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ON THE BIRTH OF LIMIT CYCLES FOR NON–SMOOTH DYNAMICAL SYSTEMS

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ABSTRACT. The main objective of this work is to develop, via Brower degree theory and regularization theory, a variation of the classical averaging method for detecting limit cycles of certain piecewise continuous dynamical systems. In fact, overall results are presented to ensure the existence of limit cycles of such systems. These results may represent new insights in averaging, in particular its relation with non smooth dynamical systems theory. An application is presented in careful detail.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The discontinuous differential systems, i.e. differential equations with discontinuous right–hand sides, is a subject that has been developed very fast these last years. It has become certainly one of the common frontiers between Mathematics, Physics and Engineering. Thus certain phenomena in control systems [2], impact and friction mechanics [6], nonlinear oscillations [1, 22], economics [13, 16], and biology [3, 18], are the main sources of motivation of their study, see for more details Teixeira [27]. A recent review appears in [30].

The knowledge of the existence or not of periodic solutions is very important for understanding the dynamics of differential systems. One of good tools for study the periodic solutions is the averaging theory, see for instance the books of Sanders and Verhulst [25] and Verhulst [29]. We point out that the method of averaging is a classical and matured tool that provides a useful means to study the behaviour of nonlinear smooth dynamical systems. The method of averaging has a long history that starts with the classical works of Lagrange and Laplace who provided an intuitive justification of the process. The first formalization of this procedure was given by Fatou in 1928 [10]. Very important practical and theoretical contributions in the averaging theory were made by Krylov and Bogoliubov [5] in the 1930s and Bogoliubov [4] in 1945. The principle of averaging has been extended in many directions for both finite- and infinite-dimensional differentiable systems. The classical results for studying the periodic orbits of differential systems need at least that those systems be of class $\mathcal{C}^2$. Recently Buica and Llibre [8] extended the averaging theory for studying periodic orbits to continuous differential systems using mainly the Brouwer degree theory.

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The main objective of this paper is to extend the averaging theory for studying periodic orbits to discontinuous differential systems using again the Brouwer degree.

Let D be an open subset of $\mathbb{R}^n$. We shall denote the points of $\mathbb{R}\times D$ as (t,x), and we shall call the variable t as the time. Let $h:\mathbb{R}\times D\to\mathbb{R}$ be a $\mathcal{C}^1$ function having the $0\in\mathbb{R}$ as a regular value, and let $\Sigma=h^{-1}(0)$. Given $p\in\Sigma$ we denote its connected component in Σ by Σ_p.

Let $X,Y:\mathbb{R}\times D\to\mathbb{R}^n$ be two continuous vector fields. Assume that the functions h, X and Y are T–periodic in the variable t. Now we define a *discontinuous piecewise differential* system

$$x'(t)=Z(t,x)=\begin{cases}X(t,x) & \text{if} \quad h(t,x)>0,\\ Y(t,x) & \text{if} \quad h(t,x)<0.\end{cases} \tag{1}$$

We concisely denote $Z=(X,Y)_h$.

Here we deal with a different formulation for the discontinuous differential system (1). Let $\text{sign}(u)$ be the sign function defined in $R\setminus\{0\}$ as

$$\text{sign}(u)=\begin{cases}1 & \text{if} \quad u>0,\\ -1 & \text{if} \quad u<0.\end{cases}$$

Then the discontinuous differential system (1) can be written using the function $\text{sign}(u)$ as

$$x'(t)=Z(t,x)=F_1(t,x)+\text{sign}(h(t,x))F_2(t,x), \tag{2}$$

where

$$F_1(t,x)=\frac{1}{2}\left(X(t,x)+Y(t,x)\right) \quad \text{and} \quad F_2(t,x)=\frac{1}{2}\left(X(t,x)-Y(t,x)\right).$$

To work with the discontinuous differential system (2) we should introduce the regularization process, where the discontinuous vector field $Z(t,x)$ is approximated by an one–parameter family of continuous vector fields $Z_\delta(t,x)$ such that $\lim_{\delta\to 0}Z_\delta=Z(t,x)$.

In [26] Sotomayor and Teixeira introduced a regularization for the discontinuous vector fields in $\mathbb{R}^2$ having a line of discontinuity and, using this technique, they proved generically that its regularization provides the same extension of the orbits through the line of discontinuity that the one given by the Filippov's rules, see [11]. Later on Llibre and Teixeira [24] studied the regularization of generic discontinuous vector fields in $\mathbb{R}^3$ having a surface of discontinuity, and proved that $\lim_{\delta\to 0}Z_\delta$ essentially agrees with Filippov's convention in dimension three. Finally, in [27] Teixeira generalized the regularization procedure to finite dimensional discontinuous vector fields.

In [21] Llibre, da Silva and Teixeira studied singular perturbations problems in dimension three which are approximations of discontinuous vector fields proving that the regularization process developed in [24] produces a singular problem for which the discontinuous set is a center manifold, moreover, they proved that the definition of sliding vector field coincides with the reduced problem of the corresponding singular problem for a class of vector fields.

In general, a transition function is used in these regularizations to average the vector fields X and Y on the set of discontinuity in order to get a family of continuous vector fields that approximates the discontinuous one.

A continuous function $\phi : \mathbb{R} \to \mathbb{R}$ is a *transition function* if $\phi(u) = -1$ for $u \leq -1$, $\phi(u) = 1$ for $x \geq 1$ and $\phi'(u) > 0$ if $u \in (-1, 1)$. The ϕ–regularization of $Z = (X, Y)_h$ is the one–parameter family of continuous functions Z_δ with $\delta \in (0, 1]$ given by

$$Z_\delta(t, x) = \frac{1}{2}\left(X(t, x) + Y(t, x)\right) + \frac{1}{2}\phi_\delta(h(t, x))\left(X(t, x) - Y(t, x)\right),$$

with

$$\phi_\delta(u) = \phi\left(\frac{u}{\delta}\right). \tag{3}$$

Note that for all $(t, x) \in (\mathbb{R} \times D)\backslash\Sigma$ we have that $\lim_{\delta\to 0} Z_\delta(t, x) = Z(t, x)$.

The formulation (2) of the discontinuous differential system (1) admits a natural regularization. Define the transition function ϕ as

$$\phi(u) = \begin{cases} 1 & \text{if} \quad u \geq 1, \\ u & \text{if} \quad -1 < u < 1, \\ -1 & \text{if} \quad u \leq -1. \end{cases} \tag{4}$$

Let $\phi_\delta : \mathbb{R} \to \mathbb{R}$ be the continuous function defined in (3). It is clear that

$$\lim_{\delta\to 0} \phi_\delta(u) = \operatorname{sign}(u), \tag{5}$$

and

$$Z_\delta(t, z) = F_1(t, x) + \phi_\delta(h(t, x))F_2(t, x),$$

is the *ϕ–regularization* of the discontinuous differential system (1).

As usual ∇h denotes the gradient of the function h, $\partial_x h$ denotes the gradient of the function h restricted to the variable x and $\partial_t h$ denotes the partial derivative of the function h with respect to the variable t.

Our main results are given in the next theorems. Its proof uses the theory of Brouwer degree for finite dimensional spaces (see the appendix A for a definition of the Brouwer degree $d_B(f, V, 0)$), and is based on the averaging theory for non–smooth differential system stated by Buica and Llibre [8] (see Appendix B).

Theorem A. *We consider the following discontinuous differential system*

$$x'(t) = \varepsilon F(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{6}$$

with

$$F(t, x) = F_1(t, x) + \operatorname{sign}(h(t, x))F_2(t, x),$$
$$R(t, x, \varepsilon) = R_1(t, x, \varepsilon) + \operatorname{sign}(h(t, x))R_2(t, x, \varepsilon),$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $R_1, R_2 : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ *and* $h : \mathbb{R} \times D \to \mathbb{R}$ *are continuous functions, T–periodic in the variable t and D is an open subset of* $\mathbb{R}^n$. *We also suppose that h is a $\mathcal{C}^1$ function having 0 as a regular value.*

Define the averaged function $f : D \to \mathbb{R}^n$ *as*

$$f(x) = \int_0^T F(t, x)dt. \tag{7}$$

We assume the following conditions.

(*i*) F_1, F_2, R_1, R_2 *and h are locally Lipschitz with respect to x;*

(*ii*) *there exists an open bounded set $C \subset D$ such that, for $|\varepsilon| > 0$ sufficiently small, every orbit starting in C reaches the set of discontinuity only at its crossing regions (crossing hypothesis).*

(*iii*) *for $a \in C$ with $f(a) = 0$, there exist a neighbourhood $U \subset C$ of a such that $f(z) \neq 0$ for all $z \in \overline{U} \backslash \{a\}$ and $d_B(f, U, 0) \neq 0$.*

Then, for $|\varepsilon| > 0$ sufficiently small, there exists a T–periodic solution $x(t, \varepsilon)$ of system (6) such that $x(0, \varepsilon) \to a$ as $\varepsilon \to 0$.

Theorem A is proved in section 2.

In order to stablish a theorem with weaker hypotheses we denote by D_0 the set of points $z \in D$ such that the map $h_z : t \in [0, T] \mapsto h(t, z)$ has only isolated zeros. Clearly $\mathrm{Int}(D_0) \neq \emptyset$.

Theorem B. *In addition to the assumptions of Theorem* (A) *unless condition* (*ii*) *we assume the following hypothesis.*

(*ii′*) *there exists an open bounded set $C \subset D_0$ such that, for $\varepsilon > 0$ sufficiently small, every orbit starting in C reaches the set of discontinuity only at its crossing regions.*

Then, for $\varepsilon > 0$ sufficiently small, there exists a T–periodic solution $x(t, \varepsilon)$ of system (6) *such that $x(0, \varepsilon) \to a$ as $\varepsilon \to 0$.*

Theorem B is proved in section 2.

Remark 1. *Assuming the hypotheses of Theorem A, we have that for $\varepsilon = 0$ its solutions starting in C, i.e. straight lines $\{(t, z) : t \in \mathbb{R}\}$ for $z \in C$, reaches the set of discontinuity only at its crossing region. This fact is not necessarily true when we assume the hypotheses of Theorem B, because the crossing hypothesis holds only for $\varepsilon > 0$. This is the main difference between Theorems A and B. Nevertheless we shall see that to prove both theorems we just have to guarantee that the map $h_z : t \in [0, T] \mapsto h(t, z)$ for $z \in C$ has only isolated zeros. After that the proof follows similarly for both theorems.*

Proposition 2. *Assume that $\partial_t h(t, x) \neq 0$ for each $(t, x) \in \Sigma$. Then hypothesis* (*ii*) *holds.*

Proposition 3. *Assume that for each $p \in \Sigma$ such that $\partial_t h(p) = 0$ there exists a continuous positive function $\xi_p : \mathbb{R} \times D \to \mathbb{R}$ for which the inequality*

$$\left(\partial_t h \langle \partial_x h, F_1 \rangle + \varepsilon \frac{\langle \partial_x h, F_1 \rangle^2 - \langle \partial_x h, F_2 \rangle^2}{2} \right) (t, x) \geq \varepsilon \xi_p(t, x)$$

holds for each $(t, x) \in \Sigma_p$. Then hypothesis (*ii′*) *holds.*

It is worthwile to say that the averaging theory appears as a very useful tool in discontinuous dynamical systems. For example, in [23], lower bounds for the maximum number of limit cycles for the m–piecewise discontinuous polynomial differential equations was provided using the averaging theory. In [9] the averaging theory was used to study the bifurcation of limit cycles from discontinuous perturbations of two and four dimensional linear center in $\mathbb{R}^n$. Also, in [20], the averaging theory was applied to study the number of limit cycles of the discontinuous piecewise linear differential systems in $\mathbb{R}^{2n}$ with two zones separated by a hyperplane.

In Theorems A and B we have extended to general discontinuous differential systems the ideas used in the previous mentioned papers for particular discontinuous differential systems.

Now an application of Theorem A to a class of discontinuous piecewise linear differential systems is given. Such systems have been studied recently by Han and

Zhang [12], and Huan and Yuang [14], among other papers. In [12] some results about the existence of two limit cycles appeared, so that the authors conjectured that the maximum number of limit cycles for this class of piecewise linear differential systems is exactly two. This conjecture is analogous to Conjecture 1 in the discussion of Tonnelier in [28]. However, by considering a specific family of discontinuous PWL differential systems with two linear zones sharing the equilibrium position, in [14] strong numerical evidence about the existence of three limit cycles was obtained, and a proof was provided by Llibre and Ponce [19]. This example represents up to now the first discontinuous piecewise linear differential system with two zones and 3 limit cycles surrounding a unique equilibrium. Now we shall provide a new proof of the existence of these three limit cycles through Theorem A.

In polar coordinates (r,θ) given by $x = r\cos\theta$ and $y = r\sin\theta$, the planar discontinuous piecewise linear differential system with two zones separated by a straight line corresponding to the system studied in the paper [19] is

$$\frac{dr}{d\theta} = F(\theta,r) = \begin{cases} \varepsilon \dfrac{19}{50}\, r & \text{if } r\cos\theta \geq 1, \\ \varepsilon \dfrac{2300\cos(2\theta) - 4623\sin(2\theta) - 300}{1500}\, r & \text{if } r\cos\theta < 1, \end{cases} \tag{8}$$

where we have multiplied the right hand side of the system of [19] by the small parameter ε. Our main contribution in this application is to provide the explicit analytic equations defining the limit cycles of the discontinuous piecewise linear differential system with two zones (8).

Theorem 4. *Any limit cycle of the discontinuous piecewise linear differential system with two zones* (8) *which intersects the straight line $x = 1$ in two points (r_0,θ_0) and (r_1,θ_1) with $-\pi/2 < \theta_0 < 0 < \theta_1 < \pi/2$, $r_k\cos\theta_k = 1$ for $k = 0,1$, and $r_0 > 1$ and θ_1 must satisfy the following two equations*

$$\begin{aligned} &\exp\left(\frac{19(\theta_1-\theta_0)}{50}\right) r_0\cos\theta_1 - 1 = 0, \\ &\frac{19(\theta_1-\theta_0)}{50} + \frac{1}{5}\arctan\left(\frac{1}{15}\sec\theta_0(23\cos\theta_0 - 100\sin\theta_0)\right) \\ &\quad -\frac{1}{5}\arctan\left(\frac{1}{15}\sec\theta_1(23\cos\theta_1 - 100\sin\theta_1)\right) \\ &\quad -\frac{1}{2}\log(|4623\cos(2\theta_0) + 2300\sin(2\theta_0) - 5377|) \\ &\quad +\frac{1}{2}\log(|4623\cos(2\theta_1) + 2300\sin(2\theta_1) - 5377|) - \frac{2\pi}{5} = 0, \end{aligned} \tag{9}$$

where $\theta_0 = \arccos(1/r_0) - \pi$, and the determination of the arctan *is in the interval $(-\pi/2,\pi/2)$ and of the* arccos *in the interval $(0,\pi)$.*

On the other hand, any limit cycle of system (8) *which intersects the straight line $x = 1$ in two points (r_0,θ_0) and (r_1,θ_1) with $0 < \theta_0 < \theta_1 < \pi/2$, $r_k\cos\theta_k = 1$ for $k = 0,1$, and $r_0 > 1$ and θ_1 must satisfy the equations* (9), *but now in both equations $\theta_0 = \arccos(1/r_0)$.*

We recall that a *limit cycle* of system (8) is an isolated periodic orbit of that system in the set of all periodic orbits of the system. It is well known that the

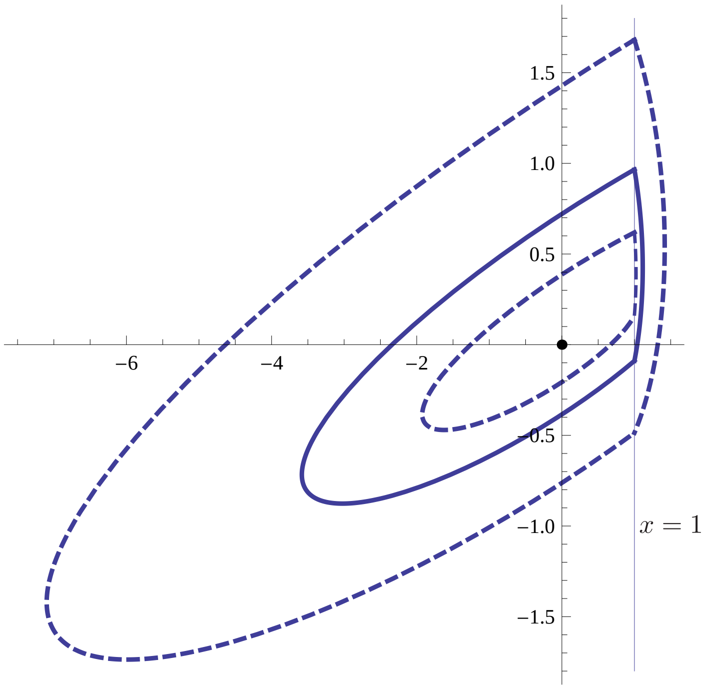

Figure 1. The three limit cycles surrounding the origin.

study of the limit cycles of the differential systems in dimension two is one of the main problems of the qualitative theory of differential systems in dimension two, see for instance the surveys of Ilyashenko [15] and Jibin Li [17].

In fact, as we shall see in the proof of Theorem 4 equations (9) have three solutions, providing the three limit cycles of Figure 1.

2. Proofs of Propositions 2 and 3 and Theorems A and B

Proof of Proposition 2. System (6) can be written as the autonomous system

$$\begin{pmatrix}\tau'\\ x'\end{pmatrix}=\begin{cases}X(\tau,x) & \text{if} \quad h(\tau,x)>0,\\ Y(\tau,x) & \text{if} \quad h(\tau,x)<0,\end{cases}$$

in $\mathbb{R}\times D$, where

$$X(\tau,x)=\begin{pmatrix}1\\ \varepsilon(F_1(\tau,x)+F_2(\tau,x))+\varepsilon^2(R_1(\tau,x,\varepsilon)+R_2(\tau,x,\varepsilon))\end{pmatrix},$$

$$Y(\tau,x)=\begin{pmatrix}1\\ \varepsilon(F_1(\tau,x)-F_2(\tau,x))+\varepsilon^2(R_1(\tau,x,\varepsilon)-R_2(\tau,x,\varepsilon))\end{pmatrix}.$$

So

$$
\begin{aligned}
(Xh)(Yh) = &\ \langle \nabla h, X\rangle \langle \nabla h, Y\rangle \\
= &\ (\partial_t h)^2 + \varepsilon 2\partial_t h \langle \nabla_x h, F_1\rangle \\
&+\varepsilon^2 \left(2\partial_t h\langle \nabla_x h, R_1\rangle + \langle \nabla_x h, F_1\rangle^2 - \langle \nabla_x h, F_2\rangle^2\right) \\
&+\varepsilon^3 2 \left(\langle \nabla_x h, F_1\rangle\langle \nabla_x h, R_1\rangle - \langle \nabla_x h, F_2\rangle\langle \nabla_x h, R_2\rangle\right) \\
&+\varepsilon^4 \left(\langle \nabla_x h, R_1\rangle^2 - \langle \nabla_x h, R_2\rangle^2\right).
\end{aligned} \tag{10}
$$

Let $x(t,z)$ be the solution of the system (6) such that $x(0,z)=z$. Fixed an open bounded subset $U\subset D_0$ we define the compact subset $K=\{(t,x(t,z)) : (t,z)\in [0,T]\times \overline{U}\}\subset [0,T]\times D$.

Hence, we can choose $|\varepsilon_0|>0$ sufficiently small such that $(Xh)(Yh)(t,x)>0$ for every $(t,x)\in K\cap\Sigma$ and $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$. Indeed $\left(\partial_t h(t,x)\right)^2$ is a continuous positive function in $\mathbb{R}\times D$ so there exists $\kappa_0>0$ such that $\left(\partial_t h(t,x)\right)^2>\kappa_0$ for every $(t,x)\in K$. □

Proof of Proposition 3. Consider the notation of the proof of Proposition 2. From (10) we also conclude that

$$
\begin{aligned}
(Xh)(Yh) = &\ \left(\partial_t h + \varepsilon^2 \langle \nabla_x h, R_1\rangle\right)^2 \\
&+2\varepsilon\left(\partial_t h\langle \nabla_x h, F_1\rangle + \varepsilon\frac{\langle \nabla_x h, F_1\rangle^2 - \langle \nabla_x h, F_2\rangle^2}{2}\right) + \varepsilon^3\mathcal{O}(1).
\end{aligned}
$$

Now, we note that $K\cap\Sigma$ has a finite number of connected components, since Σ is a regular manifold in $[0,T]\times D$. So we can choose a finite subset $\{p_1,p_2,\ldots,p_m\}\subset\Sigma$ such that $\partial_t h(p_i)=0$ for $i=1,2,\ldots,m$, $\Sigma_{p_i}\cap\Sigma_{p_j}=\emptyset$ for $i\neq j$, and $\partial_t h(t,x)\neq 0$ for every $(t,x)\in\Sigma\backslash\left(\Sigma_{p_1}\cup\Sigma_{p_2}\cdots\Sigma_{p_m}\right)$. Thus for $i=1,2,\ldots,m$

$$
(Xh)(Yh)(t,x)\geq \varepsilon\xi_{p_i}(t,x)+\varepsilon^3\mathcal{O}(1),
$$

for every $(t,x)\in\Sigma_{p_i}$. We can choose then $\varepsilon_i>0$ sufficiently small such that $(Xh)(Yh)(t,x)>0$ for every $(t,x)\in K\cap\Sigma_{p_i}$ and $\varepsilon\in(0,\varepsilon_i)$. Indeed ξ_{p_i} is a continuous positive function in $\mathbb{R}\times D$ so there exists $\kappa_i>0$ such that $\xi_{p_i}(t,x)>\kappa_i$ for every $(t,x)\in K$. Moreover, by the proof of Proposition 2, we can choose $|\varepsilon_0|>0$ sufficiently small such that $(Xh)(Yh)(t,x)>0$ for every $(t,x)\in K\cap \Sigma\backslash\left(\Sigma_{p_1}\cup\Sigma_{p_2}\cup\cdots\cup\Sigma_{p_m}\right)$ and $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$.

Hence, choosing $\bar\varepsilon=\min\{\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_m\}$ we conclude that $(Xh)(Yh)(t,x)>0$ for every $(t,x)\in K\cap\Sigma$ and $\varepsilon\in(0,\bar\varepsilon)$ □

For proving Theorems A and B we need some preliminary lemmas. As usual μ denotes the *Lebesgue Measure*.

The hypotheses (ii) and (ii') of Theorem A and B respectively make assumptions on the Brouwer degree of the averaged function f. So we need to show that the function f is continuous in order that the Brouwer degree will be well defined, for more details see Appendix A.

Lemma 5. *The averaged function* (7) *is continuous in* C.

Proof. First of all we note that either the hypothesis (ii) Theorem A or (ii') of Theorem B implies that the map $h_z : t\mapsto h(t,z)$ has only isolated zeros for $z\in C$. Because the constant function $t\mapsto z$ is the solution of system (6) for $\varepsilon=0$, thus, from hypothesis (ii) of Theorem A, it reaches the set of discontinuity only at its crossing region. From hypothesis (ii') of Theorem B this conclusion is immediately.

For $z \in C$ we define the sets $A_z^+ = \{t \in [0,T] : h(t,z) > 0\}$, $A_z^- = \{t \in [0,T] : h(t,z) < 0\}$, and $A_z^0 = \{t \in [0,T] : h(t,z) = 0\}$. We note that $\mu\left(A^0(z)\right) = 0$, since the map $h_z : t \mapsto h(t,z)$ has only isolated zeros for $z \in C$. Moreover $[0,T] = \overline{A^+(z) \cup A^-(z)}$.

Now, fix $z_0 \in C$, for $z \in C$ in some neighborhood of z_0, we estimate

$$
\begin{aligned}
|f(z) - f(z_0)| \leq\ & \int_0^T |F_1(t,z_0) - F_1(t,z)|dt \\
& + \int_0^t |\text{sign}(h(t,z_0))F_2(t,z_0) - \text{sign}(h(t,z))F_2(t,z)|dt \\
\leq\ & TL|z_0 - z| \\
& + \int_{A_{z_0}^+ \cap A_z^+} |F_2(t,z_0) - F_2(t,z)|dt + \int_{A_{z_0}^- \cap A_z^-} |F_2(t,z_0) - F_2(t,z)|dt \\
& + \int_{A_{z_0}^+ \cap A_z^-} |F_2(t,z_0) + F_2(t,z)|dt + \int_{A_{z_0}^- \cap A_z^+} |F_2(t,z_0) + F_2(t,z)|dt \\
\leq\ & 3T|z_0 - z| + \left(\mu\left(A_{z_0}^+ \cap A_z^-\right) + \mu\left(A_{z_0}^- \cap A_z^+\right)\right) M,
\end{aligned}
$$

where L is the Lipschitz constant of F_1 and $M = \max\{F_2(t,z) : (t,z) \in [0,T] \times \overline{U}\} < \infty$. It is easy to see that $\mu\left(A_{z_0}^+ \cap A_z^-\right) \to 0$ and $\mu\left(A_{z_0}^- \cap A_z^+\right) \to 0$ when $z \to z_0$. So the lemma is proved. □

Let a be the point in hypothesis (iii) of Theorems A and B. By Lemma 5 there exists a neighborhood $U \subset C$ of a such that f is continuous in U. Hence, by Theorems 10 and 11 (see Appendix A), there exists a unique map that satisfies the properties of the Brouwer degree for the function $f(z)$ with $z \in \overline{U}$, because $0 \notin f(\partial U)$. This map is denoted by $d_B(f,U,0)$.

Lemma 6. *For $|\varepsilon| > 0$ (or $\varepsilon > 0$) sufficiently small the solutions of system* (6) *(in the sense of Filippov) starting in C are uniquely defined.*

To prove Lemma 6 we will need the following proposition, that has been proved in Corollary 1 of section 10 of chapter 1 of [11]. Define

$$
\begin{aligned}
S^+ &= \{(t,x) \in \mathbb{R} \times D : h(t,x) > 0\}, \\
S^- &= \{(t,x) \in \mathbb{R} \times D : h(t,x) < 0\}.
\end{aligned}
$$

Note that $\mathbb{R} \times D = S^- \cup \Sigma \cup S^+$.

Proposition 7. *For every point of the manifold Σ where $(Xh)(Yh) > 0$, there is a unique solution passing either from S^- into S^+, or from S^+ into S^-.*

Proof of Lemma 6. The proof follows immediately from Proposition 7 and hypothesis (ii). □

Instead of working with the discontinuous differential system (6) we shall work with the continuous differential system

$$
x'(t) = \varepsilon F_\delta(t,x) + \varepsilon^2 R_\delta(t,x,\varepsilon), \tag{11}
$$

where
$$F_\delta(t,x) = F_1(t,x) + \phi_\delta(h(t,x))F_2(t,x),$$
$$R_\delta(t,x,\varepsilon) = R_1(t,x,\varepsilon) + \phi_\delta(h(t,x))R_2(t,x,\varepsilon),$$
and $\phi_\delta : \mathbb{R} \to \mathbb{R}$ is the continuous function defined in (3) and (4), and satisfying (5).

For system (11) the averaged function is defined as
$$f_\delta(z) = \int_0^T F_\delta(t,x)dt.$$

We need to guarantee that hypothesis (i) of Theorem 12 (see appendix A) holds for the functions F_δ and R_δ. For this purpose we prove the following lemma.

Lemma 8. *For $\delta \in (0,1]$ the function $\phi_\delta : \mathbb{R} \to \mathbb{R}$ defined in (3) with ϕ given by (4) is globally $1/\delta$–Lipschitz; i.e. for all $u_1, u_2 \in \mathbb{R}$ we have that $|\phi_\delta(u_1) - \phi_\delta(u_2)| \leq (1/\delta)|u_1 - u_2|$.*

Proof. If $u_1 \leq -\delta < \delta \leq u_2$, then $|\phi_\delta(u_1) - \phi_\delta(u_2)| = 2 = (1/\delta)2\delta \leq (1/\delta)|u_1 - u_2|$.

If $u_1, u_2 \leq -\delta$ or $u_1, u_2 \geq \delta$, then $|\phi_\delta(u_1) - \phi_\delta(u_2)| = 0 \leq (1/\delta)|u_1 - u_2|$.

Assume that $u_1 \in (-\delta, \delta)$. If $|u_2| < \delta$, then $|\phi_\delta(u_1) - \phi_\delta(u_2)| = (1/\delta)|u_1 - u_2|$; and if $|u_2| \geq \delta$, then $|\phi_\delta(u_1) - \phi_\delta(u_2)| \leq \max\{|1/\delta|, |1/u_2|\}|u_1 - u_2| \leq (1/\delta)|u_1 - u_2|$. This completes the proof of the lemma. □

Proposition 9. *For $\delta \in (0,1]$ the functions F_δ and R_δ are locally Lipschitz with respect to the variable x.*

Proof. Let $K \subset D$ be a compact subset. Denote $M = \sup\{|F_2(t,x)| : (t,x) \in [0,T] \times K\}$, which is well defined by continuity of the function $(t,x) \mapsto |F_2(t,x)|$ and compactness of the set $[0,T] \times K$. For x_1 and x_2 in K where F_1 and h are locally Lipschitz and by Lemma 8, we have
$$\begin{aligned}
|F_\delta(t,x_1) - F_\delta(t,x_2)| &= |F_1(t,x_1) - F_1(t,x_2) \\
&\quad + \phi_\delta(h(t,x_1))F_2(t,x_1) - \phi_\delta(h(t,x_2))F_2(t,x_2)| \\
&\leq |F_1(t,x_1) - F_1(t,x_2)| \\
&\quad + |\phi_\delta(h(t,x_1))F_2(t,x_1) - \phi_\delta(h(t,x_2))F_2(t,x_2)| \\
&\leq L|x_1 - x_1| + |\phi_\delta(h(t,x_1))||F_2(t,x_1) - F_2(t,x_2)| \\
&\quad + |F_2(t,x_2)||\phi_\delta(h(t,x_1)) - \phi_\delta(h(t,x_2))| \\
&\leq 2L|x_1 - x_2| + \frac{M}{\delta}|h(t,x_1) - h(t,x_2)| \\
&\leq \left(2L + \frac{ML}{\delta}\right)|x_1 - x_2| = L_\delta|x_1 - x_2|.
\end{aligned}$$
Here L is the maximum between the Lipschitz constant of the functions F_1 and F_2.

The proof for R_δ is analogous. □

Now we are ready to prove Theorems A and B. We shall prove only theorem A. The proof of Theorem B is completely analogous.

Proof of Theorem A. We will study the Poincaré maps for the discontinuous differential system (6) and for the continuous differential system (11). For each $z \in C$, let $x(t,z,\varepsilon)$ denote the solution (in the sense of Filippov) of system (6)

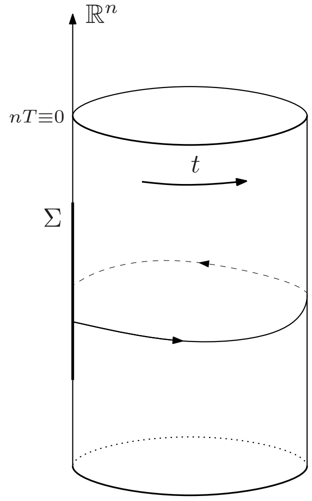

Figure 2. Generalized cylinder.

such that $x(0, z, \varepsilon) = z$; and let $x_\delta(t, z, \varepsilon)$ denote the solution of system (11) such that $x_\delta(0, z, \varepsilon) = z$. Since all solutions starting in C reaches the set of discontinuity at its crossing region for $|\varepsilon| > 0$ (or $\varepsilon > 0$) sufficiently small, it follows that $x_\delta(t, z, \varepsilon) \to x(t, z, \varepsilon)$ when $\delta \to 0$ for every $(t, z) \in [0, T] \times C$ and for $|\varepsilon| > 0$ (or $\varepsilon > 0$) sufficiently small.

Since the differential system (11) is T–periodic in the variable t, we can consider system (11) as a differential system defined on the generalized cylinder $\mathbb{S}^1 \times D$ obtained by identifying $\Sigma = \{(\tau, x) : \tau = 0\}$ with $\{(\tau, x) : \tau = T\}$, see Figure 2. On this cylinder Σ is a section for the flow. Moreover, if $z \in C$ is the coordinate of a point on Σ, then we consider the Poincaré map $P_\delta^\varepsilon(z) = x_\delta(T, z, \varepsilon)$ for the points z such that $x_\delta(T, z, \varepsilon)$ is defined.

Observe that there exists $\varepsilon_0 > 0$ such that, whenever $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, the solution $x_\delta(t, z, \varepsilon)$ is uniquely defined on the interval $[0, T]$. Indeed, if (t_z^-, t_z^+) is the maximal open interval for which the solution passing through $(0, z)$ is defined. Now we shall apply the local existence and uniqueness theorem for the solutions of these differential, see for example Theorem 1.2.2 of [25]. Note that we can apply that theorem due to the result of Proposition 9. Hence, by the local existence and uniqueness theorem we have that $t_z^+ > h_z$ and $h_z = \inf\{T, d\backslash m(\varepsilon)\}$ where $m(\varepsilon) \geq |\varepsilon F_\delta(t, x) + \varepsilon^2 R_\delta(t, x, \varepsilon)|$ for all $t \in [0, T]$, for each x with $|x - z| \leq d$ and for every $z \in C$. When $|\varepsilon| > 0$ (or $\varepsilon > 0$) is sufficiently small, $m(\varepsilon)$ can be arbitrarily large, in such a way that $h_z = T$ for all $z \in C$. Hence, for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, the Poincaré map of system (11) is well defined and continuous for every $z \in C$.

From the definition of the Poincaré map $P_\delta^\varepsilon(z)$ its fixed points correspond to periodic orbits of period T of the differential system (11) defined on the cylinder.

We can define in a similar way the Poincaré map $P^\varepsilon(z) = x(T, z, \varepsilon)$ of the discontinuous differential system (6). The referred Poincaré map is the composition of the Poincaré maps of the continuous differential systems, so for $|\varepsilon| > 0$ (or $\varepsilon > 0$) sufficiently small it is well defined and continuous for every $z \in C$. Again the fixed points of $P^\varepsilon(z)$ correspond to periodic orbits of the discontinuous differential system (6).

Clearly (from above considerations), for $z \in C$ and for $|\varepsilon| > 0$ (or $\varepsilon > 0$) sufficiently small, the pointwise limit of the Poincaré map $P_\delta^\varepsilon(z)$ of system (11), when $\delta \to 0$ is the Poincaré map $P^\varepsilon(z)$ of system (6).

By definition the continuous differential system (11) is $\mathcal{C}^2$ in the variable ε. So we do the Taylor expansion of the Poincaré map of system (11) around ε up to order two, and we get

$$P_\delta^\varepsilon(z) = z + \varepsilon f_\delta(z) + \mathcal{O}(\varepsilon^2), \tag{12}$$

where $f_\delta(z)$ is the averaged function of the continuous differential system (11), for more details see for instance [8]. Due to (5) we obtain that the pointwise limit in Σ of the function f_δ, when $\delta \to 0$ is the function f.

Let $a \in U$ be the point satisfying hypotheses (ii) of Theorem A. Therefore $f(z) \neq 0$ for all $z \in \overline{V}\backslash\{a\}$. Define $f_0 = f|_{\overline{V}}$, we know that f_0 is continuous by Lemma 5. Then, we consider the continuous homotopy $\{f_\delta|_{\overline{V}}, 0 \leq \delta \leq 1\}$. We claim that there exists a $\delta_0 \in (0,1]$ such that $0 \notin f_\delta(\partial V)$ for all $\delta \in [0, \delta_0]$. Now we shall prove the claim.

As usual $\mathbb{N}$ denotes the set of positive integers. Suppose that there exists a sequence $(z_m)_{m\in\mathbb{N}}$ in ∂U such that $f_{\frac{1}{m}}(z_m) = 0$. As the sequence (z_m) is contained in the compact set ∂U, so there exists a subsequence $(z_{m_\ell})_{\ell\in\mathbb{N}}$ such $z_{m_\ell} \to z_0 \in \partial U$. Consequently we obtain that $f(z_0) = 0$, in contradiction with the hypotheses (ii) of Theorem A. Hence, the claim is proved.

From the above claim and the property (iii) of Theorem 10 (see Appendix A) we conclude that $d_B(f_\delta, V, 0) \neq 0$ for $0 \leq \delta \leq \delta_0$. Therefore, by the property (i) of Theorem 10 we obtain that $0 \in f_\delta(V)$, so there exists $a_\delta \in U$ such that $f_\delta(a_\delta) = 0$. Since, by continuity, there exists the $\lim_{\delta\to 0} a_\delta$ and it is a zero of the function $f_0 = f|_U$. This limit is the point a of the hypotheses (ii) of Theorem A, because a is the unique zero of f_0 in U.

In summary, in order that for every $\delta \in (0, \delta_0]$ the averaged function f_δ satisfy the assumptions (ii) of Theorem 12 (see Appendix B). So it only remains to show that in U we have that $f_\delta(z) \neq 0$ for all $z \in \overline{V} \setminus \{a_\delta\}$. But this can be achieved in a complete similar way as we proved the above claim. Hence, by Proposition 9 for every $\delta \in (0, \delta_0]$ the continuous differential system (11) satisfies all the assumptions of Theorem 12. Hence, for $|\varepsilon|$ sufficiently small there exists a periodic solution $x_\delta(t,\varepsilon)$ of the continuous differential system (11) such that $z_{(\delta,\varepsilon)} := x_\delta(0,\varepsilon) \to a_\delta$ when $\varepsilon \to 0$.

Now, from (12) the point $z_{(\delta,\varepsilon)}$ is a fixed point of the Poincaré map $P_\delta^\varepsilon(z)$, i.e. $P_\delta^\varepsilon(z_{(\delta,\varepsilon)}) = z_{(\delta,\varepsilon)}$. Since $\lim_{\delta\to 0} P_\delta^\varepsilon(z) = P^\varepsilon(z)$, it follows that $z_\varepsilon = \lim_{\delta\to 0} z_{(\delta,\varepsilon)}$ is a fixed point of the Poincaré map $P^\varepsilon(z)$. So, the discontinuous differential system (6) has a periodic solution $x(t,\varepsilon)$ such that $z_\varepsilon = x(0,\varepsilon) \to a$ as $\varepsilon \to 0$. Therefore the theorem is proved. □

3. Application

In this section we shall prove Theorem 4, by applying Theorem A to the discontinuous differential system (8). So, we must compute the integral (7), which for system (8) becomes

$$f(r) = \int_0^{2\pi} F(\theta, r)\, d\theta, \tag{13}$$

where the function $F(\theta, r)$ is given in (8).

The solution of the differential system (8) in the half–plane $x = r\cos\theta \geq 1$ starting at the point (r_0, θ_0) with $r_0 \cos\theta_0 = 1$ and $\theta_0 \in (-\pi/2, 0)$ is

$$r(\theta) = \exp\left(\frac{19(\theta - \theta_0)}{50}\right) r_0.$$

Therefore, at the point (r_1, θ_1) with $r_1 \cos\theta_1 = 1$ and $\theta_1 \in (0, \pi/2)$ we have that

$$\exp\left(\frac{19(\theta_1 - \theta_0)}{50}\right) r_0 \cos\theta_1 = 1.$$

This equation coincides with the first equation of (9).

Now computing the integral (13) we obtain exactly the right hand side of the second equation of (9) multiplied by r. According to Theorem A we must find the zeros of this last expression. Since r cannot be zero the equation for the zeros is reduced exactly to the second equation of (9). In short, by Theorem A we have proved that a periodic orbit of system (8) intersects the straight line $x = 1$ in two points (r_0, θ_0) and (r_1, θ_1) with $\theta_0 \in (-\pi/2, 0)$, $\theta_1 \in (0, \pi/2)$, $r_k \cos\theta_k = 1$ for $k = 0, 1$, and $r_0 > 1$ and θ_1 must satisfy the equations (9).

In [19] it is proved that the discontinuous differential equation (8) has three limit cycles (i) and that the their points (r_0, θ_0) and (r_1, θ_1) are approximately for the inner limit cycle of Figure 1

$$r_0 = 1.013330663139.., \quad \theta_0 = 0.162383740477.., \quad \theta_1 = 0.5541676264624..; \tag{14}$$

for the middle limit cycle of Figure 1

$$r_0 = 1.003945075086.., \quad \theta_0 = -0.088680876377.., \quad \theta_1 = 0.768002346543..; \tag{15}$$

for the external limit cycle of Figure 1

$$r_0 = 1.111870463116.., \quad \theta_0 = -0.452434880837.., \quad \theta_1 = 1.034197922817... \tag{16}$$

It is easy to check that (14), (15) and (16) satisfies the two equations (9). Hence, Theorem 4 is proved.

Appendix A: Basic results on the Brouwer degree

In this appendix we present the existence and uniqueness result from the degree theory in finite dimensional spaces. We follow the Browder's paper [7], where are formalized the properties of the classical Brouwer degree.

Theorem 10. *Let $X = \mathbb{R}^n = Y$ for a given positive integer n. For bounded open subsets V of X, consider continuous mappings $f : \overline{V} \to Y$, and points y_0 in Y such that y_0 does not lie in $f(\partial V)$ (as usual ∂V denotes the boundary of V). Then to each such triple (f, V, y_0), there corresponds an integer $d(f, V, y_0)$ having the following three properties.*

- (i) *If $d(f, V, y_0) \neq 0$, then $y_0 \in f(V)$. If f_0 is the identity map of X onto Y, then for every bounded open set V and $y_0 \in V$, we have*

$$d\left(f_0\big|_V, V, y_0\right) = \pm 1.$$

(*ii*) (*Additivity*) *If* $f:\overline{V}\to Y$ *is a continuous map with* V *a bounded open set in* X*, and* V_1 *and* V_2 *are a pair of disjoint open subsets of* V *such that*

$$y_0 \notin f(\overline{V}\backslash(V_1\cup V_2)),$$

then,

$$d\,(f_0,V,y_0)=d\,(f_0,V_1,y_0)+d\,(f_0,V_1,y_0)\,.$$

(*iii*) (*Invariance under homotopy*) *Let* V *be a bounded open set in* X*, and consider a continuous homotopy* $\{f_t:0\le t\le 1\}$ *of maps of* $\overline{V}$ *in to* Y*. Let* $\{y_t:0\le t\le 1\}$ *be a continuous curve in* Y *such that* $y_t\notin f_t(\partial V)$ *for any* $t\in[0,1]$*. Then* $d(f_t,V,y_t)$ *is constant in* t *on* $[0,1]$*.*

Theorem 11. *The degree function* $d(f,V,y_0)$ *is uniquely determined by the three conditions of Theorem 10.*

For the proofs of Theorems 10 and 11 see [7].

Appendix B: Basic results on averaging theory

In this appendix we present the basic result from the averaging theory that we shall need for proving the main results of this paper. For a general introduction to averaging theory see for instance the book of Sanders and Verhulst [25].

Theorem 12. *We consider the following differential system*

$$x'(t)=\varepsilon\,F(t,x)+\varepsilon^2\,R(t,x,\varepsilon), \tag{17}$$

where $F:\mathbb{R}\times D\to\mathbb{R}^n$ *and* $R:\mathbb{R}\times U\times(-\varepsilon_f,\varepsilon_f)\to\mathbb{R}^n$ *are continuous functions,* T*-periodic in the first variable and* D *is an open subset of* $\mathbb{R}^n$*. We define the averaged function* $f:D\to\mathbb{R}^n$ *as*

$$f(x)=\int_0^T F(s,x)ds, \tag{18}$$

and assume that

(*i*) F *and* R *are locally Lipschitz with respect to* x*;*
(*ii*) *for* $a\in D$ *with* $f(a)=0$*, there exist a neighborhood* V *of* a *such that* $f(z)\neq 0$ *for all* $z\in\overline{V}\backslash\{a\}$ *and* $d_B(f,V,0)\neq 0$*.*

Then, for $|\varepsilon|>0$ *sufficiently small, there exist a* T*–periodic solution* $x(t,\varepsilon)$ *of the system* (17) *such that* $x(0,\varepsilon)\to a$ *as* $\varepsilon\to 0$*.*

Theorem 12 for studying the periodic orbits of continuous differential systems has weaker hypotheses than the classical result for studying the periodic orbits of smooth differential systems, see for instance Theorem 11.5 of Verhulst [29], where instead of (*i*) is assumed that

(*j*) F, R, D_xF, D_x^2F and D_xR are defined, continuous and bounded by a constant M (independent of ε) in $[0,\infty)\times D$, $-\varepsilon_f<\varepsilon<\varepsilon_f$;

and instead of (*ii*) it is required that

(*jj*) for $a\in D$ with $f(a)=0$ we have that $J_f(a)\neq 0$, where $J_f(a)$ is the Jacobian matrix of the function f at the point a.

For a proof of Theorem 12 see [8] section 3.

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