND STATEMENT OF RESULTS

We study polynomial differential systems in $\mathbb{R}^2$ defined by

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{aligned} \tag{1}$$

where P, Q are polynomials with real coefficients such that the maximum degree of P and Q is at most 2, i.e.

$$P(x,y) = \sum_{i,j=0}^{2} P_{i,j} x^i y^j,$$
$$Q(x,y) = \sum_{i,j=0}^{2} Q_{i,j} x^i y^j.$$

We call these differential systems simply *quadratic systems*, eventually they can be linear. The dot denotes derivative with respect to the independent variable t, which is called here the *time*. Associated with the systems (1) we have the quadratic polynomial vector fields $\mathcal{X}$ with

$$\mathcal{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

We also will refer to $\mathcal{X}$ as a quadratic system. For general properties on the quadratic systems see [10] and [36].

An isolated singular point of a quadratic system is called a *center* if it has a neighborhood such that every solution passing through one of

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its points other than the singularity is a non-trivial periodic solution. A center of a quadratic system will be called *quadratic center*. The set of all periodic orbits surrounding a given center is called the *period annulus* of that center.

A *limit cycle* of a differential system (1) is a periodic orbit isolated in the set of all periodic orbits of system (1).

In spite of over a thousand papers published on quadratic systems, questions about limit cycles remain hard to tackle. In particular the second part of Hilbert's 16th problem stated by Hilbert [19] in his address in Paris in 1900 for general polynomial differential systems, remains open even for the quadratic class.

In recent years some interesting subclasses of the quadratic class were studied globally using an interplay of a variety of methods: algebraic, geometric, analytic and numerical (for example see [26, 1]). This work has brought to light interesting facts about the relationship between limit cycles and centers. Systems with quadratic centers seem to be *the main engine* in producing limit cycles (in their quadratic perturbations) as well as in producing the maximum number of four limit cycles so far obtained in concrete examples for this class.

We are mainly interested here in quadratic centers and in the limit cycles which can bifurcate from their periodic orbits when we perturb them inside the class of all quadratic systems. We give below some of the main results on quadratic centers and related notions.

The quadratic centers are characterized in the next result, see Kapteyn [20] and [21], and Bautin [2], see also Schlomiuk [31, 32] for more details about this result and quadratic centers.

Theorem 1 (Kapteyn–Bautin Theorem). *Any quadratic system candidate to have a center can be written in the form*

$$\dot{x} = -y - bx^2 - Cxy - dy^2, \qquad \dot{y} = x + ax^2 + Axy - ay^2. \tag{2}$$

This system has a center at the origin if and only if one of the following conditions holds

(i) $C = a = 0$,
(ii) $b + d = 0$,
(iii) $C + 2a = A + 3b + 5d = a^2 + bd + 2d^2 = 0$,
(iv) $A - 2b = C + 2a = 0$.

All quadratic systems with center are integrable, more precisely they have a non-constant first integral defined on the complement of an algebraic curve. For the history of this integrability result see [33]. As for quadratic systems with a center we have algebraic curves which are unions of solutions, this integrability result can be proved (see [32, 12]) by the elegant method of Darboux [11] involving the notion of *invariant algebraic curve* of the vector field. Darboux' work was for complex differential systems. As each real differential system yields a complex one when the variables range over $\mathbb{C}$, Darboux's theory apply to our systems. Furthermore from the complex first integral obtained by the method of Darboux, one can fabricate a real first integral for systems (1) with a center, see for instance [32].

Let $f = f(x, y)$ be a polynomial over $\mathbb{C}$. We say that the algebraic curve $f = 0$ is *invariant* for the quadratic system $\mathcal{X}$ if $\mathcal{X}f = kf$ for some polynomial $k = k(x, y)$ over $\mathbb{C}$. Consider the inner product $< \cdot, \cdot >$ of $\mathbb{C}^2$. Since $\mathcal{X}f =< (P, Q), (f_x, f_y) >= 0$ on the points (x, y) of $f = 0$, the vector field $\mathcal{X}$ is tangent to the curve $f = 0$, consequently the curve $f = 0$ is formed by orbits of $\mathcal{X}$. This justifies the name of invariant algebraic curve.

We have indicated above that invariant algebraic curve play an important role in questions of integrability. *But do these mainly algebraic objects also play a role in the production of such subtle and elusive analytic objects as the limit cycles*?

In this paper we give an affirmative answer to this question.

A relevant ingredient in this paper will be the notion of inverse integrating factor. Let U be an open subset of $\mathbb{R}^2$. A differentiable function $V : U \to \mathbb{R}$ is an *inverse integrating factor* of the differential system (1) if V verifies the partial differential equation

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} - \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) V = 0 \tag{3}$$

in U. Here we only consider inverse integrating factors defined in the whole plane, i.e. $U = \mathbb{R}^2$. The fact that V is an inverse integrating factor means that in $\Sigma = \mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 : V(x, y) = 0\}$ a function $H = H(x, y)$ can be defined such that

$$\begin{aligned} \dot{x} &= \frac{P(x, y)}{V(x, y)} = -\frac{\partial H}{\partial y}(x, y), \\ \dot{y} &= \frac{Q(x, y)}{V(x, y)} = \frac{\partial H}{\partial x}(x, y). \end{aligned} \tag{4}$$

Of course in Σ the function H is a *first integral* of the system (1) having the inverse integrating factor V, i.e. H is constant on the solutions of (1) which are contained in Σ.

The quadratic systems with centers have always a polynomial inverse integrating factor of degree at most 5. More precisely there is the following result due to Ndiaye [29].

Proposition 2 (Ndiaye Proposition)**.** *The polynomial inverse integrating factor V of a quadratic center* (2) *satisfying one of the conditions (i)-(iv) in the conclusion of Theorem 1 is:*

$$\begin{array}{ll} (i) & (1+Ay)\Big(d-A-b+b(A^2+3Ab+2b^2)x^2+ \\ & \quad 2b(A+b-d)y+bd(A+2b)y^2\Big), \\ (ii) & (1+Ay)\Big((1-by)^2+C(1-by)x-b(A+b)x^2\Big), \\ (iii) & \Big(d^2+(a^2+d^2)(2dy+(dy-ax)^2)\Big) \\ & \Big(d^2+(a^2+d^2)(3dy(1-ax+dy)+(dy-ax)^3)\Big), \\ (iv) & -\frac{1}{2}(x^2+y^2)-bx^2y+axy^2-\frac{1}{3}(dy^3+ax^3). \end{array}$$

The quadratic systems with centers satisfying condition (iv) of Proposition 2 are the Hamiltonian ones, i.e. they have a polynomial first integral of degree 3. We remark that all quadratic centers of Proposition 2 except some of those satisfying condition (iv), (more precisely except those with a cubic integrating factor irreducible over $\mathbb{R}$) have a quadratic factor. Note that for the inverse integrating factors V of Proposition 2, we have that $V=0$ is an invariant algebraic curve, this follows easily from the definition (4) of inverse integrating factor and from the definition of invariant algebraic curve. Therefore always the quadratic centers (i), (ii) and (iii) have an invariant conic, and occasionally also the quadratic centers (iv). When this conic is an ellipse it can be a periodic orbit of the period annulus associated to the center. In short, we can have *a periodic solution of a quadratic center whose phase curve is an ellipse, which is in the zero set of a polynomial inverse integrating factor*.

On the other hand, the inverse integrating factor has an interesting property described in the next theorem. This theorem actually holds for C^1 differential systems in $\mathbb{R}^2$, but here we only state it for quadratic systems.

Theorem 3. *Let $V:\mathbb{R}^2\to\mathbb{R}$ be an inverse integrating factor of a quadratic system* (1)*. If γ is a limit cycle of X, then γ is contained in* $\{(x,y)\in\mathbb{R}^2 : V(x,y)=0\}$.

Theorem 3 is due to Giacomini, Llibre and Viano see [17]; for an easier proof see Llibre and Rodríguez [25].

The following is a natural question: *Is it true that a periodic solution of a system with center whose phase curve is an ellipse and which is in* *the zero set of a polynomial inverse integrating factor could generate a limit cycle when the system is perturbed inside the class of all quadratic systems*?

One can think of giving an answer to this question by computing the generalized Abelian integrals associated to the non-Hamiltonian centers having a periodic orbit whose phase curve is an ellipse and which is in the zero set of a polynomial inverse integrating factor. But it is well known that such generalized Abelian integrals are in general so far very difficult or impossible to compute. For more details on Abelian integrals and generalized Abelian integrals see the second part of the book of Christopher and Li [6]. The main result of this paper gives an answer to the previous question without the use of generalized Abelian integrals. First we recall the definition of algebraic limit cycle. A limit cycle is *algebraic* if it is contained in an irreducible algebraic curve over $\mathbb{C}$; the degree of the algebraic curve is the *degree* of the algebraic limit cycle.

Theorem 4. *Assume that we have a quadratic or linear center whose period annulus has a periodic solution with an ellipse as phase curve. Then the following statements hold.*

(a) *There exists a real affine change of variables and a rescaling of the time such that the quadratic or linear system with a such center can be written:*

(a.1) *Either as*

$$\dot{x} = -1 + cy + x^2 + (b+1)y^2, \qquad \dot{y} = -cx - bxy, \tag{5}$$

with $b^2 < c^2$. For such a quadratic system the ellipse is $x^2 + y^2 = 1$, and it has the polynomial inverse integrating factor $V = (c + by)(x^2 + y^2 - 1)$. For the qualitative phase portraits in the Poincaré disc see Figure 1.

(a.2) *Or as*

$$\dot{x} = y(x + 4), \qquad \dot{y} = -x(x + 4), \tag{6}$$

For such a quadratic system the ellipse is $x^2 + y^2 = 1$. Moreover this quadratic system has all the ellipses $x^2 + y^2 = r^2$ with $0 < r < 2$ invariant, and it has the polynomial inverse integrating factor $V = (x + 4)(x^2 + y^2 - 1)$.

(a.3) *Or as*

$$\dot{x} = y, \qquad \dot{y} = -x, \tag{7}$$

For such a linear system the ellipse is $x^2 + y^2 = 1$. Moreover this linear system has all the ellipses $x^2 + y^2 = r^2$ with $r > 0$ invariant, and it has the polynomial inverse integrating factor $V = x^2 + y^2 - 1$.

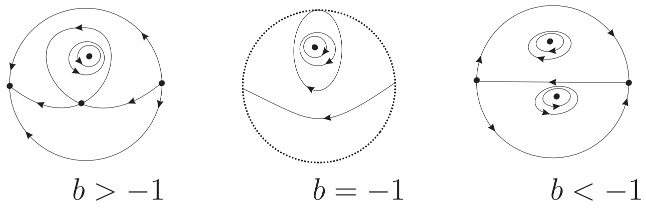

FIGURE 1. Qualitative phase portraits of quadratic systems with a center whose period annuli have a periodic orbit with an ellipse as phase curve.

(b) *When we perturb a quadratic center of the families* (5), (6) *or* (7) *inside the class of all quadratic systems there is at least one* 1*–parameter family of quadratic systems having a limit cycle bifurcating from the periodic orbit* $x^2+y^2 = 1$*. Moreover this family is formed by algebraic limit cycles.*

For more details on the Poincaré disc see [12].

We point out that there are quadratic centers possessing an invariant ellipse whose phase portraits are different than those in Figure 1 but only those indicated in Figure 1 have a periodic solution whose phase curve is an ellipse. The others possess singularities on their invariant ellipses.

Note that all quadratic systems with centers whose period annuli have a periodic solution with an ellipse $f = 0$ as phase curve, have f as a factor of a polynomial inverse integrating factor of degree at most 3 of the system.

The key point in the proof of Theorem 4 is the use of algebraic limit cycles given by ellipses.

Algebraic limit cycles for quadratic systems started to be studied in 1958. Up to now we know seven different families of algebraic limit cycles in quadratic systems, one of degree 2 (see [30] and [36]), four of degree 4 (see [35], [16], [5] and [3]), one of degree 5 and one of degree 6 (see [7]). It was proved that quadratic systems have no algebraic limit cycle of degree 3 (see [13]–[15] and [4]). It is unknown if there are algebraic limit cycles of quadratic systems of degree larger than 6. Also there is the conjecture that a quadratic system can have at most one algebraic limit cycle, the conjecture is proved under generic assumptions and some non-generic ones (see [22], [23], [24] and [34]). Additional information about the algebraic limit cycles of quadratic systems can be find in [27] and [28].

With respect to algebraic limit cycles of polynomial vector fields not necessarily quadratic, we can mention that Hilbert's 16-th problem restricted to the maximum number of algebraic limit cycles has been solved recently in the generic case see [23], and in some particular non–generic cases see [24] and [34]. Moreover it was proved in [25] that any topological configuration of finitely many limit cycles is realized with algebraic limit cycles for a convenient polynomial vector field.

Finally we mention that it is known that when a quadratic system has a polynomial inverse integrating factor, then it has no limit cycles, see [9].

Another natural question intensively studied in the quadratic systems is: *How many limit cycles can appear from the period annulus surrounding a quadratic center when we perturb it inside the class of all quadratic systems*? The sharp upper bound of the number of such limit cycles is called the *cyclicity of the period annulus*. For more details about this cyclicity and a survey of the results on it, see again [6] and the big number of references quoted there. The following result will follow easily from Theorem 4, a proof is given at the end of the next section.

Corollary 5. *Assume that we have a quadratic center whose period annulus has a periodic solution with an ellipse as phase curve. Then the cyclicity of that period annulus is at least* 1.

2. Proof of Theorem 4

We recall that an algebraic curve $f = f(x, y) = 0$ is non–singular if there is no point at which f and its first derivatives f_x and f_y are all zero. The following result is proved in [8] (see Lemma 6 of [8]).

Lemma 6. *Assume that a polynomial differential system has a non–singular invariant algebraic curve $f = 0$. If $gcd(f_x, f_y) = 1$, then the polynomial differential system can be written in the form*

$$\begin{aligned} \dot{x} &= Af + Cf_y, \\ \dot{y} &= Bf - Cf_x, \end{aligned} \tag{8}$$

where A, B and C are suitable polynomials over $\mathbb{R}$.

Proof of statement (a) of Theorem 4. It is well known that by a real affine change of coordinates any ellipse takes the form $f = x^2 + y^2 - 1 = 0$. By Lemma 6 all quadratic systems having $f = 0$ as invariant algebraic curve can be written as in (8) with $A, B \in \mathbb{R}$ and $C = (ax + by + c)/2$, i.e.

$$\begin{aligned} \dot{x} &= A(x^2 + y^2 - 1) + y(ax + by + c), \\ \dot{y} &= B(x^2 + y^2 - 1) - x(ax + by + c). \end{aligned} \tag{9}$$

Clearly we may consider $B \geq 0$ as otherwise we change the sign of all coefficients A, B, a, b and c which is equivalent to change the sign of the time in the differential system.

Now let us consider an arbitrary rotation with angle θ.

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

In the new coordinates X, Y computations give:

$$\begin{aligned} \dot{X} &= \overline{A}(X^2+Y^2-1) - Y(\overline{a}X + \overline{b}Y + \overline{c}), \\ \dot{Y} &= \overline{B}(X^2+Y^2-1) - X(\overline{a}X + \overline{b}Y + \overline{c}), \end{aligned} \tag{10}$$

with $\overline{A}, \overline{B}, \overline{a}, \overline{b}, \overline{c}$ new constants with $\overline{B} = A\sin(\theta) + B\cos(\theta)$. In general $\overline{B} \neq 0$ but it is zero when θ is a solution of the equation $A\sin(\theta) + B\cos(\theta) = 0$. If $\overline{B} \neq 0$, dividing by $\sqrt{A^2+B^2}$ we see that we can take θ such that $\cos(\theta) = A/\sqrt{A^2+B^2}$ and $\sin(\theta) = -b/\sqrt{A^2+B^2}$ so $\theta = \arccos\left(A/\sqrt{A^2+B^2}\right)$. The quadratic system in the new variables becomes

$$\begin{aligned} \dot{X} &= \overline{A}(X^2+Y^2-1) - Y(\overline{a}X + \overline{b}Y + \overline{c}), \\ \dot{Y} &= X(\overline{a}X + \overline{b}Y + \overline{c}), \end{aligned} \tag{11}$$

where

$$\overline{A} = \sqrt{A^2+B^2}, \quad \overline{a} = \frac{aB - bA}{\sqrt{A^2+B^2}}, \quad \overline{b} = -\frac{aA + bB}{\sqrt{A^2+B^2}}, \quad \overline{c} = c.$$

Note that renaming the coefficients the quadratic system (11) is the quadratic system (9) with $B = 0$. Hence all quadratic systems having an ellipse invariant can be written in the form (9) with $B = 0$.

Case 1: $B = 0$ and $A \neq 0$. Then rescaling the time (if necessary) we can assume that $A = 1$. So from now on we shall work with the quadratic system

$$\begin{aligned} \dot{x} &= x^2 + y^2 - 1 + y(ax + by + c), \\ \dot{y} &= -x(ax + by + c), \end{aligned} \tag{12}$$

which has the unit circle as invariant algebraic curve.

We must choose from the quadratic systems (12) those with a quadratic center whose period annulus has a periodic orbit with an ellipse as phase curve, now the unit circle. We now study the singular points of a system (12). We consider first $b \neq -1$. In this case the system has four singular points:

$$p_{\pm} = \left(0, \frac{-c \pm \sqrt{c^2 + 4(b+1)}}{2(b+1)}\right),$$
$$\left(\frac{-ac \mp b\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}, \frac{-bc \pm a\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}\right).$$

If $b = -1$ the system has only three singular points:

$$p = \left(0, \frac{1}{c}\right),$$
$$\left(\frac{-ac \pm \sqrt{a^2 + 1 - c^2}}{a^2 + 1}, \frac{c \pm a\sqrt{a^2 + 1 - c^2}}{a^2 + 1}\right).$$

Since in both cases the last two points are on the unit circle and we want that such a circle be a periodic orbit, we must assume that $a^2 + b^2 < c^2$.

It is well known that in the region bounded by a periodic solution, a quadratic system must have a unique singular point, which is either a focus or a center, see for instance [10]. Therefore if $b \neq -1$ then $c^2 + 4(b+1) \geq 0$, otherwise the quadratic system would not have real singular points. But this last condition always is forced by the previous condition $a^2 + b^2 < c^2$, because this implies $|b| < |c|$, and consequently

$$c^2 + 4(b+1) > c^2 + 4(-|c| + 1) = (|c| - 2)^2 \geq 0.$$

Note that when $b = -1$ we have $|c| > 1$, and consequently the singular point $(0, 1/c)$ always is in the interior of the unit circle.

The real part of the eigenvalues of the linear part of the quadratic system (12) at the singular points $p_{\pm}$ are

$$\frac{-ac \mp a\sqrt{c^2 + 4b + 4}}{4(b+1)},$$

and at the singular point p are a/c. In both cases this real part is zero if and only if $a = 0$. So in order that the quadratic systems (12) have a center a must vanish, and we get system (5).

Finally the quadratic system (5) satisfying $b^2 < c^2$ has a center in the region bounded by the unit circle, because it has the first integral

$$H = (c + b\,y)^2 \left(x^2 + y^2 - 1\right)^b,$$

which is well defined at the singular point which is in the interior of the region bounded by the unit circle, and consequently such a singular point cannot be a focus. This completes the proof of statement (a.1) of Theorem 4.

Case 2: $B = A = 0$ and $ax + by + c \neq c$ with $c \neq 0$. Then it can easily be checked that a system (9), after a rotation and a rescaling of the time, can be written as system (6). This proves statement (a.2) of Theorem 4.

Case 3: $B = A = 0$ and $ax + by + c = c \neq 0$. Then system (9) after a rescaling of the time can be written as system (7). This shows statement (a.3) of Theorem 4. □

Proof of statement (b) of Theorem 4. We consider the following perturbation of the quadratic system of statement (a.1) of Theorem 4

$$\begin{aligned} \dot{x} &= -1 + cy + x^2 + (b+1)y^2 + \varepsilon xy, \\ \dot{y} &= -cx - bxy - \varepsilon x^2, \end{aligned} \tag{13}$$

satisfying $b^2 < c^2$, where ε is a small parameter. Note that the systems (13) coincide with systems (12) with $a = \varepsilon$. So for any ε, the systems (13) have $x^2 + y^2 - 1 = 0$ as an invariant algebraic curve and for $\varepsilon \neq 0$ sufficiently small, this curve is a periodic orbit because it has no singular point on it.

Now we shall see that the singular point which is contained in the interior region bounded by the unit circle is a focus. Indeed an easy computation shows that for $\varepsilon \neq 0$ sufficiently small the real part of the eigenvalues at such singular point is

$$-\frac{c + \sqrt{c^2 + 4(b+1)}}{2(b+1)}\varepsilon + O(\varepsilon^2) \quad \text{if } c > 0 \text{ and } b \neq -1;$$

$$\frac{-c + \sqrt{c^2 + 4(b+1)}}{2(b+1)}\varepsilon + O(\varepsilon^2) \quad \text{if } c < 0 \text{ and } b \neq -1; \text{ and}$$

$$\frac{1}{2c}\varepsilon \quad \text{if } b = -1;$$

and consequently it is a focus.

In short the periodic orbit $x^2 + y^2 - 1 = 0$ of the quadratic system (13) encircles a focus for $\varepsilon \neq 0$ sufficiently small. We show that it is a limit cycle. Indeed consider a segment S starting at the focus and ending at a point p of the unit circle, closed at both ends. The Poincaré map defined on S is analytic because the quadratic system is an analytic differential system, see for instance [18]. If the singular point is a focus, such a Poincaré map cannot be the identity, and consequently it cannot have fixed points accumulating at p. Since a fixed point of this Poincaré map correspond to a periodic orbit of the system (13), it follows the unit circle cannot be an accumulation of periodic orbits. Hence it is a limit cycle. Of course this last argument about the Poincaré map is very well known.

In summary for $\varepsilon \neq 0$ sufficiently small we have that $x^2 + y^2 - 1 = 0$ is always an algebraic limit cycle of the quadratic system (13). So this family of limit cycles clearly tends to the periodic orbit $x^2 + y^2 - 1 = 0$ of the quadratic center (5) when $\varepsilon \to 0$. Consequently statement (b) of Theorem 4 for the system of statement (a.1) is proved.

Now we consider the following perturbation of the quadratic system of statement (a.2) of Theorem 4

$$\begin{aligned} \dot{x} &= \varepsilon(x^2 + y^2 - 1) + y(x + 4), \\ \dot{y} &= -x(x + 4), \end{aligned} \tag{14}$$

where ε is a small parameter. Note that system (14) is a particular system (9) with $B = 0$. So for any ε a system (14) has $x^2 + y^2 - 1 = 0$ as invariant algebraic curve and for $\varepsilon \neq 0$ sufficiently small, this curve is a periodic orbit because it has no singular point on it. In fact the two singular points of this system are

$$p_\pm = \left(0, \frac{-2 \pm \sqrt{4 + \varepsilon^2}}{\varepsilon}\right).$$

Now we shall see that the singular point p_+ which is contained in the interior region bounded by the unit circle is a focus. Indeed an easy computation shows that for $\varepsilon \neq 0$ sufficiently small the eigenvalues at this singular point are

$$\frac{\varepsilon}{8} \pm 4i + O(\varepsilon^2),$$

and consequently it is a focus.

The proof that the periodic orbit $x^2 + y^2 = 1$ is a limit cycle for $\varepsilon \neq 0$ sufficiently small follows as in the proof of the case (a.1). In short for $\varepsilon \neq 0$ sufficiently small we have that $x^2 + y^2 - 1 = 0$ is always an algebraic limit cycle of any quadratic system (14). So this family of limit cycles clearly tends to the periodic orbit $x^2 + y^2 - 1 = 0$ of the quadratic center (6) when $\varepsilon \to 0$. Consequently statement (b) of Theorem 4 for the system of statement (a.2) is proved.

Finally we consider the following perturbation of the quadratic system of statement (a.3) of Theorem 4

$$\begin{aligned} \dot{x} &= \varepsilon(x^2 + y^2 - 1) + y(\varepsilon x + 1), \\ \dot{y} &= -x(\varepsilon x + 1), \end{aligned} \tag{15}$$

where ε is a small parameter. Note that a system (15) is a particular system (9) with $B = 0$. So for any ε system (15) has $x^2 + y^2 - 1 = 0$ as invariant algebraic curve and for $\varepsilon \neq 0$ sufficiently small, this curve is a periodic orbit because it has no singular point on it. In fact the two singular points of this system are

$$q_{\pm} = \left(0, \frac{-1 \pm \sqrt{1+4\varepsilon^2}}{2\varepsilon}\right).$$

Now we shall see that the singular point q_+ which is contained in the interior region bounded by the unit circle is a focus. Indeed an easy computation shows that for $\varepsilon \neq 0$ sufficiently small the eigenvalues at this singular point are

$$\frac{\varepsilon^2}{2} \pm (1+\varepsilon^2)i + O(\varepsilon^3),$$

and consequently it is a focus.

The proof that the periodic orbit $x^2+y^2=1$ is a limit cycle for $\varepsilon \neq 0$ sufficiently small follows as in the proof of the case (a.1). In short for $\varepsilon \neq 0$ sufficiently small we have that $x^2+y^2-1=0$ is always an algebraic limit cycle of the quadratic system (15). So this family of limit cycles clearly tends to the periodic orbit $x^2+y^2-1=0$ of the quadratic center (7) when $\varepsilon \to 0$. Consequently statement (b) of Theorem 4 for the system of statement (a.3) is proved. □

Proof of Corollary 5. From statement (b) of Theorem 4 if follows immediately that the cyclicity is at least one. This cyclicity cannot increase by a possible Hopf bifurcation when we perturb the singular point which is a center because it is known that when a quadratic system has an ellipse as limit cycle, this is the unique limit cycle of the system, see for instance [3]. □

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