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A NOTE ON THE DZIOBECK CENTRAL CONFIGURATIONS

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Abstract. For the Newtonian n–body problem in $\mathbb{R}^{n-2}$ with $n \geq 3$ we prove that the following two statements are equivalent.

(a) Let x be a Dziobek central configuration having one mass located at the center of mass.

(b) Let x be a central configurations formed by $n-1$ equal masses located at the vertices of a regular $(n-2)$–simplex together with an arbitrary mass located at its barycenter.

1. Introduction and statement of the main results

The main problem of the classical Celestial Mechanics is the *n-body problem*; i.e. the description of the motion of n particles of positive masses under their mutual Newtonian gravitational forces. This problem is completely solved only when $n = 2$, and for $n > 2$ there are only few partial results.

Consider the Newtonian n–body problem in the d–dimensional space $\mathbb{R}^d$, i.e.

$$\ddot{x}_i = \sum_{j=1,\ j\neq i}^{n} \frac{m_j(x_j - x_i)}{r_{ij}^3}, \quad \text{for} \quad i = 1, \ldots, n.$$

Here m_i are the masses of the bodies, $x_i \in \mathbb{R}^d$ are their positions, and $r_{ij} = |x_i - x_j|$ are their mutual distances. The vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^{nd}$ will be called the *configuration* of the system. The differential equations are well–defined if the configuration is of non–collision type, i.e, $r_{ij} \neq 0$ when $i \neq j$. The dimension of any non–collision configuration of $n \geq 2$ bodies satisfies $1 \leq \delta(x) \leq n-1$.

We define the *dimension* $\delta(x)$ *of a configuration* x to be the dimension of the smallest affine subspace of $\mathbb{R}^d$ which contains all of the points x_i. As usual configurations with $\delta(x) = 1, 2, 3$ will be called *collinear, planar* and *spatial*, respectively.

The *total mass* and the *center of mass* of the n bodies are

$$M = m_1 + \ldots + m_n, \qquad c = \frac{1}{M}\left(m_1x_1 + \cdots + m_nx_n\right),$$

respectively. A configuration x is a *central configuration* if the acceleration vectors of the bodies satisfy

$$\sum_{j=1,\ j\neq i}^{n} \frac{m_j(x_j - x_i)}{r_{ij}^3} + \lambda(x_i - c) = 0, \quad \text{for} \quad i = 1, \ldots, n, \tag{1}$$

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Central configurations started to be studied in the second part of the 18th century, there is an extensive literature concerning these solutions. For a classical background, see the sections on central configurations in the books of Wintner [17] and Hagihara [6]. For a modern background see, for instance, the papers of Albouy and Chenciner [2], Albouy and Kaloshin [3], Hampton and Moeckel [7], Moeckel [9], Palmore [13], Saari [14], Schmidt [15], Xia [18], ... One of the reasons why central configurations are important is that they allow to obtain the unique explicit solutions in function of the time of the n–body problem known until now, the *homographic solutions* for which the ratios of the mutual distances between the bodies remain constant. They are also important because the total collision or the total parabolic escape at infinity in the n–body problem is asymptotic to central configurations, see for more details Dziobek [5] and [14]. Also if we fix the total energy h and the angular momentum c of the n–body problem, then some of the bifurcation points (h, c) for the topology of the level sets with energy h and angular momentum c are related with the central configurations, see Meyer [11] and Smale [16] for a full background on these topics.

Moulton [12] proved that for a fixed mass vector $m = (m_1, \ldots, m_n)$ and a fixed ordering of the bodies along the line, there exists a unique collinear central configuration, up to translation and scaling.

At the other extreme of the dimension range, Lagrange [8] showed that for $n = 3$, the only central configuration x with $\delta(x) = n-1 = 2$ is the equilateral triangle, and it is central for all choices of the masses. An analogous result of Lagrange‘s result holds for all n. Thus, it is well known that for $n \geq 3$ the only central configuration x with $\delta(x) = n - 1$ of the n–body problem is formed by the vertices of a regular $(n-1)$–simplex, which is central for all choices of the masses. Of course, a 1–simplex is a closed interval, a 2–simplex is a closed equilateral triangle, a 3–simplex is a closed regular tetrahedron, and so on.

For other values of the dimension $\delta(x)$ the problem of finding or even counting the central configurations x of the n–body problem is very difficult, see for instance Moeckel [10]. The dimension $d(x) = 2$ is of course the most interesting of all, because planar central configurations give rise to physically realistic periodic orbits. For $n = 4$, Dziobek [5] formulated the planar central configuration problem in terms of mutual distances r_{ij} and obtained algebraic equations characterizing the central configurations. His approach has been adopted and developed by Albouy [1] in his study of the central configurations with four equal masses, see also Albouy and Llibre [4]. The natural generalization to higher n is the case $\delta(x) = n-2$. Following [1] and [10] we call such central configurations *Dziobek configurations.*

The goal of this paper is to prove the following result.

Theorem 1. *The following two statements are equivalent for the n–body problem with $n \geq 3$.*

(a) *Let x be a central configuration with $\delta(x) = n - 2$ (i.e. a Dziobek central configurations) having one mass located at the center of mass.*
(b) *Let x be a central configurations formed by $n-1$ equal masses located at the vertices of a regular $(n-2)$–simplex together with an arbitrary mass located at its barycenter.*

Of course, statement (b) of Theorem 1 implies immediately statement (a). The converse implication is proved in section 3.

If the central configuration is not Dziobek, then the equivalence of Theorem 1 does not hold. Thus, for instance, consider a regular polygon with $n-1>3$ equal masses located at its vertices and an arbitrary mass located at its barycenter. This is a non–Dziobek central configuration having one mass located at the center of mass, and since $n>4$ it is different from the configuration formed by the vertices of a regular $(n-2)$–simplex together with its barycenter.

2. Equations for the Dziobek central configurations

Since we want to study the Dziobek central configurations we consider the n–body problem in $\mathbb{R}^{n-2}$. To each configuration $x=(x_1,\ldots,x_n)\in\mathbb{R}^{n(n-2)}$ we associate the $n\times n$ matrix

$$X=\begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ 0 & \cdots & 0 \end{pmatrix}.$$

Let X_k be the $(n-1)\times(n-1)$ matrix obtained delating from the matrix X its k–th column and its last row. Then define $\Delta_k=(-1)^{k+1}\det(X_k)$ for $k=1,\ldots,n$.

Dziobek [5] (see also equations (8) and (16) of Moeckel [10]) reduces the equations for the central configurations (1) of the n–body problem to the following system of

$$N=\frac{n(n-1)}{2}+n+2$$

equations and N unknowns:

$$\begin{aligned} &\frac{1}{r_{ij}^3}=c_1+c_2\frac{\Delta_i\Delta_j}{m_im_j}, \\ &t_i-t_j=0, \end{aligned} \tag{2}$$

for $1\le i<j\le n$, with

$$t_i=\sum_{j=1,\,j\neq i}^{n}\Delta_j\,r_{ij}^2.$$

The N unknowns in equations (2) are the $n(n-1)/2$ mutual distances r_{ij}, the n variables Δ_i, and the two constants c_k.

3. Proof of Theorem 1

In order to complete the proof of Theorem 1, we must prove that statement (a) of that theorem implies statement (b).

We assume that $x=(x_1,\ldots,x_n)\in\mathbb{R}^{n(n-2)}$ is a Dziobek central configuration having one mass located at the center of masses. Without loss of generality we can suppose:

(i) the center of mass is at the origin of coordinates;
(ii) the mass m_n is located at the center of mass, i.e. $x_n=0$;
(ii) the unit of mass it taken in such a way that $m_{n-1}=1$. Then

$$x_{n-1}=-\sum_{i=1}^{n-2}m_ix_i.$$

Since x is a Dziobek central configuration it satisfies the equations (2). Easy computations using the properties of the determinants show that

$$(3)\qquad \begin{aligned} \frac{\Delta_i \Delta_j}{m_i m_j} &= \det(x_1 \cdots x_{n-2})^2 \quad \text{for} \quad 1 \le i < j \le n-1, \\ \frac{\Delta_i \Delta_n}{m_i m_n} &= \frac{M - m_n}{m_n} \det(x_1 \cdots x_{n-2})^2 \quad \text{for} \quad i = 1, \ldots, n-1. \end{aligned}$$

Then, the first equations of (2) become

$$(4)\qquad \begin{aligned} \frac{1}{r_{ij}^3} &= c_1 + c_2 \det(x_1 \cdots x_{n-2})^2 \quad \text{for} \quad 1 \le i < j \le n-1, \\ \frac{1}{r_{in}^3} &= c_1 + c_2 \frac{M - m_n}{m_n} \det(x_1 \cdots x_{n-2})^2 \quad \text{for} \quad i = 1, \ldots, n-1. \end{aligned}$$

The equations $t_i = t_j$ of (2) are trivially satisfied by direct computations, but they are not relevant in this proof.

From equations (4) we obtain that

$$(5)\qquad \begin{aligned} r_{ij} &= k_1 \quad \text{for} \quad 1 \le i < j \le n-1, \\ r_{in} &= k_2 \quad \text{for} \quad i = 1, \ldots, n-1, \end{aligned}$$

where k_1 and k_2 are constants. Therefore, from the first equations of (5) it follows that the masses m_k for $k = 1, \ldots, n-1$ are at the vertices of a regular $(n-2)$–simplex, and from the second equations of (5) we obtain that the mass m_n is at the barycenter of this regular $(n-2)$–simplex. Moreover, since the barycenter must be the center of mass of the $n-1$ masses located at the vertices of a regular $(n-2)$–simplex, this forces that these $n-1$ masses must be equal. Of course, the mass m_n located at the barycenter is arbitrary. This completes the proof of the theorem.

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