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ON THE INTEGRABILITY AND POLYNOMIAL INTEGRABILITY OF THE EULER EQUATIONS

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ABSTRACT. In this paper we show that the Euler equations in the Manakov or product cases are completely integrable. But our main results prove that the Euler equations either satisfies the Manakov condition, or have at most four functionally independent polynomial first integrals. Also we provide necessary conditions in order that the Euler equations can have a fourth functionally independent polynomial first integral.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Given a system of ordinary differential equations depending on parameters, in general is very difficult to recognize for which values of the parameters the equations have first integrals. Except for some simple cases, this problem is very hard and there are no satisfactory methods to solve it.

In this paper we study the first integrals of the Euler differential equations in $\mathbb{R}^6$ depending on six parameters. Because these equations are used here exclusively as an interesting and nontrivial example of a multiparameter family of ordinary differential equations, we consider them without any explanations of their origin, which can be found for instance in the references [1, 2, 3, 10, 11, 14]. Here we discuss neither their physical origin, nor their relevant information that can be found in the quoted references.

Before introducing the Euler equations we first recall some basic definitions on the analytic and polynomial integrability of the polynomial differential systems of the form

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \mathbf{P}(\mathbf{x}), \qquad \mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n, \tag{1}$$

with $\mathbf{P}(\mathbf{x}) = (P_1(\mathbf{x}), \cdots, P_n(\mathbf{x}))$ and $P_i \in \mathbb{R}[x_1, \cdots, x_n]$ for $i = 1, \cdots, n$. As usual $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{R}[x_1, \cdots, x_n]$ denotes the polynomial ring over $\mathbb{R}$ in the variables $x_1, \cdots, x_n$.

A non–constant function $H(x_1, \ldots, x_n)$ is a *first integral* of system (1) on an open subset Ω of $\mathbb{R}^n$ if it is constant on all solution curves $(x_1(t), \cdots, x_n(t))$ of system (1) contained in Ω. If H is C^1 on Ω, then H is a first integral of

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system (1) if and only if

$$\sum_{i=1}^{n} P_i(x)\frac{\partial H}{\partial x_i}(x) \equiv 0, \qquad \text{for all } x \in \Omega.$$

If H is a first integral of (1) and is analytic (resp. polynomial), then it is called an *analytic* (resp. a *polynomial*) first integral.

The first integrals $H_1, \ldots, H_r$ of the differential system (1) with $r < n$ are *functionally independent* if the $r \times n$ matrix

$$\begin{pmatrix} \dfrac{\partial H_1}{\partial x_1} & \cdots & \dfrac{\partial H_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \dfrac{\partial H_r}{\partial x_1} & \cdots & \dfrac{\partial H_r}{\partial x_n} \end{pmatrix}(\mathbf{x})$$

has rank r at all points $\mathbf{x} \in \mathbb{R}^n$ where they are defined with the exception (perhaps) of a zero Lebesgue measure set. Moreover, the differential system (1) is *completely integrable* if it has $H_1, \ldots, H_{n-1}$ first integrals functionally independent.

The Euler equations studied in this paper are

$$\begin{aligned} \frac{dx_1}{dt} &= (\lambda_3 - \lambda_2)x_2x_3 + (\lambda_6 - \lambda_5)x_5x_6 = P_1(x_1, x_2, x_3, x_4, x_5, x_6), \\ \frac{dx_2}{dt} &= (\lambda_1 - \lambda_3)x_1x_3 + (\lambda_4 - \lambda_6)x_4x_6 = P_2(x_1, x_2, x_3, x_4, x_5, x_6), \\ \frac{dx_3}{dt} &= (\lambda_2 - \lambda_1)x_1x_2 + (\lambda_5 - \lambda_4)x_4x_5 = P_3(x_1, x_2, x_3, x_4, x_5, x_6), \\ \frac{dx_4}{dt} &= (\lambda_3 - \lambda_5)x_3x_5 + (\lambda_6 - \lambda_2)x_2x_6 = P_4(x_1, x_2, x_3, x_4, x_5, x_6), \\ \frac{dx_5}{dt} &= (\lambda_4 - \lambda_3)x_3x_4 + (\lambda_1 - \lambda_6)x_1x_6 = P_5(x_1, x_2, x_3, x_4, x_5, x_6), \\ \frac{dx_6}{dt} &= (\lambda_2 - \lambda_4)x_2x_4 + (\lambda_5 - \lambda_1)x_1x_5 = P_6(x_1, x_2, x_3, x_4, x_5, x_6), \end{aligned} \tag{2}$$

defined in $\mathbb{R}^6$. Note that these differential equations depend on six parameters $\lambda_1, \ldots, \lambda_6$.

It is well known that the Euler equations (2) have always the following three polynomial first integrals of degree 2:

$$H_1 = x_1x_4 + x_2x_5 + x_3x_6, \qquad H_2 = \sum_{i=1}^{6} x_i^2, \qquad H_3 = \frac{1}{2}\sum_{i=1}^{6} \lambda_i x_i^2,$$

which are functionally independent unless $\lambda_1 = \ldots = \lambda_6$.

One additional fourth polynomial first integral of degree 2 functionally independent of the first three is known for the Euler equations if the parameters satisfy some conditions, more precisely in the *product condition* when

$$\lambda_4 = \lambda_1, \qquad \lambda_5 = \lambda_2, \qquad \lambda_6 = \lambda_3,$$

or in the *Manakov condition* when

$$M = \lambda_1\lambda_4(\lambda_{23} + \lambda_{56}) + \lambda_2\lambda_5(\lambda_{31} + \lambda_{64}) + \lambda_3\lambda_6(\lambda_{12} + \lambda_{45}) = 0. \tag{3}$$

Thus in the product case the Euler equations have the first integral

$$H_5 = \lambda_1 x_1 x_4 + \lambda_2 x_2 x_5 + \lambda_3 x_3 x_6.$$

and in the Manakov case the Euler equations have the first integral

$$H_4 = \lambda_{16}\lambda_{24}x_4^2 + \lambda_{15}\lambda_{26}x_5^2 + \lambda_{16}\lambda_{26}x_6^2,$$

here and in what follows for simplifying the notation we denote

$$\lambda_{ij} = \lambda_i - \lambda_j \quad \text{for} \quad i, j \in \{1, \dots, 6\}.$$

Maciejewski, Popov and Strelcyn in [9] provided explicitly all values of the parameters inside the Manakov and the product cases for which the Euler equations have a fourth polynomial first integral functionally independent of the three known integrals. In [12] Popov and Strelcyn proved that if the Euler equations have a rational first integral functionally independent of the known three ones, then they have a polynomial first integral that is also functionally independent of them. Popov, Respondek and Strelcyn in [13] presented a simpler proof of the result of [12].

Our first result on the integrability of the Euler equations is the next one.

Theorem 1. *The Euler equations satisfying either the Manakov or the product conditions are completely integrable.*

Theorem 1 is proved in section 2.

The complete characterization of all polynomial first integrals of the Euler equations (2) is not known, but Popov and Strelcyn in [12] stated the following:

Conjecture: *The Euler equations have a fourth polynomial first integral functionally independent of H_1, H_2 and H_3 only either in the Manakov case or in the product case.*

This conjecture remains open. Related to it we have the following result.

Theorem 2. *The Euler equations* (2) *either satisfy the Manakov condition, or have at most four functionally independent polynomial first integrals.*

Theorem 2 is proved in section 3.

We remark that from the proof of Theorem 2, the result of this theorem can be stated for analytic first integrals instead of polynomial first integrals.

We consider the system of polynomial equations

$$P_i(c_1, c_2, c_3, c_4, c_5, c_6) + c_i = 0, \qquad \text{for } i = 1, \dots, 6, \tag{4}$$

where $P_1, \dots, P_6$ are defined in (2). The non–zero solutions (real or complex) of this system of polynomial equations defines the *balances* of the Euler equations (2) with the weight exponent $(1, 1, 1, 1, 1, 1)$. The precise definitions of balances and weight exponents are given in section 4.

We define

$$L_1 = \lambda_{15}\lambda_{16}\lambda_{56}, \quad L_2 = \lambda_{34}\lambda_{35}\lambda_{45}, \quad L_3 = \lambda_{42}\lambda_{26}\lambda_{46}, \quad L_4 = \lambda_{12}\lambda_{13}\lambda_{23}.$$

Theorem 3. *Assume that* $L_1L_2L_3L_4 \neq 0$. *Then the Euler equations have* 16 *different balances* C_{ij} *for* $i, j = 1, \dots, 4$ *given in* (20).

(a) *A necessary condition in order that the Euler equations have a polynomial first integral* H *of degree* $m > 2$ *functionally independent of the first integrals* H_1, H_2 *and* H_3 *is that the gradient* $\mathrm{grad}\, H$ *satisfies that either it is identically zero at all the* 16 *balances* C_{ij}, *or that it is not identically zero at some balance* C_{i_0,j_0} *and* $M/L_{i_0} = m^2 - m$. *Of course, if* $\mathrm{grad}\, H$ *is not identically zero at two balances* C_{i_1,j_1} *and* C_{i_2,j_2}, *then* $L_{i_1} = L_{i_2} = M/(m^2 - m)$.

(b) *A necessary condition in order that the Euler equations have a polynomial first integral* H *of degree* 2 *functionally independent of the first integrals* H_1, H_2 *and* H_3 *is that* $\mathrm{grad}\, H$ *satisfies that either it is linearly dependent on* $\mathrm{grad}\, H_1$, $\mathrm{grad}\, H_2$ *and* $\mathrm{grad}\, H_3$ *at all the* 16 *balances* C_{ij}, *or that it is linearly independent of* $\mathrm{grad}\, H_1$, $\mathrm{grad}\, H_2$ *and* $\mathrm{grad}\, H_3$ *at some balance* C_{i_0,j_0} *and* $M/L_{i_0} = 2$. *Of course, if* $\mathrm{grad}\, H$ *is linearly independent of* $\mathrm{grad}\, H_1$, $\mathrm{grad}\, H_2$ *and* $\mathrm{grad}\, H_3$ *at two balances* C_{i_1,j_1} *and* C_{i_2,j_2}, *then* $L_{i_1} = L_{i_2} = M/2$.

The proof of Theorem 3 is given in section 4.

We remark that in the Manakov case the gradient of the first integral H_4 is not identical zero at the balances C_{1j}, C_{2j}, C_{3j}, and it is identically zero at the balances C_{4j}.

As we shall see in section 4 in the Manakov case the gradient of the first integral H_4 is linearly dependent on the gradients of H_1, H_2 and H_3 at all the balances C_{ij} for $i, j \in \{1, \dots, 4\}$. But in the product case the gradient of the first integral H_5 is linearly independent of the gradients of H_1, H_2 and H_3 at all the balances C_{ij} for $i, j \in \{1, \dots, 4\}$.

We remark that Theorems 2 and 3 also hold when the Euler equations are in $\mathbb{C}^6$ instead of $\mathbb{R}^6$, because their proofs also work in $\mathbb{C}^6$.

Finally we note that if $L_1L_2L_3L_4 = 0$, working in a similar way as in the proof of Theorem 3 we can get a similar result to the one stated in that theorem. But since the statement of this new theorem is too long (because it must consider many different possibilities), and the results are not more interesting than the results of Theorem 3, we decide do not include these results in this paper.

2. Proof of Theorem 1

For proving Theorem 1 we need to recall some classical results on integrability due to Jacobi and Whittaker, see for instance the book of Goriely [7].

Let $J = J(\mathbf{x})$ be a non–negative C^1 function non–identically zero on any open subset of $\mathbb{R}^n$, then J is a *Jacobi multiplier* of the differential system (1) if

$$\int_\Omega J(\mathbf{x})d\mathbf{x} = \int_{\varphi_t(\Omega)} J(\mathbf{x})d\mathbf{x}, \tag{5}$$

where Ω is any open subset of $\mathbb{R}^n$ and φ_t is the flow defined by the differential system (1).

The following result is due to Jacobi, for a proof see Theorem 2.7 of [7].

Theorem 4. *Consider the differential system* (1) *in* $\mathbb{R}^n$*, and assume that it admits a Jacobi multiplier* $J = J(\mathbf{x})$ *and* $n-2$ *first integrals functionally independent. Then the system admits an additional first integral functionally independent with the previous* $n-2$ *first integrals. That is the differential system* (1) *is completely integrable.*

We recall that the *divergence* of the differential system (1) is

$$\sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i}.$$

In general given a function J it is not easy to verify (5) for knowing if it is a Jacobi multiplier. However we have the following result of Whittaker [16], which plays a main role for detecting Jacobi multipliers.

Proposition 5. *Let* $J = J(\mathbf{x})$ *be a non–negative* C^1 *function non–identically zero on any open subset of* $\mathbb{R}^n$*. Then* J *is a Jacobi multiplier of the differential system* (1) *if and only if the divergence of the differential system*

$$\dot{x}_i = J(\mathbf{x})P_i(\mathbf{x}), \qquad \text{for } i = 1, \ldots, n,$$

is zero.

From Theorem 4 and Proposition 5 it follows immediately the following result.

Corollary 6. *Consider the differential system* (1) *in* $\mathbb{R}^n$*, and assume that it has zero divergence and* $n-2$ *first integrals functionally independent. Then the system is completely integrable.*

Proof of Theorem 1. It is immediate to verify that the Euler equations (2) in $\mathbb{R}^6$ have zero divergence, because every P_i does not depend on x_i. In the Manakov or product cases the Euler equations have $4 = 6 - 2$ first integrals functionally independent. So in this two cases the Euler equations satisfy the assumptions of Corollary 6. Therefore in the Manakov or product cases the Euler equations are completely integrable. □

3. Proof of Theorem 2

We denote by $\mathbb{Z}_+$ the set of non–negative integers. The following result, due to Zhang [20], will be used in the proof of Theorem 2.

Theorem 7. *For an analytic vector field* (1) *in* $(\mathbb{R}^n, 0)$ *with* $\mathbf{P}(\mathbf{0}) = \mathbf{0}$*, let* $\lambda_1, \ldots, \lambda_n$ *be the eigenvalues of* $D\mathbf{P}(\mathbf{0})$*. Set*

$$\mathcal{G} = \left\{ (k_1, \ldots, k_n) \in (\mathbb{Z}_+)^n : \sum_{i=1}^n k_i \lambda_i = 0, \sum_{i=1}^n k_i > 0 \right\}.$$

Assume that system (1) *has* $r < n$ *functionally independent local analytic first integrals* $\Phi_1(x), \ldots, \Phi_r(x)$ *at* $(\mathbb{R}^n, 0)$*. If the* $\mathbb{Z}$*–linear space generated by* $\mathcal{G}$ *has dimension* r*, then any nontrivial analytic first integral of system* (1) *is an analytic function of* $\Phi_1(x), \ldots, \Phi_r(x)$*.*

We call each element $(k_1, \ldots, k_n)$ of $\mathcal{G}$ a *resonant lattice* of the eigenvalues $\lambda_1, \ldots, \lambda_n$. Theorem 7 implies that the number of functionally independent local analytic first integrals of system (1) at the singularity $\mathbf{0}$ does not exceed the number of linearly independent resonant lattices of $\lambda_1, \ldots, \lambda_n$.

Direct calculations show that the Euler equations have the following nine planes of singularities

$$\begin{gathered}
(x_1, 0, 0, x_4, 0, 0)\,, \\
(0, x_2, 0, 0, x_5, 0)\,, \\
(0, 0, x_3, 0, 0, x_6)\,, \\
\left(\frac{\lambda_{34}\lambda_{46}x_4}{\sqrt{\lambda_{31}\lambda_{34}\lambda_{16}\lambda_{46}}}, 0, \frac{\sqrt{\lambda_{16}\lambda_{46}}x_6}{\sqrt{\lambda_{31}\lambda_{34}}}, x_4, 0, x_6 \right), \\
\left(-\frac{\lambda_{34}\lambda_{46}x_4}{\sqrt{\lambda_{31}\lambda_{34}\lambda_{16}\lambda_{46}}}, 0, -\frac{\sqrt{\lambda_{16}\lambda_{46}}x6}{\sqrt{\lambda_{31}\lambda_{34}}}, x_4, 0, x_6 \right), \\
\left(\frac{\lambda_{24}\lambda_{45}x_4}{\sqrt{\lambda_{21}\lambda_{24}\lambda_{15}\lambda_{45}}}, \frac{\sqrt{\lambda_{15}\lambda_{45}}x_5}{\sqrt{\lambda_{21}\lambda_{24}}}, 0, x_4, x_5, 0 \right), \\
\left(-\frac{\lambda_{24}\lambda_{45}x_4}{\sqrt{\lambda_{21}\lambda_{24}\lambda_{15}\lambda_{45}}}, -\frac{\sqrt{\lambda_{15}\lambda_{45}}x_5}{\sqrt{\lambda_{21}\lambda_{24}}}, 0, x_4, x_5, 0 \right), \\
\left(0, \frac{\lambda_{35}\lambda_{56}x_5}{\sqrt{\lambda_{32}\lambda_{35}\lambda_{26}\lambda_{56}}}, \frac{\sqrt{\lambda_{26}\lambda_{56}}x_6}{\sqrt{\lambda_{32}\lambda_{35}}}, 0, x_5, x_6 \right), \\
\left(0, -\frac{\lambda_{35}\lambda_{56}x_5}{\sqrt{\lambda_{32}\lambda_{35}\lambda_{26}\lambda_{56}}}, -\frac{\sqrt{\lambda_{26}\lambda_{56}}x_6}{\sqrt{\lambda_{32}\lambda_{35}}}, 0, x_5, x_6 \right).
\end{gathered}$$

Of course the existence of these last six planes of singularities depends on the values of the parameters λ_i's.

At the singularities $S_1 = (x_1, 0, 0, x_4, 0, 0)$, the 6–tuple of eigenvalues $\mu = (\mu_1, \ldots, \mu_6)$ of the linear part of the Euler equations are

$$\left(0,\, 0,\, -\sqrt{\frac{a_1-\sqrt{b_1}}{2}},\, \sqrt{\frac{a_1-\sqrt{b_1}}{2}},\, -\sqrt{\frac{a_1+\sqrt{b_1}}{2}},\, \sqrt{\frac{a_1+\sqrt{b_1}}{2}}\right), \tag{6}$$

where

$$\begin{aligned} a_1 &= -\left(\lambda_{12}\lambda_{13}+\lambda_{15}\lambda_{16}\right)x_1^2+\left(\lambda_{24}\lambda_{46}+\lambda_{34}\lambda_{45}\right)x_4^2,\\ b_1 &= -4\Delta_1+a_1^2, \end{aligned}$$

with

$$\Delta_1 = \left(\lambda_{12}\lambda_{15}x_1^2+\lambda_{24}\lambda_{45}x_4^2\right)\left(\lambda_{13}\lambda_{16}x_1^2+\lambda_{34}\lambda_{46}x_4^2\right).$$

From Theorem 7 we know that the number of functionally independent analytic first integrals of the Euler equations in a neighborhood of the singularities S_1 is no more than the number of linearly independent elements of the set

$$\mathcal{G}_1 = \left\{(k_1,\ldots,k_6)\in\mathbb{Z}_+^6;\ \sum_{i=1}^{6}k_i\mu_i = 0,\ \sum_{i=1}^{6}k_i\geq 1\right\},$$

Consequently the number of the functionally independent polynomial first integrals of the Euler equations are no more than the number of the linearly independent elements of $\mathcal{G}_1$.

According to the eigenvalues (6) the resonant lattices satisfy

$$\sqrt{a_1-\sqrt{b_1}}\,(k_3-k_4)+\sqrt{a_1+\sqrt{b_1}}\,(k_5-k_6)=0. \tag{7}$$

This last equation has the following linearly independent non–negative solutions $(k_1, \ldots, k_6)$:

$$(1,0,0,0,0,0),\ (0,1,0,0,0,0),\ (0,0,1,1,0,0)\ \text{ and }\ (0,0,0,0,1,1).$$

In order that equation (7) has other linearly independent non–negative integer solutions different from the above list, we must have

(i) either $(a_1-\sqrt{b_1})(a_1+\sqrt{b_1})=0$;

(ii) or $(a_1-\sqrt{b_1})(a_1+\sqrt{b_1})\neq 0$ and $\sqrt{a_1-\sqrt{b_1}}/\sqrt{a_1+\sqrt{b_1}}$ a rational number. Then $\Delta_1\neq 0$ and $a_1\neq 0$ (otherwise $\sqrt{-\sqrt{b_1}}/\sqrt{\sqrt{b_1}}$ cannot be a rational number). Set

$$\sqrt{a_1-\sqrt{b_1}}/\sqrt{a_1+\sqrt{b_1}} = m/n, \qquad m,n\in\mathbb{Z}\setminus\{0\} \text{ coprime}.$$

This last equality can be written in an equivalent way as

$$\frac{\Delta_1}{a_1^2} = \frac{m^2}{(m+n)^2},$$

where we have used the fact that $b_1 = a_1^2 - 4\Delta_1$.

In the case (i) we have the twelve independent conditions:

- $\lambda_1 = \lambda_2$ and $\lambda_4 = \lambda_5$;
- $\lambda_1 = \lambda_5$ and $\lambda_2 = \lambda_4$;
- $\lambda_1 = \lambda_3$ and $\lambda_4 = \lambda_6$;
- $\lambda_1 = \lambda_6$ and $\lambda_3 = \lambda_4$;
- $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_5 = \lambda_6$;
- $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6$;
- $\lambda_1 = \lambda_2 = \lambda_4$;
- $\lambda_1 = \lambda_4 = \lambda_5$;
- $\lambda_1 = \lambda_3 = \lambda_4$;
- $\lambda_1 = \lambda_4 = \lambda_6$;
- $\lambda_1 = \lambda_3 = \lambda_6$ and $\lambda_2 = \lambda_4 = \lambda_5$;
- $\lambda_1 = \lambda_2 = \lambda_5$ and $\lambda_3 = \lambda_4 = \lambda_6$.

Here we have used the fact that we consider all the planes of singularities S_1 under any given condition on the parameters of Euler equations (2). Some calculations show that the first six ones are inside the Manakov condition, i.e. under such condition we have $M = 0$. Under the last six conditions we have

- $M|_{\lambda_1=\lambda_2=\lambda_4} = \lambda_{13}\lambda_{15}\lambda_{16}$,
- $M|_{\lambda_1=\lambda_4=\lambda_5} = \lambda_{12}\lambda_{13}\lambda_{16}$,
- $M|_{\lambda_1=\lambda_3=\lambda_4} = \lambda_{21}\lambda_{15}\lambda_{16}$,
- $M|_{\lambda_1=\lambda_4=\lambda_6} = \lambda_{21}\lambda_{13}\lambda_{15}$,
- $M|_{\lambda_1=\lambda_3=\lambda_6,\ \lambda_2=\lambda_4=\lambda_5} = \lambda_{12}^3$,
- $M|_{\lambda_1=\lambda_2=\lambda_5,\ \lambda_3=\lambda_4=\lambda_6} = \lambda_{12}^3$,

respectively.

Lemma 8. *Under the conditions either $\lambda_1 = \lambda_2 = \lambda_4$, or $\lambda_1 = \lambda_4 = \lambda_5$, or $\lambda_1 = \lambda_3 = \lambda_4$, or $\lambda_1 = \lambda_4 = \lambda_6$, or $\lambda_1 = \lambda_3 = \lambda_6$ and $\lambda_2 = \lambda_4 = \lambda_5$, or $\lambda_1 = \lambda_2 = \lambda_5$ and $\lambda_3 = \lambda_4 = \lambda_6$, the Euler equations either satisfy the Manakov condition or have some singularities whose eigenvalues do not have a fifth linearly independent resonant lattice.*

Proof. We shall show that some of the planar singularities $(0, x_2, 0, 0, x_5, 0)$ and $(0, 0, x_3, 0, 0, x_6)$ are suitable for the choice of the lemma in the first four cases. The singularities $\left(0, \dfrac{\lambda_{35}\lambda_{56}x_5}{\sqrt{\lambda_{32}\lambda_{35}\lambda_{26}\lambda_{56}}}, \dfrac{\sqrt{\lambda_{26}\lambda_{56}}x_6}{\sqrt{\lambda_{32}\lambda_{35}}}, 0, x_5, x_6\right)$ are suitable for the choice of the lemma in the last two cases.

At the singularities $S_2 = (0, x_2, 0, 0, x_5, 0)$, the 6–tuple of eigenvalues of the linear part of the Euler equations are

$$\left(0,\, 0,\, -\sqrt{\frac{a_2-\sqrt{b_2}}{2}},\, \sqrt{\frac{a_2-\sqrt{b_2}}{2}},\, -\sqrt{\frac{a_2+\sqrt{b_2}}{2}},\, \sqrt{\frac{a_2+\sqrt{b_2}}{2}}\right), \tag{8}$$

where

$$\begin{aligned} a_2 &= (\lambda_{12}\lambda_{23} - \lambda_{24}\lambda_{26})\, x_2^2 + (\lambda_{15}\lambda_{56} - \lambda_{35}\lambda_{45})\, x_5^2, \\ b_2 &= 4\Delta_2 + a_2^2, \end{aligned}$$

with

$$\Delta_2 = \left(\lambda_{12}\lambda_{24}x_2^2 + \lambda_{15}\lambda_{45}x_5^2\right)\left(\lambda_{23}\lambda_{26}x_2^2 + \lambda_{35}\lambda_{56}x_5^2\right).$$

Direct calculations show that

$$\begin{aligned} \Delta_2|_{\lambda_1=\lambda_2=\lambda_4} &= \lambda_{15}^2 x_5^2\left(\lambda_{13}\lambda_{16}x_2^2 + \lambda_{35}\lambda_{56}x_5^2\right), \\ \Delta_2|_{\lambda_1=\lambda_4=\lambda_5} &= \lambda_{12}^2 x_2^2\left(\lambda_{32}\lambda_{26}x_2^2 + \lambda_{13}\lambda_{16}x_5^2\right). \end{aligned}$$

Hence under one of the conditions $\lambda_1 = \lambda_2 = \lambda_4$ or $\lambda_1 = \lambda_4 = \lambda_5$ we are either in the Manakov condition or $a_2^2 - b_2 \neq 0$, i.e. $\Delta_2 \neq 0$. In the latter, working in a similar way as for the singularities S_1 for studying if there is a fifth linearly independent resonant lattice at S_2, we need to check if $\sqrt{a_2 - \sqrt{b_2}}/\sqrt{a_2 + \sqrt{b_2}}$ is a rational number.

For S_2 we are now in the conditions $\lambda_1 = \lambda_2 = \lambda_4$ and $\lambda_{13}\lambda_{15}\lambda_{16} \neq 0$. Set

$$\sqrt{a_2 - \sqrt{b_2}}/\sqrt{a_2 + \sqrt{b_2}} = m_2/n_2, \qquad m_2, n_2 \in \mathbb{Z} \setminus \{0\} \text{ coprime.}$$

This last equation can be written as

$$\frac{\Delta_2}{a_2^2} = -\frac{m_2^2}{(n_2 + m_2)^2}. \tag{9}$$

Clearly we have $a_2 \neq 0$, otherwise $\sqrt{a_2 - \sqrt{b_2}}/\sqrt{a_2 + \sqrt{b_2}}$ is not a rational number. Since

$$\frac{\Delta_2}{a_2^2} = \frac{\lambda_{13}\lambda_{16}x_2^2}{(\lambda_{35} - \lambda_{56})^2 x_5^2} + \frac{\lambda_{35}\lambda_{56}}{(\lambda_{35} - \lambda_{56})^2},$$

this shows that there always exist infinitely many singularities S_2 which cannot satisfy condition (9). At these singularities S_2 the eigenvalues do not have a fifth linearly independent resonant lattice.

In the conditions $\lambda_1 = \lambda_4 = \lambda_5$ and $\lambda_{12}\lambda_{13}\lambda_{16} \neq 0$, we also set

$$\sqrt{a_2 - \sqrt{b_2}}/\sqrt{a_2 + \sqrt{b_2}} = m_2/n_2.$$

This can be written as (9). Since $a_2 \neq 0$ and

$$\frac{\Delta_2}{a_2^2} = \frac{\lambda_{13}\lambda_{16}x_5^2}{(\lambda_{23} + \lambda_{36})^2 x_2^2} + \frac{\lambda_{32}\lambda_{26}}{(\lambda_{23} + \lambda_{26})^2},$$

this also shows that there always exist infinitely many singularities S_2 which cannot satisfy condition (9). At these singularities S_2 the eigenvalues do not have a fifth linearly independent resonant lattice.

At the singularities $S_3 = (0, 0, x_3, 0, 0, x_6)$, the 6–tuple of eigenvalues of the linear part of the Euler equations are

$$\left(0,\, 0,\, -\sqrt{\frac{a_3 - \sqrt{b_3}}{2}},\, \sqrt{\frac{a_3 - \sqrt{b_3}}{2}},\, -\sqrt{\frac{a_3 + \sqrt{b_3}}{2}},\, \sqrt{\frac{a_3 + \sqrt{b_3}}{2}}\right), \tag{10}$$

where

$$\begin{aligned} a_3 &= -\left(\lambda_{13}\lambda_{23} + \lambda_{34}\lambda_{35}\right)x_3^2 - \left(\lambda_{16}\lambda_{56} + \lambda_{26}\lambda_{46}\right)x_6^2, \\ b_3 &= -4\Delta_3 + a_3^2, \end{aligned}$$

with

$$\Delta_3 = \left(\lambda_{13}\lambda_{34}x_3^2 + \lambda_{16}\lambda_{46}x_6^2\right)\left(\lambda_{23}\lambda_{35}x_3^2 + \lambda_{26}\lambda_{56}x_6^2\right).$$

Direct calculations show that

$$\begin{aligned} \Delta_3|_{\lambda_1=\lambda_3=\lambda_4} &= \lambda_{16}^2 x_6^2\left(\lambda_{21}\lambda_{15}x_3^2 + \lambda_{26}\lambda_{56}x_6^2\right), \\ \Delta_3|_{\lambda_1=\lambda_4=\lambda_6} &= \lambda_{13}^2 x_3^2\left(\lambda_{32}\lambda_{35}x_3^2 + \lambda_{21}\lambda_{15}x_6^2\right). \end{aligned}$$

Hence under one of the conditions $\lambda_1 = \lambda_3 = \lambda_4$ or $\lambda_1 = \lambda_4 = \lambda_6$ we are either in the Manakov condition or $a_3^2 - b_3 \neq 0$, i.e. $\Delta_3 \neq 0$. In the latter, working in a similar way as for the singularities S_1 for studying if there is a fifth linearly independent resonant lattice at S_3, we need to check if $\sqrt{a_3 - \sqrt{b_3}}/\sqrt{a_3 + \sqrt{b_3}}$ is a rational number.

For S_3 we are now in the conditions respectively $\lambda_1 = \lambda_3 = \lambda_4$ and $\lambda_{12}\lambda_{15}\lambda_{16} \neq 0$, or $\lambda_1 = \lambda_4 = \lambda_6$ and $\lambda_{12}\lambda_{13}\lambda_{15} \neq 0$. Set

$$\sqrt{a_3 - \sqrt{b_3}}/\sqrt{a_3 + \sqrt{b_3}} = m_3/n_3, \qquad m_3, n_3 \in \mathbb{Z}\setminus\{0\} \text{ coprime}.$$

This last equation can be written as

$$\frac{\Delta_3}{a_3^2} = \frac{m_3^2}{(n_3 + m_3)^2}. \tag{11}$$

Since $a_3 \neq 0$ and under the above two conditions we have respectively

$$\frac{\Delta_3}{a_3^2} = \frac{\lambda_{21}\lambda_{15}x_3^2}{(\lambda_{26} + \lambda_{56})^2 x_6^2} + \frac{\lambda_{26}\lambda_{56}}{(\lambda_{26} + \lambda_{56})^2},$$

or

$$\frac{\Delta_3}{a_3^2} = \frac{\lambda_{21}\lambda_{15}x_6^2}{(\lambda_{23} - \lambda_{35})^2 x_3^2} + \frac{\lambda_{32}\lambda_{35}}{(\lambda_{23} - \lambda_{35})^2}.$$

Hence there are infinitely many singularities S_3 at which the eigenvalues do not have a fifth linearly independent resonant lattice.

We can check that Δ_2 and Δ_3 are zero under the conditions $\lambda_1 = \lambda_3 = \lambda_6$ and $\lambda_2 = \lambda_4 = \lambda_5$, or $\lambda_1 = \lambda_2 = \lambda_5$ and $\lambda_3 = \lambda_4 = \lambda_6$. So in these two cases the singularities S_2 and S_3 are not suitable for the lemma. We consider the singularities

$$S_4 = \left(0,\, \frac{\lambda_{35}\lambda_{56}x_5}{\sqrt{\lambda_{32}\lambda_{35}\lambda_{26}\lambda_{56}}},\, \frac{\sqrt{\lambda_{26}\lambda_{56}}x_6}{\sqrt{\lambda_{32}\lambda_{35}}},\, 0,\, x_5,\, x_6\right).$$

Under any one of the conditions $\lambda_1 = \lambda_3 = \lambda_6$ and $\lambda_2 = \lambda_4 = \lambda_5$, and $\lambda_1 = \lambda_2 = \lambda_5$ and $\lambda_3 = \lambda_4 = \lambda_6$, the expression of S_4 can be simplified to $S_2 = (0, -x_5, x_6, 0, x_5, x_6)$. The 6–tuple of eigenvalues of the linear part of the Euler equations at these singularities S_4 are

$$\left(0,\, 0,\, -\sqrt{\frac{a_4-\sqrt{b_4}}{2}},\, \sqrt{\frac{a_4-\sqrt{b_4}}{2}},\, -\sqrt{\frac{a_4+\sqrt{b_4}}{2}},\, \sqrt{\frac{a_4+\sqrt{b_4}}{2}}\right), \tag{12}$$

where

$$\begin{aligned} a_4 &= \begin{cases} -2(\lambda_1-\lambda_2)^2\left(x_5^2+x_6^2\right), & \text{if } \lambda_1=\lambda_3=\lambda_6,\ \lambda_2=\lambda_4=\lambda_5,\\ -2(\lambda_1-\lambda_3)^2\left(x_5^2+x_6^2\right), & \text{if } \lambda_1=\lambda_2=\lambda_5,\ \lambda_3=\lambda_4=\lambda_6, \end{cases}\\ b_4 &= 4\Delta_4 + a_4^2, \end{aligned}$$

with

$$\Delta_4 = \begin{cases} -8(\lambda_1-\lambda_2)^4 x_5^2 x_6^2, & \text{if } \lambda_1=\lambda_3=\lambda_6,\ \lambda_2=\lambda_4=\lambda_5,\\ -8(\lambda_1-\lambda_3)^4 x_5^2 x_6^2, & \text{if } \lambda_1=\lambda_2=\lambda_5,\ \lambda_3=\lambda_4=\lambda_6. \end{cases}$$

For the Euler system to be not trivial under the two conditions mentioned above, we must have $\lambda_1 \neq \lambda_2$ or $\lambda_1 \neq \lambda_3$. Under these conditions we have $\Delta_4 \neq 0$, i.e. $a_4^2 - b_4 \neq 0$. In order that there is a fifth linearly independent resonant lattice at S_4, the ratio $\sqrt{a_4-\sqrt{b_4}}/\sqrt{a_4+\sqrt{b_4}}$ should be a rational number. That is,

$$\frac{\Delta_4}{a_4^2} = -\frac{m_4^2}{(n_4+m_4)^2}, \tag{13}$$

for some $m_4, n_4 \in \mathbb{N}$. But it follows from the expressions of a_4 and Δ_4 that there are infinitely many singularities S_4 at which (13) cannot hold. So at these singularities S_4 the eigenvalues do not have a fifth linearly independent resonant lattice. This completes the proof of the lemma. □

From Lemma 8 and Theorem 7 we have proved that in the case (i) the Euler equations either satisfy the Manakov condition or have at most four functionally independent polynomial first integrals.

Next we consider the case (ii). In order that at all the planar singularities S_1, Δ_1/a_1^2 has the form $m^2/(n+m)^2$ with $m, n \in \mathbb{Z}\setminus\{0\}$ coprime, it follows from the expressions of Δ_1 and a_1 that Δ_1 should be a square of $\lambda_{12}\lambda_{15}x_1^2 + \lambda_{24}\lambda_{45}x_4^2$ or of $\lambda_{13}\lambda_{16}x_1^2 + \lambda_{34}\lambda_{46}x_4^2$, without loss of generality we set

$$\lambda_{13}\lambda_{16}x_1^2 + \lambda_{34}\lambda_{46}x_4^2 = L^2\left(\lambda_{12}\lambda_{15}x_1^2 + \lambda_{24}\lambda_{45}x_4^2\right),$$

and that $a_1/\left(\lambda_{12}\lambda_{15}x_1^2 + \lambda_{24}\lambda_{45}x_4^2\right)$ is a constant. Set

$$a_1 = K\left(\lambda_{12}\lambda_{15}x_1^2 + \lambda_{24}\lambda_{45}x_4^2\right).$$

Then we must have

$$\lambda_{13}\lambda_{16} = \lambda_{12}\lambda_{15}L^2, \tag{14}$$

$$\lambda_{34}\lambda_{46} = \lambda_{24}\lambda_{45}L^2, \tag{15}$$

$$-\lambda_{12}\lambda_{13} - \lambda_{15}\lambda_{16} = \lambda_{12}\lambda_{15}K, \tag{16}$$

$$\lambda_{24}\lambda_{46} + \lambda_{34}\lambda_{45} = \lambda_{24}\lambda_{45}K, \tag{17}$$

where $L/K = m/(n+m)$.

If $\lambda_5 = \lambda_6$, equations (14) and (15) have two solutions $\lambda_2 = (L^2\lambda_1 + \lambda_{31})/L^2$ and $\lambda_6 = \lambda_4$, and $\lambda_2 = (L^2\lambda_4 + \lambda_{34})/L^2$ and $\lambda_6 = \lambda_1$. We can check easily that these conditions are inside the Manakov case. This shows that under the condition $\lambda_5 = \lambda_6$ the Euler equations are either inside the Manakov condition or have at most four functionally independent polynomial first integrals.

If $\lambda_5 \neq \lambda_6$, equations (14) and (15) has a unique solution

$$\lambda_2 = -\frac{\lambda_{16}\lambda_{46} - L^2(\lambda_1\lambda_{46} - \lambda_6\lambda_{45})}{L^2\lambda_{56}},$$

$$\lambda_3 = -\frac{\lambda_1\lambda_{45} - \lambda_5\lambda_{46} - L^2\lambda_{15}\lambda_{45}}{\lambda_{56}}.$$

We can check that this condition is also inside the Manakov one. Hence in the case *(ii)* we have proved that the Euler equations are either inside the Manakov condition or have at most four functionally independent polynomial first integrals. This completes the proof of Theorem 2.

4. Proof of Theorem 3

First we recall some basic results on the quasi–homogeneous polynomial differential systems that we shall need later on in the proof of Theorem 3.

We say that the polynomial differential system (1) is *quasi–homogeneous* if there exist $\mathbf{s} = (s_1, \cdots, s_n) \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$ such that for arbitrary $\alpha \in \mathbb{R}^+ = \{a \in \mathbb{R},\, a > 0\}$,

$$P_i(\alpha^{s_1}x_1, \cdots, \alpha^{s_n}x_n) = \alpha^{s_i - 1 + d}P_i(x_1, \cdots, x_n),$$

for $i = 1, \ldots, n$. We call $\mathbf{s} = (s_1, \cdots, s_n)$ the *weight exponent* of system (1), and d the *weight degree* with respect to the weight exponent $\mathbf{s}$. In the particular case that $\mathbf{s} = (1, \cdots, 1)$ system (1) is the classical *homogeneous polynomial differential system of degree d.*

We remark that if system (1) is quasi–homogeneous with weight exponent $\mathbf{s}$ and weight degree $d > 1$, then the system is invariant under the change of variables $x_i \to \alpha^{w_i}x_i$, $t \to \alpha^{-1}t$, where $w_i = s_i/(d-1)$.

Recently the integrability of quasi–homogeneous polynomial differential systems have been investigated by several authors. Probably the best results have been provided by Yoshida [17, 18, 19], Furta [4] and Goriely [6, 7], see also Gonzalez–Gascon [5], Tsygvintsev [15] and Llibre and Zhang [8].

Given an analytic function F we can split it in the form $F = \sum_i F_i$, where F_i is a *quasi–homogeneous polynomial of weight degree* i *with respect to the weight exponent* $\mathbf{s}$, i.e.

$$F_i(\alpha^{s_1}x_1, \dots, \alpha^{s_n}x_n) = \alpha^i F_i(x_1, \dots, x_n) \quad \text{for all } \alpha \in \mathbb{R}^+.$$

The following result is well known, see for instance Proposition 1 of [8].

Proposition 9. *Let F be an analytic function and let $F = \sum_i F_i$ be its decomposition into weight–homogeneous polynomials of weight degree i with respect to the weight exponent $\mathbf{s}$. Then F is an analytic first integral of the quasi–homogeneous polynomial differential system* (1) *with weight exponent $\mathbf{s}$ if and only if each weight–homogeneous part F_i is a first integral of system* (1) *for all i.*

Suppose that system (1) is a quasi–homogeneous polynomial differential system of weight degree d with respect to the weight exponent $\mathbf{s}$. Then we define $\mathbf{w} = \mathbf{s}/(d-1)$. The interest for the quasi–homogeneous polynomial differential systems is based in the existence of the particular solutions of the form

$$(x_1(t), \dots, x_n(t)) = \left(c_1 t^{-w_1}, \dots, c_n t^{-w_n}\right),$$

where the coefficients $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{C}^n \backslash \{\mathbf{0}\}$ are given by the polynomial system of equations

$$P_i(c_1, \dots, c_n) + w_i c_i = 0 \quad \text{for} \quad i = 1, \dots, n. \tag{18}$$

For a given $(w_1, \dots, w_n)$ there may exist different $\mathbf{c}$'s, called the *balances*.

For each balance $\mathbf{c}$ we introduce a matrix

$$K(\mathbf{c}) = D\mathbf{P}(\mathbf{c}) + \operatorname{diag}(w_1, \dots, w_n), \tag{19}$$

where as usual $D\mathbf{P}(\mathbf{c})$ denotes the differential of $\mathbf{P}$ evaluated at $\mathbf{c}$, and $\operatorname{diag}(w_1, \dots, w_n)$ denotes the matrix whose diagonal is equal to $(w_1, \dots, w_n)$ and zeros in the rest.

The eigenvalues of $K(\mathbf{c})$ are called the *Kowalevsky exponents* of the balance $\mathbf{c}$. Sophia Kowalevskaya was the first to introduce the matrix K to compute the Laurent series solutions of the rigid body motion. It can be shown that there always exists a Kowalevsky exponent equal to -1 related to the arbitrariness of the origin of the parametrization of the solution by the time. The eigenvector associated to the eigenvalue -1 is $(w_1c_1, \dots, w_nc_n)$, for more details see [17] or [4].

Probably the best results in order to know if a weighted homogeneous polynomial of weight degree d with respect to the weight exponent $\mathbf{s}$ is a first integral of a quasi–homogeneous polynomial differential system (1) with weight degree m with respect to the weight exponent $\mathbf{s}$ is essentially due to Yoshida [17], and are the following two theorems.

Theorem 10. *Let $H(x_1, \dots, x_n)$ be a weighted homogeneous polynomial first integral of weight degree m with respect to the weight exponent $\mathbf{s}$ of the* *quasi–homogeneous polynomial differential system* (1) *with weight degree* d *with respect to the weight exponent* $\mathbf{s}$. *Suppose the gradient of* H *evaluated at a balance* $\mathbf{c}$ *is finite and not identically zero. Then* $m/(d-1)$ *is a Kowalevsky exponent of the balance* $\mathbf{c}$.

Theorem 11. *Let* r *be a positive integer such that* $1 < r < n$, *and let* $H_k(x_1, \ldots, x_n)$ *for* $k = 1, \ldots, r$ *be weighted homogeneous polynomial first integrals of weight degree* m *with respect to the weight exponent* $\mathbf{s}$ *of the quasi–homogeneous polynomial differential system* (1) *with weight degree* d *with respect to the weight exponent* $\mathbf{s}$. *Suppose that the gradients of* H_k *for* $k = 1, \ldots, r$ *evaluated at a balance* $\mathbf{c}$ *are finite, not identically zero and linearly independent. Then* $m/(d-1)$ *is a Kowalevsky exponent of the balance* $\mathbf{c}$ *with multiplicity at least* r.

As we mentioned after Theorem 3 that in the Manakov case the gradient of the first integral H_4 is linearly dependent on the gradients of H_1, H_2 and H_3 at each of the balances C_{ij} for $i, j \in \{1, \ldots, 4\}$. So H_4 does not satisfy the condition of Theorem 11. In fact, direct computations show that the Kowalevsky exponents at all the 16 balances are $(-1, 2, 2, 2, 1, 0)$ in the Manakov case. This Manakov case shows that the linearly independent condition in Theorem 11 cannot be erased.

Proof of Theorem 3. For the Euler equations we take the weight exponent $\mathbf{s} = (1, 1, 1, 1, 1, 1)$. Then the Euler equations have a weight homogenous polynomial differential system of weight degree 2 with the exponent $\mathbf{s}$.

If $\lambda_{12}\lambda_{13}\lambda_{15}\lambda_{16}\lambda_{23}\lambda_{24}\lambda_{26}\lambda_{34}\lambda_{35}\lambda_{45}\lambda_{46}\lambda_{56} \neq 0$, with the help of the algebraic manipulator Mathematica we compute the balances $C = (c_1, \ldots, c_6)$ of the Euler equations and we get the following 16 balances:

$$
\begin{aligned}
C_1 &= \left(\frac{\delta\lambda_{56}}{\sqrt{\lambda_{15}\lambda_{56}}}\frac{\gamma}{\sqrt{\lambda_{61}\lambda_{56}}}, 0, 0, 0, \frac{\gamma}{\sqrt{\lambda_{15}\lambda_{56}}}, \frac{\delta}{\sqrt{\lambda_{61}\lambda_{56}}}\right),\\
C_2 &= \left(0, 0, \frac{\gamma}{\sqrt{\lambda_{53}\lambda_{34}}}, \frac{\gamma\lambda_{53}}{\sqrt{\lambda_{53}\lambda_{34}}}\frac{\delta}{\sqrt{\lambda_{53}\lambda_{45}}}, \frac{\delta}{\sqrt{\lambda_{53}\lambda_{45}}}, 0\right),\\
C_3 &= \left(0, \frac{\gamma}{\sqrt{\lambda_{62}\lambda_{24}}}, 0, \frac{\gamma\lambda_{26}}{\sqrt{\lambda_{62}\lambda_{24}}}\frac{\delta}{\sqrt{\lambda_{62}\lambda_{46}}}, 0, \frac{\delta}{\sqrt{\lambda_{62}\lambda_{46}}}\right),\\
C_4 &= \left(\frac{\delta\lambda_{23}}{\sqrt{\lambda_{23}\lambda_{12}}}\frac{\gamma}{\sqrt{\lambda_{31}\lambda_{23}}}, \frac{\delta}{\sqrt{\lambda_{23}\lambda_{12}}}, \frac{\gamma}{\sqrt{\lambda_{31}\lambda_{23}}}, 0, 0, 0\right),
\end{aligned}
\tag{20}
$$

where $\delta, \gamma \in \{1, -1\}$. We mention that each C_i contains four balances for $(\delta, \gamma) = (1, 1)$, $(1, -1)$, $(-1, 1)$, $(-1, -1)$, denoted by $C_{i1}, C_{i2}, C_{i3}, C_{i4}$ for $i = 1, 2, 3, 4$, respectively. Also these balances are complex numbers.

Direct calculations using Mathematica show that the Kowalevsky exponents associated to the four balances C_1 are equal and given by

$$
\left(-1, 2, 2, 2, \frac{1-\sqrt{A_1}}{2}, \frac{1+\sqrt{A_1}}{2}\right),
$$

with

$$A_1 = \frac{K_1\lambda_{56} + 4(\lambda_1\lambda_2 + \lambda_3\lambda_6)\lambda_{45} - 4(\lambda_1\lambda_3 + \lambda_2\lambda_5)\lambda_{46}}{\lambda_{15}\lambda_{16}\lambda_{56}},$$

where $K_1 = (\lambda_1\lambda_{15} + \lambda_6\lambda_{51} + 4(\lambda_1\lambda_4 + \lambda_2\lambda_3))$. The Kowalevsky exponents associated to C_2 are equal and given by

$$\left(-1, 2, 2, 2, \frac{1-\sqrt{A_2}}{2}, \frac{1+\sqrt{A_2}}{2}\right),$$

with

$$A_2 = \frac{K_2\lambda_{45} + 4(\lambda_1\lambda_6 + \lambda_2\lambda_5)\lambda_{34} - 4(\lambda_1\lambda_4 + \lambda_2\lambda_6)\lambda_{35}}{\lambda_{34}\lambda_{35}\lambda_{45}},$$

where $K_2 = (\lambda_3\lambda_{34} + \lambda_5\lambda_{43} + 4(\lambda_1\lambda_2 + \lambda_3\lambda_6))$. The Kowalevsky exponents associated to C_3 are equal and given by

$$\left(-1, 2, 2, 2, \frac{1-\sqrt{A_3}}{2}, \frac{1+\sqrt{A_3}}{2}\right),$$

with

$$A_3 = \frac{K_3\lambda_{46} - 4(\lambda_1\lambda_2 + \lambda_3\lambda_6)\lambda_{45} - 4(\lambda_1\lambda_4 + \lambda_2\lambda_3)\lambda_{56}}{\lambda_{24}\lambda_{26}\lambda_{46}},$$

where $K_3 = (\lambda_2\lambda_{24} + \lambda_6\lambda_{42} + 4(\lambda_1\lambda_3 + \lambda_2\lambda_5))$. The Kowalevsky exponents associated to C_4 are equal and given by

$$\left(-1, 2, 2, 2, \frac{1-\sqrt{A_4}}{2}, \frac{1+\sqrt{A_4}}{2}\right),$$

with

$$A_4 = \frac{K_4\lambda_{23} - 4(\lambda_1\lambda_5 + \lambda_3\lambda_6)\lambda_{24} + 4(\lambda_1\lambda_6 + \lambda_2\lambda_5)\lambda_{34}}{\lambda_{12}\lambda_{13}\lambda_{23}},$$

where $K_4 = (\lambda_1\lambda_{12} + \lambda_3\lambda_{21} + 4(\lambda_1\lambda_4 + \lambda_5\lambda_6))$. Furthermore, we can prove that

$$A_k = 1 + \frac{4M}{L_k}, \qquad \text{for } k = 1, 2, 3, 4,$$

where M was defined in (3) for the Manakov condition and the L_k's are defined in the statement of Theorem 3.

Assume that the Euler equations have a polynomial first integral H of degree m which is functionally independent of the first integrals H_1, H_2 and H_3.

Suppose that $m > 2$. If the gradient of H at some balance C_{ij}, $i, j \in \{1, 2, 3, 4\}$, is not identically zero, then we get from Theorem 10 that the degree m of H must be equal to the Kowalevsky exponent $\left(1 \pm \sqrt{A_k}\right)/2$. This forces that $A_k = (2m-1)^2$, and consequently we have $M/L_k = m^2 - m$. This proves statement (a).

Suppose now that $m = 2$. If the gradient of H at some balance C_{ij}, $i, j \in \{1, 2, 3, 4\}$, is linearly independent of the gradients of H_1, H_2 and H_3, then we get from Theorem 11 that the Kowalevsky exponent 2 must be of multiplicity 4. Hence we must have $\left(1 + \sqrt{A_k}\right)/2 = 2$. This forces that

$M/L_k = 2$. This proves statement (b). We complete the proof of Theorem 3. □

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