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THE GENERALIZED LIÉNARD POLYNOMIAL DIFFERENTIAL SYSTEMS $x' = y,\ y' = -g(x) - f(x)y$ WITH $\deg g = \deg f + 1$ ARE NOT LIOUVILLIAN INTEGRABLE

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Abstract. We prove the nonexistence of Liouvillian first integrals for the generalized Liénard polynomial differential systems of the form $x' = y$, $y' = -g(x) - f(x)y$, where $g(x)$ and $f(x)$ are arbitrary polynomials such that $\deg g = \deg f + 1$.

1. Introduction and statement of the main result

One of the more classical problems in the qualitative theory of planar differential systems depending on parameters is to characterize the existence or not of first integrals.

We consider the system

$$x' = y, \quad y' = -g(x) - f(x)y, \tag{1}$$

called the *generalized Liénard differential system*, where x and y are complex variables and the prime denotes derivative with respect to the time t, which can be either real or complex. Such differential systems appear in several branches of the sciences, such as biology, chemistry, mechanics, electronics, etc. For $g(x) = x$ the Liénard differential systems (1) are called the *classical Liénard systems*. In MathScinet now it appears more than 450 articles published such that in their title appears the words "Liénard system".

Let $U \subset \mathbb{C}^2$ be an open set. We say that the non–locally constant function $H\colon \mathbb{C}^2 \to \mathbb{C}$ is a *first integral* of the polynomial vector field X on U, if $H(x(t), y(t)) =$ constant for all values of t for which the solution $(x(t), y(t))$ of X is defined on U. Clearly H is a first integral of X on U if and only if $XH = 0$ on U.

A *Liouvillian first integral* is a first integral H which is a Liouvillian function, that is, roughly speaking which can be obtained "by quadratures" of elementary functions. For a precise definition see [15]. The study of the Liouvillian first integrals is a classical problem of the integrability theory of the differential equations which goes back to Liouville, see for details again [15].

As far as we know all the Liouvillian first integrals of some multi-parameter family of planar polynomial differential systems has only been classified for the Lotka-Volterra system, see [1, 7, 11, 12, 13, 14].

The main objective of this paper is to study the *Liouvillian first integrals* of systems (1) depending on the polynomial functions $f(x)$ and $g(x)$. We denote by m and n the degrees of g and f, respectively. The Liouvillian integrability for the generalized Liénard polynomial differential systems has been studied when

(i) $m = 1$ and n is arbitrary (see Theorem 1 done in [9]), and
(ii) $2 \le m \le n$ with m and n arbitrary (see Theorem 2 done in [10]).

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Theorem 1 ([9]). *The unique Liouvillian first integrals $H = H(x, y)$ of the Liénard polynomial differential system*

$$x' = y, \quad y' = -cx - f(x)y.$$

are:

(a) $H = cx^2 + y^2$ *if* $f(x) = 0$;
(b) $H = y + \int f(x)\,dx$ *if* $c = 0$;
(c) $H = H = \Big(\dfrac{1}{2}(1 - \sqrt{1+4\alpha})x + y\Big)^{-1+\sqrt{1+4\alpha}} \Big(\dfrac{1}{2}(1 + \sqrt{1+4\alpha})x + y\Big)^{1+\sqrt{1+4\alpha}}$ *with* $\alpha = -c/f(0)^2$ *if* $f(x) = f(0) \neq 0$; *and*
(d) $H = e^{f'(0)^2x^2 + 2f'(0)y}(c + f'(0)y)^{-2c}$ *if* $f(x) = f'(0)x \neq 0$.

Theorem 2 ([10]). *The unique Liouvillian first integrals $H = H(x, y)$ of the generalized Liénard polynomial differential system* (1) *with* $2 \le m \le n$ *are* $H = (a + y)e^{-(y+F(x))/a}$ *if* $g(x) = af(x)$ *where* $a \in \mathbb{C} \setminus \{0\}$.

Clearly, for completing the study of the Liouvillian integrability for the generalized Liénard polynomial differential systems it remains to study the case

(iii) $n + 1 \le m$ with m and n arbitrary.

We recall the notion of *invariant algebraic curve.* Let $h(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}$. As usual $\mathbb{C}[x, y]$ denotes the ring of all complex polynomials in the variables x and y. We say that $v = 0$ is an *invariant algebraic curve* of the vector field X associated to system (1) if it satisfies

$$y\frac{\partial h}{\partial x} - (g(x) + f(x)y)\frac{\partial h}{\partial y} = Kh,$$

the polynomial $K = K(x, y) \in \mathbb{C}[x, y]$ is called the *cofactor* of $h = 0$ and has degree at most n.

When $m > n + 1$ and n is arbitrary, it is unknown the characterization of the invariant algebraic curves of the generalized Liénard polynomial differential systems. This characterization is necessary for studying the existence of Liouvillian first integrals for this class of polynomial differential systems.

For $m = n + 1$ and n arbitrary Hayashi in [6] has characterized the invariant algebraic curves for the generalized Liénard polynomial differential systems as follows.

Proposition 3 ([6]). *System* (1) *with* $\deg g = \deg f + 1$ *has an invariant algebraic curve* $h = h(x, y)$ *if and only if* $h = y - P(x)$ *and* P *satisfies* $g(x) = -(f(x) + P'(x))P(x)$ *where:*

(a) *either* $P(x)$ *has degree one,*
(b) *or* $P(x)$ *is such that* $P(x) + \displaystyle\int f(x)\,dx$ *is a polynomial of degree one.*

In both cases the cofactor K *of* h *is* $K = -f(x) - P'(x)$.

Proposition 3 allows to classify in this paper the existence or non–existence of Liouvillian first integrals for the generalized Liénard polynomial differential systems satisfying $m = n+1$ and n arbitrary. Thus our main result is.

Theorem 4. *There are no Liouvillian first integrals for the generalized Liénard polynomial differential system* (1) *with* $2 \le \deg g = \deg f + 1$.

The proof of Theorem 4 is given in section 2.

2. Proof of Theorem 4

For proving Theorem 4 we need some preliminary results.

If $h(x, y) = 0$ with $h(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}$ is an invariant algebraic curve, then h is called a *Darboux polynomial* of system (1). Note that a polynomial first integral is a Darboux polynomial with zero cofactor.

The Darboux polynomials are important because a sufficient number of them forces the existence of a first integral. This result is the basis of the Darboux theory of integrability, see for instance [4, 8].

An *exponential factor* E of system (1) is a function of the form $E = e^{u/v} \notin \mathbb{C}$ with $u, v \in \mathbb{C}[x, y]$ satisfying that

$$y\frac{\partial E}{\partial x} - (g(x) + f(x)y)\frac{\partial E}{\partial y} = LE, \tag{2}$$

for some polynomial $L = L(x, y)$ of degree at most n, called the *cofactor* of E.

Proposition 5. *The following statements hold.*

(a) *If $E = \exp(u/v)$ is an exponential factor for the polynomial differential system* (1) *and v is not a constant polynomial, then $v = 0$ is an invariant algebraic curve.*
(b) *Eventually $E = \exp(u)$ can be exponential factors coming from the multiplicity of the infinite invariant straight line.*

The following result given in [3] characterizes the algebraic multiplicity of an invariant algebraic curve using the number of exponential factors of system (1) associated with the invariant algebraic curve.

Proposition 6. *Given an irreducible invariant algebraic curve $v = 0$ of degree k of system* (1)*, it has algebraic multiplicity d if and only if the vector field associated to system* (1) *has $d - 1$ exponential factors $\exp(u_i/v^i)$ where u_i is a polynomial of degree at most ik and $(u_i, v) = 1$ for $i = 1, \dots, d - 1$.*

In view of Proposition 6 if we prove that $e^{u/v}$ is not an exponential factor with $\deg u \le \deg v$, there are no exponential factors associated to the invariant algebraic curve $v = 0$. The existence of exponential factors $\exp(u/v)$ is due to the fact that the multiplicity of the invariant algebraic curve $v = 0$ is larger than 1, for more details see [3].

In 1992 Singer [15] proved that a polynomial differential system has a Liouvillian first integral, if and only if it has an inverse integrating factor of the form

$$e^{\left(\int U_1(x,y)dx + \int U_2(x,y)dy\right)} \tag{3}$$

where U_1 and U_2 are rational functions which verify $\partial U_1/\partial y = \partial U_2/\partial x$. In 1999 Christopher [2] improved the results of Singer showing that the inverse integrating factor (3) can be written in the form

$$e^{u/v}\prod_{i=1}^{k} h_i^{\lambda_i} \tag{4}$$

where u, v and h_i are polynomials and $\lambda_i \in \mathbb{C}$.

From (4) and the Darboux theory of integrability (see [5, 8]) we have the following result.

Theorem 7. *The polynomial differential system* (1) *has a Liouvillian first integral if and only if system* (1) *has an integrating factor of the form* (4)*, or equivalently there exist p invariant algebraic curves $h_i = 0$ with cofactors K_i for $i = 1, \dots, p$, q exponential factors $E_j = e^{u_j/v_j}$ with cofactors L_j for $j = 1, \dots, q$ and $\lambda_j, \mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -\text{divergence of (1)} = f(x).$$

In short, for proving Theorem 4 we need to characterize the Darboux polynomials and the exponential factors of system (1). We start with some preliminaries. We write

$$f(x)=\sum_{j=0}^{n}a_jx^j \quad \text{and hence} \quad F(x)=\sum_{j=0}^{n}\frac{a_j}{j+1}x^{j+1}, \quad g(x)=\sum_{j=0}^{n+1}b_jx^j \tag{5}$$

with $n\geq 1$.

The first result that we will prove is the following.

Proposition 8. *Systems (1) with $2\leq m=n+1$ has the exponential factors e^{x^k} with cofactors $kx^{k-1}y$ for $k=1,\ldots,n$.*

Proof. In view of Proposition 3 system (1) has an irreducible Darboux polynomial $h=y-P(x)$ if and only if $g(x)=-(f(x)+P'(x))P(x)$ with $P(x)$ either of degree one or $P(x)+\displaystyle\int f(x)\,dx$ of degree one. Hence, from Proposition 6 if we prove that there are no exponential factors of the form $e^{u/h}$ with u and h coprime, then the unique possible exponential factors would be of the form e^u with $u\in\mathbb{C}[x,y]\setminus\mathbb{C}$.

Let $E=\exp(u/h)$ with $u=u(x,y)$ of degree at most one and coprime with h. We write $P(x)=\alpha_0+\alpha_1x$ with $\alpha_0,\alpha_1\in\mathbb{C}$ and $g(x)=-(f(x)+\alpha_1)(\alpha_0+\alpha_1x)$. If $u=\gamma_0+\gamma_1x+\gamma_2y$ with $\gamma_i\in\mathbb{C}$, $i=0,1,2$, then equation (2) becomes

$$\gamma_0(\alpha_1+f(x))+\gamma_1(\alpha_1x+y+xf(x))+\gamma_2\big(\alpha_0\alpha_1+\alpha_1^2x+\alpha_1y+(\alpha_0+\alpha_1x)f(x)\big)=L(y-\alpha_0-\alpha_1x),$$

where $L=L(x,y)$ is a polynomial of degree n. Evaluating it on $y=\alpha_0+\alpha_1x$ we get

$$\gamma_0(\alpha_1+f(x))+\gamma_2(\alpha_0+\alpha_1x)(2\alpha_1+f(x))+\gamma_1(\alpha_0+2\alpha_1x+xf(x))=0. \tag{6}$$

The coefficient of degree $n+1$ in equation (6) becomes

$$a_n(\gamma_1+\alpha_1\gamma_2)=0, \quad \text{that is,} \quad \gamma_1=-\alpha_1\gamma_2.$$

Then (6) becomes

$$(f(x)+\alpha_1)(\gamma_0+\gamma_2\alpha_0) \quad \text{that is} \quad \gamma_0=-\alpha_0\gamma_2.$$

Hence,

$$u=-\alpha_0\gamma_2-\alpha_1\gamma_2x+\gamma_2y=\gamma_2(y-\alpha_0-\alpha_1x),$$

which is not possible since u and $y-P(x)=y-\alpha_0-\alpha_1x$ are coprime.

Let $E=\exp(u/(y-P(x))$ with $u=u(x,y)$ of degree at most $n+1$ and $g(x)=-(f(x)+P'(x))P(x)$ with $P(x)=-\int f(x)\,dx+\beta_0+\beta_1x$ with $\beta_0,\beta_1\in\mathbb{C}$. We also assume that u and $y-P(x)$ are coprime. In this case equation (2) becomes

$$y\frac{\partial u}{\partial x}-(f(x)y-f(x)P(x)-P'(x)P(x))\frac{\partial u}{\partial y}+\beta_1u=L\Big(y-\beta_0-\beta_1x+\int f(x)\,dx\Big),$$

where $L=L(x,y)$ is a polynomial of degree n. Let $\bar{u}=\bar{u}(x)$ be the restriction of u to $y=\beta_0+\beta_1x-\int f(x)\,dx$. By hypothesis u and $y-\beta_0-\beta_1x+\int f(x)\,dx$ are coprime and thus $\bar{u}\neq 0$. Furthermore, since $y-\beta_0-\beta_1x+\int f(x)\,dx=0$ is an invariant algebraic curve, we get that $\bar{u}$ satisfies

$$\frac{d\bar{u}}{dx}\Big(\beta_0+\beta_1x-\int f(x)\,dx\Big)+\beta_1\bar{u}=0.$$

Solving this linear differential equation we get

$$\bar{u}=\bar{K}\exp\Bigg(-\int\frac{\beta_1}{\beta_0+\beta_1x-\displaystyle\int f(x)\,dx}\Bigg), \quad \bar{K}\in\mathbb{C}\setminus\{0\}.$$

Since

$$\int f(x)\,dx = \sum_{i=0}^{n} \frac{a_i}{i+1} x^{i+1}, \quad a_n \neq 0, n \geq 1$$

has degree $n+1$, we get that $\bar{u}$ cannot be a polynomial, which is not possible.

In short, the unique exponential factors are of the form e^u with $u = u(x,y)$ a polynomial. Let $E = e^u$ be an exponential factor of system (1) and

$$L = L(x,y) = \sum_{k=0}^{n} \sum_{l=0}^{k} \beta_{k-l,l} x^{k-l} y^l, \quad \beta_{k-l,l} \in \mathbb{C}$$

be its cofactor. We have

$$y\frac{\partial u}{\partial x} - (g(x) + f(x)y)\frac{\partial u}{\partial y} = \sum_{k=0}^{n} \sum_{l=0}^{k} \beta_{k-l,l} x^{k-l} y^l. \tag{7}$$

We write u as a polynomial in the variable y as $u = \sum_{j=0}^{r} u_j(x) y^j$. Without loss of generality we can assume that $u_r(x) \neq 0$. We rewrite (7) as

$$\begin{aligned} &\sum_{j=0}^{r} u_j'(x) y^{j+1} - g(x) \sum_{j=1}^{r} j u_j(x) y^{j-1} - f(x) \sum_{j=0}^{r} j u_j(x) y^j = \\ &\sum_{k=0}^{n} \sum_{l=0}^{k} \beta_{k-l,l} x^{k-l} y^l = \sum_{l=0}^{n} \sum_{k=l}^{n} \beta_{k-l,l} x^{k-l} y^l = \\ &\sum_{k=0}^{n} \beta_{k,0} x^k + y \sum_{k=1}^{n} \beta_{k-1,1} x^{k-1} + \sum_{l=2}^{n} y^l \sum_{k=l}^{n} \beta_{k-l,l} x^{k-l}. \end{aligned} \tag{8}$$

Now we assume that $r \geq n$ and we shall arrive to a contradiction. Computing in (8) the coefficient of y^{r+1} we get that $u_r'(x) = 0$, that is, $u_r(x) = \gamma_r \in \mathbb{C} \setminus \{0\}$. Now we will show by induction on j that if we write

$$u = \gamma_r y^r + \sum_{j=1}^{r} u_{r-j}(x) y^{r-j},$$

then

$$u_{r-j}(x) = \gamma_r \frac{a_n^j}{j!(n+1)^j} x^{j(n+1)} \prod_{i=0}^{j-1} (r-i) + l.o.t., \quad \text{for } j = 1, \ldots, r, \tag{9}$$

where $l.o.t$ means lower order terms in x.

For $j = 1$ computing the coefficient of y^r in (8), we get that

$$u_{r-1}'(x) - f(x) r u_r(x) = 0, \quad \text{if } r > n,$$

and

$$u_{r-1}'(x) - f(x) r u_r(x) = \beta_{0,r}, \quad \text{if } r = n.$$

Integrating it we obtain

$$u_{r-1}(x) = \gamma_r r F(x) + \text{constant} = \gamma_r \frac{a_n}{n+1} r x^{n+1} + l.o.t., \quad \text{if } r > n,$$

and

$$u_{r-1}(x) = \gamma_r r F(x) + \beta_{0,r} x + \text{constant} = \gamma_r \frac{a_n}{n+1} r x^{n+1} + l.o.t., \quad \text{if } r = n.$$

These last two expressions coincide with (9) for $j = 1$.

Now we assume that (9) is true for $j = 0, \ldots, J$ with $J < r$ and we will prove it for $j = J + 1$. Computing the coefficient of y^{r-J} in (8) we get

$$u'_{r-J-1}(x) = g(x)(r - J + 1)u_{r-J+1}(x) + f(x)(r - J)u_{r-J}(x) + \sum_{k=r-J}^{n} \beta_{k-r+J,r-J} x^{k-r+J}.$$

Now using the induction hypothesis and since $x^{(J-1)(n+1)+n+1}$ belongs to the lower terms in comparison with $x^{J(n+1)+n}$, we obtain that

$$\begin{aligned} u'_{r-J-1}(x) &= f(x)(r - J)\frac{\gamma_r a_n^J}{J!(n + 1)^J} x^{J(n+1)} \prod_{i=0}^{J-1}(r - i) + l.o.t. \\ &= \frac{\gamma_r a_n^{J+1}}{J!(n + 1)^J} x^{J(n+1)+n} \prod_{i=0}^{J}(r - i) + l.o.t. \end{aligned}$$

Now integrating the previous equation we obtain

$$\begin{aligned} u_{r-J-1}(x) &= \frac{\gamma_r a_n^{J+1}}{J!(n + 1)^J (J + 1)(n + 1)} x^{(J+1)(n+1)} \prod_{i=0}^{J}(r - i) + l.o.t. \\ &= \frac{\gamma_r a_n^{J+1}}{(J + 1)!(n + 1)^{J+1}} x^{(J+1)(n+1)} \prod_{i=0}^{J}(r - i) + l.o.t., \end{aligned}$$

which is equation (9) with $j = J + 1$. This completes the proof of (9).

From (9) with $j = r - 1$ we obtain

$$u_1(x) = \frac{\gamma_r a_n^{r-1}}{(r - 1)!(n + 1)^{r-1}} x^{(r-1)(n+1)} \prod_{i=0}^{r-2}(r - i) + l.o.t.$$

Then computing the coefficient of y^0 in (8) we get

$$-g(x)u_1(x) = \sum_{k=0}^{n} \beta_{k,0} x^k,$$

or equivalently

$$-g(x)u_1(x) = -\frac{b_{n+1}\gamma_r a_n^{r-1}}{(r - 1)!(n + 1)^{r-1}} x^{(r-1)(n+1)+n+1} \prod_{i=0}^{r-2}(r - i) + l.o.t. = \sum_{k=0}^{n} \beta_{k,0} x^k. \tag{10}$$

Since $r \geq n \geq 1$ we have that $(r-1)(n+1)+n+1 > n$, from (10) we have a contradiction. Hence $r \leq n - 1$.

We first assume that $r \geq 2$ and again we will reach a contradiction. We claim that (8) becomes

$$\begin{aligned} &\sum_{j=0}^{r} u'_j(x)y^{j+1} - g(x)\sum_{j=1}^{r} ju_j(x)y^{j-1} - f(x)\sum_{j=0}^{r} ju_j(x)y^j = \\ &\sum_{k=0}^{n} \beta_{k,0}x^k + y\sum_{k=1}^{n} \beta_{k-1,1}x^{k-1} + \sum_{l=2}^{r+1} y^l \sum_{k=l}^{n} \beta_{k-l,l}x^{k-l}. \end{aligned} \tag{11}$$

Indeed, since all the coefficients with y^l for $l = r + 2, \ldots, n$ in (8) only appears in the right-hand we have that

$$\sum_{l=r+2}^{n} y^l \sum_{k=l}^{n} \beta_{k-l,l}x^{k-l} = 0.$$

This implies that (11) holds, and consequently the claim is proved.

Computing the coefficient of y^{r+1} in (11) we get that

$$u_r'(x) = \sum_{k=r+1}^{n} \beta_{k-r-1,r+1} x^{k-r-1} \quad \text{i.e,} \quad u_r(x) = c_r + \sum_{k=r+1}^{n} \frac{\beta_{k-r-1,r+1}}{k-r} x^{k-r} = \sum_{k=r}^{n} \tilde{\beta}_k x^{k-r},$$

where $\tilde{\beta}_r = c_r \in \mathbb{C}$ and $\tilde{\beta}_k = \beta_{k-r-1,r+1}/(k-r)$ for $k = r+1,\ldots,n$. Without loss of generality and since $u_r(x) \neq 0$ we denote by k^* the greatest integer of $\{r,\ldots,n\}$ such that $\tilde{\beta}_{k^*} \neq 0$. Then it is clear that

$$u_r(x) = \tilde{\beta}_{k^*} x^{k^*-r} + l.o.t.$$

We claim that

$$u_{r-j}(x) = \frac{\tilde{\beta}_{k^*} a_n^j}{\prod_{i=1}^{j}(i(n+1)+k^*-r)} x^{j(n+1)+k^*-r} \prod_{i=0}^{j-1}(r-i) + l.o.t., \tag{12}$$

for $j = 1,\ldots,r-1$.

Computing the coefficient of y^r in (11) we get

$$u_{r-1}'(x) - f(x) r u_r(x) = \sum_{k=r}^{n} \beta_{k-r,r} x^{k-r}.$$

Since $k^* \geq r \geq 2$, the terms x^{n-r} belongs to the lower terms in comparison with x^{n-r+k^*}. Then we obtain that

$$u_{r-1}'(x) = a_n r \tilde{\beta}_{k^*} x^{n+k^*-r} + l.o.t.$$

Integrating this last expression we get

$$u_{r-1}(x) = \frac{\tilde{\beta}_{k^*} a_n}{n+1+k^*-r} x^{n+1+k^*-r} r + l.o.t.,$$

which coincides with (12) with $j = 1$.

Now we assume that (12) is true for $j = 1,\ldots,J$ with $1 \leq J < r-1$ and we will prove it for $j = J+1$. Computing the coefficient of y^{r-J} in (11) we get

$$u_{r-J-1}'(x) - g(x)(r-J+1)u_{r-J+1}(x) - f(x)(r-J)u_{r-J}(x) = \sum_{k=r-J}^{n} \beta_{k-r+J,r-J} x^{k-r+J}.$$

Now using the induction hypothesis and since $x^{(J-1)(n+1)+n+1+k^*-r}$ and x^{n-r+J} belong to the lower terms in comparison with $x^{J(n+1)+k^*-r+n}$ (note that $J \geq 1$), from the last equation we obtain that

$$\begin{aligned} u_{r-J-1}'(x) &= a_n x^n (r-J) \frac{\tilde{\beta}_{k^*} a_n^J}{\prod_{i=1}^{J}(i(n+1)+k^*-r)} x^{J(n+1)+k^*-r} \prod_{i=0}^{J-1}(r-i) + l.o.t. \\ &= \frac{\tilde{\beta}_{k^*} a_n^{J+1}}{\prod_{i=1}^{J}((i(n+1)+k^*-r)} x^{J(n+1)+k^*-r+n} \prod_{i=0}^{J}(r-i) + l.o.t. \end{aligned}$$

Now integrating the previous equation we obtain

$$\begin{aligned} u_{r-J-1}(x) &= \frac{\tilde{\beta}_{k^*} a_n^{J+1} x^{(J+1)(n+1)+k^*-r}}{((J+1)(n+1)+k^*-r)\prod_{i=1}^{J}(i(n+1)+k^*-r)} \prod_{i=0}^{J}(r-i) + l.o.t. \\ &= \frac{\tilde{\beta}_{k^*} a_n^{J+1}}{\prod_{i=1}^{J+1}(i(n+1)+k^*-r)} x^{(J+1)(n+1)+k^*-r} \prod_{i=0}^{J}(r-i) + l.o.t., \end{aligned}$$

which is equation (12) with $j = J+1$. This proves the claim done in (12).

From (12) with $j = r-1$ we obtain

$$u_1(x) = \frac{\tilde{\beta}_{k^*} a_n^{r-1}}{\prod_{i=1}^{r-1}(i(n+1)+k^*-r)} x^{(r-1)(n+1)+k^*-r} \prod_{i=0}^{r-2}(r-i) + l.o.t.,$$

Then we have that the coefficient of y^0 in (11) satisfies

$$-g(x)u_1(x) = \sum_{k=0}^{n} \beta_{k,0} x^k,$$

or equivalently

$$-g(x)u_1(x) = -\frac{b_{n+1}\tilde{\beta}_{k^*} a_n^{r-1}}{\prod_{i=1}^{r-1}(i(n+1)+k^*-r)} x^{(r-1)(n+1)+k^*-r+n+1} \prod_{i=0}^{r-2}(r-i) + l.o.t. = \sum_{k=0}^{n} \beta_{k,0} x^k.$$

Since $r \geq 2$ we have that $(r-1)(n+1)+k^*-r = (r-1)n+k^*-1 \geq n+k^*-1 \geq n+r-1 \geq n+1$, we have a contradiction. Hence $r < 2$, that is $r \leq 1$.

We write $u(x,y) = u_0(x) + yu_1(x)$. From (8) we have

(13)
$$yu_0'(x) + y^2 u_1'(x) - (g(x)+f(x)y)u_1(x) = \sum_{k=0}^{n} \beta_{k,0} x^k + y\sum_{k=1}^{n} \beta_{k-1,1} x^{k-1} + y^2 \sum_{k=2}^{n} \beta_{k-2,2} x^{k-2}.$$

Computing the coefficient of y^2 in (13) we get

(14)
$$u_1'(x) = \sum_{k=2}^{n} \beta_{k-2,2} x^{k-2}, \quad \text{i.e.} \quad u_1(x) = \beta^* + \sum_{k=2}^{n} \frac{\beta_{k-2,2}}{k-1} x^{k-1} \quad \text{with} \quad \beta^* \in \mathbb{C}.$$

Furthermore the coefficient of y in (13) gives

$$u_0'(x) - f(x)u_1(x) = \sum_{k=1}^{n} \beta_{k-1,1} x^{k-1}.$$

Integrating we have

(15)
$$\begin{aligned} u_0(x) &= \int f(x) \left(\beta^* + \sum_{k=2}^{n} \frac{\beta_{k-2,2}}{k-1} x^{k-1} \right) dx + \sum_{k=1}^{n} \frac{\beta_{k-1,1}}{k} x^k \\ &= \beta^* F(x) + \sum_{k=2}^{n} \frac{\beta_{k-2,2}}{k-1} \int f(x) x^{k-1}\, dx + \sum_{k=1}^{n} \frac{\beta_{k-1,1}}{k} x^k. \end{aligned}$$

Finally the coefficient of y^0 in (13) gives

(16)
$$-g(x)\left(\beta^* + \sum_{k=2}^{n} \frac{\beta_{k-2,2}}{k-1} x^{k-1} \right) = \sum_{k=0}^{n} \beta_{k,0} x^k.$$

Since $g(x) = \sum_{j=0}^{n+1} b_j x^j$ and $b_{n+1} \neq 0$ with $n \geq 1$ we have that

(17)
$$\beta^* = 0 \quad \text{and} \quad \beta_{k-2,2} = 0 \quad \text{for } k \geq 2.$$

Thus from (16) we get that $\beta_{k,0} = 0$ for $k = 0, \ldots, n$. From (14), (15) and (17) we get

$$u_1 = 0, \quad u_0 = \sum_{k=1}^{n} \frac{\beta_{k-1,1}}{k} x^k,$$

and thus

$$u(x,y) = \sum_{k=1}^{n} \frac{\beta_{k-1,1}}{k} x^k.$$

and

$$L = y \sum_{k=1}^{n} \beta_{k-1,1} x^{k-1}. \tag{18}$$

This completes the proof of the proposition. □

Proof of Theorem 1. The exponential factors are $e^{u(x,y)} = e^{x^k}$ with cofactor $kx^{k-1}y$, see Proposition 8. We consider two cases.

Case 1*:* Assume that the polynomial g does not satisfy the assumptions of Proposition 3. Then in order that system (1) has a Liouvillian first integral, by Theorem 7 and Propositions 3 and 8, we have

$$\sum_{l=1}^{n} \mu_l x^l y = f(x), \tag{19}$$

which is not possible since the right-hand side of (19) is independent of y. This ends the proof of Theorem 1 in this case.

Case 2*:* Assume that the polynomial g satisfies the assumptions of Proposition 3. Therefore in order that system (1) has a Liouvillian first integral, by Theorem 7 and Propositions 3 and 8, we have

$$-\lambda(f(x) + P'(x)) + y \sum_{l=1}^{n} \mu_l x^l = f(x),$$

which implies $\mu_l = 0$ for $l = 1, \ldots, n$ and

$$(1 + \lambda) f(x) + \lambda P'(x) = 0. \tag{20}$$

Assume that statement (a) of Proposition 3 holds. Then $P(x)$ has degree one. Using that f has degree $n \geq 1$, we get from (20) that $a_n(1 + \lambda) = 0$, i.e. $\lambda = -1$, and then $P'(x) = 0$ which is not possible.

Assume that statement (b) of Proposition 3 holds. Therefore $P(x) + \int f(x)\, dx$ has degree one. Consequently $P'(x) = -f(x) + b$ where b is a non-zero constant. In this case, (20) becomes

$$f(x) + \lambda b = 0,$$

which is again not possible. This ends the proof of Theorem 1. □

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References

[1] L. Cairó, H. Giacomini and J. Llibre, *Liouvillian first integrals for the planar Lotka-Volterra system*, Rend. Circ. Mat. Palermo **52** (2003), 389–418.

[2] C.J. Christopher, Liouvillian first integrals of second order polynomial differential equations, Electron. J. Differential Equations **1999** (1999), pp7.

[3] C. Christopher, J. Llibre and J.V. Pereira, *Multiplicity of invariant algebraic curves and Darboux integrability*, Pacific J. Math., **229** (2007), 63–117.

[4] G. Darboux, *Mémoire sur les équations différentielles algèbriques du premier ordre et du premier degré* (Mélanges), Bull. Sci. Math. 2ème. série 2 (1878), 60–96; 123–144; 151–200.

[5] F. Dumortier, J. Llibre and J.C. Artés, *Qualitative theory of planar differential systems*, Universitext, Springer-Verlag, Berlin, 2006.

[6] M. Hayashi, *On polynomial Liénard systems which have invariant algrebraic curves*, Funkcial. Ekvac. **39** (1996), 403–408.
[7] S. Labrunie, *On the polynomial first integrals of the* (a, b, c) *Lotka–Volterra system*, J. Math. Phys. **37** (1996), 5539–5550.
[8] J. Llibre, *Integrability of polynomial differential systems*, Handbook of Differential Equations, Ordinary Differential Equations, Eds. A, Canada, P. Drabek and A. Fonda, Elsevier (2004), pp. 437–533.
[9] J. Llibre and C. Valls, *Liouvillian first integrals for Liénard polynomial differential systems*, Proc. Amer. Math. Soc. **138** (2010), 3229–3239.
[10] J. Llibre and C. Valls, *Liouvillian first integrals for generalized Liénard polynomial differential systems*, to appear in Advanced Nonlinear Studies.
[11] J. Moulin Ollagnier, *Polynomial first integrals of the Lotka–Volterra system*, Bull. Sci. Math. **121** (1997), 463–476.
[12] J. Moulin Ollagnier, *Rational integration of the Lotka–Volterra system*, Bull. Sci. Math. **123** (1999), 437–466.
[13] J. Moulin Ollagnier, *Liouvillian Integration of the Lotka-Volterra system*, Qual. Theory Dyn. Syst. **2** (2001), 307–358.
[14] J. Moulin Ollagnier, *Corrections and complements to: "Liouvillian integration of the Lotka-Volterra system" [Qual. Theory Dyn. Syst.* **2** *(2001), 307–358]*, Qual. Theory Dyn. Syst. **5** (2004), 275–284.
[15] M. F. Singer, *Liouvillian first integrals of differential systems*, Trans. Amer. Math. Soc. **333** (1992), 673–688.

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