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UNIQUENESS OF LIMIT CYCLES FOR SEWING PLANAR PIECEWISE LINEAR SYSTEMS

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Abstract. This paper proves the uniqueness of limit cycles for sewing planar piecewise linear systems with two zones separated by a straight line, Σ, and only one Σ-singularity of monodromic type. The proofs are based in an extension of Rolle's Theorem for dynamical systems on the plane.

1. Introduction and main results

Usually the models used in many problems related with engineering, biology, control theory, design of electric circuits, mechanical systems, economics science, and medicine are differential systems that are not analytic, nor differentiable. A good tool to describe the dynamics of these models is the study of piecewise differential systems. See [1, 8] for a wide selection of models and real applications. Typically this class of systems is obtained using two or more linear vector fields that are defined on different regions separated by discontinuity boundaries. In particular, a circuit having an ideal switch with state feedback can be modeled with a planar piecewise linear system where the discontinuity boundary is defined by a straight line, see Sec. 1.1.7 of [1].

Planar linear differential systems are completely understood using only linear algebra and they do not present isolated periodic orbits, so called *limit cycles*. This is not the case for piecewise linear differential systems. The classification of the different phase portraits or the study of the maximum number of limit cycles are still open problems, even when the number of regions is small, two in our case, or the boundaries are straight lines. The existence of real and/or virtual singularities, connection of separatrices, isolated periodic orbits,... increase, in comparison with the linear one, the number of possible phase portraits in the class of piecewise linear differential systems. We focus our attention only on the number of isolated periodic solutions. More concretely, we focus on the maximum number of limit cycles that a class of piecewise linear systems can present. This is a very intricate problem and there are few studies that provide a complete answer, even when we restrict it for a concrete class of planar systems. It can be considered as a generalization of the classical 16th Hilbert problem, see [16]. In our approach the study of uniqueness of limit cycles relates to the uniqueness of intersection points of algebraic curves. One of the main tools used in this paper is the extension of Rolle's Theorem for dynamical systems on the plane, introduced by Khovanskii in 1982, see [17].

Lum and Chua conjectured in 1990 that continuous piecewise linear systems in two zones have at most one limit cycle, see [21]. This was proved in [11]. Few

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years ago, as an application of a Liénard criterion, another piecewise linear family with at most one limit cycle was studied, see [20]. All these cases have no sliding regions. Recently, examples of piecewise linear systems with a sliding segment that present two and three limit cycles are provided, see among others [3, 13, 14, 15] and [5, 7, 19], respectively.

Before explaining our results with more detail we introduce some notation. Let 0 be a regular value of a function $h:\mathbb{R}^2\to\mathbb{R}$. We denote the discontinuity boundary by $\Sigma=h^{-1}(0)$ and the two regions by $\Sigma^{\pm}=\{\pm h(x,y)>0\}$. With this notation we consider the Σ-piecewise vector field $\mathcal{X}=(X^+,X^-)$ defined by

$$\mathcal{X}(q)=\begin{cases} X^+(q), & h(q)>0,\\ X^-(q), & h(q)<0,\end{cases}$$

where $X^{\pm}$ are planar vector fields. The singularities $p^{\pm}$ of $X^{\pm}$ are called *real* or *virtual* if $p^{\pm}\in\Sigma^{\pm}$ or $p^{\pm}\in\Sigma^{\mp}$, respectively.

The vector field $\mathcal{X}$ is defined on Σ following Filippov's terminology, see [10] and Figure 1. The points in Σ where both vectors fields $X^{\pm}$ simultaneously point outward or inward from Σ define the *escaping* or *sliding region*, the complement in Σ defines the *sewing region*. In fact the boundary of both regions is defined by the tangential points of $X^{\pm}$ in Σ. The *sewing points* in $\Sigma=h^{-1}(0)$ of $\mathcal{X}=(X^+,X^-)$ satisfy $X^+h(p)\cdot X^-h(p)>0$, where X^+h denote the derivative of the function h in the direction of the vector X^+ i.e., $X^+h(p)=\langle\nabla h(p),X^+(p)\rangle$. Equivalently for X^-h.

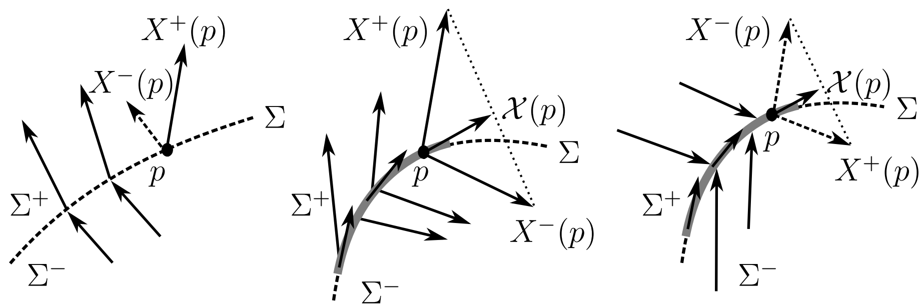

FIGURE 1. Definition of the vector field on Σ following Filippov's convention in the sewing, escaping, and sliding regions.

The point p in Σ is a *tangential point* of $X^{\pm}$ if $X^{\pm}h(p)=0$ and we say that p is a Σ-*singularity* of $\mathcal{X}$ if $p\in\Sigma$ and it is either a tangential point or a singularity of X^+ or X^-. We call $p\in\Sigma$ an *invisible fold* of $X^{\pm}$ if p is a tangential point of $X^{\pm}$ and $(X^{\pm})^2h(p)<0$. Moreover, $p\in\Sigma$ is a Σ-*monodromic singularity* of $\mathcal{X}=(X^+,X^-)$ if p is a Σ-singularity of $\mathcal{X}$ and there exists a neighborhood of the p such that the solutions of $\mathcal{X}$ turn around it either in forward or in backward time. See Figure 2.

The objective of this paper is to prove the uniqueness of limit cycle for a class of planar piecewise linear systems where Σ is formed by sewing regions with a unique Σ-singularity that is of monodromic type. As the vector fields $X^{\pm}$ are linear, all the isolated periodic solutions should cross the discontinuity boundary Σ.

More concretely, we consider $\mathcal{X}=(X^+,X^-)$ and $\Sigma=h^{-1}(0)$, where $X^{\pm}$ are linear vector fields, h is also linear and p is the unique Σ-monodromic singularity. The

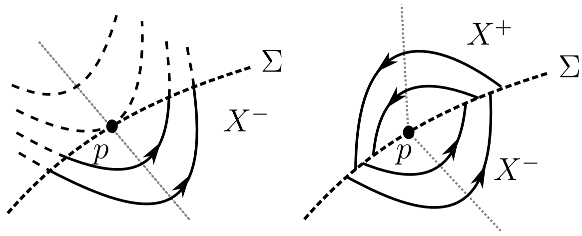

FIGURE 2. Invisible fold in p for the vector field X^- on Σ^- in the left picture and a Σ-monodromic singularity at p on the right one. The gray lines show that p is a tangential point.

vector field $\mathcal{X}$, after a rotation and a translation if necessary, can be expressed by

$$\mathcal{X}(x,y) = \begin{cases} (a^+x + b^+y + c^+, d^+x + e^+y + f^+), & y > 0, \\ (a^-x + b^-y + c^-, d^-x + e^-y + f^-), & y < 0, \end{cases} \tag{1}$$

where $h(x, y) = y$.

As we will see, system (1) is of sewing type if it can be written as the following family:

$$\mathcal{X}(x,y) = \begin{cases} (\mu_0^+ + \mu_1^+x + \mu_2^+y, x), & y > 0, \\ (\mu_0^- + \mu_1^-x + \mu_2^-y, x), & y < 0. \end{cases} \tag{2}$$

Here Σ is the x-axis and the Σ-singularity is located at the origin. We remark that the nonexistence of a sliding region allows us to extend, from the Filippov's convention, the vector fields $X^\pm$ up to full Σ. Other canonical forms can be obtained from [12], see Section 2.

The uniqueness problem in the focus-focus case, using an equivalent expression for (2), is studied in [20]. In this work, the authors also present necessary conditions for the existence of periodic orbits for the cases focus-node and node-node. Their proof uses an equivalent Liénard type form for (2) and an extension of a classical uniqueness criterion of limit cycles from these type of equations for the smooth case, see [22]. This extension only works in the focus-focus case. The saddle-saddle and node-node cases are done [14, 15] also using a Liénard type form. The completely different approach used in this paper allow us to prove the uniqueness result for the full family except the focus-focus case.

Theorem 1.1. *The vector field* (2) *having a* $\Sigma-$*monodromic singularity has no limit cycles when* $\mu_1^+\mu_1^- \geq 0$ *and has at most one limit cycle when* $\mu_1^+\mu_1^- < 0$. *Moreover there is a choice of the parameters for which the limit cycle exists.*

Observe that (2) is also a refracted vector field because it satisfies $X^+h(q) = X^-h(q)$ for all $q \in \Sigma$, see [4] for more details on refracted vector fields. In fact the above result shows that the refracted piecewise linear vector fields, when they have a Σ-monodromic point, have at most one limit cycle.

This paper is organized as follows. In Section 2 we present the canonical form of piecewise linear systems defined in two zones without sliding and a unique Σ-monodromic point. Section 3 deals with the characterization of their singular points and the existence of a Hopf bifurcation at the origin. Conditions for the existence and nonexistence of limit cycles are provided in Section 4. In particular, we extend the divergence property, which determines the stability of a periodic orbit, to piecewise vector fields defined in two zones. Finally, in Section 5 we prove Theorem 1.1. More concretely, firstly we provide a convenient expression for the upper and lower half return maps and secondly we use the extension of Rolle's Theorem for dynamical systems on the plane given by Khovanskii in [17].

2. Canonical forms

Next result provides a canonical form for system (1) with a Σ-monodromic singularity. First, we transform (1), without sliding nor scaping, to (2) and second, we discuss the necessary conditions to have a Σ-monodromic singularity. The singularities of $X^{\pm}$ that are also Σ-singularities of focus or center type are denoted by Σ-CF.

Lemma 2.1. *Let $\mathcal{X}$ be the Σ-piecewise linear vector field defined by* (1) *without sliding in Σ and a unique Σ-singularity p. If p is a Σ-monodromic singularity, then it is the unique Σ-singularity and after a change of coordinates, p can be translated to the origin and the vector field $\mathcal{X}$ can be written as* (2) *and satisfying one of the following conditions:*

(i) $\mu_0^+ < 0$, $\mu_0^- > 0$ *(invisible fold/invisible fold);*
(ii) $\mu_0^+ = 0$, $\mu_2^+ < -(\mu_1^+/2)^2$, *and* $\mu_0^- > 0$ *(Σ-CF/invisible fold);*
(iii) $\mu_0^+ = \mu_0^- = 0$, $\mu_2^+ < -(\mu_1^+/2)^2$, *and* $\mu_2^- < -(\mu_1^-/2)^2$ *(Σ-CF/Σ-CF).*

Under the hypothesis of the above lemma, we remark that the singularities of $X^{\pm}$ are real saddles or virtual focus or nodes.

Proof of Lemma 2.1. Consider the vector field (1). As all regions in Σ are of sewing type we have that $X^+h(p)\,X^-h(p) = (d^+x + f^+)\,(d^-x + f^-) \geq 0$ for all $p \in \Sigma$. So, there is a unique point such that $X^+h(p)\,X^-h(p) = 0$. After a translation, and without loss of generality, we can assume that $p = 0$. Consequently, we can consider $f^+ = f^- = 0$ and $d^+d^- \neq 0$.

Doing the twin change of variables given by

$$(u,v) = \begin{cases} \varphi^+(x,y) = (x + (b^+/d^+)y, (1/d^+)y) \text{ in } \Sigma^+, \\ \varphi^-(x,y) = (x + (b^-/d^-)y, (1/d^-)y) \text{ in } \Sigma^-, \end{cases}$$

and in the sequel rescaling time we can write the Σ-piecewise linear vector field as (2). Note that the for all $q \in \Sigma$ we get $\varphi^+(q) = \varphi^-(q)$.

Next, we observe that as the origin is a Σ-monodromic singularity, then it is a singularity for X^+ (resp. X^-) of focus or center type, or it is an invisible tangency for X^+ (resp. X^-). More concretely, when the origin is a center or focus for X^+ (resp. X^-) then $\mu_2^+ < -(\mu_1^+)^2/4$ (resp. $\mu_2^- < -(\mu_1^-)^2/4$). Whereas the origin is an invisible fold for X^+ (resp. X^-) then $(X^+)^2h(0,0) = \mu_0^+$ (resp. $(X^-)^2h(0,0) = \mu_0^-$).

Finally, doing the changes of variables and time $(x,y,t) \to (x,-y,-t)$, $(x,y,t) \to (-x,y,-t)$, or $(x,y,t) \to (-x,-y,-t)$ if necessary, the assumption of signs of $\mu_0^{\pm}$ follows. □

The equivalent canonical form used in [14, 15, 20] can be obtained from the twin change of variables $(u,v) = (-y, x - \mu_1^+ y)$ and $(u,v) = (-y, x - \mu_1^- y)$ in $\Sigma^{\pm}$, respectively. Hence the Liénard piecewise form of vector field (2) is

$$L = \begin{cases} (\mu_1^+ u - v, -\mu_2^+ u + \mu_0^+), & u > 0, \\ (\mu_1^- u - v, -\mu_2^- u + \mu_0^-), & u < 0. \end{cases}$$

3. Stability and classification of singularities

The singularities in $\Sigma^{\pm}$, when they exist, can be easily classified from their trace and determinant because the vector field (2) is piecewise linear. The unique point that remain to be classified is the origin, which is the unique Σ-singularity. The following result provides conditions to determine its stability for the case $\mu_0^{\pm} \neq 0$ where a Hopf bifurcation occurs. Consequently, the existence of a limit cycle in Theorem 1.1 is guaranteed.

Proposition 3.1. *Consider the vector field* (2) *when* $\mu_0^{\pm} \neq 0$ *and the origin is a* Σ*-monodromic singularity. Then the origin is asymptotically stable (unstable) if* $\mu_0^+\mu_0^-(\mu_1^+\mu_0^- - \mu_1^-\mu_0^+) < 0$ (> 0) *or* $\mu_1^+\mu_0^- - \mu_1^-\mu_0^+ = 0$ *and* $\mu_0^+\mu_1^+(\mu_2^+(\mu_0^-)^2 - \mu_2^-(\mu_0^+)^2) < 0$ (> 0), *respectively. Moreover, the origin is a center if and only if* $\mu_1^-\mu_0^+ - \mu_1^+\mu_0^- = \mu_1^+(\mu_2^+(\mu_0^-)^2 - \mu_2^-(\mu_0^+)^2) = 0$.

Proof. The stability conditions of the statement follow from the computation of the Lyapunov constants, see [6], because of the monodromic property. These constants can be computed from the return map defined as the composition of the two half return maps $\Pi(x_0) = \Pi^-((\Pi^+)(x_0))$. Notice that this return map is analytic. Equivalently we can also look for the difference of both maps $\Delta(x_0) = \pi^-(x_0) - \pi^+(x_0)$, where $\pi^+(x) = \Pi^+(x)$ and $\pi^-(x) = (\Pi^-(x))^{-1}$. For the vector field (2), when $\mu_0^{\pm} \neq 0$, we have

$$\begin{aligned}\pi^{\pm}(x_0) = &- x_0 + \frac{2\mu_1^{\pm}}{3\mu_0^{\pm}}x_0^2 - \frac{4(\mu_1^{\pm})^2}{9(\mu_0^{\pm})^2}x_0^3 \\ &+ \frac{2(22(\mu_1^{\pm})^2 + 9\mu_2^{\pm})\mu_1^{\pm}}{135(\mu_0^{\pm})^3}x_0^4 - \frac{4(\mu_1^{\pm})^2(26(\mu_1^{\pm})^2 + 27\mu_2^{\pm})}{405(\mu_0^{\pm})^4}x_0^5 + \cdots .\end{aligned}$$

Consequently $\Delta(x_0) = \sum_{k=1}^{\infty} V_k x_0^k$ where $V_1 = 0$ and

$$\begin{aligned}V_2 &= \frac{2}{3}\frac{\mu_1^-\mu_0^+ - \mu_1^+\mu_0^-}{\mu_0^-\mu_0^+}, \\ V_3 &= 0, \text{ when } V_2 = 0, \\ V_4 &= \frac{2}{15}\frac{\mu_1^+}{(\mu_0^-)^2(\mu_0^+)^3}(-\mu_2^+(\mu_0^-)^2 + \mu_2^-(\mu_0^+)^2), \text{ when } V_2 = 0.\end{aligned} \tag{3}$$

The proof finishes vanishing V_2 and V_4 simultaneously and showing that the center, using the change $(x, y, t) \to (x, -y, -t)$, is reversible. □

An immediate consequence of the expressions (3) is the next corollary. Notice that, because of there are only two nonvanishing center conditions, only one bifurcated limit cycle from the origin can be guaranteed.

Corollary 3.2. *When* $\mu_0^{\pm} \neq 0$, *the maximum order of a weak-focus of the vector field* (2) *is one. Moreover, if* $\mu_0^+\mu_1^+(\mu_2^+(\mu_0^-)^2 - \mu_2^-(\mu_0^+)^2) < 0$ (> 0) *there exists* $\varepsilon > 0$ (< 0) *small enough such that for* $\mu_1^- = (\mu_1^+\mu_0^+(\mu_0^-)^2 - \varepsilon)/((\mu_0^+)^2\mu_0^-)$ *a stable (unstable) limit cycle bifurcates from the origin by a Hopf bifurcation.*

Proof. From the proof of Proposition 3.1, using (3), there exist values of the parameters, $\mu_0^{\pm} \neq 0$, such that $V_2 = 0$, and $V_4 \neq 0$. Consequently, the vector field (2) has a weak focus of fixed order. Hence moving only μ_1^- as in the statement a limit cycle bifurcates from the origin. The stability condition comes also from (3). □

4. Existence, nonexistence and stability of limit cycles

For smooth differential systems the integral of the divergence along a periodic orbit determines its stability, see [9]. This property cannot be used when the periodic orbit cuts the discontinuity boundary Σ. We call it a Σ-periodic orbit. For that reason we need to extend this well-known result to our class of vector fields. Below we consider, using the notation of Section 1, a general piecewise differential system $\mathcal{X} = (X^+, X^-)$. Here X^+ and X^- are two smooth vectors fields defined on $\Sigma^+ = h^{-1}(0) > 0$ and $\Sigma^- = h^{-1}(0) < 0$, respectively. At the end of this section we apply this extension to the vector field (2).

Consider $\gamma(t) = (\varphi(t), \psi(t))$ a solution of the planar vector field $X(x, y) = (f(x, y), g(x, y))$ such that $\gamma(0) = p$ and $\Sigma = (\alpha(s), \beta(s)), s \in I \subset \mathbb{R}$ a regular curve given by $h^{-1}(0)$, where h is a function of the plane and 0 is a regular value of Σ. Here, $(\alpha(0), \beta(0)) = p$. Let $\mathcal{T}_0, \mathcal{T}_1 \subset \Sigma$ be transversal sections to γ at $p = \gamma(0) \in \mathcal{T}_0$ and $q = \gamma(\tau) \in \mathcal{T}_1$, respectively, where τ is the smallest value such that $\gamma(\tau) \cap \mathcal{T}_1 = q$ for $t > 0$.

Following the steps of the study of the stability of a periodic orbit described in [2] we have the next result.

Proposition 4.1. *Let Π be the Poincaré map between the transversal sections $\mathcal{T}_0$ and $\mathcal{T}_1$ of the orbit $\gamma(t) = (\varphi(t), \psi(t)), \gamma(0) = p$ for $t \in [0, \tau]$ of $X(x, y) = (f(x, y), g(x, y))$. Then the derivative of Π at $p \in \mathcal{T}_0 \cap \gamma$ is given by*

$$\Pi'(p) = \frac{\nabla(0,0)}{\nabla(\tau,0)} \exp\Big(\int_0^\tau \operatorname{div} X(\gamma(s))\, ds \Big),$$

where $\nabla(t, 0) = \begin{vmatrix} \varphi'(t) & \psi'(t) \\ \alpha'(0) & \beta'(0) \end{vmatrix}$.

Theorem 4.2. *Let $\mathcal{X} = (X^+, X^-) = ((f^+, g^+), (f^-, g^-))$ be a Σ-piecewise vector field on $\mathbb{R}^2$, $\gamma^\pm$ two solutions of $X^\pm$ such that $\gamma = \gamma^+ \cup \gamma^-$ is a Σ-periodic orbit that cuts transversally Σ in $p^\pm$. Then the derivative of the Poincaré map at $p = p^+$ is*

$$\Pi'(p) = \frac{X^+h(p^+)}{X^-h(p^+)} \frac{X^-h(p^-)}{X^+h(p^-)} \exp \int_\gamma \operatorname{div} \mathcal{X},$$

where $\operatorname{div} \mathcal{X} = \operatorname{div} X^\pm$ on $\Sigma^\pm$.

Proof. Let $\mathcal{T}_0, \mathcal{T}_1 \subset \Sigma$ be the transversal sections to γ at p^+ and p^-, respectively. Let $\Pi^\pm$ be the Poincaré maps associated to $\gamma^\pm$, respectively. So, we define the Poincaré map associated to γ by the composition $\Pi = \Pi^-(\Pi^+)$. Notice that the derivative at p is obtained multiplying the derivatives of Π^+ and Π^-, i.e., $\Pi'(p) = (\Pi^-)'(p^-)(\Pi^+)'(p)$, where $p^- = \Pi^+(p)$ and $p^+ = p$.

Consider a parametrization of Σ given by $(\alpha(s), \beta(s))$, $s \in I \subset \mathbb{R}$ such that $(\alpha(0), \beta(0)) = p$. We have that $h(\alpha(s), \beta(s)) = 0$ and $(\alpha'(s), \beta'(s)) = \lambda\,(-h_y, h_x)$ for a given constant λ which depends on the parametrization. Hence, from Proposition 4.1, we know that

$$(\Pi^\pm)'(p^\pm) = \frac{\begin{vmatrix} f^\pm(p^\pm) & g^\pm(p^\pm) \\ -h_y(p^\pm) & h_x(p^\pm) \end{vmatrix}}{\begin{vmatrix} f^\pm(p^\mp) & g^\pm(p^\mp) \\ -h_y(p^\mp) & h_x(p^\mp) \end{vmatrix}} \exp \int_{\gamma^\pm} \operatorname{div} X^\pm.$$

Notice that the constant λ disappears because it is in the numerator and denominator of the above expression.

The proof finishes observing that

$$\begin{vmatrix} f^{\pm}(p^{\pm}) & g^{\pm}(p^{\pm}) \\ -h_y(p^{\pm}) & h_x(p^{\pm}) \end{vmatrix} = \langle X^{\pm}, \nabla h(p^{\pm})\rangle = X^{\pm}h(p^{\pm}),$$

and $\begin{vmatrix} f^{\pm}(p^{\mp}) & g^{\pm}(p^{\mp}) \\ -h_y(p^{\mp}) & h_x(p^{\mp}) \end{vmatrix} = \langle X^{\pm}, \nabla h(p^{\mp})\rangle = X^{\pm}h(p^{\mp}).$ □

As a consequence of the theorem above, we can write the derivative of the return map of a Σ-periodic orbit. In fact, the second statement of the corollary below applies to (2).

Corollary 4.3. *Let $\mathcal{X} = (X^+, X^-)$ be a Σ-piecewise vector field as in Theorem 4.2 and $p \in \Sigma$.*

(i) If $h(x, y) = y$ and $p = p^+$, then

$$\Pi'(p) = \frac{g^+(p^+)}{g^-(p^+)}\frac{g^-(p^-)}{g^+(p^-)} \exp \int_\gamma \operatorname{div} \mathcal{X}.$$

(ii) If $\mathcal{X}$ is continuous in the second coordinate then

$$\Pi'(p) = \exp \int_\gamma \operatorname{div} \mathcal{X}.$$

We remember that the linearity of the vectors fields $X^{\pm}$ that define the class of systems studied in this paper rules out the possibility of limit cycles not crossing the discontinuity line Σ. More concretely, the conditions for the parameters in (2) ensures that the limit cycles surround the origin. So, we get the following result.

Proposition 4.4. *Every limit cycle of the vector field* (2) *has the origin in its interior.*

Corollary 4.3.(ii) gives conditions for determining the stability of a limit cycle. In particular, the second part establishes that the Dulac criterion, see [9], for the nonexistence of periodic solution is also valid for (2). This is useful to prove next result.

Proposition 4.5. *Let γ and $\tau^{\pm}$ be a limit cycle of the vector field* (2) *and the respective flight times in $\Sigma^{\pm}$. If $\mu_1^+\tau^+ + \mu_1^-\tau^- < 0$ (> 0), then γ is hyperbolic and stable (unstable). Moreover, if $\mu_1^+\mu_1^- \geq 0$ or $\mu_0^{\pm} = 0$ there are no limit cycles. Hence, necessary conditions for the existence of limit cycles for* (2) *are $\mu_1^+\mu_1^- < 0$ and $\mu_0^+ \neq 0$ or $\mu_0^- \neq 0$.*

Proof. The hyperbolicity and stability properties and the conditions for $\mu_1^{\pm}$ follows directly from Corollary 4.3 and the Dulac criterion. When $\mu_0^{\pm} = 0$ the vector field (2) is homogeneous and the existence of a periodic orbit implies the existence of a continuum. Hence, there are no limit cycles for this second case. □

5. Uniqueness of limit cycles

Proposition 4.5 gives conditions for the existence of limit cycles for system (2), see Figure 3. We start this section simplifying the parameter space where these limit cycles exist, see Lemma 5.1. Next we provide a new expression for the return map in the half plane $\{y \geq 0\}$ determined by the different type of phase portraits: (i) saddle, (ii) invisible fold (no singularity), (iii) node, (iv) degenerate node, and (v) focus, see Lemma 5.2. The propositions of this section prove the uniqueness of the limit cycle when it exists for the different possibilities that vector field X^+ can present. Hence, adding the Hopf bifurcation studied in Section 3, the proof of Theorem 1.1 is done.

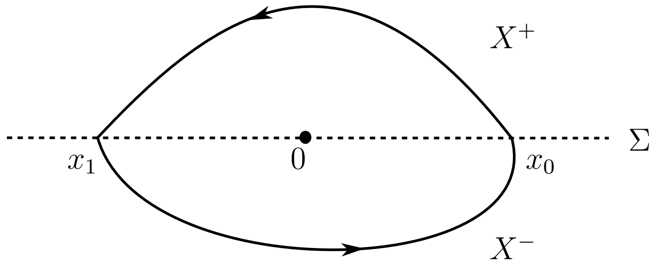

FIGURE 3. Drawing of a limit cycle and the relative position of the intersection points with the discontinuity line, $y = 0$.

Lemma 5.1. *Let $\mathcal{X}$ be the vector field* (2) *having a $\Sigma-$monodromic singularity. If $\mathcal{X}$ has a limit cycle γ, it is not restrictive to assume $\mu_1^+ = 1$, $\mu_1^- = -1$, and $\mu_0^- = 1$. Consequently, if $A^{\pm}$ denote the areas of γ in $\Sigma^{\pm}$, respectively, then $A^+ = A^-$.*

Proof. Proposition 4.5 provides the necessary conditions for the existence of a limit cycle for (2). The change of variables $(x, y, t) \to (x, -y, -t)$ if necessary, as in the proof of Lemma 2.1, allow us to assume that $\mu_0^- \neq 0$. The proof of the first part of the statement finishes doing different rescalings in $\Sigma^{\pm}$ for the variable y and the time but the same for the variable x. Indeed the qualitative behavior of the orbits does not change and the orbits that connect in Σ also connect after the rescalings. The proof of the second part of the statement follows directly applying Green's formula. □

Now, given a point $(x_0, 0) \in \Sigma$, we define $(x_1, 0) \in \Sigma$ as the first point such that the solution γ of the vector field that begins in $(x_0, 0)$ intersects Σ, see Figure 3. In the result below, we obtain new relations from γ involving just x_0, x_1 and the parameters of the vector field, i.e., not depending of the time. It provides the intersection points of any periodic orbit, in fact the half return maps, with the line Σ, depending on the matrix that define the linear vector field. Here we only consider the vector field X on Σ^+.

Lemma 5.2. *Let μ_0, μ_2 be real numbers. For $\mu_0 \leq 0$, consider $x_1 < 0 < x_0$ such that there exists the smallest positive real number τ satisfying that the boundary value problem*

$$(x', y') = (\mu_0 + x + \mu_2 y, x),\ (x(0), y(0)) = (x_0, 0),\ (x(\tau), y(\tau)) = (x_1, 0) \tag{4}$$

has a unique solution. Denoting $s = \sqrt{1 + 4\mu_2}$, $\tilde{s} = \sqrt{-1 - 4\mu_2}$, and $\alpha = (s - 1)/(s + 1)$, then the next conditions hold:

(i) When $\mu_2 > 0$ then $s > 1$, $\alpha \in (0, 1)$. Considering

$$z = \frac{2\mu_0 + x_0(1 + s)}{2\mu_0 + x_1(1 + s)}, \quad w = \frac{2\mu_0 + x_1(1 - s)}{2\mu_0 + x_0(1 - s)}, \tag{5}$$

we get the relation $w = z^{\alpha}$, where $z, w \in (0, 1]$.

(ii) For $\mu_2 = 0$, we consider

$$z = \frac{\mu_0 + x_0}{\mu_0 + x_1}, \quad w = \frac{x_0 - x_1}{\mu_0}, \tag{6}$$

and we obtain that $\alpha = 0$ and $w = \log z$, where $z \in (0, 1]$, and $w \in (-\infty, 0)$.

(iii) When $-1/4 < \mu_2 < 0$ then $\alpha \in (-1, 0)$. Considering z and w defined as in (5), we get the relation $w = z^\alpha$, where $z \in (0, 1]$ and $w \in [1, \infty)$.

(iv) When $\mu_2 = -1/4$ then $\alpha = -1$. Let z, w be given by

$$z = \frac{2\mu_0 + x_0}{2\mu_0 + x_1}, \quad w = \frac{2\mu_0(x_0 - x_1)}{(2\mu_0 + x_0)(2\mu_0 + x_1)}. \tag{7}$$

So, $w = \log z$ where $z \in (0, 1]$, and $w \in (-\infty, 0]$.

(v) For $\mu_2 < -1/4$ and $\mu_0 < 0$ we consider

$$z = \frac{x_1^2\tilde{s}^2 + (x_1 + 2\mu_0)^2}{x_0^2\tilde{s}^2 + (x_0 + 2\mu_0)^2}, \quad w = \frac{2\mu_0\tilde{s}(x_1 - x_0)}{2\mu_0(x_1 + x_0 + 2\mu_0) + x_1x_0(1 + \tilde{s}^2)} \tag{8}$$

only when $2\mu_0(x_1+x_0+2\mu_0)+x_1x_0(1+\tilde{s}^2) \neq 0$ and we obtain that $\arctan(w) = \log z^{\tilde{s}/2} - (\operatorname{sgn}(w) - 1)\pi/2$ where $z \in [1, \exp(2\pi/\tilde{s}))$, and $w \in [0, \infty)$. If $\mu_2 < -1/4$ and $\mu_0 = 0$, we get that $z = \tilde{s}^2x_1^2/x_0^2$ and $w \in \mathbb{R}$.

Proof. The proof will be done in a case by case study. Notice that because of the definition of τ and (4) $y(\tau)$ does not change its sign for the orbit joining the points $(x_0, 0)$ and $(x_1, 0)$, being $x_1 < 0 < x_0$.

(i) The condition $\mu_2 > 0$ implies that $s > 1$ and $0 < \alpha < 1$. Solving the boundary value problem (4), we can write

$$\begin{aligned} \exp\left(\frac{1+s}{2}\tau\right) &= \frac{2\mu_0 + x_1(1+s)}{2\mu_0 + x_0(1+s)} = \frac{1}{z}, \\ \exp\left(\frac{1-s}{2}\tau\right) &= \frac{2\mu_0 + x_1(1-s)}{2\mu_0 + x_0(1-s)} = w, \end{aligned} \tag{9}$$

and the relation (5) is done. Then equating the flight time τ in the above equalities and using the definition of α we obtain the relations of the statement for z, w.

The singular point, that is of saddle type, $(0, -\mu_0/\mu_2)$ belongs in $\Sigma^+ = \{y > 0\}$ and the intersections of the invariant straight lines with the x-axis are the points $(\tilde{x}_0, 0)$ and $(\tilde{x}_1, 0)$ where $\tilde{x}_0 = -2\mu_0/(1 + s) > 0$ and $\tilde{x}_1 = -2\mu_0/(1 - s) < 0$. Hence, $\tilde{x}_1 < x_1 \leq 0 \leq x_0 < \tilde{x}_0$, and $0 < (2\mu_0 + x_0(1 + s))/(2\mu_0 + x_1(1 + s)) = z \leq 1$. Similarly we get $0 < w \leq 1$. See Figure 4(a) for an illustration of all the points that play a special role.

(ii) The condition $\mu_2 = 0$ gives $s = 1$ and $\alpha = 0$. Arguing similarly to the preceding case the solution writes as

$$\exp(-\tau) = z, \quad -\tau = w, \tag{10}$$

where z and w satisfy (6). There are no singular points and $x = -\mu_0 = \tilde{x}_0$ is an invariant straight line. Hence, $x_1 \leq 0 \leq x_0 < \tilde{x}_0$, and, from (10), $z \in (0, 1]$ and $w \in (-\infty, 0]$. See Figure 4(b).

(iii) The condition $\mu_2 \in (-1/4, 0)$ gives $s \in (0, 1)$ and $\alpha \in (-1, 0)$. The solutions of the differential equation (4) write as (9). In this case the singular point $(0, -\mu_0/\mu_2)$ is a node and belongs in $\Sigma^- = \{y < 0\}$. The intersections of the invariant straight lines with the x-axis are the points $(\tilde{x}_0, 0)$ and $(\tilde{x}_1, 0)$ where $\tilde{x}_0 = -2\mu_0/(1 + s) > 0$

and $\tilde{x}_1 = -2\mu_0/(1-s) > \tilde{x}_0 > 0$. Using (9) we have that $z \in (0,1]$ and $w \in [1,\infty)$. See Figure 4(c).

(iv) For $\mu_2 = -1/4$, we get that $s = 0$, $\alpha = -1$, $(0, 4\mu_0) \in \Sigma^-$ is a degenerate node and the intersection of the invariant straight line with the x-axis is the point $(\tilde{x}_0, 0)$ where $\tilde{x}_0 = -2\mu_0 > 0$. Relation (7) follows, solving the differential equation (4). That is

$$\begin{aligned} \exp\left(\frac{\tau}{2}\right) &= \frac{2\mu_0 + x_1}{2\mu_0 + x_0} = \frac{1}{z}, \\ -\frac{\tau}{2} &= \frac{2\mu_0(x_0 - x_1)}{4\mu_0^2 + 2\mu_0(x_0 + x_1) + x_0 x_1} = w. \end{aligned} \tag{11}$$

Now, using (11) we get $z \in (0,1]$ and $w \in (-\infty, 0]$. See Figure 4(d).

(v) Finally we consider $\mu_2 < -1/4$, $\mu_0 < 0$ and $\tilde{s} > 0$. The solutions of the differential equation (4) write as

$$\exp(\tau) = z, \quad \arctan(w) = \frac{\tilde{s}\tau}{2},$$

where z and w are defined in (8). In this last case $(0, -\mu_0/\mu_2) \in \Sigma^-$ is a focus and, using the nonvanishing condition of the statement, we get $z \in [1, \exp(2\pi/\tilde{s}))$ and $w \in [0, \infty)$. See Figure 4(e). If $\mu_0 = 0$ then the origin is a focus and the flight time, $\tau = 2\pi/\tilde{s}$, is constant. Hence the orbit moves to the straight line, in (z,w)-coordinates, $z = \exp(2\pi/\tilde{s})$. □

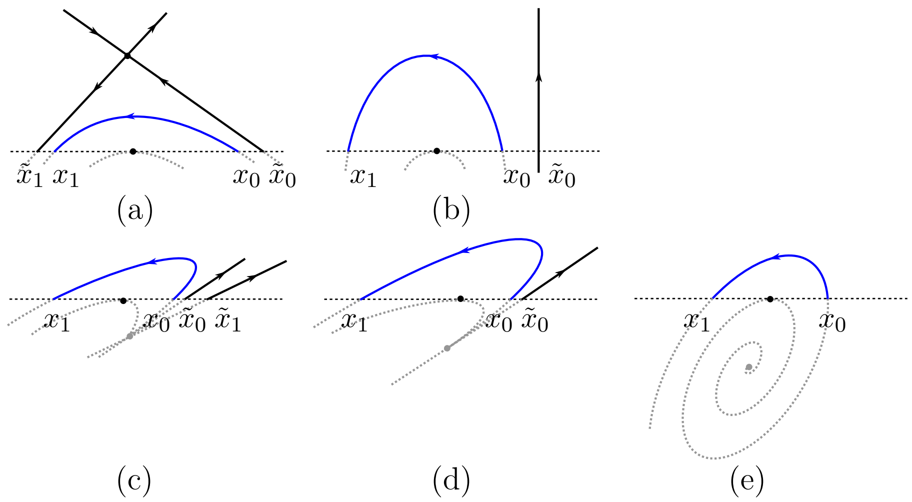

FIGURE 4. Phase portraits of vector field X only considered on Σ^+.

Before proving the uniqueness of limit cycles we introduce two technical results. The first one provides, for the vector fields associated to the half return map $w = \varphi(z)$ that relates the points x_0 and x_1 of the above lemma, the change of variables that gives the analog of the Flow Box Theorem in a half plane. The second one is an extension of Rolle's Theorem for curves that intersect an orbit of a vector field. See Figure 5. Although we do not use the first in our proofs, we present the conjugations for all cases to better understanding the relative position of the curves. See for instance Figure 6. In fact we illustrate how this conjugation acts for the first case in the proof of Proposition 5.5.

Lemma 5.3. *Let ξ be a real number. Next statements hold:*

(i) The change of variables $(u,v)=(z,z^{-\xi}w)$ conjugates $(z',w')=(z,\xi w)$ defined in $z>0$ with $(u',v')=(u,0)$ defined in $u>0$. Moreover, the curve $w-z^{\xi}=0$ moves to the straight line $v-1=0$.

(ii) The change of variables $(u,v)=(z,\log z-w)$ conjugates $(z',w')=(z,1)$ defined in $z>0$ with $(u',v')=(u,0)$ defined in $u>0$. Moreover, the curve $w-\log z=0$ moves to the straight line $v=0$.

(iii) The change of variables $(u,v)=(z,\xi\log z-\arctan w)$ conjugates $(z',w')=(z,\xi(1+w^2))$ defined in $z>0$ with $(u',v')=(u,0)$ defined in $u>0$. Moreover, the curve $w-\tan\log z^{\xi}=0$ moves to the straight line $v=0$.

Theorem 5.4 ([17])**.** *Let X be a $\mathcal{C}^1$ planar vector field without singular points in an open region $\Omega\subset\mathbb{R}^2$. If a $\mathcal{C}^1$ curve, $\zeta\subset\Omega$, intersects an integral curve of X at two points then, in between these points, there exists a point of tangency between ζ and X. See Figure 5.*

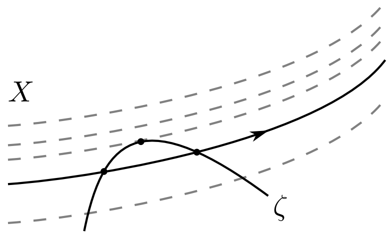

FIGURE 5. Extension of Rolle's Theorem for dynamical systems on the plane.

Finally, in the next propositions and following Lemma 5.2, we will prove the uniqueness result when the vector field X^+ has a saddle, a node, an invisible fold (no singularity), a degenerate node, or a focus, respectively.

Proposition 5.5. *The vector field* (2) *with $\mu_2^+>0$, $\mu_1^+=1$, $\mu_1^-=-1$, and $\mu_0^-=1$ has at most one limit cycle when $\mu_0^+<0$ and has no limit cycles when $\mu_0^+=0$.*

Proof. As $\mu_2^+>0$, the singularity of X^+ is in Σ^+ and it is a real saddle. When $\mu_0^+=0$ the vector field in Σ^+ has two invariant straight lines passing trough the origin, that is a singularity. Hence there are no limit cycles. From now on we consider only $\mu_0^+<0$.

Following the notation introduced in Lemma 5.2, the different cases to be studied will be labeled as (i^+,i^-), (i^+,ii^-), (i^+,iii^-), (i^+,iv^-), and (i^+,v^-). Notice that these cases correspond to the different phase portraits of the vector fields (X^+,X^-).

We start proving the saddle-saddle case in detail, hence $\mu_2^->0$. Using the boundary value problem of Lemma 5.2 we know that, when $y>0$, doing the inverse of the change of variables (5) we obtain

$$\begin{aligned} x_0 &=2\mu_0^+\frac{-1+s^+-2s^+z+(1+s^+)zw}{(1-(s^+)^2)(1-zw)},\\ x_1 &=2\mu_0^+\frac{-1-s^++2s^+w+(1-s^+)zw}{(1-(s^+)^2)(1-zw)}, \end{aligned} \tag{12}$$

with $s^+ = \sqrt{1+4\mu_2^+}$. We remark that (12) is well defined because $z, w \in (0,1)$ and $s^+ > 1$. Finally we can write the solution that goes from $(x_0, 0)$ to $(x_1, 0)$ with the flow X^+ as $w = z^\alpha$ with $\alpha = (s^+ - 1)/(s^+ + 1) \in (0,1)$. Hence the curve $C_f = \{f(z,w) = 0\}$, associated to the upper half return map, is well defined in the open square $S = (0,1) \times (0,1)$, where $f(z,w) = w - z^\alpha$.

When $y < 0$, doing the change $(x, y, t) \to (x, -y, -t)$, we use again Lemma 5.2 with $\mu_0 := -\mu_0^- = -1$ and $\mu_2 := \mu_2^-$ for writing the solution that goes from $(x_1, 0)$ to $(x_0, 0)$ with the flow X^- as $W = Z^\beta$ where

$$Z = \frac{2 + x_0(-1 - s^-)}{2 + x_1(-1 - s^-)}, \quad W = \frac{2 + x_1(s^- - 1)}{2 + x_0(s^- - 1)}, \tag{13}$$

$s^- = \sqrt{1+4\mu_2^-}$, and $\beta = (s^- - 1)/(s^- + 1)$. Straightforward computations show that there exist real numbers z_β and w_β such that the lower half return map writes as the curve $C_F = \{F(z,w) = 0\}$, with $F(z,w) = W(z,w) - Z(z,w)^\beta$, and it is well defined in the open square $\tilde{S} = (z_\beta, 1) \times (w_\beta, 1)$.

From (12) and (13) the limit cycles of (2), under the hypotheses of the statement, correspond with the intersections of the curves C_f and C_F. The proof of the proposition follows proving that $(1,1)$ is an intersection point where both curves are tangent and, when they are not coincident, there is at most one intersection point in the open square $S \cap \tilde{S}$.

When $z \to 1^-$ we have that $w \to 1^-$, $x_0 \to 0^-$, and $x_1 \to 0^+$. Consequently also $Z \to 1$ and $W \to 1$ and both curves pass through the point $(1,1)$.

The drawing of C_f and C_F with respect to the vector field $(z, \alpha w)$ and the transformation to the (u,v)-plane is done in Figure 6.

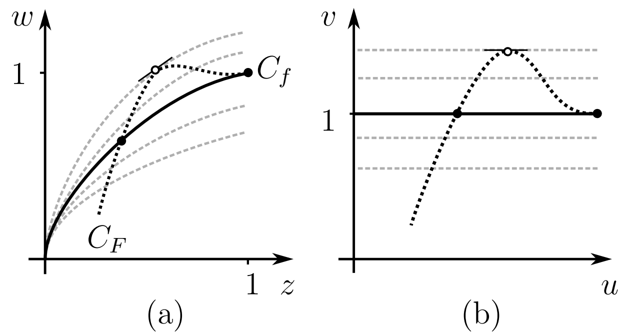

FIGURE 6. Drawing of curves C_f and C_F in (z,w) and (u,v) planes.

We will prove the uniqueness of the intersection points of the curves C_f and C_F in $S \cap \tilde{S}$ by contradiction. First we prove the result when there exist two simple intersection points ι_1 and ι_2 in $S \cap \tilde{S}$, afterwards we deal with the case when they are tangent intersection points. Hence by Lemma 5.3(i) and Theorem 5.4 there exist two tangential points ι_3 and ι_4 in S such that they are solutions of system

$$\{F(z,w) = 0, G(z,w) = 0\}, \tag{14}$$

with

$$G(z,w) = (\nabla F(z,w) \cdot (z, \alpha w))|_{\{F(z,w)=0\}}. \tag{15}$$

In particular, ι_3 and ι_4 are in the arcs of C_F defined by ι_1 and ι_2, and ι_2 and $(1, 1)$, respectively. See Figure 7 for an illustration of the relative position of all these points with respect to the curves C_f and C_F.

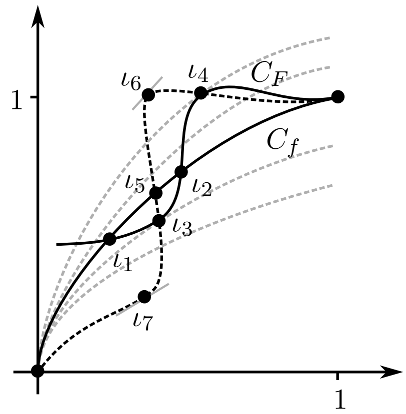

FIGURE 7. Curves C_f and C_F having two intersection points in S.

The function G in (15) writes

$$G(z,w) = \left(\frac{\partial W}{\partial z} - \beta\frac{W}{Z}\frac{\partial Z}{\partial z}\right) z + \left(\frac{\partial W}{\partial w} - \beta\frac{W}{Z}\frac{\partial Z}{\partial w}\right)\alpha w.$$

Let

$$\delta = (1 - \beta - \mu_0^+\beta + \mu_0^+\alpha\beta)(-1 + \beta - \mu_0^+ + \mu_0^+\alpha) \tag{16}$$

be a real number. If $\delta = 0$ then $G(z, w) \equiv 0$ so equation (2) has a center at the origin. Otherwise we get $G(z, w) = f_1(z, w)f_2(z, w)$ where

$$\begin{aligned} f_1(z,w) &= \alpha^2(\beta+1)(\alpha+1)\beta\delta\frac{x_0x_1\widehat{x}_0^2}{Z\widehat{Z}^2\widehat{W}^2}, \\ f_2(z,w) &= (1-z)^2w + \lambda z(1-w)^2, \end{aligned} \tag{17}$$

and

$$\lambda = (-\alpha + \alpha\beta + \mu_0^+ - \mu_0^+\alpha)(-\alpha + \alpha\beta - \mu_0^+\beta + \mu_0^+\alpha\beta)\alpha^{-2}\delta^{-1}. \tag{18}$$

Here $\widehat{x}_0$, $\widehat{Z}$, and $\widehat{W}$ are the denominators of x_0, $Z(z, w)$, and $W(z, w)$, respectively. Their explicit expressions in these new coordinates are

$$\begin{aligned} \widehat{x}_0 =& \alpha(zw-1), \\ \widehat{Z} =& \alpha(\alpha\mu_0^+ + \beta - \mu_0^+ - 1)zw - \mu_0^+(\alpha^2-1)w - \alpha\beta + \alpha\mu_0^+ + \alpha - \mu_0^+, \\ \widehat{W} =& (-\alpha\beta(1+\mu_0^+) + \beta\mu_0^+ + \alpha)zw + \beta\mu_0^+(\alpha^2-1)z - \alpha(1 - \beta + \mu_0^+(\alpha-1)\beta). \end{aligned}$$

We remark that from (13) and the interval of definition of x_0 and x_1 in Lemma 5.2 all these denominators do not vanish. Consequently f_1 and f_2 are well defined in $S \cap \tilde{S}$. Moreover, as f_2 is a polynomial it is also well defined in $\mathbb{R}^2$.

Hence, as $f_1(z, w)$ does not vanish in the open square $S \cap \tilde{S}$, the solutions of system (14) coincide with the solutions of system $\{F(z, w) = 0, f_2(z, w) = 0\}$. Moreover, the points $(0, 0)$, ι_3, ι_4 and $(1, 1)$ are in the curve $C_{f_2} = \{f_2(z, w) = 0\}$. This implies that the curves C_f and C_F are tangent at the point $(1, 1)$ and there exist ι_5 in

$C_f \cap C_{f_2}$. Using Theorem 5.4 again, there exist two tangential points ι_6, ι_7 in C_{f_2} such that they are solutions of

$$\{f_2(z,w)=0, f_3(z,w)=0\}, \qquad (19)$$

where $f_3(z,w) = (\nabla f_2(z,w)\cdot(z,\alpha w))|_{\{f_2(z,w)=0\}} = \lambda(2\alpha+1)zw^2 + (\alpha+2)z^2w - 2(\lambda+1)(\alpha+1)zw + \lambda z + \alpha w$.

The solution of system (19) can be viewed as the intersection of the algebraic curves C_{f_2} and $C_{f_3} = \{f_3(z,w)=0\}$, that are well defined in the full region S. Using Gröbner basis theory, see for instance [18], system (19) is equivalent to

$$\{w^2 + \frac{2(2+(\alpha^2+1)\lambda)}{\lambda(\alpha^2-1)}w + 1 = 0, z + \alpha\lambda w + \frac{(\alpha-1)(\alpha\lambda-1)}{\alpha+1} = 0\},$$

that only has one solution in S. This contradicts the existence of two intersection points in $\{C_f \cap C_F\}$.

When the intersection points ι_1 and ι_2 of C_f and C_F, see Figure 7, are tangent and have odd multiplicity, the proof done for simple intersection points is also valid. If there are two different intersection points with $z \in (0,1)$ and even multiplicity, the corresponding points ι_1, ι_2, ι_3, and ι_5, coincide and, consequently, the tangent point ι_5 belongs to the curves C_f, C_F and C_{f_2}, see again Figure 7. This also contradicts the fact that the curves C_{f_2} and C_{f_3} have no more than one intersection point in the open region S and, consequently, the case (i^+, i^-) is proved.

The proof of uniqueness of limit cycle for all remaining cases follows in a similar way as it is done for (i^+, i^-). We will only indicate the differences. The curve C_F is defined from the corresponding equation (13), according with Lemma 5.2. Consequently the domain $\tilde{S}$ also changes. Straightforward computations show that the expressions for $f_1(z,w)$, λ, and δ given in (16), (17), and (18) change for the cases ii^-, iv^-, and v^- and they are done in the next table:

	$f_1(z,w)$	λ	δ
ii^-	$(\alpha+1)\alpha^2\delta\dfrac{x_0x_1\widehat{x}_0^2}{Z\widehat{W}^2\widehat{Z}^2}$	$\dfrac{\mu_0^+(\alpha-1)+\alpha}{\alpha\delta}$	$\mu_0^+(\alpha-1)-1$
iv^-	$(\alpha+1)\alpha^2\delta^2\dfrac{x_0x_1\widehat{x}_0^2}{\widehat{W}^2}$	$-\dfrac{(\mu_0^+(\alpha-1)+2\alpha)^2}{\alpha^2\delta^2}$	$\mu_0^+(\alpha-1)-2$
v^-	$-(\alpha+1)\alpha^2\delta(\tilde{s}^2+1)\tilde{s}\dfrac{x_0x_1\widehat{x}_0^2}{\widehat{W}^2}$	$\dfrac{4(1-\alpha^2)(\mu_0^++1)-\delta}{\alpha^2\delta}$	$(\mu_0^+(\alpha-1)-2)^2$ $+\tilde{s}^2(\alpha-1)^2(\mu_0^+)^2$

As in the case i all the denominators do not vanish because of the interval of definition of the change introduced in Lemma 5.2. □

Proposition 5.6. *The vector field* (2) *with* $-1/4 < \mu_2^+ < 0$, $\mu_1^+ = 1$, $\mu_1^- = -1$, *and* $\mu_0^- = 1$ *has at most one limit cycle when* $\mu_0^+ < 0$ *and has no limit cycles when* $\mu_0^+ = 0$.

Proof. The proof is equivalent to the proof of Proposition 5.5 because the expressions for x_0, x_1 in Lemma 5.2 for cases i and iii are exactly the same. In this case the working region is $S = (0,1)\times(1,\infty)$. □

Proposition 5.7. *The vector field* (2) *with* $\mu_2^+ = 0$, $\mu_1^+ = 1$, $\mu_1^- = -1$, *and* $\mu_0^- = 1$ *has at most one limit cycle when* $\mu_0^+ < 0$ *and has no limit cycles when* $\mu_0^+ = 0$.

Proof. As the procedure is the same used in the proof of Proposition 5.5 we will only indicate the main differences, that are only for $\mu_0^+ < 0$. We remark that the cases (ii^+, i^-) and (ii^+, iii^-) are equivalent to (i^+, ii^-) and (iii^+, ii^-), respectively. So the cases (ii^+, ii^-), (ii^+, iv^-), and (ii^+, v^-) are the ones that remain to be studied.

Lemma 5.2 provides the new expressions for the curves C_f and C_F. Consequently the region $S = (0,1) \times (-\infty, 0)$. According Lemma 5.3 equation (15) and the system of equations (14) write as

$$\{F(z,w) = 0, \nabla F(z,w) \cdot (z,1) = 0\} \tag{20}$$

and $G(z,w) = f_1(z,w) f_2(z,w)$ with $f_1 \neq 0$ and f_2 are the functions given in the next table:

	$f_1(z,w)$	$f_2(z,w)$	λ	δ
ii^-	$-(\mu_0^+ + 1)\dfrac{x_0 x_1 \widehat{x}_0^2}{Z\widehat{Z}^2}$	1		
iv^-	$\delta^2 \dfrac{x_0 x_1 \widehat{x}_0^2}{\widehat{W}^2}$	$(z-1)^2 + \lambda z w^2$	$-\dfrac{(\mu_0^+)^2}{\delta^2}$	$\mu_0^+ + 2$
v^-	$-\delta \tilde{s}(\tilde{s}^2+1)\dfrac{x_0 x_1 \widehat{x}_0^2}{\widehat{W}^2}$	$(z-1)^2 + \lambda z w^2$	$-\dfrac{(\mu_0^+)^2(\tilde{s}^2+1)}{\delta}$	$(\mu_0^+)^2 \tilde{s}^2 + (\mu_0^+ + 2)^2$

Finally, the proof that (20) has only one intersection point for $z \in (0,1)$ follows similarly as in the proof of Proposition 5.5 for the equivalent polynomial equation (19) and the polynomial vector field $(z', w') = (z, 1)$. As the working region S is noncompact, we can consider the intersection problem in the Poincaré disc, see [9]. The corresponding figure is equivalent to Figure 7. We do not have indicate the region $\tilde{S}$ because the respective curves C_{f_2} and C_{f_3} are well defined in S. □

Proposition 5.8. *The vector field* (2) *with* $\mu_2^+ = -1/4$, $\mu_1^+ = 1$, $\mu_1^- = -1$, *and* $\mu_0^- = 1$ *has at most one limit cycle when* $\mu_0^+ < 0$ *and has no limit cycles when* $\mu_0^+ = 0$.

Proof. As in the proofs of Propositions 5.6 and 5.7, we will only indicate the main differences with the proof of Proposition 5.5. In fact, by the equivalences (iv^+, i^-), (iv^+, ii^-), and (iv^+, iii^-) with (i^+, iv^-), (ii^+, iv^-), and (iii^+, iv^-), respectively, the remaining cases to prove are (iv^+, iv^-) and (iv^+, v^-).

Lemma 5.2 provides the new expressions for the curves C_f and C_F and the region $S = (0,1) \times (-\infty, 0)$. According Lemma 5.3 equation (15) and the system of equations (14) write as

$$\{F(z,w) = 0, \nabla F(z,w) \cdot (z,1) = 0\} \tag{21}$$

and $G(z,w) = f_1(z,w) f_2(z,w)$. Here $f_2(z,w) = (z+1)w + \lambda(z-1)$ and $f_1 \neq 0$ and λ are given in the next table:

	$f_1(z,w)$	λ	δ
iv^-	$-\delta^2\dfrac{z(1-z)x_0x_1\widehat{x}_0^2}{4\widehat{W}^2}$	$-\dfrac{2\mu_0^+}{\delta}$	$\mu_0^+ + 1$
v^-	$\delta\tilde{s}(\tilde{s}^2+1)\dfrac{z(1-z)x_0x_1\widehat{x}_0^2}{4\widehat{W}^2}$	$-\dfrac{2\mu_0^+(\mu_0^+(\tilde{s}^2+1)+1)}{\delta}$	$(\mu_0^+)^2\tilde{s}^2+(\mu_0^++1)^2$

As in the proof of Proposition 5.7 we show that (21) has only one intersection point for $z \in (0,1)$. In fact, the relative position of the curves and the intersection points, in the contradiction argument, is also equivalent to Figure 7. □

Proposition 5.9. *The vector field* (2) *with* $\mu_2^+ < -1/4$, $\mu_1^+ = 1$, $\mu_1^- = -1$, $\mu_0^- = 1$ *and* $\mu_0^+ \leq 0$ *has at most one limit cycle.*

Proof. The cases (v^+, i^-), (v^+, ii^-), (v^+, iii^-) and, (v^+, iv^-) were studied in the Propositions 5.5, 5.7, 5.6, and, 5.8, respectively. The remaining case (v^+, v^-) with $\mu_0^+ < 0$ was studied in [20] and for $\mu_0^+ = 0$ in [12]. An alternative proof when $\mu_0^+ = 0$ comes also from Lemma 5.2. In fact, the change of variables moves every orbit of X^+ to a vertical straight line. Hence the corresponding curves in that coordinates can only cut in at most one point. So, we conclude this proof. □

We remark that the key point for the proofs of the above propositions is that the expressions for z, w as a function of x_0, x_1, given in Lemma 5.2, are birrational for all cases, except when the singular point is a focus (case v). This is the reason why the procedure detailed above does not work for the case (v^+, v^-), i.e. the focus-focus case, and $\mu_0^+\mu_0^- < 0$.

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