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EXPLICIT TRAVELING WAVES AND INVARIANT ALGEBRAIC CURVES

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Abstract. In this paper we introduce a precise definition of algebraic traveling wave solution for general n-th order partial differential equations. All examples of explicit traveling waves known by the authors fall in this category. Our main result proves that algebraic traveling waves exist if and only if an associated n-dimensional first order ordinary differential system has some invariant algebraic curve. As a paradigmatic application we prove that, for the celebrated Fisher-Kolmogorov equation, the only algebraic traveling waves solutions are the ones found in 1979 by Ablowitz and Zeppetella. To the best of our knowledge, this is the first time that this type of results have been obtained.

1. Introduction and Main Results

Mathematical modelling of dynamical processes in a great variety of natural phenomena leads in general to non-linear partial differential equations. There is a particular class of solutions for these non-linear equations that are of considerable interest. They are the traveling wave solutions [10, 12, 13, 31]. Such a wave is a special solution of the governing equations, that may be localised or periodic, which does not change its shape and which propagates at constant speed. In the case of linear equations the profile is usually arbitrary. In contrast, a non-linear equation will normally determine a restricted class of profiles, as the result of a balance between nonlinearity and dissipation. These waves appear in fluid dynamics [16, 20], chemical kinetics involving reactions [10, 21], mathematical biology [13, 28], lattice vibrations in solid state physics [24], plasma physics and laser theory [15], optical fibers [3], etc. In these systems the phenomena of dispersion, dissipation, diffusion, reaction and convection are the fundamental physical common facts.

There is an increasing interest in finding explicit exact solutions for these traveling waves. There are several standard methods for obtaining such solutions, as the inverse scattering transformation [1, 5], the Backlund transformation [1, 5], the Painlevé method [11] and the Hirota's bilinear method [16].

The inverse scattering transformation is a non-linear analog of the Fourier transform used for solving linear equations. This method allows certain non-linear problems, called integrable, to be treated by what are essentially linear methods.

The Backlund transformation allows to find solutions to a non-linear partial differential equation from either a known solution to the same equation or from a solution to another equation. This can enable one to find more complex solutions from a simple one, e.g. a multi-soliton solution from a single soliton solution.

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The Painlevé method is a procedure to detect integrable differential equations. The Lie group method is applied to a partial differential equation for finding group-invariant solutions that satisfy ordinary differential equations. Then the Painlevé property is tested for these reduced equations. An ordinary differential equation is said to have the Painlevé property if the general solution has no movable critical singularities. Movable refers to the arbitrary position of the solutions singularities in complex time. For any solution the presence and position of movable singularities is given by the initial conditions. The other type of singularities that can be found are fixed singularities.

The Hirota's direct method is employed for constructing multi-soliton solutions to integrable non-linear evolution equations. The method is based on introducing a transformation into new variables, so that in these new variables multi-soliton solutions appear in a particularly simple form. In fact they appear as polynomials of simple exponentials in the new variables. This transformation requires sometimes the introduction of new dependent and sometimes even independent variables. Expressed in the new variables the equation will be quadratic in the dependent variables (the so-called Hirota's bilinear form) and the derivatives must only appear in combinations that can be expressed using Hirota's differential operator.

We consider in this work general n-th order partial differential equations of the form

$$\frac{\partial^n u}{\partial x^n} = F\Big(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial t}, \frac{\partial^2 u}{\partial t^2}, \ldots, \frac{\partial^{n-1} u}{\partial x^{n-1}}, \frac{\partial^{n-1} u}{\partial x^{n-2}\partial t}, \ldots, \frac{\partial^{n-1} u}{\partial x \partial t^{n-2}}, \frac{\partial^{n-1} u}{\partial t^{n-1}}\Big), \tag{1}$$

where x and t are real variables and F is a smooth map. The traveling wave solutions (TWS) of (1) are particular solutions of the form $u = u(x,t) = U(x-ct)$, where $U(s)$ satisfies the boundary conditions

$$\lim_{s\to-\infty} U(s) = a \quad \text{and} \quad \lim_{s\to\infty} U(s) = b, \tag{2}$$

where a and b are solutions, not necessarily different, of $F(u,0,\ldots,0)=0$. Plugging $u(x,t) = U(x-ct)$ into (1) we get that $U(s)$ has to be a solution, defined for all $s \in \mathbb{R}$, of the n-th order ordinary differential equation

$$\begin{aligned} U^{(n)} = F\big(&U, U', -cU', U'', -cU'', c^2U'', \ldots, \\ &U^{(n-1)}, -cU^{(n-1)}, \ldots, (-c)^{n-2}U^{(n-1)}, (-c)^{n-1}U^{(n-1)}\big), \end{aligned} \tag{3}$$

where $U = U(s)$ and the derivatives are taken with respect to s. The parameter c is called the *speed* of the TWS.

We remark that although in this paper we restrict our attention to TWS associated to only one partial differential equation and $x \in \mathbb{R}$, our approach can be extended to systems of partial differential equations, with $\mathbf{u} \in \mathbb{R}^d$ and $\mathbf{x} \in \mathbb{R}^m$. In this situation, we would search for TWS of the form $u_j(\mathbf{x},t) = U_j(\mathbf{k}\cdot\mathbf{x} - ct)$, $j = 1,\ldots,d$, for some $\mathbf{k} \in \mathbb{R}^m$ and $c \in \mathbb{R}$.

Definition. We will say that $u(x,t) = U(x-ct)$ is an *algebraic TWS* if $U(s)$ is a non constant function that satisfies (2) and (3) and there exists a polynomial $p \in \mathbb{R}[z,w]$ such that $p(U(s), U'(s)) = 0$.

All the explicit TWS known by the authors are algebraic when F is a polynomial. Let us present several well-known examples.

Consider at first the Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - a\frac{\partial^2 u}{\partial x^2} = 0,$$

where $a \neq 0$ is an arbitrary constant. This equation appears in the modeling of acoustic and hydrodynamic waves, gas dynamics and traffic flow (see [30]) and has the one-parametric family of solutions

$$u(x,t) = c\Big(1 - \tanh\big(\frac{c}{2a}(x-ct)\big)\Big),$$

where c, the speed of the wave, is an arbitrary constant. For this case $p(U,U') = 2aU' + (2c-U)U$.

The famous Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

appears in several domains of physics, non-linear mechanics, water waves, etc (see [1, 5, 18, 30]). It has a one-parametric family of solutions given by

$$u(x,t) = \frac{-c}{2\cosh^2(\frac{\sqrt{c}}{2}(x-ct))},$$

where c is an arbitrary positive parameter. For this second example $p(U,U') = (U')^2 - (c+2U)U^2$.

Consider now the Boussinesq equation

$$\frac{\partial^2 u}{\partial t^2} + u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x^2} + \Big(\frac{\partial u}{\partial x}\Big)^2 - \frac{\partial^4 u}{\partial x^4} = 0.$$

This equation describes surface water waves (see [1, 16]) and has the two-parametric family of solutions

$$u(x,t) = (1-8k^2-c^2) + 12k^2\tanh^2(k(x-ct)),$$

where k and c are arbitrary constants. Here we have

$$\begin{aligned} p(U,U') =& 3(U')^2 - U^3 - 3\,(c-1)\,(1+c)\,U^2 - 3\left(c^2-1+4k^2\right)\left(c^2-1-4k^2\right)U \\ & - \left(c^2-1+8k^2\right)\left(c^2-1-4k^2\right)^2. \end{aligned}$$

We consider now the so-called improved modified Boussinesq equation

$$\frac{\partial^2 u}{\partial t^2} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x^2} - \Big(\frac{\partial u}{\partial x}\Big)^2 - \frac{\partial^4 u}{\partial x^2 \partial t^2} = 0.$$

This equation appears in the modeling of non-linear waves in a weakly dispersive medium (see for instance [17]) and has a three-parametric family of TWS given by

$$u(x,t) = c^2 - 1 + 4c^2k^2 - 8c^2mk^2 + 12c^2mk^2\,\mathrm{cn}^2(k(x-ct),m),$$

where c, k and m are arbitrary constants and $\mathrm{cn}(x,m)$ is the Jacobi elliptic function of elliptic modulus m that reduces to $\cos(x)$ when $m=0$. In this equation, this family of traveling waves and many others have been found in [39]. For this case

$$\begin{aligned} p(U,U') =& 3c^2(U')^2 + U^3 + 3\left(1-c^2\right)U^2 \\ &+ \left(48c^4\left(m-m^2-1\right)k^4 + 3\left(1-c^2\right)^2\right)U \\ &+ 64c^6\left(-1+2m\right)\left(m+1\right)\left(m-2\right)k^6 \\ &+ 48c^4\left(1-c^2\right)\left(m-m^2-1\right)k^4 + \left(1-c^2\right)^3. \end{aligned}$$

Notice also that the class of TWS given by $U(s) = q(e^{\lambda s})$ for some real number $\lambda \neq 0$ and some rational function $q \in \mathbb{R}(z)$, that are usually obtained with the so-called exp-function method ([14]), are always algebraic TWS. In this case $U'(s) = \lambda q'(e^{\lambda s})$. Write $U(s) = q_1(z)/q_2(z)$, and $U'(s) = q_3(z)/q_4(z)$, with $z = e^{\lambda s}$, for some polynomials $q_j \in \mathbb{R}[z]$, $j = 1, \ldots, 4$. Then, define

$$p(U,U') = \operatorname{Res}\big(q_2(z)U - q_1(z), q_4(z)U' - q_3(z), z\big),$$

where $\operatorname{Res}(M(z), N(z), z)$ denotes the resultant of the polynomials M and N with respect to z; see [36, p.45]. Then, clearly $p(U(s), U'(s)) = 0$ for some polynomial p, as we wanted to see.

It is known that the TWS correspond to homoclinic ($a = b$) or heteroclinic ($a \neq b$) solutions of an associated n-dimensional system of ordinary differential equations, see also the proof of Theorem 1.1. In many cases, the critical points where these invariant manifolds start and end are hyperbolic. When F is regular we get, using for instance normal form theory, that in a neighborhood of each of these points, this manifold can be parameterized as $\varphi(e^{\lambda s})$, for some smooth function φ, where λ is one of the eigenvalues of the critical points. This fact, together with the above list of examples, motivate our definition of algebraic TWS.

Our main result, which is proved in Section 2, is:

Theorem 1.1. *The partial differential equation* (1) *has an algebraic traveling wave solution with speed c if and only if the first order differential system*

$$\left\{ \begin{array}{ccl} y_1' & = & y_2, \\ y_2' & = & y_3, \\ \vdots & & \vdots \\ y_{n-1}' & = & y_n, \\ y_n' & = & G_c(y_1, y_2, \ldots, y_n), \end{array} \right. \tag{4}$$

where

$$\begin{aligned} G_c(y_1, y_2, \ldots, y_n) = F(&y_1, y_2, -cy_2, y_3, -cy_3, c^2y_3, \ldots, \\ &y_n, -cy_n, \ldots, (-c)^{n-2}y_n, (-c)^{n-1}y_n), \end{aligned}$$

has an invariant algebraic curve containing the critical points $(a, 0, \ldots, 0)$ *and* $(b, 0, \ldots, 0)$ *and no other critical points between them.*

Recall that, as usual, we will say that a differential system has an invariant algebraic curve C if this curve is invariant by the flow and moreover it is contained in the intersection of $n-1$ functionally independent algebraic varieties of codimension one. We remark that these varieties do not need to be necessarily invariant by the flow of the system.

When F is a polynomial, the condition for the existence of an algebraic TWS is that a certain polynomial differential system must have an algebraic invariant curve. The problem of determining necessary conditions for the existence of algebraic invariant curves for polynomial differential systems goes back to the work of Poincaré. This problem has been extensively investigated in the last years for the case $n = 2$, see for instance [4, 6, 22] and references therein, but for $n > 2$ the research is only beginning, see for instance [8, 23]. As a consequence, for second order partial differential equations of the form (1), our result translates the question of the existence of algebraic TWS to a related problem for which many tools are available.

We remark that explicit TWS have also been searched for by using several direct methods, such as the exp-function method and the tanh-function method and its variants, see for instance [14, 24, 25, 26, 39]. These methods are essentially based on the following idea: fix a class of functions with several free parameters and then impose conditions on the parameters to find some particular cases satisfying the corresponding equations. For instance, the four examples of algebraic TWS given above can be obtained by applying these direct methods.

On the contrary, our approach gives necessary and sufficient conditions for a partial differential equation to have explicit algebraic TWS. To the best of our knowledge, this is the first time that this type of results have been obtained. As a paradigmatic example, we apply our method to the celebrated Fisher-Kolmogorov reaction-diffusion partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u\,(1-u), \tag{5}$$

introduced in 1937 in the classical papers [7, 19] to model the spreading of biological populations; see also [9] for some recent results. For this equation $a = 1$ and $b = 0$ in (2). Moreover, from [7, 19], it is also known that the traveling waves only exist for $c \geq 2$. We prove:

Theorem 1.2. *The Fisher-Kolmogorov equation* (5) *has algebraic traveling wave solutions only when the speed is $c = 5/\sqrt{6}$ and they are the ones given by Ablowitz and Zeppetella (* [2]*):*

$$u(x,t) = \frac{1}{\left(1 + k e^{\frac{1}{\sqrt{6}}\left(x - \frac{5}{\sqrt{6}} t\right)}\right)^2}, \quad k > 0.$$

These explicit TWS have been found by applying the Painlevé method; see [11] for an introduction to this method.

Notice that for (5), the above function is an algebraic TWS, because the corresponding $U(s)$ satisfies

$$p(U, U') = 3(U')^2 + 2\sqrt{6}UU' + 2(1-U)U^2 = 0.$$

We remark that this family of TWS only exists for a fixed value of the speed c, while for the other examples given above the speed c is arbitrary. This can also be seen in the corresponding associated systems (4), because in all these cases, for all values of c, the system possesses an invariant algebraic curve. In fact, in the first two equations (Burgers and Korteweg-de Vries) all the solutions of the vector fields are contained in algebraic curves.

Another family exhibiting algebraic TWS for a given speed c is presented in Section 3. It includes the so-called Nagumo equation; see [27].

Our approach can be applied to characterize the existence of algebraic TWS for many other polynomial partial differential equations, like for instance the Newell-Whitehead-Segel equation([29, 34]), the Zeldovich equation([38]) or some of the equations considered in [11, 13, 32, 33, 37].

2. Proof of Theorem 1.1

In this section we prove Theorem 1.1 and give some of its consequences. Furthermore, we introduce an algebraic characterization of the planar invariant algebraic curves.

Proof of Theorem 1.1. Assume first that the partial differential equation (1) has an algebraic TWS, $u(x,t) = U(x-ct)$, with $p(U(s), U'(s)) = 0$ for some polynomial p. For the sake of notation we define $p_1 := p$ and

$$p_2(U(s), U'(s), U''(s)) := \mathcal{D}_1\, p_1(U(s), U'(s))\, U'(s) + \mathcal{D}_2\, p_1(U(s), U'(s))\, U''(s),$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ indicate partial derivatives with respect to the first and second variables of $p_1(U, U')$, respectively, and $p_2 \in \mathbb{R}[u, v, w]$. Notice that since $p_1(U(s), U'(s)) = 0$ it holds that $p_2(U(s), U'(s), U''(s)) = 0$. Doing successive derivatives we obtain $n-3$ new polynomials p_j, $j = 3, \ldots, n-1$, for which

$$p_j(U(s), U'(s), U''(s), \ldots, U^{(j)}(s)) = 0.$$

Using all the above equalities, and the fact that U gives a TWS, we obtain that the vector function

$$(y_1(s), y_2(s), \ldots, y_n(s)) = \big(U(s), U'(s), \ldots, U^{(n-1)}(s)\big)$$

is a parametric representation of a curve $\mathcal{C}$ in the phase space of the system (4) associated to (1). In fact, C is an algebraic curve, because it is contained in the intersection of the $n-1$ functionally independent algebraic hypersurfaces $p_j(y_1, \ldots, y_j) = 0, j = 1, 2 \ldots, n-1$, that is,

$$\mathcal{C} \subset \bigcap_{j=1}^{n-1} \big\{p_j(y_1, \ldots, y_j) = 0\big\}.$$

As U satisfies (2) the system has no critical points on this curve between $(a, 0, \ldots, 0)$ and $(b, 0, \ldots, 0)$. Hence the first part of the theorem follows.

Assume, to prove the converse implication, that system (4) has an algebraic invariant curve. Let

$$\mathbf{y}(s) = \big(U(s), U'(s), U''(s), \ldots, U^{(n-1)}(s)\big)$$

be the solution of system (4) associated to this curve and joining the critical points $(a, 0, \ldots, 0)$ and $(b, 0, \ldots, 0)$. By definition, this curve is included in the intersection of $n-1$ codimension one functionally independent algebraic hypersufaces $q_j(y_1, y_2, \ldots, y_n) = 0$, $j = 1, 2, \ldots, n-1$. Therefore, $U(s)$ must satisfy the $n-1$ polynomial differential equations

$$q_j(U(s), U'(s), \ldots, U^{(n-1)}(s)) = 0, \quad j = 1, 2, \ldots, n-1.$$

Doing successive resultants, we obtain that U satisfies all the resulting lower order polynomial differential equations. This procedure arrives to a polynomial first order equation $q(U(s), U'(s)) = 0$. This equation proves that the TWS is algebraic. $\square$

In view of our result, we need a method to detect when a polynomial system of ordinary differential equations has algebraic invariant curves to determine whether some polynomial partial differential equation can have algebraic TWS.

Although, as we have already explained in the introduction, there are some works dealing with this problem in the n-dimensional setting [8, 23], the planar case is the most developed one.

Consider a planar differential system,

$$\left\{ \begin{array}{lcl} x' & = & P(x,y), \\ y' & = & Q(x,y), \end{array} \right. \tag{6}$$

where P and Q are polynomials of degree at most N, and assume that there is a polynomial $g(x,y)$ such that the set $\{g(x,y) = 0\}$ is non-empty and invariant by the flow of (6). If g is not irreducible in $\mathbb{C}[x,y]$ then there exist several irreducible polynomials, $\widetilde{g}_j, j = 1, \ldots, k$, such that for each j, the corresponding set $\{\widetilde{g}_j(x,y) = 0\}$ is also non-empty and invariant by the flow of the system and $\{g(x,y) = 0\} = \cup_{j=1}^{k} \{\widetilde{g}_j(x,y) = 0\}$.

For irreducible polynomials we have the following algebraic characterization of invariant algebraic curves, which is the one that we will use in Section 4. Given an irreducible polynomial of degree n, $f(x,y)$, then $f(x,y) = 0$ is an *invariant algebraic curve* for the system if there exists a polynomial of degree at most $N-1$, $k(x,y)$, called the *cofactor of f*, such that

$$P(x,y)\frac{\partial f(x,y)}{\partial x} + Q(x,y)\frac{\partial f(x,y)}{\partial y} - k(x,y)f(x,y) = 0. \tag{7}$$

For a proof of this result see [4, 6, 22]. The above characterization is also used for n-dimensional systems to determine codimension one invariant algebraic varieties; see for instance [23]. For finding invariant algebraic curves the cofactor is then exchanged for a $(n-1) \times (n-1)$ matrix of cofactors, see [8].

3. Proof of Theorem 1.2 and other examples

Our proof of Theorem 1.2 is based on the following result, which will be proved in the next section.

Theorem 3.1. *Consider the system*

$$\left\{ \begin{array}{lcl} x' & = & -y, \\ y' & = & -x - cy + x^2, \end{array} \right. \tag{8}$$

with $c \geq 2$. Assume that it has an irreducible invariant algebraic curve that passes through the origin. Then $c = 5/\sqrt{6}$ and this curve is

$$y^2 + 2\sqrt{\frac{2}{3}}(1-x)y + \frac{2}{3}x(1-x)^2 = 0. \tag{9}$$

Proof of Theorem 1.2. Assume that (5) has an algebraic TWS, $u(x,t)=U(x-ct)$. By the results of [7, 19] we already know that $c\geq 2$. Moreover, by Theorem 1.1, the planar system

$$\begin{cases} y_1' &= y_2, \\ y_2' &= -cy_2 - y_1(1-y_1), \end{cases}$$

should have an invariant algebraic curve, $g(y_1,y_2)=0$, containing the critical points $(0,0)$ and $(1,0)$. Moreover, without loss of generality, we can assume that it is irreducible.

Taking $x=1-y_1$ and $y=y_2$ we obtain system (8). Then, it should also have an irreducible invariant algebraic curve $f(x,y)=0$, with $f(0,0)=f(1,0)=0$. By Theorem 3.1 we get that $c=5/\sqrt{6}$ and f has to be

$$f(x,y)=y^2+2\sqrt{\frac{2}{3}}(1-x)y+\frac{2}{3}x(1-x)^2.$$

The branch of $f(x,y)=0$ that contains the origin is

$$y=A(1-\sqrt{1-x})(x-1),$$

where $A:=\sqrt{6}/3$. Using the first equation of (8), that in this case is $x'=-y$, we obtain that

$$x'(s)=A(1-\sqrt{1-x(s)})(1-x(s)).$$

Returning to the function $U(s)=y_1(s)=1-x(s)$ we get the differential equation

$$U'(s)=-A\big(1-\sqrt{U(s)}\big)U(s).$$

Introducing $W(s)=\sqrt{U(s)}$ we obtain that W satisfies the logistic differential equation

$$W'(s)=-\frac{A}{2}\big(1-W(s)\big)W(s).$$

Its non-constant solutions that are defined for all $s\in\mathbb{R}$ are

$$W(s)=\frac{1}{1+ke^{\frac{A}{2}s}},\quad k>0.$$

Hence

$$U(s)=\frac{1}{(1+ke^{\frac{A}{2}s})^2}=\frac{1}{(1+ke^{\frac{1}{\sqrt{6}}s})^2}$$

and

$$u(x,t)=\frac{1}{\left(1+ke^{\frac{1}{\sqrt{6}}\left(x-\frac{5}{\sqrt{6}}t\right)}\right)^2},\quad k>0,$$

as we wanted to prove. $\square$

3.1. **A simple family with algebraic TWS.** In this subsection we consider the family of second order reaction-diffusion equations

$$\frac{\partial u}{\partial t} = -d\,f(u)(f'(u)+r) + d\frac{\partial^2 u}{\partial x^2}, \tag{10}$$

where f is a polynomial function and $d > 0$ and r are real constants. As we will see, studying its algebraic TWS we recover some of the results presented in [27, Ch.11]. In particular we will find some algebraic TWS for the Nagumo equation, related with the FitzHugh-Nagumo model for the nerve action potentials.

The planar system (4) associated to (10) is

$$\left\{\begin{array}{rcl} x' & = & y, \\ y' & = & -\dfrac{c}{d}y + f(x)(f'(x)+r). \end{array}\right. \tag{11}$$

It is easy to obtain one invariant algebraic curve for it for some particular values of the parameters.

Lemma 3.2. *When $r = c/d$, system* (11) *has the invariant algebraic curve $y - f(x) = 0$.*

Proof. Consider the algebraic curve $H(x,y) = y - f(x)$ and $r = c/d$. Then

$$\begin{aligned} y\frac{\partial H(x,y)}{\partial x} &+ \Big(-\frac{c}{d}y + f(x)\big(f'(x)+\frac{c}{d}\big)\Big)\frac{\partial H(x,y)}{\partial y} \\ &= -\,yf'(x) + \Big(-\frac{c}{d}y + f(x)\big(f'(x)+\frac{c}{d}\big)\Big) \\ &= -\Big(f'(x)+\frac{c}{d}\Big)(y-f(x)) = -\Big(f'(x)+\frac{c}{d}\Big)H(x,y). \end{aligned}$$

Hence the result follows. □

As a corollary of this lemma and the results of the previous section we have:

Corollary 3.3. *The solutions of the polynomial ordinary differential equation $p(U(s),U'(s)) = U'(s) - f(U(s)) = 0$, with adequate boundary conditions, give the algebraic TWS of equation* (10)*, $u(x,t) = U(x-ct)$ with speed $c = rd$.*

Let us apply this corollary to find algebraic TWS for the partial differential equations:

$$\frac{\partial u}{\partial t} = a(u-u_1)(u_2-u)(u-u_3) + d\frac{\partial^2 u}{\partial x^2}, \tag{12}$$

$$\frac{\partial u}{\partial t} = u^{q+1}(1-u^q) + \frac{\partial^2 u}{\partial x^2}, \tag{13}$$

where $a > 0, d > 0$, $u_1 < u_2 < u_3$ are given real constants and $q \in \mathbb{N}^+$.

Equation (12) is the Nagumo equation. Notice that it is of the type (10) since the following equality holds

$$a(u-u_1)(u_2-u)(u-u_3) = -df(u)\Big(f'(u) + \sqrt{\frac{a}{2d}}(u_1-2u_2+u_3)\Big),$$

where $f(u) = \sqrt{\frac{a}{2d}}(u-u_1)(u-u_3)$. Hence, using Corollary 3.3, we obtain that, when

$$c = \sqrt{\frac{ad}{2}}(u_1-2u_2+u_3),$$

equation (12) has the algebraic TWS, $u(x,t) = U(x-ct)$, where U satisfies

$$U'(s) = \sqrt{\frac{a}{2d}}(U(s)-u_1)(U(s)-u_3),$$

which is a logistic equation. Its non-constant solutions that are defined for all s are

$$U(s) = \frac{u_3 + ku_1 e^{\alpha(u_3-u_1)s}}{1+ke^{\alpha(u_3-u_1)s}}, \quad \text{with} \quad k>0 \quad \text{and} \quad \alpha = \sqrt{\frac{a}{2d}}.$$

Similarly, we have the equality

$$u^{q+1}(1-u^q) = -f(u)\Big(f'(u) + \frac{1}{\sqrt{q+1}}\Big),$$

where $f(u) = \frac{1}{\sqrt{q+1}}u(u^q-1)$. Applying again Corollary 3.3, with $d=1$, we obtain that when

$$c = \frac{1}{\sqrt{q+1}},$$

equation (13) has the algebraic TWS, $u(x,t) = U(x-ct)$, where U satisfies

$$U'(s) = \frac{1}{\sqrt{q+1}}U(s)(U^q(s)-1).$$

Its non-constant solutions that are defined for all s are

$$U(s) = \Big(1+ke^{\frac{q}{\sqrt{q+1}}s}\Big)^{-\frac{1}{q}}, \quad \text{with} \quad k>0.$$

We remark that studying all the invariant algebraic curves of the planar system (11) we could know whether the corresponding partial differential equation (10) does or does not have algebraic TWS with speed different from rd.

4. Algebraic invariant curves for system (8)

This section is devoted to the proof of Theorem 3.1. We need some preliminary results. The first one collects some well-known properties of the Gamma function, and also relates it with the Pochhammer symbol, $x^{[m]} := x(x+1)(x+2)\cdots(x+m-1)$.

Lemma 4.1. *For $x, y \in \mathbb{R}$ and $p, q, m \in \mathbb{N}$,*

(i) $\Gamma(x+1) = x\Gamma(x)$,

(ii) $\prod_{j=p}^{q}(x+j) = \dfrac{\Gamma(x+q+1)}{\Gamma(x+p)}$,

(iii) $\sum_{j=0}^{m}\binom{m}{j}\Gamma(x+j)\Gamma(y+m-j) = \dfrac{\Gamma(x)\Gamma(y)\Gamma(x+y+m)}{\Gamma(x+y)}$,

(iv) $\sum_{j=0}^{m}\binom{m}{j}(m-j)\Gamma(x+j)\Gamma(y+m-j) = \dfrac{my\Gamma(x)\Gamma(y)\Gamma(x+y+m)}{(x+y)\Gamma(x+y)}$,

(v) $\dfrac{\Gamma(x+m)}{\Gamma(x)} = x^{[m]}$.

The next results reduce the set of possible invariant curves and cofactors.

Proposition 4.2. *If the quadratic system* (8) *has an irreducible invariant algebraic curve of degree n, then its cofactor $k(x,y)$ must be constant, i.e. $k(x,y) \equiv c_0$, and its degree has to be even.*

Proof. Since the system (8) is quadratic ($N = 2$), the cofactor of an invariant algebraic curve of degree n, $f_n(x,y) = 0$, with

$$f_n(x,y) = h_n(x)y^n + h_{n-1}(x)y^{n-1} + \cdots + h_1(x)y + h_0(x), \tag{14}$$

is linear, i.e $k(x,y) = c_0 + c_1x + c_2y$. Then, equation (7) writes as

$$-y\frac{\partial f_n(x,y)}{\partial x} + (-x - cy + x^2)\frac{\partial f_n(x,y)}{\partial y} - (c_0 + c_1x + c_2y)f_n(x,y) = 0. \tag{15}$$

Imposing that the higher order term in y of the above equation vanishes we get the differential equation

$$c_2h_n(x) + h_n'(x) = 0.$$

Since h_n has to be a polynomial we obtain that $c_2 = 0$ and that $h_n(x)$ is a constant. Hence, without loss of generality, we can assume that $h_n(x) \equiv 1$. Then, equality (15) is equivalent to the following set of linear differential equations

$$h_{j-2}'(x) = jx(x-1)h_j(x) - ((j-1)c + c_0 + c_1x)h_{j-1}(x), \quad j = n+1, n, \ldots, 2, 1, \tag{16}$$

where $h_n(x) \equiv 1$ and $h_{n+1}(x) \equiv h_{-1}(x) \equiv 0$.

If $c_1 \neq 0$, using (16) we can obtain the degrees of the functions h_j. They are:

$$\deg(h_{n-k}) = 2k, \quad k = 0, 1, \ldots, n-1, n.$$

In particular $\deg(h_1) = 2n - 2$ and $\deg(h_0) = 2n$. From (16), for $j = 1$, we obtain that

$$-c_0h_0(x) - c_1xh_0(x) - xh_1(x) + x^2h_1(x) = 0. \tag{17}$$

Studying the higher order terms in x of this equation we get that relation (17) can never be satisfied. As a consequence $c_1 = 0$ and so $k(x,y) = c_0$ as we wanted to prove.

Consider now equation (16) with $c_1 = 0$. Assume, to arrive to a contradiction, that n is odd. Studying again the degrees of the functions h_j we get that

$$\deg(h_{n-2k}) = 3k \quad \text{and} \quad \deg(h_{n-(2k+1)}) \leq 3k + 1, \quad k = 0, 1, \ldots (n-1)/2.$$

In particular, $\deg(h_0) \leq (3n-1)/2$ and $\deg(h_1) = 3(n-1)/2$. Again, as in the case $c_1 \neq 0$, the higher order terms in x corresponding to equation (17) can not cancel. Therefore n must be even, as we wanted to prove. $\square$

Proposition 4.3. *Let*

$$f_n(x,y) = h_n(x)y^n + h_{n-1}(x)y^{n-1} + \cdots + h_1(x)y + h_0(x) = 0$$

be an irreducible invariant algebraic curve of system (8) *with even degree, $n = 2m$. Then*

$$h_0(x) = \left(\frac{2}{3}\right)^m x^{3m} + O\big(x^{3m-1}\big), \tag{18}$$

$$h_1(x) = \frac{1}{5}\left(\frac{2}{3}\right)^m \left(5c_0 - \big(5c_0 + 6mc\big)\frac{\left(\frac{5}{6}\right)^{[m]}}{\left(\frac{1}{3}\right)^{[m]}}\right) x^{3m-2} + O\big(x^{3m-3}\big), \tag{19}$$

where $x^{[m]} = x(x+1)(x+2)\cdots(x+m-1)$.

Proof. We proceed as in the proof of Proposition 4.2. The coefficients h_j of f_n must satisfy the differential equations (16), with $c_1 = 0$. Arguing as in that proof we obtain the degrees of each h_j. We can write

$$h_j(x) = a_j(2m)x^{\deg(h_j)} + O\big(x^{\deg(h_j)-1}\big),$$

where,

$$\deg(h_j) = \begin{cases} 3k-2, \text{ when } j = 2m-(2k-1), \\ 3k, \text{ when } j = 2m-2k, \end{cases}$$

for $k = 0, 1, \ldots, m$ and $a_{2m}(2m) = 1$. Let us determine these functions.

Plugging the above expressions in (16) we obtain that the terms $a_j = a_j(2m)$ satisfy the following recurrences

$$a_{2m-2k} = \frac{2m-(2k-2)}{3k} a_{2m-(2k-2)}, \quad k = 1, 2, \ldots, m, \tag{20}$$

$$a_{2m-(2k+1)} = \frac{(2m-(2k-1))a_{2m-(2k-1)} + h(2m-2k)a_{2m-2k}}{3k+1}, \; k = 1, 2, \ldots, m-1, \tag{21}$$

where $h(j) = -(c_0 + jc)$ and the initial conditions are

$$a_{2m} = 1 \quad \text{and} \quad a_{2m-1} = h(2m) = -(c_0 + 2mc).$$

The even terms a_{2j} can be easily obtained from (20). We get

$$a_{2m-2j} = \binom{m}{j}\left(\frac{2}{3}\right)^j \tag{22}$$

and in particular $a_0 = (2/3)^m$ as we wanted to prove. It remains to obtain the general expression of the last odd term $a_1 = a_1(2m)$. We take advantage of the linearity of the problem with respect to the initial condition a_{2m-1} and write

$$a_1(2m) = -\big(\widehat{a}_1(2m)c_0 + \widetilde{a}_1(2m)c\big),$$

where $\widehat{a}_1$ and $\widetilde{a}_1$ are the solution of the recurrences (20)-(21) with initial conditions $a_{2m} = 1$ and

$$a_{2m-1} = 1 \quad \text{or} \quad a_{2m-1} = 2m,$$

respectively.

Substituting expression (22) in (21) and developing the recurrent expressions we arrive at

$$\widehat{a}_1(2m) = \sum_{j=0}^{m-1} \binom{m}{j}\left(\frac{2}{3}\right)^j \prod_{k=0}^{m-j-1}(2k+1)\prod_{k=j}^{m-1}\frac{1}{3k+1},$$

$$\widetilde{a}_1(2m) = 2\sum_{j=0}^{m}(m-j)\binom{m}{j}\left(\frac{2}{3}\right)^j \prod_{k=0}^{m-j-1}(2k+1)\prod_{k=j}^{m-1}\frac{1}{3k+1}.$$

We introduce the following auxiliary functions

$$\alpha(m) = \Gamma\Big(\frac{1}{2}\Big)\Gamma\Big(\frac{1}{3}+m\Big), \quad \beta(m) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{3})\Gamma(\frac{5}{6}+m)}{\Gamma(\frac{5}{6})},$$

$$\gamma(m) = \frac{\beta(m)}{\alpha(m)} = \frac{\Gamma(\frac{1}{3})\Gamma(\frac{5}{6}+m)}{\Gamma(\frac{5}{6})\Gamma(\frac{1}{3}+m)} = \frac{\left(\frac{5}{6}\right)^{[m]}}{\left(\frac{1}{3}\right)^{[m]}},$$

where in the last equality we have used (v) of Lemma 4.1. Let us simplify the expressions of $\widehat{a}_1$ and $\widetilde{a}_1$ using the above functions and the other equalities given in Lemma 4.1.

$$\begin{aligned}
\widehat{a}_1(2m) &= \Big(\frac{2}{3}\Big)^m \sum_{j=0}^{m-1} \binom{m}{j} \prod_{k=0}^{m-j-1} \Big(\frac{1}{2}+k\Big) \prod_{k=j}^{m-1} \frac{1}{\frac{1}{3}+k} \\
&= \frac{1}{\alpha(m)}\Big(\frac{2}{3}\Big)^m \sum_{j=0}^{m-1} \binom{m}{j} \Gamma\Big(\frac{1}{2}+m-j\Big)\Gamma\Big(\frac{1}{3}+j\Big) \\
&= \frac{1}{\alpha(m)}\Big(\frac{2}{3}\Big)^m \left(\sum_{j=0}^{m} \binom{m}{j} \Gamma\Big(\frac{1}{2}+m-j\Big)\Gamma\Big(\frac{1}{3}+j\Big) - \Gamma\Big(\frac{1}{2}\Big)\Gamma\Big(\frac{1}{3}+m\Big)\right) \\
&= \frac{1}{\alpha(m)}\Big(\frac{2}{3}\Big)^m \left(\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{3})\Gamma(\frac{5}{6}+m)}{\Gamma(\frac{5}{6})} - \alpha(m)\right) \\
&= \frac{1}{\alpha(m)}\Big(\frac{2}{3}\Big)^m (\beta(m)-\alpha(m)) = \Big(\frac{2}{3}\Big)^m (\gamma(m)-1).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\widetilde{a}_1(2m) &= \frac{2}{\alpha(m)}\Big(\frac{2}{3}\Big)^m \sum_{j=0}^{m} (m-j)\binom{m}{j} \Gamma\Big(\frac{1}{2}+m-j\Big)\Gamma\Big(\frac{1}{3}+j\Big) \\
&= \frac{2}{\alpha(m)}\Big(\frac{2}{3}\Big)^m \frac{\frac{1}{2}\Gamma(\frac{1}{2})\Gamma(\frac{1}{3})\Gamma(\frac{5}{6}+m)}{\frac{5}{6}\Gamma(\frac{5}{6})} m \\
&= \frac{1}{\alpha(m)}\Big(\frac{2}{3}\Big)^m \frac{6}{5}\beta(m)m = \Big(\frac{2}{3}\Big)^m \frac{6}{5}\gamma(m)m.
\end{aligned}$$

Hence

$$\begin{aligned}
a_1(2m) &= -\big(\widehat{a}_1(2m)c_0 + \widetilde{a}_1(2m)c\big) \\
&= -\Big(\frac{2}{3}\Big)^m \left((\gamma(m)-1)c_0 + \frac{6}{5}\gamma(m)mc\right) \\
&= \frac{1}{5}\Big(\frac{2}{3}\Big)^m \Big(5c_0 - (5c_0+6mc)\gamma(m)\Big) \\
&= \frac{1}{5}\Big(\frac{2}{3}\Big)^m \left(5c_0 - (5c_0+6mc)\frac{\left(\frac{5}{6}\right)^{[m]}}{\left(\frac{1}{3}\right)^{[m]}}\right),
\end{aligned}$$

as we wanted to prove. □

When an invariant algebraic curve passes by an elementary critical point, in many cases, the value of the cofactor at this point can be obtained. These type of results, based on previous works of Seidenberg ([35]), are proved in [4]. In the next proposition, which is included in [4, Thm 14], we state one of these cases.

Proposition 4.4. *Let $f(x,y)=0$ be an invariant algebraic curve of a planar system with corresponding cofactor $k(x,y)$. Assume that it contains a critical point of the system, (x_0,y_0), and that it is a hyperbolic saddle with eigenvalues $\lambda^-<0<\lambda^+$. Then $k(x_0,y_0)\in\{\lambda^+,\lambda^-,\lambda^++\lambda^-\}$.*

Proof of Theorem 3.1. By Propositions 4.2 and 4.3 we know that the invariant curve has even degree $n=2m, m\in\mathbb{N}$, and it can be written as

$$f(x,y)=h_n(x)y^n+h_{n-1}(x)y^{n-1}+\cdots+h_1(x)y+h_0(x)=0,$$

where h_0 and h_1 satisfy (18) and (19). Moreover its cofactor is constant, $k(x,y)=c_0$. Using that h_0 and h_1 must satisfy (17) we get the identity

$$-c_0h_0(x)-xh_1(x)+x^2h_1(x)\equiv 0.$$

Using Proposition 4.3 we obtain that

$$-c_0h_0(x)-xh_1(x)+x^2h_1(x)=-\frac{5c_0+6mc}{5}\left(\frac{2}{3}\right)^m\frac{\left(\frac{5}{6}\right)^{[m]}}{\left(\frac{1}{3}\right)^{[m]}}x^{3m}+O\big(x^{3m-1}\big).$$

Therefore,

$$5c_0+6mc=0. \tag{23}$$

The origin of (8) is a saddle point with eigenvalues $\lambda^\pm=\dfrac{-c\pm\sqrt{c^2+4}}{2}$, where $\lambda^-<0<\lambda^+$. Since, by hypothesis, $f(0,0)=0$ we can apply Proposition 4.4 to determine $c_0=k(0,0)$. We obtain that $c_0\in\{\lambda^+,\lambda^-,-c\}$. When $c_0=-c$, equation (23) gives $(6m-5)\,c=0$, which is in contradiction with the hypothesis $c\geq 2$. Therefore, if the system has an algebraic invariant curve under the above hypotheses, then $c_0\in\{\lambda^+,\lambda^-\}$. Take $c_0=\lambda^\pm$. Hence, equation (23) writes as $6mc+5\lambda^\pm=0$, or equivalently,

$$c=\mp\frac{5}{\sqrt{6}}\frac{1}{\sqrt{m(6m-5)}}.$$

Imposing that $c\geq 2$ we get that the only possibility is $c_0=\lambda^-$ and $m=1$. Then, $c=5/\sqrt{6}$ as we wanted to prove. Finally, simple computations give (9) and the theorem follows. ☐

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References

[1] M. J. Ablowitz and P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, London Mathematical Society Lectures Notes **149**, Cambridge University Press (1991).

[2] M. J. Ablowitz and A. Zeppetella, *Explicit solutions of Fisher's equation for a special wave speed*, Bull. Math. Biol. **41** (1979), 835–840.

[3] N. Akhmediev and A. Ankiewicz ed., *Dissipative Solitons: from Optics to Biology and Medicine*, Lecture Notes in Physics **751**, Springer-Verlag, Berlin (2008).
[4] J. Chavarriga, H. Giacomini and M. Grau, *Necessary conditions for the existence of invariant algebraic curves for planar polynomial systems*, Bull. Sci. Math. **129** (2005), 99–126.
[5] P. G. Drazin and R. S. Johnson, *Solitons: An Introduction*, Cambridge University Press (1989).
[6] F. Dumortier, J. Llibre and J. C. Artes, *Qualitative Theory of Planar Differential Systems*, Universitext. Springer-Verlag, Berlin (2006).
[7] R. A. Fisher, *The wave of advance of advantageous genes*, Ann. Eugenics **7** (1937), 355–369.
[8] A. Gasull, H. Giacomini and M. Grau, *On the stability of periodic orbits for differential systems in* $\mathbb{R}^n$, Discrete Contin. Dyn. Syst. Ser. B **10** (2008), 495–509.
[9] A. Gasull, H. Giacomini and J. Torregrosa, *Explicit upper and lower bounds for the traveling wave solutions of Fisher-Kolmogorov type equations*, Discrete Contin. Dyn. Syst. **33** (2013), 3567–3582.
[10] B. H. Gilding and R. Kersner, *Travelling Waves in Nonlinear Diffusion Convection Reaction*, BirkhÃuser (2004).
[11] A. Goriely, *Integrability and Nonintegrability of Dynamical Systems*, Advanced Series in Nonlinear Dynamics, **19**, World Scientific Publishing Co., Inc., River Edge, NJ (2001).
[12] G. W. Griffiths and W. E. Schiesser, *Traveling Wave Solutions of Partial Differential Equations: Numerical and Analytical Methods with Matlab and Maple*, Academic Press (2011).
[13] P. Grindrod, *Patterns and Waves, The Theory and Applications of Reaction-Diffusion Equations*, Clarendon Press (1991).
[14] J. H. He and X. H. Wu, *Exp-function method for nonlinear wave equations*, Chaos Solitons and Fractals **30** (2006), 700–708.
[15] E. Infield and G. Rowlands, *Nonlinear Waves, Solitons and Chaos*, 2nd Edition, Cambridge University Press (2000).
[16] R. S. Johnson, *A Modern Introduction to the Mathematical Theory of Water Waves*, Cambridge University Press (1999).
[17] T. Kano and T. Nishida, *A mathematical justification for Korteweg-de Vries equation and Boussinesq equation of water surface waves*, Osaka J. Math. **23** (1986), 389–413.
[18] R. A. Knobel, *An Introduction to the Mathematical Theory of Waves*, American Mathematical Society (2000).
[19] A. Kolmogorov, I. Petrovskii and N. Piskunov, *A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem*, In V. M. Tikhomirov, editor, *Selected Works of A. N. Kolmogorov I*, 248–270. Kluwer 1991. Translated by V. M. Volosov from Bull. Moscow Univ., Math. Mech. **1** (1937), 1–25.
[20] A. Kundu Ed., *Tsunami and Nonlinear Waves*, Springer (2007).
[21] A. W. Liehr, *Dissipative Solitons in Reaction Diffusion Systems, Mechanism, Dynamics, Interaction*, Volume **70** of Springer Series in Synergetics, Springer, Berlin-Heidelberg (2013).
[22] J. Llibre, *Integrability of Polynomial Differential Systems*, Handbook of Differential Equations, 437–532, Elsevier/North-Holland, Amsterdam (2004).
[23] J. Llibre and X. Zhang, *Invariant algebraic surfaces of the Lorenz system*, J. Math. Phys. **43** (2002), 1622–1645.
[24] W. Mafliet, *Solitary waves solutions of nonlinear wave equations*, Am. J. Physics **60** (1992), 650–654.
[25] W. Mafliet and W. Hereman, *The tanh method I-exact solutions of nonlinear evolution wave equations*, Physica Scripta **54** (1996), 563–568.
[26] W. Mafliet and W. Hereman, *The tanh method II-exact solutions of nonlinear evolution wave equations*, Physica Scripta **54** (1996), 569–575.
[27] J. D. Murray, *Mathematical Biology. I. An Introduction*, Third edition. Interdisciplinary Applied Mathematics, **17**. Springer-Verlag, New York (2002).
[28] J. D. Murray, *Mathematical Biology. II. Spatial Models and Biomedical Applications*, Third edition, Interdisciplinary Applied Mathematics, **18**, Springer-Verlag, New York (2003).
[29] A. C. Newell and J. A. Whitehead, *Finite bandwidth, finite amplitude convection*, J. Fluid Mech. **38** (1969), 279–303.

[30] A. D. Polyanin and V. F. Zaitsev, *Handbook of Nonlinear Partial Differential Equations*, Chapman and Hall/CRC, Boca Raton (2004).

[31] M. R. Rodrigo and R. M. Miura, *Exact and approximate traveling waves of reaction-diffusion systems via a variational approach*, Anal. Appl. (Singap.) **9** (2011), 187–199.

[32] F. Sánchez-Garduño and P. K. Maini, *Travelling wave phenomena in some degenerate reaction-diffusion equations*, J. Differential Equations **117** (1995), 281–319.

[33] F. Sánchez-Garduño and P. K. Maini, *Travelling wave phenomena in non-linear diffusion degenerate Nagumo equations*, J. Math. Biol. **35** (1997), 713–728.

[34] L. A. Segel, *Distant sidewalls cause slow amplitude modulation of cellular convection*, J. Fluid Mech. **38** (1969), 203-224.

[35] A. Seidenberg, *Reduction of singularities of the differential equation* $Ady = Bdx$, Amer. J. Math. **90** (1968), 248–269.

[36] B. Sturmfels, *Solving Systems of Polynomial Equations.* CBMS Regional Conference Series in Mathematics, **97**, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2002. viii+152

[37] J. Xin, *Front propagation in heterogeneous media*, SIAM Rev. **42** (2000), 161–230.

[38] Y. B. Zeldovich and D. A. Frank-Kamenetskii, *A theory of thermal propagation of flame*, Acta Physicochimica URSS **9** (1938), 341–350. English translation: In Dynamics of curved fronts, editor R. Pelcé, Perspectives in Physics Series, Academic Press, New York (1988), 131–140.

[39] Z. Yang and B. Y.C. Hon, *An improved modified extended tanh function method*, Z. Naturforsch **61** (2006), 103–115.

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