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PERIODIC ORBITS FOR REAL PLANAR POLYNOMIAL VECTOR FIELDS OF DEGREE n HAVING n INVARIANT STRAIGHT LINES TAKING INTO ACCOUNT THEIR MULTIPLICITIES

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Abstract. We study the existence and non–existence of periodic orbits and limit cycles for planar polynomial differential systems of degree n having n real invariant straight lines taking into account their multiplicities. The polynomial differential systems with $n = 1, 2, 3$ are completely characterized.

1. Introduction and statement of the main results

The study of the periodic solutions and of the limit cycles of real polynomial differential equations in the plane $\mathbb{R}^2$ is one of the main problems of the qualitative theory of the differential systems in dimension two during the last century and the present one, see for instance the 16–th Hilbert problem [6, 8, 10].

Let P and Q be two polynomials in the variables x and y with real coefficients, then we say that

$$\mathcal{X} = \mathcal{X}(x, y) = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y} \tag{1}$$

is a *polynomial vector field* of degree n or simply a *vector field* of degree n if the maximum of the degrees of the polynomials P and Q is n. The differential polynomial system

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \tag{2}$$

associated to the polynomial vector field $\mathcal{X}$ of degree n is called a *polynomial differential system* of degree n.

A *limit cycle* of a differential system (2) is a periodic orbit isolated in the set of all periodic orbits of system (2).

Polynomial vector fields of degree 2 have been investigated intensively, and more than one thousand papers have been published about them (see for instance [13, 15, 14]), but in general the problem of counting their limit cycles and finding their maximum number remains open. There are some results for polynomial vector fields of degree 3 but not too much.

In this paper we study the existence or non–existence of periodic orbits for real polynomial vector fields of degree n having n real invariant straight lines taking into account their multiplicities.

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We recall the definition of an invariant straight line and also of the multiplicity of an invariant straight line.

Let $f = ax + by + c = 0$ be a straight line of $\mathbb{R}^2$. The straight line $f = 0$ is *invariant* for the polynomial differential system (2) (i.e. it is formed by solutions of system (2)) if for some polynomial $K \in \mathbb{R}[x, y]$ we have

$$\mathcal{X}f = P(x, y)\frac{\partial f}{\partial x} + Q(x, y)\frac{\partial f}{\partial y} = Kf. \tag{3}$$

The polynomial K is called the *cofactor* of the invariant straight line $f = 0$.

For the polynomial differential system (2) we define the polynomial

$$R(x, y) = \det \begin{pmatrix} 1 & x & y \\ 0 & P & Q \\ 0 & PP_x + QP_y & PQ_x + QQ_y \end{pmatrix}.$$

We say that the invariant straight line $f = 0$ has *multiplicity* k for the polynomial differential system (2) if the polynomial f^k divides the polynomial $R(x, y)$ and the polynomial f^{k+1} does not divide the polynomial $R(x, y)$. Roughly speaking if an invariant straight line L has multiplicity k for a given polynomial differential system, this means that it is possible to perturb such polynomial differential system in such a way that they appear k different invariant straight lines bifurcating from L. For more details on the multiplicity see [3].

For polynomial differential systems of degree 2 the following result is known.

Theorem 1. *For a polynomial differential system* (2) *of degree* 2 *having two real invariant straight lines taking into account their multiplicities the following statements hold.*

(a) *Assume that the two invariant straight lines are real and intersect in a point. Then system* (2) *has no limit cycles. It can have periodic solutions.*

(b) *Assume that the two invariant straight lines are real and parallel. Then system* (2) *has no periodic solutions.*

(c) *Assume that the system has a unique invariant straight line of multiplicity* 2. *Then system* (2) *has no periodic solutions.*

Since statement (a) of Theorem 1 is due to Bautin [2] and its proof is short, and statements (b) and (c) are very easy to prove we shall provide in section 2 a proof of all statements of Theorem 1. In fact we cannot find in the literature the proofs of statements (b) and (c) of Theorem 1.

Of course the results of Theorem 1 for polynomial differential systems (2) of degree 2 can be extended to polynomial differential systems (2) of degree 1, i.e. to linear differential systems in $\mathbb{R}^2$. More precisely we have the following well known result.

Theorem 2. *A polynomial differential system* (2) *of degree* 1 *having one real invariant straight line has no periodic solutions.*

It is well known that the unique linear differential systems in $\mathbb{R}^2$ having periodic solutions are the ones having a center, and those have no invariant straight lines. Hence Theorem 2 is proved.

In short, Theorems 1 and 2 characterize the existence or non-existence of periodic solutions and limit cycles for the polynomial differential systems (2) of degree n

having n real invariant straight lines taking into account their multiplicities when $n = 1, 2$. From these two theorems if follows immediately the next result.

Corollary 3. *The polynomial differential systems* (2) *of degree* n *having* n *real invariant straight lines taking into account their multiplicities when* $n = 1, 2$ *have no limit cycles.*

We shall see that Corollary 3 cannot be extended to $n > 2$, because polynomial differential systems of degree 3 having 3 invariant straight lines taking into account their multiplicities can have limit cycles for some configurations of their invariant straight lines.

Before characterizing for $n = 3$ the existence or non-existence of periodic solutions and limit cycles for the polynomial differential systems of degree 3 having 3 invariant straight lines taking into account their multiplicities according with the different configurations of their three invariant straight lines we present a general result for degree n.

Theorem 4. *A polynomial differential system* (2) *of degree* $n > 2$ *having* n *real parallel invariant straight lines taking into account their multiplicities has no periodic solutions.*

Theorem 4 is proved in section 3.

Theorem 5. *For a polynomial differential system* (2) *of degree* 3 *having* 3 *real invariant straight lines taking into account their multiplicities the following statements hold.*

(a) *If these* 3 *invariant straight lines taking into account their multiplicities are parallel, system* (2) *has no periodic solutions.*

(b) *We assume that the system has* 3 *different invariant straight lines, two of them are parallel and intersects the third one, then the cubic polynomial differential system* (2) *can have limit cycles.*

(c) *We assume that the system has only* 2 *different invariant straight lines which are not parallel. Then the cubic polynomial differential system* (2) *can have limit cycles.*

(d) *We assume that the system has* 3 *different invariant straight lines intersect at a unique point. Then the cubic polynomial differential system* (2) *can have limit cycles.*

(e) *We assume that the system has* 3 *different invariant straight lines intersect in three different points. Then the cubic polynomial differential system* (2) *can have limit cycles.*

Theorem 5 is proved in section 4.

We must mention that Kooij in [9] studied the existence and non-existence of periodic orbits and limit cycles of the cubic polynomial differential systems with 4 real invariant straight lines, while in Theorem 6 for the same cubic polynomial differential systems but with only 3 real invariant straight lines taking into account their multiplicities we study the existence and non-existence of periodic orbits and limit cycles. We shall use some of the ideas of Kooij for proving Theorem 6.

2. Proof of Theorem 1

In this section we shall prove Theorem 1. We shall need the next three well known results.

Theorem 6. *Let R be a simply connected region of $\mathbb{R}^2$, and assume that the differential system (2) is $\mathcal{C}^1$ in R, i.e. $P, Q : R \to \mathbb{R}$ are $\mathcal{C}^1$ maps. Then any periodic orbit of system (2) in R must surround an equilibrium point of this system.*

For a proof of Theorem 6 see for instance Theorem 1.31 of [5].

Let U be a dense open subset of $\mathbb{R}^2$. Then a *first integral* $H : U \to \mathbb{R}$ for system (2) is a non–locally constant function which is constant on the orbits of the system contained in U. In other words, if $\mathcal{X}$ given in (1) is the vector field associated to system (2), then H is a first integral if and only if $\mathcal{X}H = 0$ in U.

Theorem 7 (Dulac's Theorem)**.** *Assume that there exists a C^1 function $D(x, y)$ in a simply connected region R of $\mathbb{R}^2$ such that for the differential system (2) either $\partial(DP)/\partial x + \partial(DQ)/\partial y \geq 0$, or $\partial(DP)/\partial x + \partial(DQ)/\partial y \leq 0$, and at most this previous expression is zero in a zero measure Lebesgue set. Then the differential system (2) has no periodic orbits in R.*

For a proof of the Dulac's Theorem see for instance Theorem 7.12 of [5].

Finally we summarize for the differential systems defined in an open subset of $\mathbb{R}^2$ some basic results characterizing the *Hopf bifurcation*. These results will be used for proving some statements of Theorem 5. For more details on the results presented on the Hopf bifurcation for differential systems in the plane see Marsden and McCracken [11].

For $u \in \mathbb{R}^2$ consider a family of autonomous systems of ordinary differential equations depending on a parameter μ

$$\frac{du}{dt} = F(u, \mu), \tag{4}$$

where $F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is C^∞ and μ is the bifurcation parameter. Suppose that $a(\mu)$ is a equilibrium point of the differential system (4) for every μ in a neighborhood U of $\mu = 0$, i.e. $F(a(\mu), \mu) = 0$ if $\mu \in U$. Assume that $DF|_{(a(\mu),\mu)}$ has eigenvalues of the form $\alpha(\mu) \pm i\beta(\mu)$.

Poincaré [12], Andronov and Witt [1] and Hopf [7] (a translation into English of Hopf's original paper can be found in section 5 of [11]) shown that for μ sufficiently small, an one–parameter family of periodic orbits of the differential system (4) arises at $(u, \mu) = (0, 0)$ if

(i) $DF|_{(0,0)}$ has eigenvalues $\pm i\beta(0) \neq 0$,
(ii) $(d\alpha/d\mu)|_{\mu=0} \neq 0$, and
(iii) we do not have a center at $u = 0$ for $\mu = 0$.

We say that $\mu = 0$ is the value of the *Hopf bifurcation*.

Proof of statement (a) of Theorem 1. After an affine change of variables we can assume that the two invariant straight lines of statement (a) of Theorem 1 of the polynomial differential system (2) of degree 2 are $x = 0$ and $y = 0$. Then, if $K_1 = K_1(x, y)$ (respectively $K_2 = K_2(x, y)$) is the cofactor of $x = 0$ (respectively $y = 0$), from the definition of invariant straight line (3) we have that $P = xK_1$ (respectively $Q = yK_2$), where K_1 and K_2 are polynomials of degree 1. Therefore,

it is sufficient to prove statement (a) of Theorem 1 for the following polynomial differential system of degree 2

$$\dot{x} = P(x,y) = x(a_1x + b_1y + c_1), \qquad \dot{y} = Q(x,y) = y(a_2x + b_2y + c_2), \tag{5}$$

where $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R}$ are arbitrary real numbers.

From Theorem 6 the unique equilibrium point of system (5) which can be surrounded by periodic orbits is

$$p = \left(\frac{b_2c_1 - b_1c_2}{a_2b_1 - a_1b_2}, \frac{a_1c_2 - a_2c_1}{a_2b_1 - a_1b_2} \right),$$

because the other equilibria are on the invariant straight lines $x = 0$ and $y = 0$. So in order that periodic orbits exist for system (5) we need to have that $a_2b_1 - a_1b_2 \neq 0$.

We consider the function $D(x,y) = x^l y^k$ where

$$l = \frac{2a_1b_2 - a_2(b_1 + b_2)}{a_2b_1 - a_1b_2}, \quad k = \frac{2a_1b_2 - (a_1 + a_2)b_1}{a_2b_1 - a_1b_2},$$

defined in the open quadrant of $\mathbb{R}^2 \setminus \{xy = 0\}$ containing the equilibrium point p. Then we have

$$\frac{\partial(DP)}{\partial x} + \frac{\partial(DQ)}{\partial y} = \frac{(a_1 - a_2)b_2c_1 + a_1(-b_1 + b_2)c_2}{a_2b_1 - a_1b_2}\, x^l y^k.$$

If $(a_1 - a_2)b_2c_1 + a_1(-b_1 + b_2)c_2 \neq 0$, by Dulac's Theorem system (5) has no periodic orbits.

Assume that $(a_1 - a_2)b_2c_1 + a_1(-b_1 + b_2)c_2 = 0$. Then the differential system $\dot{x} = DP$, $\dot{y} = DQ$ is Hamiltonian, and

$$H = x^{\frac{(a_2 - a_1)b_2}{a_1b_2 - a_2b_1}}\, y^{\frac{a_1(b_1 - b_2)}{a_1b_2 - a_2b_1}} ((b_1 - b_2)c_2 + (a_1 - a_2)b_2x + (b_1 - b_2)b_2y)$$

is a first integral of system (5), consequently this system cannot have limit cycles in the quadrant containing the equilibrium point p.

In order to complete the proof of this statement we only need to show that under convenient conditions system (5) has periodic solutions. Under the assumption $(a_1 - a_2)b_2c_1 + a_1(-b_1 + b_2)c_2 = 0$, if

$$(a_2b_1 - a_1b_2)(-a_2b_2c_1^2 + a_2b_1c_1c_2 + a_1b_2c_1c_2 - a_1b_1c_2^2) < 0,$$

then the equilibrium point p has purely imaginary eigenvalues. Hence, since additionally the system has the first integral H it follows that the singular point p is a center, consequently under these assumptions system (5) has periodic orbits. This completes the proof statement (a) of Theorem 1. □

Proof of statement (b) of Theorem 1. Doing an affine change of variables we can suppose that the two parallel invariant straight lines of statement (b) of Theorem 1 of the polynomial differential system (2) of degree 2 are $x - 1 = 0$ and $x + 1 = 0$. Then, if $K_1 = K_1(x,y)$ (respectively $K_2 = K_2(x,y)$) is the cofactor of $x - 1 = 0$ (respectively $x + 1 = 0$), from the definition of invariant straight line (3) we obtain that $P = a(x - 1)(x + 1)$, where $K_1 = a(x + 1)$ and $K_2 = a(x - 1)$. Therefore, it is sufficient to prove statement (b) of Theorem 1 for the following polynomial differential system of degree 2:

$$\dot{x} = P(x,y) = a(x - 1)(x + 1), \qquad \dot{y} = Q(x,y), \tag{6}$$

where $a \in \mathbb{R}$ and $Q(x,y)$ an arbitrary polynomial of degree 2.

If $a = 0$ then all the straight lines $x =$ constant are invariant and consequently system (6) has no periodic solutions. Assume that $a \neq 0$, then the solution $x(t, x_0)$ of the first equation of system (6) such that $x(0, x_0) = x_0$ is

$$x(t, x_0) = \frac{x_0 \cosh(at) - \sinh(at)}{\cosh(at) - x_0 \sinh(at)}.$$

Since the function $x(t, x_0)$ is not periodic, it follows that system (6) has no periodic orbits. This completes the proof statement (b) of Theorem 1. □

Proof of statement (c) of Theorem 1. Repeating the proof of statement (b) of Theorem 1 for the invariant straight lines $x - \varepsilon = 0$ and $x + \varepsilon = 0$, we get the following polynomial differential system of degree 2:

$$\dot{x} = P(x, y) = a(x - \varepsilon)(x + \varepsilon), \qquad \dot{y} = Q(x, y), \tag{7}$$

where $a \in \mathbb{R}$ and $Q(x, y)$ an arbitrary polynomial of degree 2. Then when $\varepsilon \to 0$ system (7) becomes

$$\dot{x} = P(x, y) = ax^2, \qquad \dot{y} = Q(x, y), \tag{8}$$

and for this system the invariant straight line $x = 0$ has multiplicity 2.

As in the previous proof if $a = 0$ system (7) has no periodic solutions. Assume that $a \neq 0$, then the solution $x(t, x_0)$ of the first equation of system (10) such that $x(0, x_0) = x_0$ is

$$x(t, x_0) = \frac{x_0}{1 - ax_0 t}.$$

Since this function is not periodic, system (6) cannot have periodic orbits. This completes the proof statement (c) of Theorem 1. □

3. Proof of Theorem 4

In this section we prove the main results of our paper, i.e. Theorem 4. For this we will use some tools presented in sections 1 and 2.

Proof of statement (a) of Theorem 4. Doing an affine change of variables we can suppose that the n parallel invariant straight lines of statement (a) of Theorem 4 of the polynomial differential system (2) of degree n are

$$x - \alpha_1 = 0, \quad x - \alpha_2 = 0, \quad \ldots \quad , x - \alpha_n = 0,$$

with $\alpha_1 < \alpha_2 < \ldots < \alpha_n$.

It follows from the definition (3) of an invariant straight line that it is sufficient to prove statement (a) of Theorem 4 for the following polynomial differential system of degree n:

$$\dot{x} = P(x, y) = a(x - \alpha_1) \cdots (x - \alpha_n), \qquad \dot{y} = Q(x, y), \tag{9}$$

where $a \in \mathbb{R}$ and $Q(x, y)$ is an arbitrary polynomial of degree n.

If $a = 0$ then all the straight lines $x =$ constant are invariant and consequently system (9) has no periodic solutions. Assume that $a \neq 0$, then all the equilibrium points of the polynomial differential system (9) are on one of the invariant straight lines $x = \alpha_i$ for $i = 1, \ldots, n$. Therefore, by Theorem 6 none of the equilibrium points of system (9) can be surrounded by periodic orbits. □

Proof of statement (*b*) *of Theorem 4.* Recall the definition of multiplicity k of an invariant straight line stated in section 1. Repeating the arguments of the beginning of the proof of statement (c) of Theorem 1 and taking into account the proof of statement (a) of Theorem 4 we see that it is sufficient to prove statement (*b*) of Theorem 4 for the following polynomial differential system of degree n:

$$\dot{x} = P(x,y) = a(x-\alpha_1)^{\beta_1}\dots(x-\alpha_k)^{\beta_k}, \qquad \dot{y} = Q(x,y), \tag{10}$$

where $\alpha_1 < \ldots < \alpha_k$ and $\beta_1 + \ldots + \beta_k = n$ where β_i is a positive integer for $i = 1,\ldots,k$. Note that for this system the invariant straight line $x = \alpha_i$ has multiplicity β_i for $i = 1,\ldots,k$.

As in the proof of the previous statement, if $a = 0$ system (10) has no periodic solutions. Assume that $a \neq 0$, then all the equilibrium points of the polynomial differential system (10) are on one of the invariant straight lines $x = \beta_i$. Again, by Theorem 6 none of the equilibrium points of system (10) can be surrounded by periodic orbits. □

4. Proof of Theorem 5

Here we prove the five statements of Theorem 5.

Proof of statement (*a*) *of Theorem 5.* It follows immediately from Theorem 4. □

For proving statement (b) of Theorem 5 we shall need to distinguish between a focus and a center. Thus we briefly describe the algorithm due to Bautin for computing the Liapunov constants. It is known that all the Liapunov constants must be zero in order to have a center, for more details see Chapter 4 of [5], and the references quoted there.

We consider a planar analytic differential equation of the form

$$\begin{aligned} \dot{x} &= -y + P(x,y) = -y + \sum_{k=2}^{\infty} P_k(x,y), \\ \dot{y} &= x + Q(x,y) = x + \sum_{k=2}^{\infty} Q_k(x,y), \end{aligned} \tag{11}$$

where P_k and Q_k are homogeneous polynomials of degree k. In a neighborhood of the origin we can also write this differential system in polar coordinates (r,θ) as

$$\frac{dr}{d\theta} = \sum_{k=2}^{\infty} S_k(\theta) r^k, \tag{12}$$

where $S_k(\theta)$ are trigonometric polynomials in the variables $\sin\theta$ and $\cos\theta$.

If we denote by $r(\theta, r_0)$ the solution of (12) such that $r(0,r_0) = r_0$ then close to $r = 0$ we have

$$r(\theta, r_0) = r_0 + \sum_{k=2}^{\infty} u_k(\theta) r_0^k,$$

with $u_k(0) = 0$ for $k \geq 2$. The Poincaré return map near $r = 0$ is given by

$$\Pi(r_0) = r(2\pi, r_0) = r_0 + \sum_{k=2}^{\infty} u_k(2\pi) r_0^k.$$

Since Π is analytic it is clear that $\Pi(r_0) \equiv r_0$ if and only if $u_n(2\pi) = 0$ for all $n > 1$, i.e. if and only if the origin of system (11) is a center. The constants $u_n(2\pi)$ for

$n > 1$ are called the *Liapunov constants*, and if some of them is not zero, then the origin of system (11) is not a center.

Proof of statement (b) of Theorem 5. We assume that two of the three invariant straight lines are parallel and intersects the other invariant straight line, and that all these invariant straight lines have multiplicity 1. Now we shall prove that the cubic polynomial differential system (2) with these three invariant straight lines can have limit cycles.

Doing an affine change of variables we can suppose that the three invariant straight lines of this statement are $x-1=0$, $x+1=0$ and $y-1=0$. Proceeding as in the proof of statement (b) of Theorem 1, we have that it is sufficient to prove this statement for the following cubic polynomial differential system

$$\begin{aligned} \dot{x} = P(x,y) = &\quad (x-1)(x+1)(a_1x+b_1y+c_1),\\ \dot{y} = Q(x,y) = &\quad (y-1)(a_2x+b_2y+c_2+d_2x^2+e_2xy+f_2y^2), \end{aligned}$$

where $a_1, b_1, c_1, a_2, b_2, c_2, d_2, e_2, f_2 \in \mathbb{R}$. In fact we shall prove this statement for the particular system

$$\begin{aligned} \dot{x} = P(x,y) = &\quad (x-1)(x+1)(a_1x+b_1y),\\ \dot{y} = Q(x,y) = &\quad (y-1)(a_2x+b_2y+d_2x^2+e_2xy+f_2y^2), \end{aligned} \tag{13}$$

where $a_1, b_1, a_2, b_2, d_2, e_2, f_2 \in \mathbb{R}$.

We recall the conditions stated in section 2 in order that an one–parameter family of periodic orbits exhibits a Hopf bifurcation at an equilibrium point. The origin $(0,0)$ is an equilibrium point of system (13), and its eigenvalues are

$$\lambda_{\pm} = -\frac{1}{2}\left(a_1+b_2 \pm \sqrt{a_1^2+b_2^2-2a_1b_2+4a_2b_1}\right). \tag{14}$$

We assume that

$$a_1^2+b_2^2-2a_1b_2+4a_2b_1 < 0. \tag{15}$$

Let

$$\mu = a_1+b_2, \quad \alpha(\mu) = -\frac{1}{2}\mu, \quad \beta(\mu,b_1,b_2,a_2) = \frac{1}{2}\sqrt{-\mu^2-4(\mu-b_2)b_2-4a_2b_1}.$$

By (15) the eigenvalues (14) are of the form $\lambda_{\pm}(\mu,b_1,b_2,a_2) = \alpha(\mu) \pm \beta(\mu,b_1,b_2,a_2)i$. So, when $\mu = 0$ they are

$$\pm\beta(0,b_1,b_2,a_2)\,i = \pm\sqrt{b_2^2-a_2b_1}\,i.$$

We assume that $b_2^2 - a_2b_1 > 0$. We also have that $(d\alpha/d\mu)|_{\mu=0} = -1/2 \neq 0$. Now we claim that the origin of system (13) with $\mu = 0$, $b_2 = 0$, $a_2 = 1$, $b_1 = -1$ and $d_2e_2 + 2f_2 + e_2f_2 \neq 0$ is not a center. Before proving the claim we note that for these values the eigenvalues (14) are $\pm i$ and the condition (15) becomes $-4 < 0$. Hence, once the claim be proved all the conditions for having a Hopf bifurcation hold, consequently there are systems (13) with limit cycles, and statement (b) will be proved.

Now we prove the claim. System (13) becomes

$$\begin{aligned} \dot{x} = P(x,y) = &\quad -(x-1)(x+1)y,\\ \dot{y} = Q(x,y) = &\quad (y-1)(x+d_2x^2+e_2xy+f_2y^2). \end{aligned} \tag{16}$$

We write this system in polar coordinates (r,θ) where $x = r\cos\theta$ and $y = r\sin\theta$, and we have

$$\begin{aligned}\dot r = &\ -\sin\theta\left(d_2\cos^2\theta + (e_2-1)\sin\theta\cos\theta + f_2\sin^2\theta\right) r^2\\ &\ -\sin\theta\left(\cos^3\theta - d_2\sin\theta\cos^2\theta - e_2\sin^2\theta\cos\theta - f_2\sin^3\theta\right) r^3,\\ \dot\theta = &\ -1 - \cos\theta\left(d_2\cos^2\theta + (e_2-1)\sin\theta\cos\theta + f_2\sin^2\theta\right) r\\ &\ +\cos\theta\sin\theta\left(d_2\cos^2\theta + (e_2+1)\sin\theta\cos\theta + f_2\sin^2\theta\right) r^2.\end{aligned}$$

This system in a neighborhood of the origin can be written as

$$\begin{aligned}\frac{dr}{d\theta} = &\ r^2\sin\theta\left(d_2\cos^2\theta + (e_2-1)\sin\theta\cos\theta + f_2\sin^2\theta\right)\\ &\ -r^3\sin\theta\Big((d_2-1)(d_2+1)\cos^5\theta + d_2(2e_2-1)\sin\theta\cos^4\theta\\ &\ +\left(e_2^2 - e_2 + 2d_2f_2\right)\sin^2\theta\cos^3\theta + (d_2 + 2e_2f_2 - f_2)\sin^3\theta\cos^2\theta\\ &\ +\left(f_2^2 + e_2\right)\sin^4\theta\cos\theta + f_2\sin^5\theta\Big) + O(r^4).\end{aligned}$$

Now using the Bautin's algorithm described we get that

$$u_1(2\pi) = 1, \quad u_2(2\pi) = 0, \quad u_3(2\pi) = -\frac{\pi}{4}(d_2e_2 + 2f_2 + e_2f_2).$$

Hence, due to the fact that $d_2e_2 + 2f_2 + e_2f_2 \neq 0$ we do not have a center at the origin of system (16). This completes the proof of statement (b). □

Proof of statement (c) of Theorem 5. Consider a polynomial differential system (2) of degree 3 with one invariant straight line with multiplicity 2 intersecting an invariant straight line with multiplicity 1. We shall show that these systems can have limit cycles.

Proceeding as in the proof of statement (c) of Theorem 1 and see also the proof of statement (b) of Theorem 5 we have that it is sufficient to prove statement (c) of Theorem 5 for the following polynomial differential system of degree 3

$$\begin{aligned}\dot x = P(x,y) = &\ (x-1)^2(a_1x + b_1y + c_1),\\ \dot y = Q(x,y) = &\ (y-1)(a_2x + b_2y + c_2 + d_2x^2 + e_2xy + f_2y^2),\end{aligned}$$

where $a_1, b_1, c_1, a_2, b_2, c_2, d_2, e_2, f_2 \in \mathbb{R}$. We consider the particular subsystem

$$\begin{aligned}\dot x = &\ (x-1)^2(a_1x + b_1y),\\ \dot y = &\ (y-1)(a_2x + b_2y + d_2x^2 + e_2xy + f_2y^2),\end{aligned} \tag{17}$$

where $a_1, b_1, a_2, b_2, d_2, e_2, f_2 \in \mathbb{R}$.

The origin $(0,0)$ is an equilibrium point of system (17), and its eigenvalues are

$$\lambda_\pm = \frac{1}{2}\left(a_1 - b_2 \pm \sqrt{a_1^2 + b_2^2 + 2a_1b_2 - 4a_2b_1}\right). \tag{18}$$

We assume that

$$a_1^2 + b_2^2 + 2a_1b_2 - 4a_2b_1 < 0. \tag{19}$$

Let

$$\mu = a_1 - b_2, \quad \alpha(\mu) = \frac{1}{2}\mu, \quad \beta(\mu, b_1, b_2, a_2) = \frac{1}{2}\sqrt{-\mu^2 + 4(\mu + b_2)b_2 + 4a_2b_1}.$$

By (19) the eigenvalues (18) are of the form $\lambda_\pm(\mu, b_1, b_2, a_2) = \alpha(\mu) \pm \beta(\mu, b_1, b_2, a_2)i$. So, when $\mu = 0$ they are

$$\pm\beta(0, b_1, b_2, a_2)\, i = \pm\sqrt{b_2^2 + a_2b_1}\, i.$$

We assume that $b_2^2 + a_2 b_1 > 0$. We also have that $(d\alpha/d\mu)|_{\mu=0} = 1/2 \neq 0$. Now we claim that the origin of system (13) with $\mu = 0$, $b_2 = 0$, $a_2 = b_1 = 1$ and $d_2 e_2 + 2f_2 + e_2 f_2 \neq 0$ is not a center. Before proving the claim we note that for these values the eigenvalues (18) are $\pm i$ and the condition (19) becomes $-4 < 0$. Hence, once the claim be proved all the conditions for having a Hopf bifurcation hold, consequently there are systems (17) with limit cycles, and statement (c) will be proved.

Now we prove the claim. Systems (17) becomes

$$\begin{aligned} \dot{x} = &\quad (x-1)^2 y, \\ \dot{y} = &\quad (y-1)(x + d_2 x^2 + e_2 xy + f_2 y^2). \end{aligned} \tag{20}$$

We write this system in polar coordinates (r, θ) where $x = r\cos\theta$ and $y = r\sin\theta$, and we have

$$\begin{aligned} \dot{r} = &\quad -\sin\theta \left((d_2+2)\cos^2\theta + (e_2-1)\sin\theta\cos\theta + f_2\sin^2\theta\right) r^2 \\ &\quad +\sin\theta\left(\cos^3\theta + d_2\sin\theta\cos^2\theta + e_2\sin^2\theta\cos\theta + f_2\sin^3\theta\right) r^3, \\ \dot{\theta} = &\quad -1 - \cos\theta\left(d_2\cos^2\theta + (e_2-1)\sin\theta\cos\theta + (f_2-2)\sin^2\theta\right) r \\ &\quad +\cos\theta\sin\theta\left(d_2\cos^2\theta + (e_2-1)\sin\theta\cos\theta + f_2\sin^2\theta\right) r^2. \end{aligned}$$

This system in a neighborhood of the origin can be written as

$$\begin{aligned} \frac{dr}{d\theta} = &\quad r^2\sin\theta\left((d_2+2)\cos^2\theta + (e_2-1)\sin\theta\cos\theta + f_2\sin^2\theta\right) \\ &\quad -r^3\sin\theta\Big(\cos^3\theta + d_2\sin\theta\cos^2\theta + e_2\sin^2\theta\cos\theta + f_2\sin^3\theta \\ &\quad +d_2(d_2+2)\cos^5\theta + 2(1+d_2)(e_2-1)\sin\theta\cos^4\theta \\ &\quad +\left(e_2^2 + 2f_2 + 2d_2 f_2 - 3 - 2d_2 - 2e_2\right)\sin^2\theta\cos^3\theta \\ &\quad +2(e_2-1)(f_2-1)\sin^3\theta\cos^2\theta + (f_2-2)f_2\sin^4\theta\cos\theta\Big) + O(r^4). \end{aligned}$$

Then using the Bautin's algorithm we get that

$$u_1(2\pi) = 1, \quad u_2(2\pi) = 0, \quad u_3(2\pi) = -\frac{\pi}{4}(d_2 e_2 + 2f_2 + e_2 f_2).$$

Hence, since $d_2 e_2 + 2f_2 + e_2 f_2 \neq 0$ we do not have a center at the origin of system (20). This completes the proof of statement (c). □

Proof of statement (d) of Theorem 5. We shall show that if the 3 invariant straight lines of multiplicity 1 intersect at a unique point, then the polynomial differential system (2) of degree 3 can have limit cycles.

Doing an affine change of variables we can suppose that the three invariant straight lines of multiplicity 1 intersecting at a point of the polynomial differential system (2) of degree 3 are $x-1=0, y-2=0$ and $y-x-1=0$. Proceeding as in the proof of some previous statements we have that it is sufficient to prove statement (d) of Theorem 5 for the following polynomial differential system of degree 3:

$$\begin{aligned} \dot{x} = P(x,y) = &\quad (x-1)(a_1 x + b_1 y + c_1 + d_1 x^2 + e_1 xy + f_1 y^2), \\ \dot{y} = Q(x,y) = &\quad (y-2)(a_2 x + b_2 y + c_2 + d_2 x^2 + e_2 xy + f_2 y^2), \end{aligned} \tag{21}$$

where $a_1, b_1, c_1, a_2, b_2, c_2, d_2, e_2, f_2 \in \mathbb{R}$ and the coefficients d_2, e_2, f_2 satisfy the following relations:

$$\begin{aligned} d_2 &= -a_1 + a_2 + c_1 - c_2 + d_1, \\ e_2 &= a_1 - a_2 - b_1 + b_2 - 2c_1 + 2c_2 + e_1, \\ f_2 &= b_1 - b_2 + c_1 - c_2 + f_1. \end{aligned}$$

We consider the particular system

$$
\begin{aligned}
\dot{x} = &\quad (x-1)(a_1x + b_1y + d_1x^2 + e_1xy + f_1y^2),\\
\dot{y} = &\quad (y-2)(a_2x + b_2y + (-a_1 + a_2 + d_1)x^2 +\\
&\quad (a_1 - a_2 - b_1 + b_2 + e_1)xy + (b_1 - b_2 + f_1)y^2),
\end{aligned}
\tag{22}
$$

where $a_1, b_1, d_1, e_1, f_1, a_2, b_2 \in \mathbb{R}$.

The origin $(0,0)$ is an equilibrium point of system (22), and its eigenvalues are

$$
\lambda_{\pm} = -\frac{1}{2}\left(a_1 + 2b_2 \pm \sqrt{a_1^2 + 4b_2^2 - 4a_1b_2 + 8a_2b_1}\right). \tag{23}
$$

We assume that

$$
a_1^2 + 4b_2^2 - 4a_1b_2 + 8a_2b_1 < 0. \tag{24}
$$

Let

$$
\mu = a_1 + 2b_2, \quad \alpha(\mu) = -\frac{1}{2}\mu, \quad \beta(\mu, b_1, b_2, a_2) = \frac{1}{2}\sqrt{-\mu^2 + 4(\mu - 2b_2)b_2 - 4a_2b_1}.
$$

By (24) the eigenvalues (23) are of the form $\lambda_{\pm}(\mu, b_1, b_2, a_2) = \alpha(\mu) \pm \beta(\mu, b_1, b_2, a_2)i$. So, when $\mu = 0$ they are

$$
\pm\beta(0, b_1, b_2, a_2)\, i = \pm\sqrt{-2b_2^2 - a_2b_1}\, i.
$$

We assume that $2b_2^2 + a_2b_1 < 0$. We also have that $(d\alpha/d\mu)|_{\mu=0} = -1/2 \neq 0$. Now we claim that the origin of system (13) with $\mu = 0$, $b_2 = 0$, $a_2 = 1$, $b_1 = -1$ and $4d_1^2 + 3e_1d_1 + 6d_1 - 4f_1^2 - 2e_1 + 3e_1f_1 + 8f_1 - 3 \neq 0$ is not a center. Before proving the claim we note that for these values the eigenvalues (23) are $\pm i$ and the condition (24) becomes $-8 < 0$. Hence, once the claim be proved all the conditions for having a Hopf bifurcation hold, consequently there are systems (22) with limit cycles, and statement (d) will be proved.

Now we prove the claim. System (22) becomes

$$
\begin{aligned}
\dot{x} = &\quad (x-1)(-y + d_1x^2 + e_1xy + f_1y^2),\\
\dot{y} = &\quad (y-2)\left(\frac{1}{2}x + \left(\frac{1}{2} + d_1\right)x^2 + \left(\frac{1}{2} + e_1\right)xy + (f_1 - 1)y^2\right).
\end{aligned}
\tag{25}
$$

We write this system in polar coordinates and we obtain

$$
\begin{aligned}
\dot{r} = &\quad \frac{1}{2}\Big(-2d_1\cos^3\theta - 2(2 + 2d_1 + e_1)\cos^2\theta\sin\theta - (1 + 4e_1 + 2f_1)\cos\theta\sin^2\theta\\
&+4(1 - f_1)\sin^3\theta\Big)r^2 + \frac{1}{2}\Big(2d_1\cos^4\theta + 2e_1\cos^3\theta\sin\theta\\
&+(1 + 2d_1 + 2f_1)\cos^2\theta\sin^2\theta + (1 + 2e_1)\cos\theta\sin^3\theta + 2(f_1 - 1)\sin^4\theta\Big)r^3,
\end{aligned}
$$

$$
\begin{aligned}
\dot{\theta} = &\quad -1 - \frac{1}{2}\Big((-2(1 + 2d_1)\cos^3\theta + (-1 + 2d_1 - 4e_1)\cos^2\theta\sin\theta\\
&+2(3 + e_1 - 2f_1)\cos\theta\sin^2\theta + 2f_1\sin^3\theta\Big)r\\
&+\frac{1}{2}\Big(\cos^3\theta\sin\theta + \cos^2\theta\sin^2\theta - 2\cos\theta\sin^3\theta)r^2.
\end{aligned}
$$

This system in a neighborhood of the origin can be written as

$$\begin{aligned}\frac{dr}{d\theta} = & \frac{1}{2}\Big(2d_1\cos^3\theta + 2(2d_1+e_1+2)\sin\theta\cos^2\theta + (4e_1+2f_1+1)\sin^2\theta\cos\theta\\ & +4(f_1-1)\sin^3\theta\Big)r^2 + \frac{1}{4}\Big(-4d_1(2d_1+1)\cos^6\theta\\ & -2\left(6d_1^2+8e_1d_1+13d_1+2e_1+4\right)\sin\theta\cos^5\theta\\ & +\big(2\left(4d_1^2-12e_1d_1-8f_1d_1+6d_1-4e_1^2-13e_1-2f_1-3\right)\sin^2\theta-4d_1\big)\cos^4\theta\\ & +\big(\left(-12e_1^2+16d_1e_1-16f_1e_1+12e_1+42d_1-24d_1f_1-26f_1+31\right)\sin^3\theta\\ & -4e_1\sin\theta\big)\cos^3\theta+\big(2\left(4e_1^2-12f_1e_1+21e_1-4f_1^2-4d_1+8d_1f_1+6f_1+5\right)\sin^4\theta\\ & -2(2d_1+2f_1+1)\sin^2\theta\big)\cos^2\theta+\big(2\left(-6f_1^2+8e_1f_1+21f_1-4e_1-12\right)\sin^5\theta\\ & -2(2e_1+1)\sin^3\theta\big)\cos\theta+8(f_1-1)f_1\sin^6\theta-4(f_1-1)\sin^4\theta\Big)r^3+O(r^4).\end{aligned}$$

Now using the Bautin's algorithm we get that

$$\begin{aligned}u_1(2\pi) = & \ 1,\\ u_2(2\pi) = & \ 0,\\ u_3(2\pi) = & -\frac{\pi}{4}\left(4d_1^2+3e_1d_1+6d_1-4f_1^2-2e_1+3e_1f_1+8f_1-3\right).\end{aligned}$$

Hence taking $4d_1^2+3e_1d_1+6d_1-4f_1^2-2e_1+3e_1f_1+8f_1-3\neq 0$ we do not have a center at the origin of system (25). This completes the proof of statement (e). □

Proof of statement (e) of Theorem 4. Assume that the three invariant straight lines of multiplicity 1 intersect pairwise in a unique point. Then the polynomial differential system (2) of degree 3 can have limit cycles. Doing an affine change of variables we can suppose that these three invariant straight lines are $x-1=0, y-1=0$ and $y+x-3=0$.

Proceeding as in the previous statements we have that it is sufficient to prove statement (e) of Theorem 5 for the following polynomial differential system of degree 3:

$$\begin{aligned}\dot{x} = P(x,y) = & \ (x-1)(a_1x+b_1y+c_1+d_1x^2+e_1xy+f_1y^2),\\ \dot{y} = Q(x,y) = & \ (y-1)(a_2x+b_2y+c_2+d_2x^2+e_2xy+f_2y^2),\end{aligned} \tag{26}$$

where $a_1,b_1,c_1,a_2,b_2,c_2,d_2,e_2,f_2\in\mathbb{R}$ and the coefficients d_2,e_2,f_2 satisfy the relations

$$\begin{aligned}d_2 &= \frac{1}{18}(6a_1-6a_2+9b_1+18e_1+9f_1),\\ e_2 &= \frac{1}{36}(-6a_1-12a_2+3b_1-12b_2-18e_1+27f_1),\\ f_2 &= \frac{1}{18}(-3b_1-6b_2-9f_1),\\ d_1 &= \frac{1}{4}(-2a_1-b_1-2e_1-f_1).\end{aligned}$$

Taking $c_1 = c_2 = 0$ we consideer the particular subsystem

$$
\begin{aligned}
\dot{x} = \;& (x-1)(a_1x + b_1y + \frac{1}{4}(-2a_1 - b_1 - 2e_1 - f_1)x^2 + e_1xy + f_1y^2),\\
\dot{y} = \;& (y-1)(a_2x + b_2y + \frac{1}{18}(6a_1 - 6a_2 + 9b_1 + 18e_1 + 9f_1)x^2 +\\
& \frac{1}{36}(-6a_1 - 12a_2 + 3b_1 - 12b_2 - 18e_1 + 27f_1)xy +\\
& \frac{1}{18}(-3b_1 - 6b_2 - 9f_1)y^2),
\end{aligned}
\tag{27}
$$

where $a_1, b_1, d_1, e_1, f_1, a_2, b_2 \in \mathbb{R}$.

The origin $(0,0)$ is an equilibrium point of system (27), and its eigenvalues are

$$
\lambda_\pm = -\frac{1}{2}\left(a_1 + b_2 \pm \sqrt{a_1^2 + b_2^2 - 2a_1b_2 + 4a_2b_1}\right). \tag{28}
$$

We assume that

$$
a_1^2 + b_2^2 - 2a_1b_2 + 4a_2b_1 < 0, \tag{29}
$$

Let

$$
\mu = a_1 + b_2, \quad \alpha(\mu) = -\frac{1}{2}\mu, \quad \beta(\mu, b_1, b_2, a_2) = \frac{1}{2}\sqrt{-\mu^2 + 4(\mu - b_2)b_2 - 4a_2b_1}.
$$

By (29) the eigenvalues (28) are $\lambda_\pm(\mu, b_1, b_2, a_2) = \alpha(\mu) \pm \beta(\mu, b_1, b_2, a_2)i$. So, when $\mu = 0$ they are

$$
\pm\beta(0, b_1, b_2, a_2)\, i = \pm\sqrt{-b_2^2 - a_2b_1}\, i.
$$

We assume that $b_2^2 + a_2b_1 < 0$. We also have that $(d\alpha/d\mu)|_{\mu=0} = -1/2 \neq 0$. Now we claim that the origin of system (27) with $\mu = 0$, $b_2 = 0$, $a_2 = 1$, $b_1 = -1$ and $(6e_1 - 3f_1 + 5)(6e_1 + 9f_1 - 5) \neq 0$ is not a center. Before proving the claim we note that for these values the eigenvalues (28) are $\pm i$ and the condition (29) becomes $-4 < 0$. Hence, once the claim be proved all the conditions for having a Hopf bifurcation hold, consequently there are systems (27) with limit cycles, and statement (e) will be proved.

Now we prove the claim. System (27) becomes

$$
\begin{aligned}
\dot{x} = \;& (x-1)\left(-y + \frac{1}{4}(1 - 2e_1 - f_1)x^2 + e_1xy + f_1y^2\right),\\
\dot{y} = \;& (y-1)\Big(x + \frac{1}{18}(18e_1 + 9f_1 - 15)x^2 + \frac{1}{36}(-18e_1 + 27f_1 - 15)xy\\
& + \frac{1}{18}(3 - 9f_1)y^2\Big).
\end{aligned}
\tag{30}
$$

We write this system in polar coordinates (r,θ) and we have

$$\begin{aligned}\dot r = &\frac{1}{12}\Big(3(2e_1+f_1-1)\cos^3\theta-2(12e_1+3f_1+1)\cos^2\theta\sin\theta\\ &+(6e_1-21f_1+17)\cos\theta\sin^2\theta+2(3f_1-1)\sin^3\theta\Big)r^2\\ &-\frac{1}{12}\Big(3(2e_1+f_1-1)\cos^4\theta-12e_1\cos^3\theta\sin\theta-2(6e_1+9f_1-5)\theta\cos^2\theta\sin^2\theta\\ &+(6e_1-9f_1+5)\cos\theta\sin^3\theta+2(3f_1-1)\sin^4\theta\Big)r^3,\\ \dot\theta = &-1-\frac{1}{6}\Big((6e_1+3f_1-5)\cos^3\theta+2(3f_1-5)\cos^2\theta\sin\theta\\ &+(-6e_1-3f_1-5)\cos\theta\sin^2\theta-6f_1\sin^3\theta\Big)r\\ &+\frac{1}{12}\Big((18e_1+9f_1-13)\cos^3\theta\sin\theta+(-18e_1+9f_1-5)\cos^2\theta\sin^2\theta\\ &-2(9f_1-1)\cos\theta\sin^3\theta\Big)r^2.\end{aligned}$$

This system in a neighborhood of the origin can be written as

$$\begin{aligned}\frac{dr}{d\theta} = &\frac{1}{12}\Big(-3(2e_1+f_1-1)\cos^3\theta+2(12e_1+3f_1+1)\cos^2\theta\sin\theta\\ &+(-6e_1+21f_1-17)\sin^2\theta\cos\theta-2(3f_1-1)\sin^3\theta\Big)r^2\\ &+\frac{1}{72}\Big(3(2e_1+f_1-1)(6e_1+3f_1-5)\cos^6\theta\\ &-8\left(18e_1^2+9f_1e_1-6e_1+3f_1-5\right)\cos^5\theta\sin\theta+\left(18(2e_1+f_1-1)\right.\\ &-2\left(54f_1^2+144e_1f_1-99f_1-150e_1+25\right)\sin^2\theta\Big)\cos^4\theta\\ &+\left(6(4e_1-2f_1+5)(6e_1+9f_1-5)\sin^3\theta-72e_1\sin\theta\right)\cos^3\theta\\ &+\Big(\left(-36e_1^2+252f_1e_1-132e_1+135f_1^2-6f_1-65\right)\sin^4\theta\\ &-12(6e_1+9f_1-5)\sin^2\theta\Big)\cos^2\theta+\left(6(6e_1-9f_1+5)\sin^3\theta\right.\\ &-2\left(-54f_1^2+36e_1f_1+63f_1-6e_1-5\right)\sin^5\theta\Big)\cos\theta-12f_1(3f_1-1)\sin^6\theta\\ &+12(3f_1-1)\sin^4\theta\Big)r^3+O(r^4).\end{aligned}$$

Now using the Bautin's algorithm we obtain that

$$\begin{aligned}u_1(2\pi) = &\ 1,\\ u_2(2\pi) = &\ 0,\\ u_3(2\pi) = &\ \frac{\pi}{144}(6e_1-3f_1+5)(6e_1+9f_1-5).\end{aligned}$$

Hence taking $(6e_1-3f_1+5)(6e_1+9f_1-5)\neq 0$ we do not have a center at the origin of system (30). This completes the proof of statement (e). □

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