

AVERAGING THEORY FOR DISCONTINUOUS PIECEWISE DIFFERENTIAL SYSTEMS

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ABSTRACT. We develop the averaging theory of first and second order for studying the periodic solutions of discontinuous piecewise differential systems in arbitrary dimension and with an arbitrary number of systems with the minimal conditions of differentiability. We do two applications.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

1.1. Introduction. The discontinuous differential systems, i.e. differential equations with discontinuous right-hand sides, is a subject that has been developed very fast these last years. It has become certainly one of the common frontiers between Mathematics, Physics and Engineering. Thus certain phenomena in control systems [2], impact and friction mechanics [7], nonlinear oscillations [1, 18], economics [12, 13], and biology [3, 15], are the main sources of motivation of their study, see for more details Teixeira [20]. A recent review appears in [22].

The knowledge of the existence or not of periodic solutions is very important for understanding the dynamics of differential systems. One of good tools for study the periodic solutions is the averaging theory, see for instance the books of Sanders and Verhulst [19] and Verhulst [21]. We point out that the method of averaging is a classical and matured tool that provides a useful means to study the behaviour of nonlinear smooth dynamical systems. The method of averaging has a long history that starts with the classical works of Lagrange and Laplace who provided an intuitive justification of the process. The first formalization of this procedure was given by Fatou in 1928 [10]. Very important practical and theoretical contributions in the averaging theory were made by Krylov and Bogoliubov [6] in the 1930s and Bogoliubov [5] in 1945. The principle of averaging has been extended in many directions for both finite- and infinite-dimensional differentiable systems. The classical results for studying the periodic orbits of differential systems need at least that those systems be of class \mathcal{C}^2 . Nevertheless Buica and Llibre [9] extended the averaging theory for studying periodic orbits to continuous differential systems using mainly the Brouwer degree theory. Recently in [16] developed the averaging theory for studying periodic orbits to discontinuous differential systems.

1.2. Preliminaries. In the following we define the necessary elements for the statement of our main results.

Let D be an open subset of \mathbb{R}^d and $I = [0, T)$ an interval of \mathbb{R} . We consider a finite set of ODEs

$$(1) \quad x'(t) = f^n(t, x), \quad (t, x) \in I \times D \quad \text{for } n = 1, 2, \dots, M,$$

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where $f^n : I \times D \rightarrow \mathbb{R}^d$ is a continuous function. Here the prime denotes derivative with respect to the time t . Let (S_n) be a set of open disjoint subset of $I \times D$ for $n = 1, 2, \dots, M$. We suppose that the boundaries of each S_n are \mathcal{C}^k embedded piecewise hypersurfaces with $k \geq 1$. Furthermore the union of all boundaries, denoted by Σ , and all S_n together cover $I \times D$. So we define a *M-Discontinuous Piecewise Differential System*, or simply a *M-DPDS* as

$$(2) \quad x'(t) = f(t, x) = \begin{cases} f^1(t, x), & (t, x) \in \overline{S_1}, \\ f^2(t, x), & (t, x) \in \overline{S_2}, \\ \vdots \\ f^M(t, x), & (t, x) \in \overline{S_M}, \end{cases}$$

where $\overline{S_k}$ denotes the closure of S_k in D .

Remark 1. *Later on in this work we shall assume that the functions f^n are Lipschitz, and the boundaries are piecewise \mathcal{C}^k embedded hypersurfaces with either $k \geq 1$ or $k \geq 2$. However the theory described in the following is developed without these assumptions.*

Let A be a subset of $I \times D$ and let $\chi_A(t, x)$ be the *characteristic function* defined as

$$\chi_A(t, x) = \begin{cases} 1 & \text{if } (t, x) \in A, \\ 0 & \text{if } (t, x) \notin A. \end{cases}$$

So system (2) can be written as

$$(3) \quad x'(t) = f(t, x) = \sum_{n=1}^M \chi_{\overline{S_n}}(t, x) f^n(t, x), \quad (t, x) \in I \times D.$$

Remark 2. *The Filippov convention for the flow of system (3) passing through a point $p \in \Sigma$ does not depend on the value $f(p)$. Therefore we can define $f(p)$ for $p \in \Sigma$ in any way, also multivalued as usual. Details on Filippov conventions for differential equations with discontinuous righthand side can be found in the book of Filippov [11].*

A point $p \in \Sigma$ is called a *generic point of discontinuity* if there exists a neighborhood G_p of p such that $\mathcal{S}_p = G_p \cap \Sigma$ is a \mathcal{C}^k embedded hypersurface. In this case we can always assume that the hypersurface \mathcal{S}_p splits $G_p \setminus \mathcal{S}_p$ in two disconnected regions, namely G_p^+ and G_p^- .

Let p be a generic point of discontinuity. We denote $f_p^+ = f|_{G_p^+}$ and $f_p^- = f|_{G_p^-}$, which are continuous vector fields. So we define $l(p)$ as the segment connecting the vectors $f_p^+(p)$ and $f_p^-(p)$, see Figures 1 and 2. The sets G_p , G_p^+ and G_p^- are not uniquely defined. Nevertheless, it is easy to see that $f_p^+(p)$, $f_p^-(p)$, and $l(p)$ do not depend of this choice.

An embedded hypersurface $\mathcal{S} \subset \Sigma$ can be decomposed as the union of the closure of its *crossing region* $\Sigma^c(\mathcal{S})$ (see Figure 1), and its *sliding region* $\Sigma^s(\mathcal{S})$ (see Figure 2), which are defined as

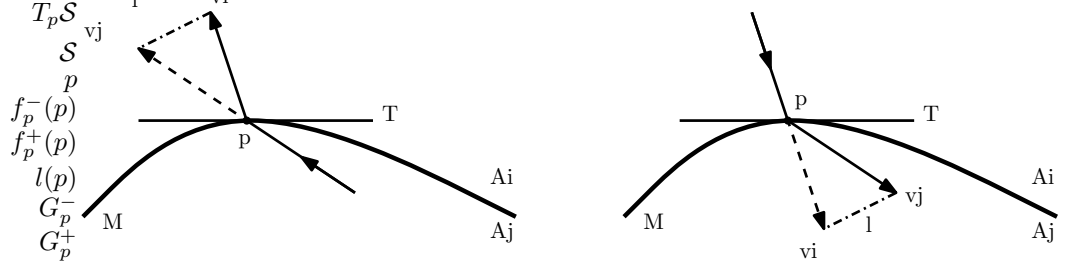
$$\Sigma^c(\mathcal{S}) = \{p \in \mathcal{S} : l(p) \cap T_p \mathcal{S} = \emptyset\} \quad \text{and} \quad \Sigma^s(\mathcal{S}) = \{p \in \mathcal{S} : l(p) \cap T_p \mathcal{S} \neq \emptyset\},$$

where as usual $T_p \mathcal{S}$ denotes the tangent space of \mathcal{S} at the point p .

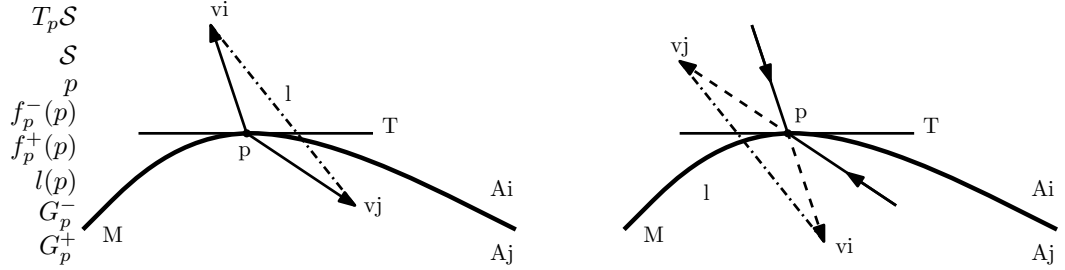
If $\mathcal{S} = h^{-1}(0)$ for some \mathcal{C}^1 function $h : I \times D \rightarrow \mathbb{R}$ which has 0 as a regular value, then the above definitions can be written as

$$\begin{aligned} \Sigma^c(\mathcal{S}) &= \{p \in \mathcal{S} : \langle \nabla h(p), f^+(p) \rangle \langle \nabla h(p), f^-(p) \rangle > 0\} \quad \text{and} \\ \Sigma^s(\mathcal{S}) &= \{p \in \mathcal{S} : \langle \nabla h(p), f^+(p) \rangle \langle \nabla h(p), f^-(p) \rangle < 0\}. \end{aligned}$$

PSfrag replacements


 FIGURE 1. Crossing region of \mathcal{S} : $\Sigma^c \mathcal{S}$.

PSfrag replacements


 FIGURE 2. Sliding region of \mathcal{S} : $\Sigma^s \mathcal{S}$.

The *crossing region* Σ^c is defined as the generic points of discontinuity p such that $p \in \Sigma^c(\mathcal{S}_p)$. Analogously, we define the *sliding region* Σ^s .

Let $\varphi_{f^n}(t, q)$ be the solution of system (1) passing through the point $q \in \mathbb{S}_n$ at time $t = 0$, i.e. $\varphi_{f^n}(0, q) = q$. The local solution $\varphi_f(t, q)$ of system (3) passing through a point $p \in \Sigma^c$ at time $t = 0$ is given by the Filippov convention, i.e. for $p \in \Sigma^c$ such that $l(p) \subset G_p^+$ and taking the origin of time at p , the trajectory through p is defined as $\varphi_f(t, p) = \varphi_{f_p^-}(t, p)$ for $t \in I_p \cap \{t < 0\}$, and $\varphi_f(t, p) = \varphi_{f_p^+}(t, p)$ for $t \in I_p \cap \{t > 0\}$. For the case $l(p) \subset G_p^-$ the definition is the same reversing the time. Here I_p is an open interval having the 0 in its interior.

The following proposition gives a condition for the existence and uniqueness of solutions of system (3).

Proposition 1. *For every point $p \in \Sigma^c$ there is a solution passing either from G_p^- into G_p^+ , or from G_p^+ into G_p^- , and uniqueness is not violated.*

For a proof of Proposition 1 see Corollary 1 of section 10 of chapter 2 of [11].

Remark 3. *The Lipschitz condition together with Proposition 1 imply, for system (3), the global uniqueness of solutions whose reach the set of discontinuity only at points of Σ^c .*

1.3. Statement of the main results. We consider the following DPDS.

$$(4) \quad x'(t) = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 R(t, x, \varepsilon),$$

with

$$F_i(t, x) = \sum_{j=1}^M \chi_{\overline{S_j}}(t, x) F_i^j(t, x), \quad \text{for } i = 1, 2, \text{ and}$$

$$R(t, x, \varepsilon) = \sum_{j=1}^M \chi_{\overline{S_j}}(t, x) R^j(t, x),$$

where $F_i^j : \mathbb{R} \times D \rightarrow \mathbb{R}^d$, $R^j : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}^d$ for $i = 1, 2$ and $j = 1, 2, \dots, M$ are continuous functions, T -periodic in the variable t and D is an open subset of \mathbb{R}^d .

Remark 4. Here the family of sets $\mathcal{P} = \{S_i : i = 1, 2, \dots, M\}$ provides a partition of $[0, T) \times D$. Clearly we can use the family \mathcal{P} for producing a partition of the intervals $[kT, (k+1)T)$ for $k \in \mathbb{Z}$. Since the righthand side of system (4) is T -periodic, a partition of $\mathbb{R} \times D$ can be obtained naturally by considering $\mathbb{R} = \cup_{k \in \mathbb{Z}} [kT, (k+1)T)$.

We denote for $i = 1, 2$

$$D_x F_i(t, x) = \sum_{j=1}^M \chi_{\overline{A_j}}(t, x) D_x F_i^j(t, x).$$

In order to state our main result we define the averaged functions $f_1, f_2 : D \rightarrow \mathbb{R}^d$ as

$$(5) \quad f_1(z) = \int_0^T F_1(t, z) dt, \quad \text{and}$$

$$(6) \quad f_2(z) = \int_0^T \left(D_z F_1(t, z) y_1(t, z) + F_2(t, z) \right) dt,$$

where

$$y_1(t, z) = \int_0^t F_1(s, z) ds.$$

Our main results on the periodic orbits of DPDS (4) are given in the next two theorems. Their proofs use the Brouwer degree theory for finite dimensional spaces (see the appendix for a definition of the Brouwer degree $d_B(f, V, 0)$).

Theorem A (First order averaging theorem for DPDS). *Assume the following conditions.*

(Ha1) For $i = 1, 2$ and $j = 1, 2, \dots, M$, the continuous functions F_i^j and R_i^j are locally Lipschitz with respect to x , and T -periodic with respect to the time t . Furthermore, for $j = 1, 2, \dots, M$, the boundaries of S_j are piecewise C^k embedded hypersurfaces with $k \geq 1$.

(Ha2) There exists an open bounded set $C \subset D$ such that, for $|\varepsilon| > 0$ sufficiently small, every orbit starting in C reaches the set of discontinuity only at its crossing regions (crossing hypothesis).

(Ha3) For $a \in C$ with $f_1(a) = 0$, there exist a neighbourhood $U \subset C$ of a such that $f_1(z) \neq 0$ for all $z \in \overline{U} \setminus \{a\}$ and $d_B(f_1, U, 0) \neq 0$.

Then for $|\varepsilon| > 0$ (or $\varepsilon > 0$) sufficiently small, there exists a T -periodic solution $x(t, \varepsilon)$ of system (4) such that $x(0, \varepsilon) \rightarrow a$ as $\varepsilon \rightarrow 0$.

Theorem A is proved in section 3.

Theorem B (Second order averaging theorem for DPDS). *Suppose that $f_1(z) \equiv 0$. In addition to the hypotheses of Theorem A assume the following conditions.*

(Hb1) For $j = 1, 2, \dots, M$, the functions $F_1^j(t, \cdot)$ are of class C^1 for all $t \in \mathbb{R}$; for $i = 1, 2$ and $j = 1, 2, \dots, M$, the functions $D_x F_1^j$ are locally Lipschitz with respect to x . Furthermore, for $j = 1, 2, \dots, M$, the boundaries of S_j are piecewise C^k embedded hypersurfaces with $k \geq 2$.

(Hb2) For $(t, z) \in \Sigma$, if $(s, y_1(t, z)) \in T_{(t,z)}\Sigma$ then $s = 0$.

(Hb3) For $a \in C$ with $f_2(a) = 0$, there exist a neighborhood $U \subset C$ of a such that $f_2(z) \neq 0$ for all $z \in \bar{U} \setminus \{a\}$ and $d_B(f_2, U, 0) \neq 0$.

Then for $|\varepsilon| > 0$ sufficiently small, there exists a T -periodic solution $x(t, \varepsilon)$ of system (4) such that $x(0, \varepsilon) \rightarrow a$ as $\varepsilon \rightarrow 0$.

Theorem B is also proved in section 3.

1.4. Discontinuous perturbation of planar linear centers. In this subsection we show how to use the Theorems A and B for studying the linear centers perturbed by DPDS systems having the set of discontinuity composed by rays passing through the origin of coordinates. In other words we shall show that the hypothesis of crossing (Ha2) of both theorems and the hypothesis (Hb2) of Theorem B always hold for such systems after a change of variables and a rescaling of the time.

Let M be a positive integer greater than 1, let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_M) \in \mathbb{T}^M$ (M -Torus) be a M -tuple of angles such that $0 \leq \alpha_1 < \alpha_2 < \dots < \alpha_M < 2\pi$ and let $\mathcal{X} = (X_1, X_2, \dots, X_M)$ be a M -tuple of locally Lipschitz vector fields defined on an open neighborhood $D \subset \mathbb{R}^2$ of the origin.

We define the set of discontinuity $\Sigma = \bigcup_{i=1}^M L_i$, where L_i for $i = 1, 2, \dots, M$, is the intersection between the ray starting at the origin and passing through the point $(\cos \alpha_i, \sin \alpha_i)$ with the set D . We note that the set Σ splits the set $D \setminus \Sigma \subset \mathbb{R}^2$ in M disjoint open sectors. We denote the sector delimited by L_i and L_{i+1} by C_i for $i = 1, 2, \dots, M$.

Now let $Z_{\mathcal{X}, \alpha}(x, y)$ be the DPDS defined in D as

$$Z_{\mathcal{X}, \alpha}(x, y) = X_i(x, y) \quad \text{if } (x, y) \in C_i.$$

Let α be a M -tuple of angles, and let \mathcal{X} and \mathcal{Y} be two M -tuples of vector fields. In this section we shall study the following DPDS.

$$(7) \quad (\dot{x}, \dot{y}) = (y, -x) + \varepsilon Z_{\mathcal{X}, \alpha}(x, y) + \varepsilon^2 Z_{\mathcal{Y}, \alpha}(x, y).$$

Now the dot denotes derivative with respect to the time t .

Using the polar coordinates $x = r \cos \theta$ and $y = r \sin \theta$, system (7) becomes

$$(8) \quad \begin{pmatrix} \dot{\theta} \\ \dot{r} \end{pmatrix} = (-1, 0) + \varepsilon A(\theta, r) + \varepsilon^2 B(\theta, r),$$

where A and B are DPDS with the set of discontinuity $\tilde{\Sigma}$ being the union of the rays $\{(\alpha_i, r) : r > 0\}$ for $i = 1, 2, \dots, M$. Moreover, for $\theta \in [0, 2\pi)$, $A(\theta, r) = A_i(\theta, r)$ and $B(\theta, r) = B_i(\theta, r)$ if $\alpha_i \leq \theta < \alpha_{i+1}$ for $i = 1, 2, \dots, M$, where $\alpha_{M+1} = \alpha_1$, and

$$A_i(\theta, r) = \begin{pmatrix} \frac{1}{r} (X_i^2(r \cos \theta, r \sin \theta) \cos \theta - X_i^1(r \cos \theta, r \sin \theta) \sin \theta), \\ X_i^1(r \cos \theta, r \sin \theta) \cos \theta + X_i^2(r \cos \theta, r \sin \theta) \sin \theta \end{pmatrix},$$

and

$$B_i(\theta, r) = \left(\frac{1}{r} (Y_i^2(r \cos \theta, r \sin \theta) \cos \theta - Y_i^1(r \cos \theta, r \sin \theta) \sin \theta), \right. \\ \left. Y_i^1(r \cos \theta, r \sin \theta) \cos \theta + Y_i^2(r \cos \theta, r \sin \theta) \sin \theta \right).$$

Here $X_i = (X_i^1, X_i^2)$ and $Y_i = (Y_i^1, Y_i^2)$ for $i = 1, 2, \dots, M$.

In the sector $\alpha_i \leq \theta \leq \alpha_{i+1}$ system (8), taking the time θ instead t can be written as the differential equation

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \frac{\varepsilon A_i^2(\theta, r) + \varepsilon^2 B_i^2(\theta, r)}{-1 + \varepsilon A_i^1(\theta, r) + \varepsilon^2 B_i^1(\theta, r)}.$$

Here $A_i = (A_i^1, A_i^2)$ and $B_i = (B_i^1, B_i^2)$ for $i = 1, 2, \dots, M$. So system (8) and consequently system (7) becomes

$$(9) \quad r' = R(\theta, r, \varepsilon),$$

where, for $i = 1, 2, \dots, M$, $R(\theta, r, \varepsilon) = R_i(\theta, r, \varepsilon)$ if $\alpha_i \leq \theta \leq \alpha_{i+1}$, and

$$R_i(\theta, r, \varepsilon) = -\varepsilon (X_i^1(r \cos \theta, r \sin \theta) \cos \theta + X_i^2(r \cos \theta, r \sin \theta) \sin \theta) \\ -\varepsilon^2 \left(\frac{1}{r} (X_i^2(r \cos \theta, r \sin \theta) \cos \theta - X_i^1(r \cos \theta, r \sin \theta) \sin \theta) \right. \\ \cdot (X_i^1(r \cos \theta, r \sin \theta) \cos \theta + X_i^2(r \cos \theta, r \sin \theta) \sin \theta) \\ \left. + (Y_i^1(r \cos \theta, r \sin \theta) \cos \theta + Y_i^2(r \cos \theta, r \sin \theta) \sin \theta) \right) + \mathcal{O}(\varepsilon^3).$$

Now the prime denotes derivative with respect to the time θ .

Proposition 2. *For system (9) the following statements hold.*

- (a) *There exists an open bounded interval $C \subset (0, \infty)$ such that, for $|\varepsilon| > 0$ sufficiently small, every orbit starting in C reaches the set of discontinuity $\tilde{\Sigma}$ only at its crossing regions.*
- (b) *If $(s, y_1(\alpha_i, r)) \in T_{(\alpha_i, r)} \tilde{\Sigma}$ then $s = 0$ for $i = 1, 2, \dots, M$.*

Proof. Let $\tilde{\Sigma}_i = \{(\alpha_i, r) : r > 0\} \subset \tilde{\Sigma}$, and let $h_i(\theta, r) = \theta - \alpha_i$. Clearly $\tilde{\Sigma} = \cup_{i=1}^M \tilde{\Sigma}_i$ and $\tilde{\Sigma}_i = h_i^{-1}(0)$. In order to study the type of $\tilde{\Sigma}_i$, according to the definitions of Section 1.2, we compute

$$\langle \nabla h_i(\alpha_i, r), (1, R_{i-1}(\alpha_i, r, \varepsilon)) \rangle \langle \nabla h_i(\alpha_i, r), (1, R_i(\alpha_i, r, \varepsilon)) \rangle = \\ \langle (1, 0), (1, R_{i-1}(\alpha_i, r, \varepsilon)) \rangle \langle (1, 0), (1, R_i(\alpha_i, r, \varepsilon)) \rangle = 1 > 0.$$

So $\tilde{\Sigma}$ has only crossing regions. Hence statement (a) is proved.

Now assume that $(s, y_1(\alpha_i, r)) \in T_{(\alpha_i, r)} \tilde{\Sigma}$. It implies that

$$0 = \langle \nabla h_i(\alpha_i, r), (s, y_1(\alpha_i, r)) \rangle = \langle (1, 0), (s, y_1(\alpha_i, r)) \rangle = s.$$

So $s = 0$, and statement (b) is proved. \square

1.5. **Example 1.** In the following example we solve a problem of type (7).

Consider $\alpha = (0, \pi/2, \pi, 3\pi/2, 2\pi) \in \mathbb{T}^4$. Thus $L_1 = \{(x, 0) : x > 0\}$, $L_2 = \{(0, y) : y > 0\}$, $L_3 = \{(x, 0) : x < 0\}$, and $L_4 = \{(0, y) : y < 0\}$. Then for $i = 1, \dots, 4$ we have that C_i is the first, second, third and fourth quadrants, respectively.

In this example we study the maximum number of limit cycles given by the averaging theory of first and second order, which can bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, perturbed inside the following class of linear DPDS:

$$(10) \quad \dot{X} = Y_i(x, y) \quad \text{if } (x, y) \in C_i, \quad i = 1, \dots, 4$$

where

$$Y_i(x, y) = \begin{pmatrix} y + \varepsilon P_i^1(x, y) + \varepsilon^2 P_i^2(x, y) \\ -x + \varepsilon Q_i^1(x, y) + \varepsilon^2 Q_i^2(x, y) \end{pmatrix},$$

with $P_i^1(x, y) = a_{0i} + a_{1i}x + a_{2i}y$, $P_i^2(x, y) = c_{0i} + c_{1i}x + c_{2i}y$, $Q_i^1(x, y) = b_{0i} + b_{1i}x + b_{2i}y$, $Q_i^2(x, y) = d_{0i} + d_{1i}x + d_{2i}y$ and $|\varepsilon| > 0$ is a small parameter.

Let \mathcal{A} denote the two conditions

$$\begin{aligned} 4a_{01} - 4(a_{02} + a_{03} - a_{04} - b_{01} - b_{02} + b_{03} + b_{04}) &= 0 \quad \text{and} \\ 2a_{21} - 2(a_{22} - a_{23} + a_{24} - b_{11} + b_{12} - b_{13} + b_{14}) &+ \\ (a_{11} + a_{12} + a_{13} + a_{14} + b_{21} + b_{22} + b_{23} + b_{24})\pi &= 0. \end{aligned}$$

Our results on the limit cycles of system (10) are stated in the next two propositions.

Proposition 3. For $|\varepsilon| > 0$ sufficiently small and using Theorem A system (10) has at most 1 limit cycle for any chosen of parameters for \mathcal{A} does not hold. Moreover we can find parameters a_{ij} , b_{ij} , c_{ij} , and d_{ij} for $i = 0, 1, 2$ and $j = 1, \dots, 4$ such that system (10) has exactly 0 or 1 limit cycle.

Proposition 4. For $|\varepsilon| > 0$ sufficiently small and using Theorem B system (10) has at most 4 limit cycles for any chosen of parameters for which the two conditions of \mathcal{A} holds. Moreover we can find parameters a_{ij} , b_{ij} , c_{ij} , and d_{ij} for $i = 0, 1, 2$ and $j = 1, \dots, 4$ such that system (10) has exactly 0, 1, 2, 3 or 4 limit cycles.

Proposition 3 and 4 are proved in Section 3.

1.6. **Example 2.** In the following example we solve a problem which is not of type (7).

Let $h(x, y) = y - x^2$. The set $\Sigma = h^{-1}(0)$ is a regular manifold which splits the set $\mathbb{R}^2 \setminus \Sigma$ in two disjoint open regions. We consider the following system

$$(11) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} x + \varepsilon P^1(x, y) + \varepsilon^2 P^2(x, y), \\ -y + \varepsilon Q^1(x, y) + \varepsilon^2 Q^2(x, y), \end{pmatrix} & \text{if } h(x, y) \geq 0, \\ \begin{pmatrix} x + \varepsilon R^1(x, y) + \varepsilon^2 R^2(x, y), \\ -y + \varepsilon S^1(x, y) + \varepsilon^2 S^2(x, y), \end{pmatrix} & \text{if } h(x, y) \leq 0, \end{cases}$$

where

$$\begin{aligned} P_i &= p_{00}^i + p_{10}^i x + p_{01}^i y + p_{20}^i x^2 + p_{11}^i xy + p_{02}^i y^2, \\ Q_i &= q_{00}^i + q_{10}^i x + q_{01}^i y + q_{20}^i x^2 + q_{11}^i xy + q_{02}^i y^2, \\ R_i &= r_{00}^i + r_{10}^i x + r_{01}^i y + r_{20}^i x^2 + r_{11}^i xy + r_{02}^i y^2, \\ S_i &= s_{00}^i + s_{10}^i x + s_{01}^i y + s_{20}^i x^2 + s_{11}^i xy + s_{02}^i y^2, \end{aligned}$$

for $i = 1, 2$.

Let \mathcal{B} denote the set of conditions

$$\begin{aligned} p_{00}^1 &= p_{10}^1 = q_{00}^1 = q_{01}^1 = q_{02}^1 = s_{00}^1 = s_{02}^1 = 0, \\ q_{10}^1 &= -p_{01}^1 - 2p_{20}^1, \quad q_{11}^1 = -p_{02}^1 - 2p_{20}^1, \\ q_{20}^1 &= -p_{11}^1, \quad s_{01}^1 = -r_{10}^1, \quad \text{and} \\ s_{20}^1 &= 3r_{10}^1 - r_{11}^1. \end{aligned}$$

Our results on the limit cycles of system (11) are given in the next two propositions.

Proposition 5. *For $|\varepsilon| > 0$ sufficiently small and using Theorem A system (11) has at most 4 limit cycles for any chosen of parameters for which the conditions of \mathcal{B} do not hold. Moreover we can find parameters p_{ij}^1 , q_{ij}^1 , r_{ij}^1 , and s_{ij}^1 for $ij = 00, 10, 01, 20, 11, 02$ such that system (11) has exactly 0, 1, 2, 3 or 4 limit cycles.*

Proposition 6. *For $|\varepsilon| > 0$ sufficiently small and using Theorem B system (11) has at most 6 limit cycles for any chosen of parameters for which the conditions of \mathcal{B} hold. Moreover we can find parameters p_{01}^1 , p_{20}^1 , p_{11}^1 , p_{02}^1 , s_{10}^1 , r_{ij}^1 , p_{ij}^2 , q_{ij}^2 , r_{ij}^2 , and s_{ij}^2 for $ij = 00, 10, 01, 20, 11, 02$ such that system (11) has exactly 0, 1, 2, 3, 4, 5 or 6 limit cycles.*

Proposition 5 and 6 are proved in Section 3.

2. PROOFS OF THEOREMS A AND B

In order to prove Theorems A and B we need some preliminary results. As usual μ denotes the *Lebesgue Measure*.

Lemma 7. *The averaged functions (5) and (6) are continuous in C .*

Proof. Let $z_0 \in C$ and let V be a neighborhood of z_0 with a compact closure contained in C . For $z \in V$ we define the sets $A_z^i(t) = \{s \in [0, T] : (s, z) \in S^i\}$, and $A_z^0(t) = \{s \in [0, T] : (s, z) \in \Sigma\}$.

From hypothesis (Ha2) we have that $\mu(A_z^0(t)) = 0$ for every $t \in [0, T]$. So

$$\begin{aligned}
|y_1(t, z_0) - y_1(t, z)| &= \left| \sum_{j=1}^M \int_{A_{z_0}^j(t)} F_1^j(s, z_0) ds - \sum_{j=1}^M \int_{A_z^j(t)} F_1^j(s, z) ds \right| \\
&\leq \sum_{j=1}^M \left| \int_{A_{z_0}^j(t)} F_1^j(s, z_0) ds - \int_{A_z^j(t)} F_1^j(s, z) ds \right| \\
&\leq \sum_{j=1}^M \int_{A_{z_0}^j(t) \cap A_z^j(t)} |F_1^j(s, z_0) - F_1^j(s, z)| ds + \sum_{j=1}^M \left| \int_{A_{z_0}^j(t) \setminus A_z^j(t)} F_1^j(s, z_0) ds - \int_{A_z^j(t) \setminus A_{z_0}^j(t)} F_1^j(s, z) ds \right| \\
&\leq MTL|z_0 - z| + \sum_{j=1}^M \left| \int_{A_{z_0}^j(t) \setminus A_z^j(t)} F_1^j(s, z_0) ds - \int_{A_z^j(t) \setminus A_{z_0}^j(t)} F_1^j(s, z) ds \right| \\
&\leq MTL|z_0 - z| + \sum_{j=1}^M L_{1,j} (\mu(A_{z_0}^j(t) \setminus A_z^j(t)) + \mu(A_z^j(t) \setminus A_{z_0}^j(t))),
\end{aligned}$$

where L is maximum of the Lipschitz constants of the functions F_i^j for $j = 1, 2, \dots, M$, and $L_{1,j} = \max\{F_1^j(s, z) : (s, z) \in [0, T] \times \bar{V}\}$ for $j = 1, 2, \dots, M$. We observe that $\mu(A_{z_0}^j(t) \setminus A_z^j(t)) \rightarrow 0$ and $\mu(A_z^j(t) \setminus A_{z_0}^j(t)) \rightarrow 0$, as $z \rightarrow z_0$ for every $t \in [0, T]$. Thus $|y_1(t, z_0) - y_1(t, z)| \rightarrow 0$, as $z \rightarrow z_0$ for every $t \in [0, T]$. So the function $y_1(t, z)$ is continuous in C for each $t \in [0, T]$. Since $f_1(z) = y_1(T, z)$, we conclude that the averaged function f_1 is continuous in C .

Given $z_0 \in C$ and repeating the computations done for y_1 , now for $\int_0^t F_2(s, z) ds$ we get that this function is continuous for $z \in C$. So for proving the continuity of the function f_2 it is sufficient to estimate the difference

$$D(z_0, z) = \left| \int_0^T (D_z F_1(t, z_0) y_1(t, z_0) - D_z F_1(t, z) y_1(t, z)) dt \right|,$$

for $z \in V$, being V the above neighborhood of z_0 with a compact closure. So

$$\begin{aligned}
D(z_0, z) &\leq \int_0^T |D_z F_1(t, z_0) - D_z F_1(t, z)| |y_1(t, z_0)| dt + \int_0^T |D_z F_1(t, z)| |y_1(t, z_0) - y_1(t, z)| dt \\
&\leq TY \int_0^T |D_z F_1(t, z_0) - D_z F_1(t, z)| dt + TL' \int_0^T |y_1(t, z_0) - y_1(t, z)| dt,
\end{aligned}$$

where $Y = \max\{|y_1(s, z)| : (s, z) \in [0, T] \times \bar{V}\}$ and $L' = \max_{j=1}^M \{ |D_z F_1^j(s, z)| : (s, z) \in [0, T] \times \bar{V} \}$.

Now from the continuity of $y_1(t, z)$ and doing similar computations as the ones done for the discontinuous function $D_z F_1(t, z)$ we conclude that $D(z_0, z) \rightarrow 0$ when $z \rightarrow z_0$, which implies the continuity of the averaged function f_2 in C . \square

Let $g : (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}^d$ be a function defined on a small interval $(-\varepsilon_0, \varepsilon_0)$. We say that

1. $g(\varepsilon) = \mathcal{O}(\varepsilon^\ell)$ for some positive integer ℓ if there exists constants $\varepsilon_1 > 0$ and $k > 0$ such that $\|g(\varepsilon)\| \leq k|\varepsilon^\ell|$ for $-\varepsilon_1 < \varepsilon < \varepsilon_1$.
2. $g(\varepsilon) = o(\varepsilon^\ell)$ for some positive integer ℓ if

$$\lim_{\varepsilon \rightarrow 0} \frac{\|g(\varepsilon)\|}{\varepsilon^\ell} = 0.$$

The symbols \mathcal{O} and o are called the *Landau's symbols* (see for instance [19]).

Lemma 8 (Fundamental lemma). *Let $x(\cdot, z, \varepsilon) : [0, t_z] \rightarrow \mathbb{R}^n$ be the solution of system (3) with $x(0, z, \varepsilon) = z$.*

- (a) *Under the hypotheses of Theorem A, the function $\varepsilon \mapsto x(t, z, \varepsilon)$ is differentiable at $\varepsilon = 0$. Moreover $x(t, z, \varepsilon) = z + \varepsilon y_1(t, z) + \mathcal{O}(\varepsilon^2)$.*
- (b) *Under the hypotheses of Theorem B, the function $\varepsilon \mapsto x(t, z, \varepsilon)$ is differentiable at $\varepsilon = 0$. Moreover the $x(t, z, \varepsilon) = z + \varepsilon y_1(t, z) + \varepsilon^2 \int_0^t (D_z F_1(s, z) y_1(s, z) + F_2(s, z)) ds + \mathcal{O}(\varepsilon^3)$.*

Proof. From hypothesis (Ha2) for $z \in C$ the function $t \in [0, t_z] \mapsto x(t, z, \varepsilon)$ is continuous and piecewise differentiable. So for $|\varepsilon| > 0$ sufficiently small we can assume that

$$x(t, z, \varepsilon) = \begin{cases} x_1(t, z, \varepsilon) & \text{if } 0 = t_\varepsilon^0 \leq t \leq t_\varepsilon^1, \\ x_2(t, z, \varepsilon) & \text{if } t_\varepsilon^1 \leq t \leq t_\varepsilon^2, \\ \vdots & \\ x_i(t, z, \varepsilon) & \text{if } t_\varepsilon^{i-1} \leq t \leq t_\varepsilon^i, \\ \vdots & \\ x_\kappa(t, z, \varepsilon) & \text{if } t_\varepsilon^{\kappa-1} \leq t \leq t_\varepsilon^\kappa = t_z \leq T, \end{cases}$$

for which we have the following recurrence

$$(12) \quad x_1(0, z, \varepsilon) = z \quad \text{and} \quad x_{i+1}(t_\varepsilon^i, z, \varepsilon) = x_i(t_\varepsilon^i, z, \varepsilon),$$

for $i = 1, 2, \dots, \kappa - 1$.

The functions $x_i(t, z, \varepsilon)$ for $i = 1, 2, \dots, \kappa - 1$ are of class \mathcal{C}^1 and satisfy the DPDS (4), i.e. there exists a subsequence (j_i) for $i = 1, \dots, \kappa$ such that

$$(13) \quad \frac{\partial}{\partial t} x_i(t, z, \varepsilon) = \varepsilon F_1^{j_i}(t, x_i(t, z, \varepsilon)) + \varepsilon^2 F_2^{j_i}(t, x_i(t, z, \varepsilon)) + \varepsilon^3 R^{j_i}(t, x_i(t, z, \varepsilon), \varepsilon).$$

The function $x_i(t, z, \varepsilon)$ is the solution of the *Cauchy Problem* defined by the differential system (13) together with the initial condition (12), i.e. $x_i(t_\varepsilon^{i-1}, z, \varepsilon) = z$ for $i = 1$, and $x_i(t_\varepsilon^{i-1}, z, \varepsilon) = x_{i-1}(t_\varepsilon^{i-1}, z, \varepsilon)$ for $i = 2, 3, \dots, \kappa - 1$.

On the other hand, for $i = 1, 2, \dots, \kappa - 1$ and for each $z \in D$, there exists $\varepsilon_i > 0$ such that if $\varepsilon \in [-\varepsilon_i, \varepsilon_i]$ then the solution $x_i(t, z, \varepsilon)$ of (13) is defined in $[0, T]$. Indeed, by the *Existence and Uniqueness Theorem* of solutions (see, for example, Theorem 1.2.4 of [19]), for $z \in D$, $x_i(t, z, \varepsilon)$ is defined for all $0 \leq t \leq \inf(T, d/M_i(\varepsilon))$, where

$$M_i(\varepsilon) \geq \left| \varepsilon F_1^{j_i}(t, x_i(t, z, \varepsilon)) + \varepsilon^2 F_2^{j_i}(t, x_i(t, z, \varepsilon)) + \varepsilon^3 R^{j_i}(t, x_i(t, z, \varepsilon), \varepsilon) \right|$$

for all $t \in [0, T]$, for each x with $|x - z| \leq d$ and for every $z \in \bar{D}$. When ε is sufficiently small we can take $d/M_i(\varepsilon)$ sufficiently large in order that $\inf(T, d/M_i(\varepsilon)) = T$ for all $z \in \bar{V}$. So for any $z \in C$ we have that the solution $x(t, z, \varepsilon)$ of system (3) is also defined for every $t \in [0, T]$, i.e. $t_z = T$.

From the continuity of the solution $x(t, z, \varepsilon)$ and by compactness of the set $[0, T] \times \bar{C} \times [-\varepsilon_1, \varepsilon_1]$, there exists a compact subset K of D such that $x(t, z, \varepsilon) \in K$ for all $t \in [0, T]$, $z \in \bar{C}$ and $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$. Now, by the piecewise continuity of the function R , $|R(s, x(s, z, \varepsilon), \varepsilon)| \leq \max\{|R(t, x, \varepsilon)|, (t, x, \varepsilon) \in [0, T] \times K \times [-\varepsilon_1, \varepsilon_1]\} = N$. Then

$$\left| \int_0^t R(s, x(s, z, \varepsilon), \varepsilon) ds \right| \leq \int_0^T |R(s, x(s, z, \varepsilon), \varepsilon)| ds = TN,$$

which implies that

$$(14) \quad \int_0^t R(s, x(s, z, \varepsilon), \varepsilon) ds = \mathcal{O}(1).$$

Now for a given $t \in (0, T)$ there exists $\bar{\kappa} \in \{1, 2, \dots, \kappa - 1\}$ such that $t \in [t_\varepsilon^{\bar{\kappa}-1}, t_\varepsilon^{\bar{\kappa}})$ and

$$\begin{aligned} x(t, z, \varepsilon) &= x_{\bar{\kappa}}(t, z, \varepsilon) \\ &= x_{\bar{\kappa}-1}(t_\varepsilon^{\bar{\kappa}-1}, z, \varepsilon) + \varepsilon \int_{t_\varepsilon^{\bar{\kappa}-1}}^t F_1(s, x(s, z, \varepsilon)) ds + \varepsilon^2 \int_{t_\varepsilon^{\bar{\kappa}-1}}^t F_2(s, x(s, z, \varepsilon)) ds + \mathcal{O}(\varepsilon^3). \end{aligned}$$

Since

$$x_{i+1}(t_\varepsilon^{i+1}, z, \varepsilon) = x_i(t_\varepsilon^i, z, \varepsilon) + \varepsilon \int_{t_\varepsilon^i}^{t_\varepsilon^{i+1}} F_1(t, x(t, z, \varepsilon)) dt + \varepsilon^2 \int_{t_\varepsilon^i}^{t_\varepsilon^{i+1}} F_2(t, x(t, z, \varepsilon)) dt + \mathcal{O}(\varepsilon^3),$$

proceeding by induction on i we obtain

$$(15) \quad x(t, z, \varepsilon) = z + \varepsilon \int_0^t F_1(s, x(s, z, \varepsilon)) ds + \varepsilon^2 \int_0^t F_2(s, x(s, z, \varepsilon)) ds + \mathcal{O}(\varepsilon^3).$$

Thus let $t^i = \lim_{\varepsilon \rightarrow 0} t_\varepsilon^i$ for $i = 1, 2, \dots, \kappa - 1$. We compute

$$(16) \quad \begin{aligned} \int_0^t F_1(s, x(s, z, \varepsilon)) ds &= \left(\sum_{i=1}^{\bar{\kappa}-1} \int_{t_\varepsilon^{i-1}}^{t_\varepsilon^i} F_1^{j_i}(s, x_i(s, z, \varepsilon)) ds \right) + \int_{t_\varepsilon^{\bar{\kappa}-1}}^t F_1^{j_{\bar{\kappa}}}(s, x_{\bar{\kappa}}(s, z, \varepsilon)) ds \\ &= \left(\sum_{i=1}^{\bar{\kappa}-1} \int_{t^{i-1}}^{t^i} F_1^{j_i}(s, x_i(s, z, \varepsilon)) ds \right) + \int_{t^{\bar{\kappa}-1}}^t F_1^{j_{\bar{\kappa}}}(s, x_{\bar{\kappa}}(s, z, \varepsilon)) ds + E(\varepsilon), \end{aligned}$$

where

$$\begin{aligned} E(\varepsilon) &= \sum_{i=1}^{\bar{\kappa}-1} \left(\int_{t_\varepsilon^{i-1}}^{t_\varepsilon^i} F_1^{j_i}(s, x_i(s, z, \varepsilon)) ds - \int_{t^{i-1}}^{t^i} F_1^{j_i}(s, x_i(s, z, \varepsilon)) ds \right) \\ &\quad + \int_{t_\varepsilon^{\bar{\kappa}-1}}^{t^{\bar{\kappa}-1}} F_1^{j_{\bar{\kappa}}}(s, x_{\bar{\kappa}}(s, z, \varepsilon)) ds. \end{aligned}$$

It is easy to see that there exists a constant \tilde{E} such that

$$(17) \quad |E(\varepsilon)| \leq \tilde{E} \sum_{i=0}^{\bar{\kappa}-1} |t^i - t_\varepsilon^i|.$$

Claim 1. *Statement (a) of Lemma 8 holds.*

Denoting $t_\varepsilon^i = t^i(\varepsilon)$ we shall prove that $t^i(\varepsilon)$ is a \mathcal{C}^k function with $k \geq 1$.

From hypothesis (Ha2) the solution $x(t, z, 0) = z$ reaches the discontinuity set only on Σ^c , i.e. $(t^i, z) \in \Sigma^c$ for every $i = 1, 2, \dots, \kappa - 1$. Particularly (t^i, z) is a generic point of Σ . So from the hypothesis (Ha1) there exists a neighbourhood $G_{(t^i, z)}$ of (t^i, z) such that $\mathcal{S}_{(t^i, z)} = G_{(t^i, z)} \cap \Sigma$ is a \mathcal{C}^k embedded hypersurface of \mathbb{R}^{d+1} with $k \geq 1$. So it is well known that $\mathcal{S}_{(t^i, z)}$ can be locally described as the inverse image of a regular value of a \mathcal{C}^k function. Thus we can find a \mathcal{C}^k function $h_i : G_{(t^i, z)} \rightarrow \mathbb{R}$ such that $\tilde{G}_{(t^i, z)} \cap \mathcal{S}_{(t^i, z)} = h_i^{-1}(0) \cap \Sigma$. Here $\tilde{G}_{(t^i, z)}$ is an open subset such that $(t^i, z) \in \tilde{G}_{(t^i, z)} \subseteq G_{(t^i, z)}$.

For $(t, z) \in \tilde{G}_{(t^i, z)}$ system (4) can be written as the autonomous system

$$\begin{pmatrix} \tau' \\ x' \end{pmatrix} = \begin{cases} X(\tau, x, \varepsilon) & \text{if } h_i(\tau, x) > 0, \\ Y(\tau, x, \varepsilon) & \text{if } h_i(\tau, x) < 0, \end{cases}$$

where

$$X(\tau, x, \varepsilon) = \begin{pmatrix} 1 \\ \varepsilon F_1^{j_{i+1}}(\tau, x) + \varepsilon^2 F_2^{j_{i+1}}(\tau, x) + \varepsilon^3 R^{j_{i+1}}(\tau, x, \varepsilon) \end{pmatrix},$$

$$Y(\tau, x, \varepsilon) = \begin{pmatrix} 1 \\ \varepsilon F_1^{j_i}(\tau, x) + \varepsilon^2 F_2^{j_i}(\tau, x) + \varepsilon^3 R^{j_i}(\tau, x, \varepsilon) \end{pmatrix}.$$

Now we define $H_i(t, \varepsilon) = h_i(t, x_i(t, z, \varepsilon))$. We have that $H_i(t^i, 0) = 0$. Moreover, from hypothesis (Ha2) we have that $Xh_i(t^i, z, \varepsilon)Yh_i(t^i, z, \varepsilon) > 0$. Since $Xh_i(t^i, z, 0)Yh_i(t^i, z, 0) > 0$ is equal to

$$\left\langle \left(\frac{\partial}{\partial t} h_i(t^i, z), \frac{\partial}{\partial z} h_i(t^i, z) \right), (1, 0) \right\rangle \left\langle \left(\frac{\partial}{\partial t} h_i(t^i, z), \frac{\partial}{\partial z} h_i(t^i, z) \right), (1, 0) \right\rangle > 0.$$

So we conclude that $((\partial/\partial t)h_i(t^i, z))^2 > 0$. Thus

$$\begin{aligned} \frac{\partial}{\partial t} H_i(t^i, 0) &= \frac{\partial}{\partial t} h_i(t, x_i(t, z, \varepsilon)) \Big|_{(t^i, 0)} \\ &= \frac{\partial}{\partial t} h_i(t^i, x_i(t^i, z, 0)) + \frac{\partial}{\partial z} h_i(t^i, x_i(t^i, z, 0)) \frac{\partial}{\partial t} x_i(t^i, z, 0) \\ &= \frac{\partial}{\partial t} h_i(t^i, x_i(t^i, z, 0)) \neq 0, \end{aligned}$$

because from (13) $(\partial/\partial t)x_i(t, z, 0) = 0$. Hence from the Implicit Function Theorem, $t^i(\varepsilon)$ is a \mathcal{C}^k with $H(t^i(\varepsilon), \varepsilon) = 0$ for every $|\varepsilon| > 0$ sufficiently small and $t^i(0) = t^i$. So $t^i(\varepsilon) = t^i + (t^i)'(0)\varepsilon + o(\varepsilon)$ for every $i = 1, 2, \dots, \kappa - 1$, because $k \geq 1$. This implies that $E(\varepsilon) = \mathcal{O}(\varepsilon)$.

Since $F_1^{j_i}$ is Lipschitz for $i = 1, 2, \dots, \kappa$ in the variable x , we have that

$$\begin{aligned} &\left| \int_{t^{i-1}}^{t^i} F_1^{j_i}(s, x_i(s, z, \varepsilon)) ds - \int_{t^{i-1}}^{t^i} F_1^{j_i}(s, z) ds \right| \leq \int_{t^{i-1}}^{t^i} |F_1^{j_i}(s, x_i(s, z, \varepsilon)) - F_1^{j_i}(s, z)| ds \\ &\leq L_{j_i} \int_{t^{i-1}}^{t^i} |x_i(s, z, \varepsilon) - z| ds \leq L_{j_i} T |x_i(s, z, \varepsilon) - z| = \mathcal{O}(\varepsilon), \end{aligned}$$

where L_{j_i} is the Lipschitz constant of the function $F_1^{j_i}$. This implies that

$$(18) \quad \int_{t^{i-1}}^{t^i} F_1^{j_i}(s, x_i(s, z, \varepsilon)) ds = \int_{t^{i-1}}^{t^i} F_1^{j_i}(s, z) ds + \mathcal{O}(\varepsilon),$$

for every $i = 1, 2, \dots, \kappa$. Going back to the inequality (16) we have

$$(19) \quad \begin{aligned} \int_0^t F_1(s, x(s, z, \varepsilon)) ds &= \left(\sum_{i=1}^{\kappa-1} \int_{t^{i-1}}^{t^i} F_1^{j_i}(s, z) ds \right) + \int_{t^{\kappa-1}}^t F_1^{j_\kappa}(s, z) ds + E(\varepsilon) + \mathcal{O}(\varepsilon) \\ &= \int_0^t F_1(s, z) ds + \mathcal{O}(\varepsilon). \end{aligned}$$

Hence from (15) and (19) we get that

$$(20) \quad x(t, z, \varepsilon) = z + \varepsilon \int_0^t F_1(s, z) ds + \mathcal{O}(\varepsilon^2).$$

Therefore the claim 1 is proved.

Claim 2. *Statement (b) of Lemma 8 holds.*

First we shall prove that $E(\varepsilon) = \mathcal{O}(\varepsilon^2)$. From hypothesis (Ha2) and following the steps of the proof of Claim 1 we can find a \mathcal{C}^k function $h_i : G_{(t^i, z)} \rightarrow \mathbb{R}$, now with $k \geq 2$, such that $\tilde{G}_{(t^i, z)} \cap \mathcal{S}_{(t^i, z)} = h_i^{-1}(0) \cap \Sigma$. Again, $\tilde{G}_{(t^i, z)}$ is an open subset such that $(t^i, z) \in \tilde{G}_{(t^i, z)} \subseteq G_{(t^i, z)}$. Applying the Inverse Function Theorem we conclude that $t^i(\varepsilon)$ is a \mathcal{C}^2 function. So

$$(21) \quad t^i(\varepsilon) = t^i + (t^i)'(0)\varepsilon + \mathcal{O}(\varepsilon^2).$$

From statement (a) and (21) the function $\varepsilon \mapsto x(t^i(\varepsilon), z, \varepsilon)$ is differentiable at $\varepsilon = 0$. Moreover $y_1(t, z) = (\partial/\partial\varepsilon)x(t, z, 0)$. Since for $|\varepsilon| > 0$ sufficiently small $h_i(t^i(\varepsilon), x(t^i(\varepsilon), z, \varepsilon)) = 0$, so

$$\begin{aligned} 0 &= \frac{\partial}{\partial\varepsilon} h(t^i(\varepsilon), x(t^i(\varepsilon), z, \varepsilon)) \\ &= \frac{\partial}{\partial t} h(t^i, z)(t^i)'(0) + \frac{\partial}{\partial z} h(t^i, z) \left(\frac{\partial}{\partial t} x(t^i, z, 0)(t^i)'(0) + \frac{\partial}{\partial\varepsilon} x(t^i, z, \varepsilon) \Big|_{\varepsilon=0} \right) \\ &= \left\langle \nabla h(t^i, z), \left((t^i)'(0), y_1(t^i, z) \right) \right\rangle, \end{aligned}$$

because $(\partial/\partial t)x(t, z, 0)$ is the vector field (4) for $\varepsilon = 0$. So $((t^i)'(0), y_1(t^i, z)) \in T_{(t^i, z)}\Sigma$. From hypothesis (Hb2) we conclude that $(t^i)'(0) = 0$. From (17) and (21) it follows that $E(\varepsilon) = \mathcal{O}(\varepsilon^2)$.

Now for $t_{i-1} \leq t \leq t^i$ we prove that

$$(22) \quad |F_1^{j_i}(t, x(t, z, \varepsilon)) - F_1^{j_i}(t, z) - \varepsilon D_z F_1^{j_i}(t, z) y_1(t, z)| = \mathcal{O}(\varepsilon^2).$$

For this goal we define

$$G_i(\lambda) = F_1^{j_i}(t, \lambda x_i(t, z, \varepsilon) + (1 - \lambda)z).$$

Computing the derivative in λ we get

$$G'(\lambda) = D_z F_1^{j_i}(t, \lambda x_i(t, z, \varepsilon) + (1 - \lambda)z)(x_i(t, z, \varepsilon) - z).$$

So from the Fundamental Theorem of Calculus and observing that for $t_{i-1} \leq t \leq t^i$, $x_i(t, z, \varepsilon) = x(t, z, \varepsilon)$ it follows that

$$G(1) - G(0) = \int_0^1 D_z F_1^{j_i}(t, \lambda x(t, z, \varepsilon) + (1 - \lambda)z)(x(t, z, \varepsilon) - z) d\lambda.$$

Thus

$$\begin{aligned}
& \frac{1}{\varepsilon^2} \left(F_1^{j_i}(t, x_i(t, z, \varepsilon)) - F_1^{j_i}(t, z) - \varepsilon D_z F_1^{j_i}(t, z) y_1(t, z) \right) = \\
& \frac{1}{\varepsilon^2} \left(G(1) - G(0) - \varepsilon D_z F_1^{j_i}(t, z) y_1(t, z) \right) = \\
& \frac{1}{\varepsilon} \int_0^1 D_z F_1^{j_i}(t, \lambda x(t, z, \varepsilon) + (1 - \lambda)z) \frac{(x(t, z, \varepsilon) - z)}{\varepsilon} d\lambda - \frac{1}{\varepsilon} D_z F_1^{j_i}(t, z) y_1(t, z) = \\
& \frac{1}{\varepsilon} \left(\int_0^1 D_z F_1^{j_i}(t, \lambda x(t, z, \varepsilon) + (1 - \lambda)z) d\lambda \right) \int_0^t F_1(s, x(s, z, \varepsilon)) ds - \\
& \frac{1}{\varepsilon} D_z F_1^{j_i}(t, z) y_1(t, z) + \mathcal{O}(1) = \\
& \frac{1}{\varepsilon} \left(\int_0^1 \left[D_z F_1^{j_i}(t, \lambda x(t, z, \varepsilon) + (1 - \lambda)z) - D_z F_1^{j_i}(t, z) \right] d\lambda \right) \int_0^t F_1(s, x(s, z, \varepsilon)) ds + \\
& \frac{1}{\varepsilon} D_z F_1^{j_i}(t, z) \left[\int_0^t F_1(s, x(s, z, \varepsilon)) - F_1(s, z) ds \right] + \mathcal{O}(1).
\end{aligned}$$

Let $B_i = \max\{|F_1^{j_i}(s, x_i(s, z, \varepsilon))| : (t, z) \in [0, T] \times \bar{C}\}$. Observing that $F_1^{j_i}$ is locally Lipschitz in the variable z , and (from (19)) that $\int_0^t F_1(s, x(s, z, \varepsilon)) - \int_0^t F_1(s, z) = \mathcal{O}(\varepsilon)$, it follows that

$$\begin{aligned}
& \left| \frac{1}{\varepsilon^2} \left(F_1^{j_i}(t, x_i(t, z, \varepsilon)) - F_1^{j_i}(t, z) - \varepsilon D_z F_1^{j_i}(t, z) y_1(t, z) \right) \right| \leq \\
& \frac{1}{\varepsilon} \int_0^1 \left| D_z F_1^{j_i}(t, \lambda x(t, z, \varepsilon) + (1 - \lambda)z) - D_z F_1^{j_i}(t, z) \right| d\lambda \left| \int_0^t F_1(s, x(s, z, \varepsilon)) ds \right| + \\
& \frac{1}{\varepsilon} \left| D_z F_1^{j_i}(t, z) \right| \left| \int_0^t F_1(s, x(s, z, \varepsilon)) - F_1(s, z) ds \right| + \mathcal{O}(1) \leq \\
& TL_i B_i \frac{|x(t, z, \varepsilon) - z|}{\varepsilon} + \mathcal{O}(1) = \mathcal{O}(1),
\end{aligned}$$

where L_i is the Lipschitz constant of the function $D_z F_1^{j_i}$. Hence for $t^{i-1} \leq t \leq t^i$ and for every $i = 1, 2, \dots, \kappa$ the equality (22) holds which implies that $F_1^{j_i}(t, x(t, z, \varepsilon)) = F_1^{j_i}(t, z) + \varepsilon D_z F_1^{j_i}(t, z) y_1(t, z) + \mathcal{O}(\varepsilon^2)$. So

$$\int_{t^{i-1}}^t F_1^{j_i}(s, x_i(s, z, \varepsilon)) ds = \int_{t^{i-1}}^t F_1^{j_i}(s, z) ds - \varepsilon \int_{t^{i-1}}^t D_z F_1^{j_i}(s, z) y_1(s, z) ds + \mathcal{O}(\varepsilon^2).$$

Going back to the inequality (16) we have

$$\begin{aligned}
(23) \quad \int_0^t F_1(s, x(s, z, \varepsilon)) ds &= \left(\sum_{i=1}^{\bar{\kappa}-1} \int_{t^{i-1}}^{t^i} F_1^{j_i}(s, z) ds \right) + \int_{t^{\bar{\kappa}-1}}^t F_1^{j_{\bar{\kappa}}}(s, z) ds \\
&- \varepsilon \left(\sum_{i=1}^{\bar{\kappa}-1} \int_{t^{i-1}}^{t^i} D_z F_1^{j_i}(s, z) y_1(s, z) ds - \int_{t^{\bar{\kappa}-1}}^t D_z F_1^{j_{\bar{\kappa}}}(s, z) y_1(s, z) ds \right) \\
&+ \mathcal{O}(\varepsilon^2) + E(\varepsilon) \\
&= \int_0^t F_1(s, z) ds - \varepsilon \int_0^t D_z F_1(s, z) y_1(s, z) ds + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Analogously to the proof of statement (a) and using that $E(\varepsilon) = \mathcal{O}(\varepsilon^2) \subset \mathcal{O}(\varepsilon)$ we can show that

$$(24) \quad \int_0^t F_2(s, x(s, z, \varepsilon)) ds = \int_0^t F_2(s, z) ds + \mathcal{O}(\varepsilon).$$

So from (15), (23) and (24) we get

$$x(t, z, \varepsilon) = z + \varepsilon \int_0^t F_1(s, z) ds - \varepsilon^2 \int_0^t (D_z F_1(s, z) y_1(s, z) + F_2(s, z)) ds + \mathcal{O}(\varepsilon^3),$$

This completes the proof of the claim 2. \square

Proof of Theorem A. Let f be the function such that $\varepsilon f(z, \varepsilon) = x(T, z, \varepsilon) - z$. From statement (a) of Lemma 8 we have that

$$f(z, \varepsilon) = f_1(z) + \mathcal{O}(\varepsilon),$$

where the function f_1 is the one defined in (5). Clearly $x(t, z, \varepsilon)$ is a T -periodic solution if and only if $f(z, \varepsilon) = 0$, because $x(t, z, \varepsilon)$ is defined for all $t \in [0, T]$.

From the Brouwer degree theory (see Proposition 12 of the appendix) and hypothesis (Ha3) we have for $|\varepsilon| > 0$ sufficiently small that

$$d_B(f_1(z), U, a) = d_B(f(z, \varepsilon), U, a) \neq 0.$$

Hence, by item (i) of Theorem 10 (see the Appendix), $0 \in f(U, \varepsilon)$ for $|\varepsilon| > 0$ sufficiently small, i.e, there exists $a_\varepsilon \in U$ such that $f(a_\varepsilon, \varepsilon) = 0$. Therefore, for $|\varepsilon| > 0$ sufficiently small, $x(t, a_\varepsilon, \varepsilon)$ is a periodic solution of (3). We can choose a_ε such that $a_\varepsilon \rightarrow a$ when $\varepsilon \rightarrow 0$, because $f(z, \varepsilon) \neq 0$ in $U \setminus \{a\}$. This completes the proof of the theorem. \square

Proof of Theorem B. Now let f be a function such that $\varepsilon^2 f(z, \varepsilon) = x(T, z, \varepsilon) - z$. From statement (b) of Lemma 8 we have that

$$f(z, \varepsilon) = f_2(z) + \mathcal{O}(\varepsilon),$$

where the function f_2 is the one defined in (6). From here the proof follows in a similar way to the proof of Theorem A. \square

3. PROOFS OF PROPOSITIONS 3, 4, 5 AND 6

Proof of Proposition 3. The linear DPDS (10) in polar coordinates (r, θ) becomes

$$\begin{aligned} \dot{r} &= \varepsilon (a_{0i} \cos \theta + a_{1i} r \cos^2 \theta + b_{0i} \sin \theta + a_{2i} r \cos \theta \sin \theta + b_{1i} r \cos \theta \sin \theta + b_{2i} r \sin^2 \theta) + \\ &\quad \varepsilon^2 (c_{0i} \cos \theta + c_{1i} r \cos^2 \theta + c_{2i} r \cos \theta \sin \theta + d_{1i} r \cos \theta \sin \theta + d_{2i} r \sin^2 \theta + d_{0i} \sin \theta), \\ \dot{\theta} &= -1 - \frac{\varepsilon}{r} (-b_{0i} \cos \theta - b_{1i} r \cos^2 \theta + a_{0i} \sin \theta + a_{1i} r \cos \theta \sin \theta - b_{2i} r \cos \theta \sin \theta + a_{2i} r \sin^2 \theta) - \\ &\quad \frac{\varepsilon^2}{r} (-d_{0i} \cos \theta - d_{1i} r \cos^2 \theta + c_{0i} \sin \theta + c_{1i} r \cos \theta \sin \theta - d_{2i} r \cos \theta \sin \theta + c_{2i} r \sin^2 \theta), \end{aligned}$$

with $i = 1$ if $0 < \theta < \pi/2$, $i = 2$ if $\pi/2 < \theta < \pi$, $i = 3$ if $\pi < \theta < 3\pi/2$ and $i = 4$ if $3\pi/2 < \theta < 2\pi$. Taking the angle θ as the new independent variable the DPDS (10) becomes

$$(25) \quad \dot{r} = \varepsilon F_{1i} + \varepsilon^2 F_{2i} + \mathcal{O}(\varepsilon^3),$$

where

$$\begin{aligned} F_{1i} &= -r(a_{0i} \cos \theta + a_{1i}r \cos^2 \theta + b_{0i} \sin \theta + a_{2i}r \cos \theta \sin \theta + b_{1i}r \cos \theta \sin \theta + b_{2i}r \sin^2 \theta), \\ F_{2i} &= \frac{1}{r}(-b_{1i}r \cos^2 \theta - b_{0i} \cos \theta + a_{1i}r \sin \theta \cos \theta - b_{2i}r \sin \theta \cos \theta + a_{2i}r \sin^2 \theta + a_{0i} \sin \theta) \\ &\quad (a_{1i}r \cos^2 \theta + a_{0i} \cos \theta + a_{2i}r \sin \theta \cos \theta + b_{1i}r \sin \theta \cos \theta + b_{2i}r \sin^2 \theta + b_{0i} \sin \theta) \\ &\quad -(c_{1i}r \cos^2 \theta + c_{0i} \cos \theta + c_{2i}r \sin \theta \cos \theta + d_{1i}r \sin \theta \cos \theta + d_{2i}r \sin^2 \theta + d_{0i} \sin \theta). \end{aligned}$$

By Proposition 3 the DPDS (25) is under the assumptions of Theorem A. Computing the averaged function f_1 we obtain

$$\begin{aligned} (26) \quad f_1(r) &= \frac{1}{4}r(-4a_{01} + 4(a_{02} + a_{03} - a_{04} - b_{01} - b_{02} + b_{03} + b_{04}) \\ &\quad - (2a_{21} - 2(a_{22} - a_{23} + a_{24} - b_{11} + b_{12} - b_{13} + b_{14}) \\ &\quad + (a_{11} + a_{12} + a_{13} + a_{14} + b_{21} + b_{22} + b_{23} + b_{24})\pi)r). \end{aligned}$$

Clearly f_1 has at most 1 zero. Moreover we can choose the coefficients a_{ij} , $i = 0, 1$, $j = 1, \dots, 4$ in such a way that f_1 will have a simple positive zero. Hence the proposition is proved. \square

Proof of Proposition 4. We choose the coefficients a_{ij} , $i = 0, 1$, $j = 1, \dots, 4$ such that the conditions \mathcal{A} hold. Then $f_1(r) \equiv 0$. From Proposition 2 the DPDS (25) is under the assumptions of Theorem B. Using some algebraic manipulator as Mathematica or Maple we obtain

$$(27) \quad f_2(r) = k_0 + k_1r + k_2r^2 + k_3r^3 + k_4r^4,$$

where k_i , $i = 0, \dots, 4$ depends freely on the coefficients a_{ij} , $i = 0, 1$, $j = 1, \dots, 4$. Function (27) is a polynomial in the variable r of degree 4. Clearly function f_2 has at most 4 zeros. Moreover we can choose coefficients a_{ij} , $i = 0, 1$, $j = 1, \dots, 4$ such that (27) has 0, 1, 2, 3 or 4 simple zeros. So Proposition 4 is proved. \square

In order to prove the Propositions 5 and 6 we have to introduce the concept of ECT-systems and prove one lemma.

Let I be a proper real interval of \mathbb{R} . We say that an ordered set of complex-valued functions $F = (f_0, f_1, \dots, f_n)$ defined on I is an *Extended Chebyshev* system or ET-system on I if and only if any nontrivial linear combination of the functions of F has at most n zeros counting multiplicities. We say that F is an *Extended Complete Chebyshev* system or an ECT-system on I if and only if for any k , $0 \leq k \leq n$, (f_0, f_1, \dots, f_k) is an ET-system. For more details, see the book of Karlin and Studden [14].

In order to prove that F is a ECT-system on I is sufficient and necessary to show that $W(f_0, f_1, \dots, f_k)(t) \neq 0$ on I for $0 \leq k \leq n$. Here $W(f_0, f_1, \dots, f_k)(t)$ denotes the Wronskians of the functions (f_0, f_1, \dots, f_k) with respect to t . We recall the definition of the Wronskian.

$$W(f_0, f_1, \dots, f_k)(t) = \begin{vmatrix} f_0(t) & f_1(t) & \cdots & f_k(t) \\ f_0'(t) & f_1'(t) & \cdots & f_k'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(k)}(t) & f_1^{(k)}(t) & \cdots & f_k^{(k)}(t) \end{vmatrix}.$$

We also recall the *Descartes Theorem* about the number of zeros of a real polynomial (for a proof see for instance either the pages 82 and 83 of [4], or the appendix of [17]).

Descartes Theorem Consider the real polynomial $p(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \cdots + a_{i_r}x^{i_r}$ with $0 \leq i_1 < i_2 < \cdots < i_r$ and $a_{i_j} \neq 0$ real constants for $j \in \{1, 2, \dots, r\}$. When $a_{i_j}a_{i_{j+1}} < 0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs is m , then $p(x)$ has at most m positive real roots.

Now consider the functions

$$\begin{aligned}
g_1(u) &= 1, \\
g_2(u) &= u^2, \\
g_3(u) &= u^4, \\
g_4(u) &= u(2+u^2) \arccos\left(\frac{u}{\sqrt{2+u^2}}\right), \\
(28) \quad g_5^1(u) &= u(2+u^2), \\
g_5^2(u) &= u(2+u^2) \left(\pi - \arccos\left(\frac{u}{\sqrt{2+u^2}}\right)\right), \\
g_6(u) &= \sqrt{2}u^6 - u(8-4u^4-u^6) \left(\frac{\pi}{2} + \arcsin\left(\frac{u}{\sqrt{2+u^2}}\right)\right), \quad \text{and} \\
g_7(u) &= -\sqrt{2}u^6 - \frac{3\pi u^3(2+u^2)^2}{2} - u(8-4u^4-u^6) \arccos\left(\frac{u}{\sqrt{2+u^2}}\right).
\end{aligned}$$

We define the sets of functions $G^1 = \{g_1, g_2, g_3, g_4, g_5^1\}$ and $G^2 = \{g_1, g_2, g_3, g_4, g_5^2, g_6, g_7\}$.

Lemma 9. The sets of functions G^1 and G^2 are ECT-systems on the interval $(0, \infty)$.

Proof. To prove the statement we compute the Wronskians

$$\begin{aligned}
W_1(u) &= g_1(u), \quad W_2(u) = W(g_1, g_2)(u), \quad W_3(u) = W(g_1, g_2, g_3)(u), \\
W_4(u) &= W(g_1, g_2, g_3, g_4)(u), \quad W_5^1(u) = W(g_1, g_2, g_3, g_4, g_5^1)(u), \\
W_5^2(u) &= W(g_1, g_2, g_3, g_4, g_5^2)(u), \quad W_6(u) = W(g_1, g_2, g_3, g_4, g_5^2, g_6)(u), \quad \text{and} \\
W_7(u) &= W(g_1, g_2, g_3, g_4, g_5^2, g_6, g_7)(u).
\end{aligned}$$

So

$$\begin{aligned}
W_1(u) &= 1, \\
W_2(u) &= 2u, \\
W_3(u) &= 16u^3, \\
W_4(u) &= \frac{16u}{(2+u^2)^2} \left(P_1(u) + P_2(u) \arccos\left(\frac{u}{\sqrt{2+u^2}}\right) \right), \\
W_5^1(u) &= -\frac{6144\sqrt{2}u^3}{(2+u^2)^3},
\end{aligned}$$

$$\begin{aligned}
W_5^2(u) &= -\frac{6144\sqrt{2}\pi u^3}{(2+u^2)^3}, \\
W_6(u) &= \frac{-12288\pi u}{(2+u^2)^6} \left(Q_1(u) + Q_2(u) \arcsin\left(\frac{u}{\sqrt{2+u^2}}\right) \right), \quad \text{and} \\
W_7(u) &= -\frac{14495514624\pi^2 u^3 (11+u^2)}{(2+u^2)^8},
\end{aligned}$$

where

$$\begin{aligned}
P_1(u) &= \sqrt{2}u(12+4u^2+3u^4), \\
P_2(u) &= 3(2-u^2)(2+u^2)^2, \\
Q_1(u) &= 144\sqrt{2}\pi - 288u - 336\sqrt{2}\pi u^2 - 20u^3 + 1656\sqrt{2}\pi u^4 + 14584u^5 + 5760\sqrt{2}\pi u^6 \\
&\quad + 14700u^7 + 4305\sqrt{2}\pi u^8 + 3780u^9 + 945\sqrt{2}\pi u^{10}, \quad \text{and} \\
Q_2(u) &= 6\sqrt{2}(2+u^2)^2(12-40u^2+175u^4+315u^6).
\end{aligned}$$

Clearly $W_1(u) \neq 0$, $W_2(u) \neq 0$, $W_3(u) \neq 0$, $W_5^1(u) \neq 0$, $W_5^2(u) \neq 0$ and $W_7(u) \neq 0$ for $u > 0$. To see that the function $W_4(u)$ does not vanish for any $u > 0$ we shall prove that

$$\tilde{P}(u) = P_1(u) + P_2(u) \arccos\left(\frac{u}{\sqrt{2+u^2}}\right)$$

is an increasing function. Computing its derivative we have

$$P'(u) = 6u \left(\sqrt{2}u(2+3u^2) + (3u^4+4u^2-4) \arcsin\left(\frac{u}{\sqrt{2+u^2}}\right) \right).$$

It is easy to see that $(3u^4+4u^2-4)$ is increasing. So $\tilde{P}'(u)$ is also a increasing function for $u > 0$, because it is sums and products of increasing functions. Since $\tilde{P}'(0) = 0$ it follows that $\tilde{P}'(u) > 0$ for every $u > 0$. This implies that $\tilde{P}(u)$ is an increasing function for $u > 0$. Again, since $\tilde{P}(0) = 0$ it follows that $\tilde{P}(u) > 0$ for every $u > 0$. Thus $W_4(u) \neq 0$ for $u > 0$.

To see that the function $W_6(u)$ does not vanish for any $u > 0$ we shall prove that

$$\tilde{Q}(u) = Q_1(u) + Q_2(u) \arcsin\left(\frac{u}{\sqrt{2+u^2}}\right)$$

is a positive function for $u > 0$. From Descartes Theorem the polynomials Q_1 and Q_2 have at most 2 zeros, and 1 minimum or maximum. Numerically we find $u_1 \approx 0.247$ and $u_2 \approx 0.269$ as the minimums for Q_1 and Q_2 respectively. So $\tilde{Q}(u)$ is an increasing function for $u > \max\{u_1, u_2\}$. Finally it is easy to see that $\tilde{Q}(u) > 0$ for $0 < u \leq \max\{u_1, u_2\}$. Thus $W_6(u) \neq 0$ for $u > 0$. Hence the lemma is proved. \square

Proof of Proposition 5. Consider system (11). Proceeding with the change of variables $x = r \cos \theta$ and $y = r \sin \theta$, and taking θ as the new time, system (11) becomes

$$(29) \quad r' = \begin{cases} A(\theta, r) & \text{if } r \sin^2 \theta + \sin \theta - r > 0, \\ B(\theta, r) & \text{if } r \sin^2 \theta + \sin \theta - r < 0. \end{cases}$$

where

$$\begin{aligned} A(\theta, r) &= -p_{20}^1 r^2 \cos^3 \theta - r \cos^2 \theta (p_{10}^1 + (p_{11}^1 + q_{20}^1) r \sin \theta) \\ &\quad - \cos \theta (p_{00}^1 + r \sin \theta (p_{01}^1 + q_{10}^1 + (p_{02}^1 + q_1^{11}) r \sin \theta)) \\ &\quad - \sin \theta (q_{00}^1 + r \sin \theta (q_{01}^1 + q_{02}^1 r \sin \theta)), \\ B(\theta, r) &= -r_{20}^1 r^2 \cos^3 \theta - r \cos^2 \theta (r_{10}^1 + (r_{11}^1 + s_{20}^1) r \sin \theta) \\ &\quad - \cos \theta (r_{00}^1 + r \sin \theta (r_{01}^1 + s_{10}^1 + (r_{02}^1 + s_1^{11}) r \sin \theta)) \\ &\quad - \sin \theta (s_{00}^1 + r \sin \theta (s_{01}^1 + s_{02}^1 r \sin \theta)). \end{aligned}$$

Clearly hypothesis (Ha1) holds for system (29).

Given

$$\theta_1(r) = \arcsin\left(\frac{u}{\sqrt{2+u^2}}\right) \quad \text{and} \quad \theta_2(r) = \pi - \arcsin\left(\frac{u}{\sqrt{2+u^2}}\right),$$

we have that for $r > 0$, $r \sin^2 \theta + \sin \theta - r > 0$ if and only if $0 \leq \theta < \theta_1(r)$ and $\theta_2(r) < \theta \leq 2\pi$; and $r \sin^2 \theta + \sin \theta - r < 0$ if and only if $\theta_1(r) < \theta < \theta_2(r)$.

Let $\tilde{h}(\theta, r) = r \sin^2 \theta + \sin \theta - r$, thus the set of discontinuity of system (29) is given by $\tilde{\Sigma} = \tilde{h}^{-1}(0) = \{(\theta_1(r), r) : r > 0\} \cup \{(\theta_2(r), r) : r > 0\}$. Since

$$\begin{aligned} \left\langle \nabla \tilde{h}(\theta_1(r), r), (1, A(\theta_1(r), r)) \right\rangle \left\langle \nabla \tilde{h}(\theta_1(r), r), (1, B(\theta_1(r), r)) \right\rangle &= \frac{(1+4r^2)(-1+\sqrt{1+4r^2})}{2r^2}, \\ \left\langle \nabla \tilde{h}(\theta_2(r), r), (1, A(\theta_2(r), r)) \right\rangle \left\langle \nabla \tilde{h}(\theta_2(r), r), (1, B(\theta_2(r), r)) \right\rangle &= \frac{(1+4r^2)(-1+\sqrt{1+4r^2})}{2r^2}, \end{aligned}$$

we conclude that $\tilde{\Sigma}$ has only crossing regions. So hypothesis (H2a) holds for system (29).

Taking $r = u\sqrt{2+u^2}/2$ and computing the averaged function f_1 we obtain

$$f_1(u) = k_1 g_1(u) + k_2 g_2(u) + k_3 g_3(u) + k_4 g_4(u) + k_5 g_5(u),$$

where

$$\begin{aligned} k_1 &= 24\sqrt{2}(q_{00}^1 - s_{00}^1), \\ k_2 &= 2\sqrt{2}(-3p_{10}^1 + 2p_{11}^1 + 3q_{01}^1 + 4q_{02}^1 + 2q_{20}^1 + 3r_{10}^1 - 2r_{11}^1 - 3s_{01}^1 - 4s_{02}^1 - 2s_{20}^1), \\ k_3 &= 6\sqrt{2}(q_{02}^1 - s_{02}^1), \\ k_4 &= -6(p_{10}^1 + q_{01}^1 - r_{10}^1 - s_{01}^1), \\ k_5 &= -3(p_{10}^1 + q_{01}^1 - r_{10}^1 - s_{01}^1). \end{aligned}$$

So from Lemma 9 and Theorem A the proof follows. \square

Proof of Proposition 6. In order to apply Theorem B to system (29) we have to guarantee that $f_1(u) \equiv 0$. By the linearity of the set of functions G^1 , $f_1(u) \equiv 0$ if and only if $k_i = 0$ for $i = 1, 2, \dots, 5$. Thus assuming that $k_i = 0$ for $i = 1, 2, \dots, 5$, it is easy to see, using some algebraic manipulator as Mathematica or Maple, that the statement $\langle \nabla h(\theta_1(r), r), (s, y_1(\theta_1(r), t)) \rangle = 0$ implies $s = 0$ holds if and only if the conditions \mathcal{B} holds. So assuming conditions \mathcal{B} the hypothesis (Hb2) holds.

Taking $r = u\sqrt{2+u^2}/2$ and computing the averaged function f_1 we obtain

$$f_2(u) = k_1 g_1(u) + k_2 g_2(u) + k_3 g_3(u) + k_4 g_4(u) + k_5 g_5^2(u) + k_6 g_6(u) + k_6 g_6(u).$$

Hence from Lemma 9 and Theorem B the proof follows. \square

APPENDIX: BASIC RESULTS ON THE BROUWER DEGREE

In this appendix we present the existence and uniqueness result from the degree theory in finite dimensional spaces. We follow the Browder's paper [8], where are formalized the properties of the classical Brouwer degree. We also present some results that we shall need for proving the main results of this paper.

Theorem 10. *Let $X = \mathbb{R}^n = Y$ for a given positive integer n . For bounded open subsets V of X , consider continuous mappings $f : \bar{V} \rightarrow Y$, and points y_0 in Y such that y_0 does not lie in $f(\partial V)$ (as usual ∂V denotes the boundary of V). Then to each such triple (f, V, y_0) , there corresponds an integer $d(f, V, y_0)$ having the following three properties.*

- (i) *If $d(f, V, y_0) \neq 0$, then $y_0 \in f(V)$. If f_0 is the identity map of X onto Y , then for every bounded open set V and $y_0 \in V$, we have*

$$d(f_0|_V, V, y_0) = \pm 1.$$

- (ii) *(Additivity) If $f : \bar{V} \rightarrow Y$ is a continuous map with V a bounded open set in X , and V_1 and V_2 are a pair of disjoint open subsets of V such that*

$$y_0 \notin f(\bar{V} \setminus (V_1 \cup V_2)),$$

then,

$$d(f_0, V, y_0) = d(f_0, V_1, y_0) + d(f_0, V_2, y_0).$$

- (iii) *(Invariance under homotopy) Let V be a bounded open set in X , and consider a continuous homotopy $\{f_t : 0 \leq t \leq 1\}$ of maps of \bar{V} in to Y . Let $\{y_t : 0 \leq t \leq 1\}$ be a continuous curve in Y such that $y_t \notin f_t(\partial V)$ for any $t \in [0, 1]$. Then $d(f_t, V, y_t)$ is constant in t on $[0, 1]$.*

Theorem 11. *The degree function $d(f, V, y_0)$ is uniquely determined by the conditions of Theorem 10.*

For the proofs of Theorems 10 and 11 see [8].

Proposition 12. *We consider the continuous functions $f_i : \bar{V} \rightarrow \mathbb{R}^n$, for $i = 0, 1, \dots, k$, and $f, g, r : \bar{V} \times [\varepsilon_0, \varepsilon_0] \rightarrow \mathbb{R}^n$, given by*

$$g(\cdot, \varepsilon) = f_1(\cdot) + \varepsilon f_2(\cdot) + \varepsilon^2 f_3(\cdot) + \dots + \varepsilon^{k-1} f_k(\cdot),$$

$$f(\cdot, \varepsilon) = g(\cdot, \varepsilon) + \varepsilon^k r(\cdot, \varepsilon).$$

Assume that $g(z, \varepsilon) \neq 0$ for all $z \in \partial V$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. If for $|\varepsilon| > 0$ sufficiently small $d_B(f(\cdot, \varepsilon), V, y_0)$ is well defined, then

$$d_B(f(\cdot, \varepsilon), V, y_0) = d_B(g(\cdot, \varepsilon), V, y_0).$$

For a proof of Proposition 12 see Lemma 2.1 in [9].

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