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ON THE SET OF PERIODS OF THE GRAPH HOMEOMORPHISMS

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Abstract. In this paper we characterize all possible sets of periods of homeomorphisms defined on some classes of finite connected compact graphs.

1. Introduction

Here a (*topological graph*) or simply a *graph* G is a compact set formed by a finite union of *vertices* (points) and *edges*, which are homeomorphic to a non–empty open interval of the real line, and are pairwise disjoint. The boundary of one edge is formed either by two vertices, or by a unique vertex. Moreover, the graphs that we consider here always are connected.

We identify a circle with the unit circle $\mathbb{S}^1$ centered at the origin of the complex plane. A *circuit* (or *loop*) of a graph G is any subset of G homeomorphic to $\mathbb{S}^1$. A *tree* is a graph without circuits. The set of vertices of a graph G will be denoted by $V(G)$. Clearly $V(G)$ is finite.

Let G be a graph and $z \in G$. Then, we consider a small open neighborhood U (in G) of z such that $\mathrm{Cl}(U)$ is a tree. The number of connected components of $U \setminus \{z\}$ is called the *valence* of z and is denoted by $\mathrm{Val}(z)$. Observe that this definition is independent of the choice of U if it is sufficiently small, and that $\mathrm{Val}(z) \neq 2$ implies that $z \in V(G)$. A vertex of valence 1 is called an *endpoint of* G and a vertex of valence larger than 2 is called a *branching point of* G.

Let $f : G \to G$ be a continuous map. A point $z \in G$ such that $f(z) = z$ is called a *fixed point* or a periodic point of period 1. The point $z \in G$ is periodic of period $m > 1$ if $f^m(z) = z$ and $f^k(z) \neq z$ for $k = 1, \ldots, m-1$. Of course, in the whole paper $f^m(z)$ denotes the m–th iterate of the point z by the map f. We denote by $\mathrm{Per}(f)$ the *set of periods of all periodic points of* f.

In this work our aim is to characterize the sets $\mathrm{Per}(f)$ when $f : G \to G$ is a homeomorphism of a given graph G. As we will see this objective is only reached for some classes of graphs, the full characterization for every graph looks as a very hard problem.

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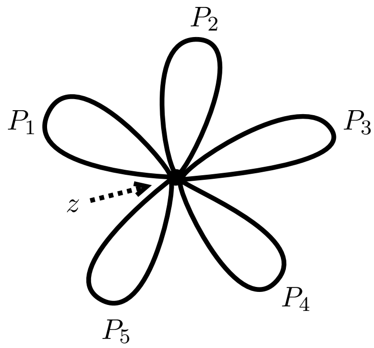

FIGURE 1. A 5–flower graph.

Probably the first result on the set of periods of a homeomorphism of a graph is the following one due to Fuller [3]. See section 2 for the definition of independent oriented loops.

Theorem 1. *Let G be a graph with c independent oriented loops and let $f : G \to G$ be a homeomorphism. Then, the following statements hold.*

(a) *If $c = 0$ (i.e. G is a tree), then $1 \in \mathrm{Per}(f)$.*

(b) *If $c > 1$, then $\mathrm{Per}(f) \cap \{1, 2, \dots, c\} \neq \emptyset$.*

In fact Fuller does not provide Theorem 1, he provided a more general result that restricted to graphs becomes Theorem 1, see for details section 2.

The characterizations of the sets of periods for the homeomorphisms on a closed interval I or on the circle $\mathbb{S}^1$ are well known for the mathematicians working in topological dynamics, see the next two theorems, but since it is not easy to find their proofs in the literature we provide a proof of these two theorems in section 3.

Theorem 2 (Interval Theorem)**.** *Let I be a non–degenerate closed interval (i.e. different from a point), and let $f : I \to I$ be a homeomorphism. Then*

$$\mathrm{Per}(f) = \begin{cases} \{1\} & \text{if } f \text{ is increasing,} \\ \{1,2\} & \text{if } f \text{ is decreasing.} \end{cases}$$

As usual $\mathbb{Q}$ and $\mathbb{R}$ denote the sets of rational and real numbers respectively. See the definition of rotation number $\rho(f) \in \mathbb{R}$ for a homeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$ which preserves the orientation in section 3.

Theorem 3 (Circle Theorem)**.** *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism.*

(a) *If f preserves the orientation, then*

$$\mathrm{Per}(f) = \begin{cases} \emptyset & \text{if } \rho(f) \notin \mathbb{Q}, \\ \{n\} & \text{if } \rho(f) = \dfrac{k}{n} \text{ with } \gcd(k,n) = 1. \end{cases}$$

(b) *If f reverses the orientation, then $Per(f)$ is either $\{1\}$, or $\{1, 2\}$.*

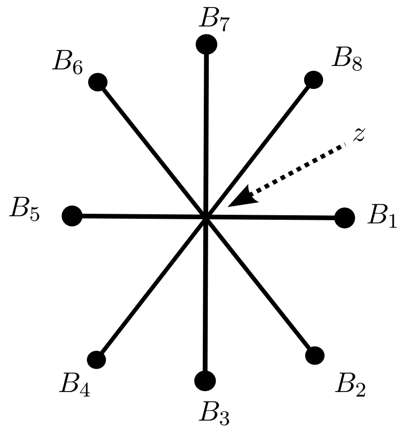

FIGURE 2. The 8−odd graph.

A *p–flower graph* is a graph with a unique branching point z and $p > 1$ edges all having a unique endpoint, the point z, equal for all of them. So, this graph has p independent loops, each one is called a *petal*. See a 5–flower graph in Figure 1.

Theorem 4 (p-Flower Theorem)**.** *Let $f : G \to G$ be a homeomorphism of a p–flower graph G with p petals $P_1, P_2, \ldots, P_p$.*

(a) *If $f(P_l) = P_l$ for $l = 1, 2, \ldots, p$, then* $\mathrm{Per}(f)$ *is either $\{1\}$, or $\{1, 2\}$.*

(b) *If $f(P_l) \neq P_l$ for some $l \in \{1, 2, \ldots, p\}$, then* $\mathrm{Per}(f)$ *is either $\{1\}$, or any subset of $\{1, n_1, n_2, \ldots, n_s, 2n_1, 2n_2, \ldots, 2n_s\}$ containing the 1, where $n_1, n_2, \ldots, n_s$ are arbitrary positive integers (non necessarily different) satisfying $1 < n_1 + n_2 + \ldots + n_s = p$.*

A graph with only one branching point z with valence $b > 2$ and b edges having every edge the vertex z and another vertex different from z as endpoints always with valence 1 is called a *b–odd graph*. See an 8–odd graph in Figure 2.

Theorem 5 (b–odd Theorem)**.** *Let $f : G \to G$ be a homeomorphism of a b–odd graph G with branching point z and edges B_1, B_2, ..., B_b. Then the set* $\mathrm{Per}(f)$ *is $\{1\}$ if $f(x) = x$ for all $x \in V(G)$, or $\{1, n_1, n_2, \ldots, n_s\}$ otherwise, where $n_1, n_2, \ldots, n_s$ are positive integers (non necessarily different) satisfying $1 < n_1 + n_2 + \ldots + n_s = b$.*

A graph with only two vertices z and w and $n > 1$ edges having every edge the vertices z and w as endpoints is called an *n–lips graph*. See a 7–lips graph in Figure 3.

Theorem 6 (n–lips Theorem)**.** *Let $f : G \to G$ be a homeomorphism of the n–lips graph G with vertices z and w, and let $e_1, e_2, \ldots, e_n$ be the edges of G. Then the set* $\mathrm{Per}(f)$ *is*

(a) *either $\{1\}$, if $f(z) = z$ and $f(e_i) = e_i$ for all $i = 1, 2, \ldots, n$;*

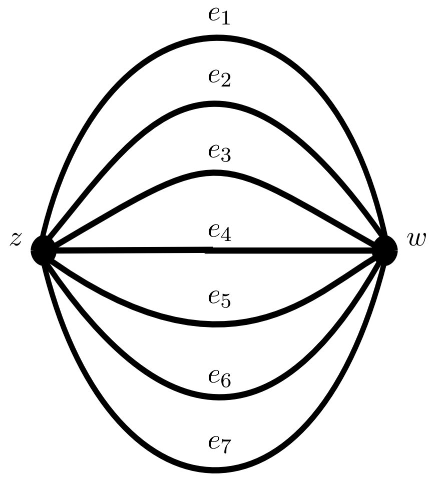

FIGURE 3. The 7–lips graph.

(b) *or any subset of* $\{1, n_1, n_2, \ldots, n_s\}$ *including the* 1*, if* $f(z) = z$ *and* $f(e_i) \neq e_i$ *for some* $i \in \{1, 2, \ldots, n\}$ (*see the restrictions of the numbers* n_i *after all the statements*)*;*

(c) *or* $\{1, 2\}$*, if* $f(z) \neq z$ *and* $f(e_i) = e_i$ *for all* $i = 1, 2, \ldots, n$*;*

(d) *or any subset of* $\{2, n_1, n_2, \ldots, n_s, 2n_1, 2n_2, \ldots, 2n_s\}$ *including the set* $\{2, n_1, n_2, \ldots, n_s\}$*, if* $f(z) \neq z$ *and* $f(e_i) \neq e_i$ *for some* $i \in \{1, 2, \ldots, n\}$*,*

where $n_1, n_2, \ldots, n_s$ *are non-negative integers* (*non necessarily different*) *satisfying* $1 < n_1 + n_2 + \ldots + n_s = n$*. The periods* $2n_i$ *for* $i = 1, 2, \ldots, s$ *only can appear if* n_i *is odd.*

A graph with $p + b$ edges, where $p \geq 1$ of them are petals and the other $b \geq 1$ are not petals, having all the edges as endpoint a point z, is called a (p, b)*–graph*. In this case the point z has valence $2p + b$, and it is called the *main branching point* of the (p, b)–graph. See a $(4, 10)$–graph in Figure 4.

Theorem 7 ((p, b)–graph Theorem)**.** *Let* $f : G \to G$ *be a homeomorphism of a* (p, b)*–graph* G *with* p *petals* P_1*,* P_2*, …,* P_p *and* b *edges* B_1*,* B_2*, …,* B_b*, which are not petals. Let* z *be the main branching point of* G*. All the biggest subgraphs of* G*, which are* n*–lips for some* n*, are grouped as follows. Let* $L^{\eta_q}_{j_q,1}, L^{\eta_q}_{j_q,2}, \ldots, L^{\eta_q}_{j_q,t_q}$ *be all the* η_q*–lips subgraphs of* G *whose two vertices are the vertex* z *and another vertex* w_k *with* $k = 1, \ldots, t_q$*, each vertex* w_k *has valence* η_q*, and* $q = 1, 2, \ldots, \rho$ (*see Figure 4*)*. Then the set* $\mathrm{Per}(f)$ *is*

(a) *either* $\{1\}$*, or* $\{1, 2\}$*, if* $f(P_l) = P_l$ *for all* $l = 1, 2, \ldots, p$*, and* $f(B_j) = B_j$ *for all* $j = 1, 2, \ldots, b$*;*

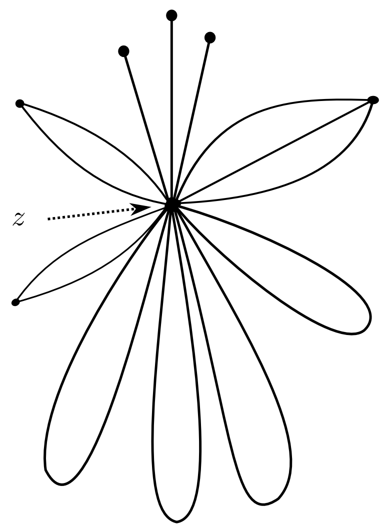

FIGURE 4. A $(4,10)$–graph with $\rho = q = 1$, $\eta_q = 3$ and $t_q = 2$.

(b) *or* $\{1, n_1, n_2, \ldots, n_s\} \bigcup \left(\bigcup_{q=1}^{\rho} \left(\bigcup_{i=1}^{v_q} r_{i,q} A_{i,q} \right) \right)$,

or $\{1, 2, n_1, n_2, \ldots, n_s\} \bigcup \left(\bigcup_{q=1}^{\rho} \left(\bigcup_{i=1}^{v_q} r_{i,q} A_{i,q} \right) \right)$,

if $f(P_l) = P_l$ *for all* $l = 1, 2, \ldots, p$, *and* $f(B_j) \neq B_j$ *for some* $j \in \{1, 2, \ldots, b\}$ *(see the restrictions on the numbers* n_i, q, $r_{i,q}$ *and* v_q *at the end of the statements);*

(c) *or* $\{1\}$, *or any subset of* $\{1, k_1, k_2, \ldots, k_u, 2k_1, 2k_2, \ldots, 2k_u\}$ *containing the* 1, *where* $k_1, k_2, \ldots, k_u$ *are arbitrary positive integers (non necessarily different) satisfying* $1 < k_1 + k_2 + \ldots + k_u = p$, *if* $f(P_l) \neq P_l$ *for some* $l \in \{1, 2, \ldots, p\}$, *and* $f(B_j) = B_j$ *for all* $j = 1, 2, \ldots, b$;

(d) *or any subset of*

$$\{1, n_1, \ldots, n_s, k_1, k_2, \ldots, k_u, 2k_1, 2k_2, \ldots, 2k_u\} \bigcup \left(\bigcup_{q=1}^{\rho} \left(\bigcup_{i=1}^{v_q} r_{i,q} A_{i,q} \right) \right),$$

including all the elements of this set except perhaps some of the elements of the set $\{k_1, k_2, \ldots, k_u, 2k_1, 2k_2, \ldots, 2k_u\}$, *if* $f(P_l) \neq P_l$ *for some* $l \in \{1, 2, \ldots, p\}$, *and* $f(B_j) \neq B_j$ *for some* $j \in \{1, 2, \ldots, b\}$;

where $n_1, n_2, \dots, n_s$, $r_{i,q}$ *for* $i = 1, 2, \dots, v_q$ *and* $q = 1, 2, \dots, \rho$, *and* $k_1, k_2, \dots, k_u$ *are positive integers (non necessarily different) satisfying*

$$1 < n_1 + n_2 + \ldots + n_s + \sum_{q=1}^{\rho} \sum_{i=1}^{v_q} \eta_q r_{i,q} = b,$$

$r_{1,q} + r_{2,q} + \ldots + r_{v_q,q} = t_q$ *and* $1 < k_1 + k_2 + \ldots + k_u = p$, *and* $A_{i,q}$ *is one of the sets of statements (a) and (b) of Theorem 6, for all* $i = 1, 2, \dots, v_q$ *and* $q = 1, \dots, \rho$.

Theorems 4, 5, 6 and 7 are proved in section 4.

2. Homology of a graph and Fuller's result

We can consider the fundamental group of a graph G, see for instance [10] for more details on the fundamental group. The elements of the fundamental group are oriented loops of G. We assume that the fundamental group of G has c independent oriented loops γ_i for $i = 1, \dots, c$, and let $f : G \to G$ be a continuous map. Then, the homology groups of G are $H_0(X, \mathbb{Q}) = \mathbb{Q}$ and $H_1(X, \mathbb{Q}) = \bigoplus_{i=1}^{c} \mathbb{Q}$, and the actions $f_{*k} : H_k(X, \mathbb{Q}) \to H_k(X, \mathbb{Q})$ for $k = 0, 1$ induced by f on these homology groups are $f_{*0} = (1)$ (because G is connected) and $f_{*1} = A$, where A is a $c \times c$ matrix with integer entries. The element a_{ij} of the matrix A is the number of times that the loop γ_i covers the loop γ_j taking into account the orientation of the covering. Therefore, the Betti's numbers of G are $B_0(G) = \dim_{\mathbb{Q}} H_0(X, \mathbb{Q}) = 1$ and $B_1(G) = \dim_{\mathbb{Q}} H_1(X, \mathbb{Q}) = c$. For more details of the homology of G see [10, 8, 11].

Fuller in [3] proved the following result; see also Halpern [4] and Brown [2].

Theorem 8 (Fuller's Theorem)**.** *Let f be a homeomorphism of a compact polyhedron X into itself. If the Euler characteristic of X is not zero, then f has a periodic point with period not greater than the maximum of* $\sum_{k \text{ odd}} B_k(X)$ *and* $\sum_{k \text{ even}} B_k(X)$, *where $B_k(X)$ denotes the k−th Betti number of X.*

Applying the Fuller's theorem to a graph G it follows Theorem 1.

For other results on the set of a continuous map from a graph into itself see for instance [1] or [6], and the references quoted there.

3. An interval and a circle

First we state a general result for the set of periods of a homeomorphism of a graph.

Proposition 9. *Let $f : G \to G$ be a homeomorphism of a graph G non homeomorphic to a circle. Then, the following statements hold.*

(a) *Let z be a vertex of valence k. Then $f(z)$ is a vertex of valence k if $k \neq 2$.*

(b) $\mathrm{Per}(f) \neq \emptyset$.

Proof. Statement (a) follows immediately from the definition of a homeomorphism. Since a graph non–homeomorphic to a circle has a vertex with valence different from 2, statement (b) follows easily from statement (a) because a graph has finitely many vertices. □

From now on we shall investigate the possible sets $\mathrm{Per}(f)$ for the homeomorphisms $f : G \to G$ of different graphs G. We shall start with the easiest graphs, as an interval and a circle, and we shall finish with more complicated graphs. The results on the set of periods for the homeomorphisms of an interval and of a circle play a main role in the study of the set of periods of the homeomorphisms of other graphs.

Proof of Theorem 2 (*Interval Theorem*). Without loss of generality, we can suppose that $I = [0, 1]$. Hence, if $f : [0, 1] \to [0, 1]$ is an orientation preserving homeomorphism (i.e. monotone increasing), by Proposition 9(a) we have $f(0) = 0$ and $f(1) = 1$. Moreover, we claim that any orbit of f, i.e. for all $x \in [0, 1]$ we have that $\{x, f(x), f^2(x), \ldots\}$, tends to a fixed point.

Firstly, we remark that if $f : [0, 1] \to [0, 1]$ is an increasing homeomorphism, then besides the fixed points 0 and 1, there can exist other fixed points into the interval $I = [0, 1]$. We restrict f to a subinterval formed by two consecutive fixed points, i.e. $[y, z]$ such that either $f(x) > x$ for all $x \in (y, z)$ or $f(x) < x$ for all $x \in (y, z)$. If $f(x) > x$ then any orbit $\{x, f(x), f^2(x), \ldots\}$ tends to the fixed point z, and if $f(x) < x$ then any orbit $\{x, f(x), f^2(x), \ldots\}$ tends to the fixed point y. More precisely, we take $x \in (y, z)$ and first we consider the case that $f(x) > x$. Then, $f^2(x) = f(f(x)) > f(x)$, because $f|_{(y,z)}$ is monotone increasing. By induction we get $f^n(x) = f(f^{n-1}(x)) > f^{n-1}(x)$. Hence, the sequence $\{f^n(x)\}_{n=0}^{\infty}$ is monotone increasing and upper bounded by z, so it converges to $\sup\{f^n(x), \text{ for } n = 0, 1, \ldots\} = z$. Therefore, the ω–limit set of the orbit of $x \in (y, z)$ is the fixed point z.

Similarly, if $f(x) < x$, then the sequence $\{f^n(x)\}_{n=0}^{\infty}$ is decreasing and lower bounded by y, and it converges to the fixed point y of f. Hence, the claim is proved.

If $f : [0, 1] \to [0, 1]$ is an orientation reversing homeomorphism (i.e. monotone decreasing), then by Proposition 9(a) we get $f(0) = 1$ and $f(1) = 0$. So, $2 \in \mathrm{Per}(f)$. By the Bolzano's Theorem also called the Intermediate Value Theorem, we get that $1 \in \mathrm{Per}(f)$. On the other hand, the second iterate f^2 is an orientation preserving homeomorphism, so from the first part we obtain $\mathrm{Per}(f^2) = \{1\}$. Therefore, $\mathrm{Per}(f) = \{1, 2\}$. □

For studying the set of periods of the homeomorphisms of the circle we need to introduce an important dynamical invariant called the *rotation number*, it was firstly introduced by Poincaré [9] in 1885.

For studying the dynamics of a continuous map $f : \mathbb{S}^1 \to \mathbb{S}^1$ it is helpful to lift the map to the straight line $\mathbb{R}$. For a such f we call a map $F : \mathbb{R} \to \mathbb{R}$ a *lifting* of f if $\pi \circ F = f \circ \pi$, where $\pi : \mathbb{R} \to \mathbb{S}^1$ is given by $\pi(x) = \exp(2\pi i x) = \cos(2\pi x) + \sin(2\pi x)i$. The *degree* of the map f is by definition the integer $F(1) - F(0)$, for more details see [1].

There are always infinitely many different liftings for a continuous map $f : \mathbb{S}^1 \to \mathbb{S}^1$. Indeed, one may easily prove that any two liftings of f differ by an integer, that is, if F_1 and F_2 are liftings, then there exists $k \in \mathbb{Z}$ such that $F_1(x) = F_2(x) + k$.

Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism. If f is orientation preserving, then its degree is 1 and, if f is orientation reversing, its degree is -1. Moreover, the lifting of a homeomorphism of the circle is a homeomorphism on the straight line.

For studying the set of periods of the orientation preserving homeomorphisms we introduce the rotation number, which is a number between 0 and 1 that roughly speaking measures the average amount of points which are rotated by an iteration of a continuous map $f : \mathbb{S}^1 \to \mathbb{S}^1$ of degree 1. Before defining the rotation number, we introduce a preliminary concept.

Let F be a lifting of an orientation–preserving homeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$ of degree 1. For $x \in \mathbb{S}^1$ we define

$$\rho_0(F, x) = \lim_{n\to\infty} \frac{F^n(x)}{n}.$$

This limit exists and does not depend upon the choice of x. For this reason we can put $\rho_0(F)$ instead of $\rho_0(F, x)$. The *rotation number of* f, $\rho(f)$, is the fractional part of $\rho_0(F)$ for any lifting F of f. That is, $\rho(f)$ is the unique number in $[0, 1)$ such that $\rho_0(F)-\rho(f)$ is an integer. For more details about the rotation number see [7, 5, 1]. We note that in [1] the rotation number is essentially defined as $\rho_0(F)$, instead of its fractional part.

Proof of Theorem 3 (*Circle Theorem*). Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism and assume that it preserves the orientation. Poincaré [9] proved that the rotation number of an orientation preserving homeomorphism is irrational if and only if it has no periodic points, see also [1]. So, for proving statement (a) we only need to prove the equality $\mathrm{Per}(f) = \{n\}$ when $\rho(f) = k/n$ with $\gcd(k, n) = 1$, and this is proved for instance in [5, 1]. So, the proof of statement (a) is completed.

Suppose that f reverses the orientation. Since continuous maps of degree -1 have fixed points, see for instance [1], we have that $1 \in \mathrm{Per}(f)$. So, there exists a point $x \in \mathbb{S}^1$ such that $f(x) = x$. Then $f^2(x) = x$ and, as f^2 is a homeomorphism that preserves the orientation, by statement (a) we get $\mathrm{Per}(f^2) = \{1\}$ and, consequently, $\mathrm{Per}(f) \subseteq \{1, 2\}$.

If we consider now the circle as the interval $[0, 1]$ with both endpoints identified, the map $f : [0, 1] \to [0, 1]$ defined by $f(x) = 1-x$ is such that f^2 is the identity. So, for this orientation reversing homeomorphism we have that

$\mathrm{Per}(f) = \{1, 2\}$. Now, there are monotone decreasing maps $g : [0, 1] \to [0, 1]$ such that $g(0) = 1$, $g(1) = 0$ and $\mathrm{Per}(g) = \{1\}$. For example, consider a decreasing map $g : [0, 1] \to [0, 1]$ such that $g(0) = 1$, $g(1) = 0$, $g(x_0) = x_0 > 1/2$, $g(x) > \frac{x_0-1}{x_0} \cdot x + 1$ for all $x \in [0, x_0]$ and $g(x) = \frac{x_0}{x_0-1} \cdot (x - 1)$ for all $x \in [x_0, 1]$, where x_0 is a fixed point of g into the interval $(\frac{1}{2}, 1)$. From the definition of g we get that $g(x) > 1-x$ for all $x \in (0, 1)$ and that these maps are orientation reversing homeomorphism such that $\mathrm{Per}(f) = \{1\}$. This completes the proof of statement (b). $\square$

4. A p-flower graph, a b-odd graph, an n–lips graph and a (p, b)-graph

In these section we shall prove Theorems 4, 5, 6 and 7.

Proof of Theorem 4 (*p-Flower Theorem*). Let G be a p–flower graph with the branching point z and p petals $P_1, P_2, \ldots, P_p$. If $f : G \to G$ is a homeomorphism, by Proposition 9(a) we have that $f(z) = z$. Then, $1 \in \mathrm{Per}(f)$.

Assume that $f(P_l) = P_l$ for all $l = 1, 2, \ldots, p$. Then, $f|_{P_l} : P_l \to P_l$ is a homeomorphism of the topological circle P_l with a fixed point z. So, from Theorem 3 it follows that $\mathrm{Per}(f) = \{1\}$ or $\mathrm{Per}(f) = \{1, 2\}$, and statement (a) is proved.

Suppose that $f(P_l) \neq P_l$ for some $l \in \{1, 2, \ldots, p\}$. Since every petal must be applied to another petal by f, there exist n_1 petals $P_{k_1}, P_{k_2}, \ldots, P_{k_{n_1}}$ such that $f(P_{k_i}) = P_{k_{i+1}}$, for all $i = 1, 2, \ldots, n_1 - 1$, and $f(P_{k_{n_1}}) = P_{k_1}$, where $1 < n_1 \leq p$. Therefore, the iterate f^{n_1} is a homeomorphism of the topological circle P_{k_1} having a fixed point. Thus, $\mathrm{Per}(f^{n_1})$ is $\{1\}$ or $\{1, 2\}$. Therefore, either $1 \in \mathrm{Per}(f)$, or $\{1, n_1\} \subset \mathrm{Per}(f)$, or $\{1, 2n_1\} \subset \mathrm{Per}(f)$, or $\{1, n_1, 2n_1\} \subset \mathrm{Per}(f)$.

Furthermore, if $n_1 < p$, there can exist other n_2 petals $P_{l_1}, P_{l_2}, \ldots, P_{l_{n_2}}$ with similar property and satisfying $1 \leq n_2 \leq p - n_1$, implying that either $1 \in \mathrm{Per}(f)$, or $\{1, n_2\} \subset \mathrm{Per}(f)$, or $\{1, 2n_2\} \subset \mathrm{Per}(f)$, or $\{1, n_2, 2n_2\} \subset \mathrm{Per}(f)$.

In short, repeating these arguments, there can exist $n_1, n_2, \ldots, n_s$ positive integers with the above properties such that either $1 \in \mathrm{Per}(f)$, or $\{1, n_i\} \subset \mathrm{Per}(f)$, or $\{1, 2n_i\} \subset \mathrm{Per}(f)$, or $\{1, n_i, 2n_i\} \subset \mathrm{Per}(f)$, for all $i = 1, 2, \ldots, s$, and satisfying $1 < n_1 + n_2 + \ldots + n_s = p$. Reordering the numbers n_i if necessary, statement (b) follows. $\square$

Proof of Theorem 5 (*b–odd Theorem*). Let G be a b–odd graph with branching point z and edges B_1, B_2, ..., B_b. If $f : G \to G$ is a homeomorphism, by Proposition 9(a) we have that $f(z) = z$. Then, $1 \in \mathrm{Per}(f)$.

If f fixes the other vertices, that is, $f(x) = x$ for all $x \in \mathrm{V}(G)$, from Theorem 2, we get that $\mathrm{Per}(f) = \{1\}$. Otherwise, since the image of an edge by the homeomorphism f is another edge, there exist n_1 edges $B_{k_1}, B_{k_2}, \ldots, B_{k_{n_1}}$ such that $f(B_{k_i}) = B_{k_{i+1}}$, for all $i = 1, 2, \ldots, n_1 - 1$, and

$f(B_{k_{n_1}}) = B_{k_1}$, where $1 < n_1 \leq b$. Therefore, the iterate f^{n_1} is a homeomorphism of the topological interval B_{k_1} having two fixed points, that is, the branching point z and the other vertex of B_{k_1}. Thus, $\text{Per}(f^{n_1}) = \{1\}$. Hence, $\{1, n_1\} \subset \text{Per}(f)$ because the vertices of B_{k_i} different from z form a periodic orbit of period n_1.

Furthermore, if $n_1 < b$, there exist other n_2 edges $B_{l_1}, B_{l_2}, \ldots, B_{l_{n_2}}$ with similar property satisfying $1 \leq n_2 \leq b - n_1$, implying that $n_2 \in \text{Per}(f)$.

In short, repeating these arguments there can exist $n_1, n_2, \ldots, n_s$ positive integers with the above properties such that $n_i \in \text{Per}(f)$, for all $i = 1, 2, \ldots, s$, and satisfying $1 < n_1 + n_2 + \ldots + n_s = b$. Reordering the numbers n_i if necessary, it follows the result. □

Proof of Theorem 6 (*n–lips Theorem*). Let G be an n–lips graph with vertices z and w, and let e_i be the edges of G for $i = 1, 2, \ldots, n$. If $f : G \to G$ is a homeomorphism and $f(z) = z$, by Proposition 9(a) we have that $f(w) = w$ and then $1 \in \text{Per}(f)$. But if $f(z) \neq z$ the Proposition 9(a) assures that $f(z) = w$ and $f(w) = z$ and hence $2 \in \text{Per}(f)$.

Assume that $f(z) = z$ and $f(e_i) = e_i$ for all $i = 1, 2, \ldots, n$. Then, for each $i = 1, 2, \ldots, n$, $f|_{e_i} : e_i \to e_i$ is an increasing homeomorphism. So, by Theorem 2, follows that $\text{Per}(f) = \{1\}$. So, statement (a) is proved.

Now assume that $f(z) = z$ and $f(e_i) \neq e_i$ for some $i \in \{1, 2, \ldots, n\}$. Since the image of the edge e_i by the homeomorphism f is another edge, there exist n_1 edges e_{k_1}, e_{k_2}, …, $e_{k_{n_1}}$ such that $f(e_{k_i}) = e_{k_{i+1}}$, for all $i = 1, 2, \ldots, n_1 - 1$, and $f(e_{k_{n_1}}) = e_1$, where $1 < n_1 \leq n$. Therefore, the iterate f^{n_1} is an increasing homeomorphism of the topological interval e_{k_1} having a fixed point. Thus, by Theorem 2 $\text{Per}(f^{n_1}|_{e_{k_1}}) = \{1\}$. Hence, $\text{Per}(f)$ contains either $\{1\}$ or $\{1, n_1\}$.

Furthermore, if $n_1 < n$, there exist other n_2 edges e_{l_1}, e_{l_2}, …, $e_{l_{n_2}}$ with similar property satisfying $1 \leq n_2 \leq n - n_1$, implying that $\text{Per}(f)$ contains either $\{1\}$, or $\{1, n_1\}$, or $\{1, n_2\}$, or $\{1, n_1, n_2\}$.

In short repeating these arguments there can exist $n_1, n_2, \ldots, n_s$ non–negative integers with the above properties such that $1 \in \text{Per}(f)$ and eventually $n_i \in \text{Per}(f)$, for all $i = 1, 2, \ldots, s$, and satisfying $1 < n_1 + n_2 + \ldots + n_s = n$. Reordering the numbers n_i if necessary, statement (b) follows.

Suppose that $f(z) \neq z$ and $f(e_i) = e_i$ for all $i = 1, 2, \ldots, n$. Then, for each $i = 1, 2, \ldots, n$, $f|_{e_i} : e_i \to e_i$ is a decreasing homeomorphism. So, by Theorem 2, follows that $\text{Per}(f) = \{1, 2\}$. So, statement (c) is proved.

In the case that $f(z) \neq z$ and $f(e_i) \neq e_i$ for some $i = 1, 2, \ldots, n$ we use the same argument that in statement (b) and we obtain a positive integer n_1 such that the iterate f^{n_1} is a homeomorphism of some topological interval e_m, where $m \in \{1, 2, \ldots, n\}$. But here, we note that if n_1 is odd, then f^{n_1} is a decreasing homeomorphism implying that $\text{Per}(f)$ contains either $\{2, n_1\}$ or

$\{2, n_1, 2n_1\}$. And if n_1 is even, then f^{n_1} is an increasing homeomorphism implying that either $2 \in \mathrm{Per}(f)$ or $\{2, n_1\} \subset \mathrm{Per}(f)$. Hence, repeating the argument used in statement (b) we conclude that there can exist $n_1, n_2, \ldots, n_s$ non–negative integers such that either $n_i \in \mathrm{Per}(f)$, or $\{n_i, 2n_i\} \subset \mathrm{Per}(f)$ for all $i = 1, 2, \ldots, s$, and satisfying $1 < n_1 + n_2 + \ldots + n_s = n$. Of course, the case $\{n_i, 2n_i\} \subset \mathrm{Per}(f)$ only can occur if n_i is odd. So, statement (d) follows. $\square$

Consider a branching point z with valence k. This valence can be decomposed as $k = 2p + b$, where $p + b > 0$, $p \geq 0$ is the number of all petals with endpoint z and $b \geq 0$ is the number of edges which are not petals with endpoint z. In this case we shall say that the branching point z is of *type* (p, b). Then every vertex of a graph has a type (p, b). For example, an endpoint is a vertex of valence $k = 1$, and hence, it is of the type $(0, 1)$. Now we can improve Proposition 9(a) as follows.

Proposition 10. *Let $f : G \to G$ be a homeomorphism of a graph G non homeomorphic to a circle. If z is a vertex of type (p, b), then $f(z)$ is a vertex of type (p, b).*

Proof. It follows immediately from the definition of a homeomorphism. $\square$

Proof of Theorem 7 ((p, b)–graph Theorem). Let G be a (p, b)–graph with the main branching point z, p petals P_1, P_2, …, P_p, and b edges $B_1, B_2 \ldots, B_b$ which are not petals. Assume that $L^{\eta_q}_{j_q,1}, L^{\eta_q}_{j_q,2}, \ldots, L^{\eta_q}_{j_q,t_q}$ are the η_q–lips for $q = 1, 2, \ldots, \rho$ contained into the (p, b)–graph described in the statement of the theorem. If $f : G \to G$ is a homeomorphism, by Proposition 10 we have that $f(z) = z$. Then $1 \in \mathrm{Per}(f)$.

Assume that $f(P_l) = P_l$ for all $l = 1, 2, \ldots, p$, and $f(B_j) = B_j$ for all $j = 1, 2, \ldots, b$. Then, for each $l = 1, 2, \ldots, p$, $f|_{P_l} : P_l \to P_l$ is a homeomorphism of the topological circle P_l with the fixed point z, and, for each $j = 1, 2, \ldots, b$, $f|_{B_j} : B_j \to B_j$ is a homeomorphism of the topological interval B_j with the fixed endpoint z. So, from Theorems 2 and 3 it follows that either $\mathrm{Per}(f) = \{1\}$, or $\mathrm{Per}(f) = \{1, 2\}$, and statement (a) is proved.

Suppose that $f(P_l) = P_l$ for all $l = 1, 2, \ldots, p$, and $f(B_j) \neq B_j$ for some $j \in \{1, 2, \ldots, b\}$. For the p petals we apply Theorem 3 and we obtain that either $\{1\} \subset \mathrm{Per}(f)$, or $\{1, 2\} \subset \mathrm{Per}(f)$. Since every edge which is not a petal must be applied to another edge which is not a petal by f, we apply Theorem 5 to the n–odd subgraph of G formed by all the edges which are not petals and are not contained into the η_q–lips $L^{\eta_q}_{j_q,k}$, for $q = 1, 2, \ldots, \rho$ and $k = 1, 2, \ldots, t_q$. We conclude that there exist $n_1, n_2, \ldots, n_s$ positive integers such that $n_i \in \mathrm{Per}(f)$, for all $i = 1, 2, \ldots, s$, and satisfying $1 < n_1 + n_2 + \ldots + n_s \leq b$. Furthermore, if $f(B_j) \neq B_j$ for some edge B_j of some η_q–lips $L^{\eta_q}_{j_q,k}$, for $q = 1, 2, \ldots, \rho$ and $k = 1, 2, \ldots, t_q$, since every η_q–lips must be applied into another η_q–lips by f there can exist $r_{1,q} \leq t_q$ η_q–lips' forming a cycle, i.e. there are $L^{\eta_q}_{j_q,m_1}, L^{\eta_q}_{j_q,m_2}, \ldots, L^{\eta_q}_{j_q,m_{r_{1,q}}}$ such that

$f(L^{\eta_q}_{j_q,m_i}) = L^{\eta_q}_{j_q,m_{i+1}}$ for all $i = 1,2,\dots,r_{1,q}-1$ and $f(L^{\eta_q}_{j_q,m_{r_{1,q}}}) = L^{\eta_q}_{j_q,m_1}$. Thus, the iterate $f^{r_{1,q}}$ is a homeomorphism from the η_q–lips $L^{\eta_q}_{j_q,m_1}$ into itself. Since the branching point z is fixed by f we get that $\mathrm{Per}(f^{r_{1,q}}|_{L^{\eta_q}_{j_q,m_1}})$ is a set $A_{1,q}$ as one of the sets of statements (a) and (b) of Theorem 6. Therefore we get that the set $r_{1,q}A_{1,q} \subset \mathrm{Per}(f)$.

Furthermore, if $r_{1,q} < t_q$ there exist others $r_{2,q}$ η_q–lips' $L^{\eta_q}_{j_q,a_1}, L^{\eta_q}_{j_q,a_2},\dots, L^{\eta_q}_{j_q,a_{r_{2,q}}}$ with similar property satisfying $1 \le r_{1,q} + r_{2,q} \le t_q$, implying that $\mathrm{Per}(f^{r_{2,q}}|_{L^{\eta_q}_{j_q,a_1}})$ is a set $A_{2,q}$ as one of the sets of statements (a) or (b) of Theorem 6. Therefore we have that $r_{2,q}A_{2,q} \subset \mathrm{Per}(f)$.

In short, repeating these arguments there can exist $r_{1,q}, r_{2,q},\dots, r_{v_q,q}$ positive integers and $A_{1,q}, A_{2,q},\dots, A_{v_q,q}$ sets being as one of the sets of statements (a) or (b) of Theorem 6 such that

$$\bigcup_{q=1}^{\rho}\left(\bigcup_{i=1}^{v_q} r_{i,q}A_{i,q}\right) \subset \mathrm{Per}(f),$$

with $r_{1,q} + r_{2,q} + \ldots + r_{v_q,q} = t_q$ and

$$n_1 + n_2 + \ldots + n_s + \sum_{q=1}^{\rho}\sum_{i=1}^{v_q} \eta_q r_{i,q} = b.$$

Reordering the numbers n_j and $r_{i,q}$ if necessary, statement (b) follows.

When $f(P_l) \neq P_l$ for some $l \in \{1,2,\dots,p\}$, and $f(B_j) = B_j$ for all $j = 1,2,\dots,b$, by applying Theorem 5 to the b edges which are not petals we get that $1 \in \mathrm{Per}(f)$. Then, by using the fact that every petal must be applied to another petal by f, we apply statement (b) of Theorem 4 to the p petals and we obtain statement (c).

In the case that $f(P_l) \neq P_l$ for some $l \in \{1,2,\dots,p\}$, and $f(B_j) \neq B_j$ for some $j \in \{1,2,\dots,b\}$, we apply statement (b) of Theorem 4 to the p petals, and Theorems 5 and statements (a) and (b) of Theorem 6 to the other b edges which are not petals, using the same arguments than in the proof of statements (b) and (c) we get statement (d). □

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