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LIOUVILLIAN FIRST INTEGRALS FOR GENERALIZED RICCATI POLYNOMIAL DIFFERENTIAL SYSTEMS

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ABSTRACT. We characterize the generalized Riccati polynomial differential systems of the form $x' = y$, $y' = a(x)y^2+b(x)y+c(x)$, where $a(x)$, $b(x)$ and $c(x)$ are arbitrary polynomials that have a Liouvillian first integrals.

1. Introduction and statement of the main results

A classical problem in the qualitative theory of planar differential equations depending on parameters is to characterize the existence or non–existence of first integrals in function of these parameters.

Let x and y be complex variables. We consider the system

$$x' = y, \qquad y' = a(x)y^2 + b(x)y + c(x), \tag{1}$$

where $a(x)$, $b(x)$ and $c(x)$ are C^1 functions on x and the prime denotes derivative with respect to the time t that can be either real or complex. In fact, if $a(x)c(x) \not\equiv 0$ these systems are called *generalized Riccati differential systems*, if $a(x) \not\equiv 0$ and $c(x) \equiv 0$ they are *linear differential systems*, and if $a(x) \equiv 0$ they are *generalized Lienard differential systems*.

Our interest is on the *generalized Riccati polynomial differential systems* , i.e. when the functions $a(x)$, $b(x)$ and $c(x)$ are polynomials and we want to study its Liouvillian integrability.

The vector field associated to system (1) is

$$X = y\frac{\partial}{\partial x} + (a(x)y^2 + b(x)y + c(x))\frac{\partial}{\partial y}.$$

The main objectives of this paper is to characterize the Liouvillian first integrals of the generalized Ricatti polynomial differential systems.

Let $U \subset \mathbb{C}^2$ be an open set. We say that the non-constant function $H: U \to \mathbb{C}$ is a *first integral* of the polynomial vector field X on U if $H(x(t), y(t))$ is constant for all values of t for which the solution $(x(t), y(t))$ of X is defined on U. Clearly H is a first integral of X on U if and only if $XH = 0$ on U.

We recall that a *Liouvillian first integral* is a first integral H which is a Liouvillian function, that is, roughly speaking which can be obtained "by quadratures" of elementary functions. For a precise definition see [4]. The study of the Liouvillian first

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integrals is a classical problem of the integrability theory of the differential equations which goes back to Liouville, see for details again [4].

For studyng of the existence of Liouvillian first integrals we need to study the so-called Darboux polynomials and exponential factors of the Riccati polynomial differential systems of the second kind.

Let $h = h(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}$. As usual $\mathbb{C}[x, y]$ denotes the ring of all complex polynomials in the variables x and y. We say that $h = 0$ is an *invariant algebraic curve* of the vector field X associated to the Riccati polynomial differential system (1) if it satisfies

$$y\frac{\partial h}{\partial x} + (a(x)y^2 + b(x)y + c(x))\frac{\partial h}{\partial y} = Kh,$$

the polynomial $K = K(x, y) \in \mathbb{C}[x, y]$ is called *the cofactor* of $h = 0$ and has degree at most

$$n = \max\{2 + \deg a(x), 1 + \deg b(x), \deg c(x)\} - 1. \tag{2}$$

When $h = 0$ is an invariant algebraic curve we also say that h is a *Darboux polynomial* of the Riccati polynomial differential system. Note that a *polynomial first integral* is a Darboux polynomial with zero cofactor.

An *exponential factor* E of system (1) is a function of the form $E = \exp(g/h) \notin \mathbb{C}$ with $g, h \in \mathbb{C}[x, y]$ satisfying $(g, h) = 1$ and

$$\frac{\partial E}{\partial x}y + \frac{\partial E}{\partial y}(a(x)y^2 + b(x)y + c(x)) = LE,$$

for some polynomial $L = L(x, y)$ of degree at most n given in (2), called the *cofactor* of E.

The existence of exponential factors $\exp(g/h)$ is due to the fact that the multiplicity of the invariant algebraic curve $h = 0$ is larger than 1, for more details see [1, 3] .

Proposition 1. *The following statements hold.*

(i) *If* $E = \exp(g/h)$ *is an exponential factor for the polynomial system* (1) *and* h *is not a constant polynomial, then* $h = 0$ *is an invariant algebraic curve.*
(ii) *Eventually* e^g *can be exponential factors, coming from the multiplicity of the infinite invariant straight line.*

For a geometrical meaning of the exponential factors and a proof of Proposition 1 see [3].

A non-constant function $R\colon U \to \mathbb{C}$ is an *integrating factor* of the polynomial vector field X on U, if one of the following three equivalent conditions holds

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}, \quad \operatorname{div}(RP, RQ) = 0, \quad XR = -R\operatorname{div}(P, Q),$$

on U where $P = y$ and $Q = a(x)y^2 + b(x)y + c(x)$. As usual the *divergence* of the vector field X is given by

$$\operatorname{div}(P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

In [1] the next result is proved.

Theorem 2. *Suppose that the polynomial vector field X of degree m defined in $\mathbb{C}^2$ admits p invariant algebraic curves $f_i = 0$ with cofactors K_i, for $i = 1, \dots, p$ and q exponential factors $E_j = \exp(g_j/h_j)$ with cofactors L_j, for $j = 1, \dots, q$. Then there exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -div(P, Q),$$

if and only if the function of Darboux type

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} E_1^{\mu_1} \cdots E_q^{\mu_q}$$

is an integrating factor of the vector field X.

The proof of the following result is given in [2, 4].

Theorem 3. *The polynomial differential system* (1) *has a Liouvillian first integral if and only if it has an integrating factor of Darboux type.*

The following is the first main result of this paper. Its proof follows by direct computations.

Theorem 4. *The following holds for the generalized Riccati polynomial differential systems* (1) *:*

(a) *Assume $b(x) \equiv 0$ then*

$$\frac{y^2}{2} \exp\Big(-2 \int a(x)\, dx \Big) - \int c(x) \exp\Big(-2 \int a(u)\, du \Big)\, dx$$

is a Liouvillian first integral.

(b) *Assume $b(x) \not\equiv 0$, $c(x) = \kappa a(x)$ and $b(x) = \kappa_1 a(x)$ with $\kappa, \kappa_1 \in \mathbb{C}$, then*

(b.1) *if $\kappa = \kappa_1^2/4$ we have that*

$$\int a(x)\, dx - \frac{\kappa_1}{\kappa_1 + 2y} - \log(\kappa_1 + 2y)$$

is a Liouvillian first integral;

(b.2) *if $\kappa \neq \kappa_1^2/4$ we have that*

$$\int a(x)\, dx - \frac{1}{2} \log(y^2 + \kappa_1 y + \kappa) + \frac{\kappa_1}{\sqrt{4\kappa - \kappa_1^2}} \arctan\Big(\frac{\kappa_1 + 2y}{\sqrt{4\kappa - \kappa_1^2}} \Big)$$

is a Liouvillian first integral.

From now on we consider the case in which $b(x) \not\equiv 0$ and $c(x)/a(x) \notin \mathbb{C}$, or $b(x) \not\equiv 0$ and $b(x)/a(x) \notin \mathbb{C}$.

Theorem 5. *The generalized Riccati polynomial differential systems* (1) *with either $b(x) \not\equiv 0$ and $c(x)/a(x) \notin \mathbb{C}$, or $b(x) \not\equiv 0$ and $b(x)/a(x) \notin \mathbb{C}$ have no Liouvillian first integrals.*

The proof of Theorem 5 is given in Section 2.

2. Proof of Theorem 5

For proving Theorem 5 we first characterize the Darboux polynomials (either with zero or with nonzero cofactor) of the generalized Ricatti polynomial differential systems.

Theorem 6. *The generalized Riccati polynomial differential systems* (1) *with either* $b(x) \not\equiv 0$ *and* $c(x)/a(x) \notin \mathbb{C}$, *or* $b(x) \not\equiv 0$ *and* $b(x)/a(x) \notin \mathbb{C}$, *have no Darboux polynomials.*

We separate the proof of Theorem 6 in several steps.

Lemma 7. *The generalized Riccati polynomial differential systems* (1) *have no polynomial first integrals.*

Proof. We proceed by contradiction. Let H be a polynomial first integral of system (1), that is

$$y\frac{\partial H}{\partial x} + (a(x)y^2 + b(x)y + c(x))\frac{\partial H}{\partial y} = 0. \tag{3}$$

We write H as a polynomial in the variable y, i.e.

$$H(x,y) = \sum_{j=0}^{m} h_j(x)y^j, \quad \text{where } h_j(x) \text{ is a polynomial in the variable } x.$$

Without loss of generality we can assume that $h_m(x) \neq 0$.

Computing the coefficient of degree $m+1$ in the variable y in (3) we get that

$$h_m'(x) + m\,a(x)h_m(x) = 0, \quad \text{that is} \quad h_m(x) = C\exp\Big(-m\int a(x)\,dx\Big) \neq 0.$$

Since $h_m(x)$ is a polynomial, we must have $m = 0$ because $a(x) \not\equiv 0$. Then $H = h_0(x)$. In view of (3) we get that H satisfies

$$H'(x) = 0, \quad \text{that is} \quad H(x) \in \mathbb{C},$$

a contradiction with the fact that H is a polynomial first integral. □

Lemma 7 states that the generalized Riccati polynomial differential systems (1) have no Darboux polynomials with zero cofactor.

The proof of the following proposition is well-known and can be found in [1].

Proposition 8. *We suppose that* $h \in \mathbb{C}[x,y]$ *and let* $h = h_1^{n_1}\cdots h_r^{n_r}$ *be its factorization in irreducible factors over* $\mathbb{C}[x,y]$. *Then for a polynomial system* (1) $h = 0$ *is an invariant algebraic curve with cofactor* K_h *if and only if* $h_i = 0$ *is an invariant algebraic curve for each* $i = 1,\ldots,r$ *with cofactor* K_{h_i}. *Moreover* $K = n_1K_{h_1} + \cdots + n_rK_{h_r}$.

In view of Proposition 8 to study the Darboux polynomials with non-zero cofactor it is enough to study the irreducible ones.

Lemma 9. *Let* $h = h(x,y)$ *be an irreducible Darboux polynomial of the generalized Riccati polynomial differential system* (1) *with cofactor* $K \neq 0$. *Then* $K = mb(x) + n'(x) - n(x)a(x) + m\,a(x)y$ *with* m *a non-negative integer and* $n \in \mathbb{C}[x]$.

Proof. The cofactor K of any irreducible Darboux polynomial of the generalized Riccati polynomial differential system (1), has degree at most n (see (2)). We write it as $K(x,y)=\sum_{j=0}^{n} K_j(x)y^j$, where $K_j=K_j(x)$ is a polynomial in the variable x and has at most degree $n-j$. Since h is a Darboux polynomial of system (1) with cofactor K it satisfies

$$y\frac{\partial h}{\partial x}+(a(x)y^2+b(x)y+c(x))\frac{\partial h}{\partial y}=\left(\sum_{j=0}^{n} K_j(x)y^j\right)h. \tag{4}$$

We write h as a polynomial in the variable y, i.e. $h(x,y)=\sum_{j=0}^{m} h_j(x)y^j$, where each $h_j(x)$ is a polynomial in the variable x. Without loss of generality we can assume that $h_m(x)\neq 0$.

Assume $n\geq 2$. Computing the coefficient of y^{n+m} in (4) we get

$$0=K_n(x)h_m(x) \quad \text{that is} \quad K_n(x)=0.$$

So $n\in\{0,1\}$ and consequently $K=K_0(x)+yK_1(x)$.

Now computing the coefficient of y^{m+1} in (4) we get

$$h'_m(x)+m\,a(x)h_m(x)=h_m(x)K_1(x),$$

that is $h'_m(x)+(m\,a(x)-K_1(x))h_m(x)=0$. Hence, $h_m(x)=C\exp\Big(-\int(m\,a(x)-K_1(x))\,dx\Big)$ with $C\in\mathbb{C}\backslash\{0\}$. Using that $h_m(x)\neq 0$ and that it must be a polynomial, we have $K_1(x)=m\,a(x)$ with m a non-negative integer and $h_m(x)=C$. Now, computing the coefficient of y^m in (4) we get

$$h'_{m-1}(x)+(m-1)a(x)h_{m-1}(x)+mb(x)h_m(x)=ma(x)h_{m-1}(x)+K_0(x)h_m(x).$$

Then since $h_m(x)=C$,

$$h'_{m-1}(x)-a(x)h_{m-1}(x)=C(K_0(x)-mb(x)).$$

Therefore we have a linear system. Solving it we get

$$h_{m-1}(x)=C_1e^{\int a(x)\,dx}+Ce^{\int a(x)\,dx}\int(K_0(x)-m\,b(x))e^{-\int a(u)\,du}\,dx.$$

Since $h_{m-1}(x)\in\mathbb{C}[x]$, we deduce that $C_1=0$ and

$$K_0(x)-m\,b(x)=n'(x)-n(x)a(x),$$

with $n\in\mathbb{C}[x]$ and $h_{m-1}(x)=Cn(x)$. This completes the proof. □

Proposition 10. *The generalized Riccati polynomial differential systems* (1) *with either* $b(x)\not\equiv 0$ *and* $c(x)/a(x)\notin\mathbb{C}$, *or* $b(x)\not\equiv 0$ *and* $b(x)/a(x)\notin\mathbb{C}$ *have no irreducible Darboux polynomials with non-zero cofactor* K.

Proof. We write the generalized Riccati polynomial differential system (1) as the differential equation

$$y\frac{dy}{dx}=a(x)y^2+b(x)y+c(x), \quad y=y(x). \tag{5}$$

Then, using Lemma 9 the Darboux polynomial $h = h(x,y) = h(x,y(x))$ satisfies

$$y\frac{dh}{dx} = y\frac{\partial h}{\partial x} + \frac{\partial h}{\partial y}(a(x)y^2 + b(x)y + c(x)) = (m\, b(x) + n'(x) - n(x)a(x) + m\, a(x)y)h,$$

where m is a non-negative integer, or equivalently

$$\log h = K + \int \frac{m\, b(x) + n'(x) - n(x)a(x) + m\, a(x)y}{y}\, dx, \quad \text{where} \quad K \in \mathbb{C}.$$

Hence

$$h = h(x,y(x)) = C\exp\Big(\int \frac{m\, b(x) + n'(x) - n(x)a(x) + m\, a(x)y}{y}\, dx\Big), \quad C \in \mathbb{C}\backslash\{0\}. \tag{6}$$

Now we write

$$a(x)y^2 + b(x)y + c(x) = (y + \Gamma_1(x))(a(x)y + \Gamma_2(x)) = (a(x)y + \tilde\Gamma_1(x))(y + \tilde\Gamma_2(x)), \tag{7}$$

where

$$\Gamma_1(x) = \frac{b(x)}{2a(x)} - \frac{\sqrt{b(x)^2 - 4a(x)c(x)}}{2a(x)}, \quad \Gamma_2(x) = \frac{b(x)}{2} + \frac{\sqrt{b(x)^2 - 4a(x)c(x)}}{2}, \tag{8}$$

and $\tilde\Gamma_1(x) = a(x)\Gamma_1(x)$, $\tilde\Gamma_2(x) = \Gamma_2(x)/a(x)$. We consider different cases.

Case 1*:* $\Gamma_1(x) = \kappa \in \mathbb{C}$. In this case, from (8) we have that

$$\frac{b(x)}{2a(x)} - \frac{\sqrt{b(x)^2 - 4a(x)c(x)}}{2a(x)} = \kappa, \quad \text{i.e.} \quad b(x) - \sqrt{b(x)^2 - 4a(x)c(x)} = 2\kappa a(x),$$

which yields

$$c(x) = \kappa(b(x) - \kappa a(x)).$$

Then, again from (8) we get

$$\Gamma_2(x) = b(x) - \kappa a(x).$$

Hence, it follows from (5) and (7) that

$$y\frac{dy}{dx} = (y+\kappa)(a(x)y + \Gamma_2(x)), \quad \text{that is} \quad \frac{a(x)y + \Gamma_2(x)}{y} = \frac{dy/dx}{y+\kappa} = \frac{d}{dx}\log(y+\kappa),$$

which implies that (6) becomes

$$\begin{aligned} h &= C\exp\Big(\int \frac{m\, b(x) + n'(x) - n(x)a(x) - m\Gamma_2(x) + m(a(x)y + \Gamma_2(x))}{y}\, dx\Big) \\ &= C\exp\Big(\int \Big(\frac{m\, b(x) + n'(x) - n(x)a(x) - m\Gamma_2(x)}{y} + m\frac{d}{dx}\log(y+\kappa)\Big)\, dx\Big) \\ &= C(y+\kappa)^m \exp\Big(\int \frac{n'(x) - n(x)a(x) + m\kappa a(x)}{y}\, dx\Big), \end{aligned}$$

with $C \in \mathbb{C}\setminus\{0\}$. Since h must be a polynomial in the variables x and y, we must have

$$n'(x) - n(x)a(x) + m\kappa a(x) = 0.$$

Solving it we get

$$n(x) = C_1 e^{\int a(x)\,dx} + m\kappa e^{\int a(x)\,dx} \int a(x) e^{-\int a(u)\,du}\,dx.$$

Since $n(x)$ is a polynomial we must have $C_1 = 0$ and $m\kappa = 0$ and thus $n(x) = 0$. If $m = 0$ then we have that $K_0 = K_1 = 0$ and thus $K = 0$ a contradiction with the fact that h is a Darboux polynomial with non-zero cofactor. If $\kappa = 0$ then $c(x) \equiv 0$ in contradiction with the fact that $c(x)/a(x) \notin \mathbb{C}$. Therefore, this case is not possible.

Case 2: $\Gamma_1(x) \notin \mathbb{C}$ *and* $\tilde{\Gamma}_2 = \Gamma_2(x)/a(x) = \kappa \in \mathbb{C}$. In this case, from (8) we have that

$$\frac{b(x)}{2a(x)} + \frac{\sqrt{b(x)^2 - 4a(x)c(x)}}{2a(x)} = \kappa, \quad \text{i.e.} \quad b(x) + \sqrt{b(x)^2 - 4a(x)c(x)} = 2\kappa a(x),$$

which yields

$$c(x) = \kappa(b(x) - \kappa a(x)) \quad \text{and} \quad b(x) = 2\kappa a(x),$$

which is not possible because then we have $\Gamma_1(x) = \kappa \in \mathbb{C}$.

Case 3: $\Gamma_1(x) \notin \mathbb{C}$ *and* $\tilde{\Gamma}_2 = \Gamma_2(x)/a(x) \notin \mathbb{C}$. In this case, from (8) we can write

$$\frac{a(x)y + \Gamma_2(x)}{y} = \frac{dy/dx}{y + \Gamma_1(x)} = \frac{d}{dx}\log(y + \Gamma_1(x)) - \frac{d\Gamma_1/dx}{y + \Gamma_1(x)}$$

and analogously,

$$\frac{a(x)y + \tilde{\Gamma}_1(x)}{y} = \frac{dy/dx}{y + \tilde{\Gamma}_2(x)} = \frac{d}{dx}\log(y + \tilde{\Gamma}_2)) - \frac{d\tilde{\Gamma}_2/dx}{y + \tilde{\Gamma}_2(x)}.$$

In the first case, from (6), we have that

$$h = C(y + \Gamma_1(x))^m \exp\left(\int \frac{m\,b(x) + n'(x) - n(x)a(x) - m\Gamma_2(x)}{y}\,dx\right)$$
$$\exp\left(-m\int \frac{d\Gamma_1/dx}{y + \Gamma_1(x)}\,dx\right),$$

which to be a polynomial in the variables x and y we must have

$$\frac{m\,b(x) + n'(x) - n(x)a(x) - m\Gamma_2(x)}{y} = m\frac{d\Gamma_1/dx}{y + \Gamma_1(x)},$$

or equivalently,

$$m\,b(x) + n'(x) - n(x)a(x) - m\Gamma_2(x) = m\frac{d\Gamma_1}{dx},$$
$$\Gamma_1(x)(m\,b(x) + n'(x) - n(x)a(x) - m\Gamma_2(x)) = 0.$$

By hypothesis we have that $\Gamma_1(x) \neq 0$ and thus $m\,b(x)+n'(x)-n(x)a(x)-m\Gamma_2(x) = 0$, but then from the first relation we have $d\Gamma_1/dx = 0$, which again is not possible because $\Gamma_1(x) \notin \mathbb{C}$. Hence this case is not possible.

In the second case we have

$$h = C(y+\tilde\Gamma_2(x))^m \exp\bigg(\int \frac{m\,b(x)+n'(x)-n(x)a(x)-m\tilde\Gamma_1(x)}{y}\,dx\bigg)$$
$$\exp\bigg(-m\int \frac{d\tilde\Gamma_2/dx}{y+\tilde\Gamma_2(x)}\,dx\bigg),$$

which again is never a polynomial in the variables x and y. Hence this case is never possible. This completes the proof of the proposition. □

Proof of Theorem 6. The proof of Theorem 6 is an immediate consequence of Lemma 7 and Proposition 10. □

Lemma 11. *Assume that* $\exp(g_1/h_1),\ldots,\exp(g_r/h_r)$ *are exponential factors of some polynomial differential system*

$$x' = P(x,y),\quad y' = Q(x,y),\quad P,Q\in\mathbb{C}[x,y], \tag{9}$$

with cofactors L_j *for* $j=1,\ldots,r$*. Then* $\exp(G)=\exp(g_1/h_1+\cdots+g_r/h_r)$ *is also an exponential factor of system* (9) *with cofactor* $L=\sum_{j=1}^r L_j$*.*

Proof. Using that for $j=1,\ldots,r$, $E_1=\exp(g_1/h_1),\ldots,E_r=\exp(g_r/h_r)$ are exponential factors of system (9) with cofactors L_j we have

$$\frac{\partial(g_j/h_j)}{\partial x}P(x,y)E_j + \frac{\partial(g_j/h_j)}{\partial y}Q(x,y)E_j = L_jE_j,$$

or equivalently,

$$\frac{\partial(g_j/h_j)}{\partial x}P(x,y) + \frac{\partial(g_j/h_j)}{\partial y}Q(x,y) = L_j.$$

Therefore if we set $G=\sum_{j=1}^r g_j/h_j$ we get that

$$\frac{\partial G}{\partial x}P(x,y)+\frac{\partial G}{\partial y}Q(x,y)=\sum_{j=1}^r L_j = L,$$

and thus if $E=\exp(G)$ we obtain

$$\frac{\partial G}{\partial x}P(x,y)E+\frac{\partial G}{\partial y}Q(x,y)E = LE.$$

This concludes the proof of the lemma. □

Proof of Theorem 5. It follows from Theorem 6 that the generalized Riccati polynomial differential system (1) has no Darboux polynomials. Hence, it follows from Proposition 1 and Theorems 2 and 3 that to have a Liouvillian first integral we must have q exponential factors $E_j=\exp(g_j)$ with cofactors L_j such that

$$\sum_{j=1}^q \mu_j L_j = -2a(x)x - b(x).$$

Let $G = \sum_{j=1}^{q} \mu_j g_j \in \mathbb{C}[x, y]$. Then $E = \exp(G) = \exp(\sum_{j=1}^{q} \mu_j g_j)$, is an exponential factor of system (1) with the cofactor $L = \sum_{j=1}^{q} \mu_j L_j$ (see Lemma 11) and E satisfies

$$y\frac{\partial E}{\partial x} + \frac{\partial E}{\partial y}(a(x)y^2 + b(x)y + c(x)) = LE,$$

that is

$$y\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}(a(x)y^2 + b(x)y + c(x)) = L = -2a(x)y - b(x). \tag{10}$$

We write G as a polynomial in the variable y as follows

$$G = \sum_{j=0}^{m} G_j(x)y^j.$$

Computing the coefficient of y^{m+1} with $m \geq 1$ in (10) we get

$$G'_m(x) + m\, a(x)G_m(x) = 0,$$

that is

$$G_m(x) = C_m \exp\Big(-m\int a(x)\,dx\Big), \quad C_m \in \mathbb{C}.$$

Since G_m must be a polymoimal we must have $G_m(x) = 0$ and thus $G = G_0(x)$. Then introducing it in (10) we obtain

$$yG'_0(x) = -2a(x)y - b(x),$$

and since $b(x) \not\equiv 0$ we get that this case is not possible. This concludes the proof. □

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