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PERIODIC SOLUTIONS OF A PERIODIC FITZHUGH-NAGUMO DIFFERENTIAL SYSTEM

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Abstract. Recently some interest has appeared for the periodic FitzHugh-Nagumo differential systems. Here we provide sufficient conditions for the existence of periodic solutions in such differential systems.

1. Introduction and statements of the main result

The FitzHugh–Nagumo (or FHN) differential systems are simplified models of the Hodgkin–Huxley differential system which is a simpler mathematical model for studying the nerve membrane [4], see for more details the articles [5, 9, 8, 10] and references therein.

In this work we study the existence of periodic solutions of the following periodic FitzHugh–Nagumo differential system

$$
\begin{aligned}
\frac{dv}{dt} &= \quad \dot{v} = -v(v-1)(v-b(t)) - w + I(t), \\
\frac{dw}{dt} &= \quad \dot{w} = av - cw.
\end{aligned} \tag{1}
$$

Here v is the analog of the nerve membrane potential, w represents ion concentrations; I is an applied current (stimulus current). This model (1) can be considered as an extension of the FHN model, because it includes a time-varying threshold given by $b(t)$ which corresponds, for example, to the threshold between electrical silence and electrical firing (see details in [3], [1] and references therein). Also this model includes a periodic forcing $I(t)$. Periodical forcing has been considered in [7]. In all this paper both functions $b(t)$ and $I(t)$ are T–periodic, i.e. $b(t+T) = b(t)$ and $I(t+T) = I(t)$ for all $t \in \mathbb{R}$.

The system with I and b constant has been considered by several authors, for example, in [6] using techniques from the ordinary differential equations is analyzed the possibility of obtaining relaxation wave solutions and the asymptotic solution having the structure of a relaxation wave has been constructed.

The system (1) with $I(t)$ periodic and $b(t)$ constant was considered in [2]. A numerical study using the Poincaré map is made, and in particular the

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authors analyzed the consequences of imposing a sinusoidal perturbation of the form $I_0 + I\cos\gamma t$ upon the base current. A similar model of FHN was studied in [7]. They proved that the phase-locking occurs independently of the magnitude of the periodic forcing. The existence of periodic solutions is showed using a classical result of second order differential equations. Faghih et al in [3] revisited the issue of the utility of the FitzHugh–Nagumo model for capturing neuron firing behaviors. They model this situation considering $b(t)$ as a periodic function and I constant.

To state our main result we consider the periodic differential system (1) defined in the open square $D = \{(v,w) : -r < v, w < r\} \subset \mathbb{R}^2$ where r is an arbitrary real number as large as we want.

Let b_0 and I_0 be the averaged value of the function $b(t)$ and $I(t)$, i.e.

$$b_0 = \frac{1}{T}\int_0^T b(t)dt, \qquad I_0 = \frac{1}{T}\int_0^T I(t)dt.$$

Now we state our main results.

Theorem 1. *Assume $c \neq 0$, $d = (a+b_0c)^2 - 4c^2I_0 > 0$ and $(a, b(t), c, I(t)) = (\varepsilon^2\alpha, \varepsilon\beta(t), \varepsilon\gamma, \varepsilon^2\lambda(t))$. Then for $\varepsilon \neq 0$ sufficiently small the FitzHugh–Nagumo model* (1) *has two T–periodic solutions $(v_\pm(t,\varepsilon), w_\pm(t,\varepsilon))$ such that*

$$(v_\pm(0,\varepsilon), w_\pm(0,\varepsilon)) = \left(\frac{a + b_0c \pm \sqrt{d}}{2c} + O(\varepsilon^2), \frac{a(a + b_0c \pm c\sqrt{d})}{2c^2} + O(\varepsilon^3)\right).$$

Theorem 2. *Assume $a+b_0c \neq 0$ and $(a, b(t), c, I(t)) = (\varepsilon^2\alpha, \varepsilon\beta(t), \varepsilon\gamma, \varepsilon^3\lambda(t))$. Then for $\varepsilon \neq 0$ sufficiently small the FitzHugh–Nagumo model* (1) *has one T–periodic solution $(v(t,\varepsilon), w(t,\varepsilon))$ such that*

$$(v(0,\varepsilon), w(0,\varepsilon)) = \left(\frac{cI_0}{a + b_0c} + O(\varepsilon^3), \frac{aI_0}{a + b_0c} + O(\varepsilon^4)\right).$$

Theorem 3. *Assume $a \neq 0$ and $(a, b(t), c, I(t)) = (\varepsilon^2\alpha, \varepsilon\beta(t), \varepsilon^2\gamma, \varepsilon^3\lambda(t))$. Then for $\varepsilon \neq 0$ sufficiently small the FitzHugh–Nagumo model* (1) *has one T–periodic solution $(v(t,\varepsilon), w(t,\varepsilon))$ such that $(v(0,\varepsilon), w(0,\varepsilon)) = (O(\varepsilon^3), I_0 + O(\varepsilon^4))$.*

Note that when $c = 0$ Theorem 3 can be obtained from Theorem 2, but Theorem 2 also holds when $c \neq 0$.

Theorem 4. *Assume $b_0c \neq 0$ and $(a, b(t), c, I(t)) = (\varepsilon^4\alpha, \varepsilon\beta(t), \varepsilon\gamma, \varepsilon^4\lambda(t))$. Then for $\varepsilon \neq 0$ sufficiently small the FitzHugh–Nagumo model* (1) *has one T–periodic solution $(v(t,\varepsilon), w(t,\varepsilon))$ such that $(v(0,\varepsilon), w(0,\varepsilon)) = (O(\varepsilon^3), O(\varepsilon^5))$.*

In the next section we prove Theorems 1, 2, 3 and 4, using the averaging theory for computing periodic solutions. A summary of the results on the averaging theory that we shall need are given in the appendix.

Taking into account Theorem 5 of the appendix, the stability or instability of the periodic orbits found in Theorems 1 and 2 can be determined studying the eigenvalues of a convenient matrix which appears in the proof of both theorems.

We note that the assumptions of the previous four theorems can be satisfied simultaneously for convenient values of the parameters, so there are FitzHugh–Nagumo models (1) exhibiting at least 5 periodic solutions.

2. Proof of the theorems

In order to apply the averaging theory given a FitzHugh–Nagumo model (1) we define

$$
\begin{gathered}
V=\varepsilon^{-m_1}v, \qquad W=\varepsilon^{-m_2}w,\\
(\alpha,\beta(t),\gamma,\lambda(t))=(\varepsilon^{-n_1}a,\varepsilon^{-n_2}b(t),\varepsilon^{-n_3}c,\varepsilon^{-n_4}I(t)),
\end{gathered}
\tag{2}
$$

with m_1, m_2, n_1, n_2, n_3 and n_4 positive integers. Let β_0 and λ_0 be the averaged value of the function $\beta(t)$ and $\lambda(t)$, i.e.

$$
\beta_0=\frac{1}{T}\int_0^T\beta(t)dt, \qquad \lambda_0=\frac{1}{T}\int_0^T\lambda(t)dt.
$$

The differential system (1) in the new variables (V,W) writes

$$
\begin{aligned}
\dot V = {}& \varepsilon[\varepsilon^{n_4-m_1-1}\lambda(t)-e^{n_2-1}\beta V-\varepsilon^{m_2-m_1-1}W\\
&+\varepsilon^{m_1-1}(1+\varepsilon^{n_2}\beta(t))V^2-\varepsilon^{2m_1-1}V^3],\\
\dot W = {}& \varepsilon[\varepsilon^{n_1+m_1-m_2-1}\alpha V-\varepsilon^{n_3-1}\gamma W],
\end{aligned}
$$

and we assume that $n_2-1\geq 0$, $m_2-m_1-1\geq 0$, $m_1-1\geq 0$, $n_4-m_1-1\geq 0$, $n_1+m_1-m_2-1\geq 0$ and $n_3-1\geq 0$.

Proof of Theorem 1. We have $n_2=n_3=1$ and $n_1=n_4=2$ and we take $m_1=1$ and $m_2=2$. Then system (1) in the new variables (V,W) becomes

$$
\begin{aligned}
\dot V = {}& \varepsilon\left(\lambda(t)-\beta(t)V-W+V^2\right)+\varepsilon^2\left(\beta(t)V^2-V^3\right),\\
\dot W = {}& \varepsilon(\alpha V-\gamma W).
\end{aligned}
\tag{3}
$$

Note that system (3) is in the normal form for applying the averaging theory of first order, see the appendix. Moreover, the functions of the right-hand side of system (3) remain bounded. So the assumptions of Theorem 5 (see the appendix) are satisfied.

The averaged system of system (3) is

$$
\begin{aligned}
\dot V = {}& \varepsilon f_1(V,W)=\varepsilon\left(\lambda_0-\beta_0V-W+V^2\right),\\
\dot W = {}& \varepsilon f_2(V,W)=\varepsilon(\alpha V-\gamma W).
\end{aligned}
$$

These system has two equilibria, namely:

$$
(V_\pm^*,W_\pm^*)=\left(\frac{\alpha+\beta_0\gamma\pm\sqrt{\delta}}{2\gamma},\frac{\alpha(\alpha+\beta_0\gamma\pm\gamma\sqrt{\delta})}{2\gamma^2}\right),
$$

which exist by assumptions.

To guarantee for system (3) the existence of periodic solutions associated to the equilibria $(V_\pm^*, W_\pm^*)$, using Theorem 5, we need that the Jacobian (9) of $f = (f_1, f_2)$ in each equilibrium point $(V_\pm^*, W_\pm^*)$ be non–zero, and this is the case because

$$\det\left(\left.\frac{\partial(f_1,f_2)}{V,W}\right|_{(V,W)=(V_\pm^*,W_\pm^*)}\right) = \mp\sqrt{\delta} \neq 0.$$

Therefore, there are two periodic solutions $(V_\pm(t,\varepsilon), W_\pm(t,\varepsilon))$ of system (3) such that $(V_\pm(0,\varepsilon), W_\pm(0,\varepsilon)) = (V_\pm^*, W_\pm^*) + O(\varepsilon)$. Going back through the re-scaling (2), the statement of the theorem follows. □

Proof of Theorems 2. We have $n_1 = 2$, $n_2 = n_3 = 1$, $n_4 = 3$ and we take $m_1 = 2$ and $m_2 = 3$. Then system (1) in the new variables (V, W) writes

$$\begin{aligned} \dot V =& \ \varepsilon\left(\lambda(t) - \beta(t)V - W\right) + \varepsilon^2 V^2 + \varepsilon^3\beta(t)V^2 - \varepsilon^4 V^3, \\ \dot W =& \ \varepsilon\left(\alpha V - \gamma W\right). \end{aligned} \tag{4}$$

We have that the averaged system of system (4) is

$$\begin{aligned} \dot V =& \ \varepsilon f_1(V,W) = \varepsilon\left(\lambda_0 - \beta_0 V - W\right), \\ \dot W =& \ \varepsilon f_2(V,W) = \varepsilon(\alpha V - \gamma W). \end{aligned}$$

The unique equilibrium point of this system is

$$(V^*, W^*) = \left(\frac{\gamma\lambda_0}{\alpha+\beta_0\gamma}, \frac{\alpha\lambda_0}{\alpha+\beta_0\gamma}\right).$$

The existence of a periodic solution of system (4) associated to the equilibrium (V^*, W^*) needs that the Jacobian (9) of $f = (f_1, f_2)$ at (V^*, W^*) be non–zero, and this is the case because

$$\det\left(\left.\frac{\partial(f_1,f_2)}{\partial(V,W)}\right|_{(V,W)=(V^*,W^*)}\right) = \alpha + \beta_0\gamma \neq 0.$$

Hence, there is one periodic solution $(V(t,\varepsilon), W(t,\varepsilon))$ of system (4) such that $(V(0,\varepsilon), W(0,\varepsilon)) = (V^*, W^*) + O(\varepsilon)$. Going back through the re-scaling (2), the theorem is proved. □

Proof of Theorem 3. We have $n_1 = n_3 = 2$, $n_2 = 1$, $n_4 = 3$ and we take $m_1 = 2$ and $m_2 = 3$. Then system (1) becomes

$$\begin{aligned} \dot V =& \ \varepsilon\left(\lambda(t) - \beta(t)V - W\right) + \varepsilon^2 V^2 + \varepsilon^3\beta(t)V^2 - \varepsilon^4 V^3, \\ \dot W =& \ \varepsilon\alpha V - \varepsilon^2\gamma W. \end{aligned} \tag{5}$$

The averaged system of system (5) is

$$\begin{aligned} \dot V =& \ \varepsilon f_1(V,W) = \varepsilon\left(\lambda_0 - \beta_0 V - W\right), \\ \dot W =& \ \varepsilon f_2(V,W) = \varepsilon\alpha V. \end{aligned}$$

This system has the unique equilibrium point $(0, \lambda_0)$, and the determinant of the linear part at it satisfies

$$\det\left(\left.\frac{\partial(f_1, f_2)}{\partial(V, W)}\right|_{(V,W)=(0,\lambda_0)}\right) = \alpha \neq 0.$$

So, there exists one periodic solution $(V(t, \varepsilon), W(t, \varepsilon))$ of system (5) such that $(V(0, \varepsilon), W(0, \varepsilon)) = (0, \lambda_0) + O(\varepsilon)$. Going back through the re-scaling (2), the theorem is proved. □

Proof of Theorem 4. We have $n_1 = n_4 = 4$, $n_2 = n_3 = 1$ and we take $m_1 = 2$ and $m_2 = 4$. Then system (1) becomes

$$\begin{aligned} \dot{V} = &\ -\varepsilon\beta(t)V + \varepsilon^2(\lambda(t) - W + V^2) + \varepsilon^3\beta(t)V^2 - \varepsilon^4 V^3, \\ \dot{W} = &\ -\varepsilon\gamma W + \varepsilon^2\alpha V. \end{aligned} \tag{6}$$

Its averaged system is

$$\begin{aligned} \dot{V} = &\ \varepsilon f_1(V, W) = -\varepsilon\beta_0 V, \\ \dot{W} = &\ \varepsilon f_2(V, W) = -\varepsilon\gamma W. \end{aligned}$$

This system has the unique equilibrium point $(0, 0)$, and the determinant of the linear part at it satisfies

$$\det\left(\left.\frac{\partial(f_1, f_2)}{\partial(V, W)}\right|_{(V,W)=(0,0)}\right) = \beta_0\gamma \neq 0.$$

So, there exists one periodic solution $(V(t, \varepsilon), W(t, \varepsilon))$ of system (5) such that $(V(0, \varepsilon), W(0, \varepsilon)) = (0, 0) + O(\varepsilon)$. Going back through the re-scaling (2), the theorem is proved. □

Appendix:Averaging theory of first order

We shall present the basic results from averaging theory that we need for proving the results of this paper.

The next theorem provides a first order approximation for the periodic solutions of a periodic differential system, for the proof see Theorems 11.5 and 11.6 of Verhulst [11].

Consider the differential equation

$$\dot{\mathbf{x}} = \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{7}$$

with $\mathbf{x} \in D$, where D is an open subset of $\mathbb{R}^n$, $t \geq 0$. Moreover we assume that both $F_1(t, \mathbf{x})$ and $F_2(t, \mathbf{x}, \varepsilon)$ are $T-$periodic in t. We also consider in D the averaged differential equation

$$\dot{\mathbf{y}} = \varepsilon f(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \tag{8}$$

where

$$f(\mathbf{y}) = \frac{1}{T}\int_0^T F_1(t, \mathbf{y})dt.$$

Under certain conditions, equilibrium solutions of the averaged equation turn out to correspond with $T-$periodic solutions of equation (7).

Theorem 5. *Consider the two initial value problems* (7) *and* (8). *Suppose:*

(i) F_1*, its Jacobian* $\partial F_1/\partial x$*, its Hessian* $\partial^2 F_1/\partial x^2$*,* F_2 *and its Jacobian* $\partial F_2/\partial x$ *are defined, continuous and bounded by a constant independent of* ε *in* $[0,\infty)\times D$ *and* $\varepsilon \in (0,\varepsilon_0]$.

(ii) F_1 *and* F_2 *are* $T-$*periodic in* t *(*T *independent of* ε*).*

Then the following statements hold.

(a) *If* p *is an equilibrium point of the averaged equation* (8) *and*

$$\det\left(\frac{\partial f}{\partial \mathbf{y}}\right)\bigg|_{\mathbf{y}=p} \neq 0, \tag{9}$$

then there exists a $T-$*periodic solution* $\varphi(t,\varepsilon)$ *of equation* (7) *such that* $\varphi(0,\varepsilon)\to p$ *as* $\varepsilon \to 0$.

(b) *The stability or instability of the limit cycle* $\varphi(t,\varepsilon)$ *is given by the stability or instability of the equilibrium point* p *of the averaged system* (8). *In fact the singular point* p *has the stability behavior of the Poincaré map associated to the limit cycle* $\varphi(t,\varepsilon)$.

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