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CENTERS AND ISOCHRONOUS CENTERS FOR GENERALIZED QUINTIC SYSTEMS

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Abstract. In this paper we classify the centers and the isochronous centers of certain polynomial differential systems in $\mathbb{R}^2$ of degree $d \geq 5$ odd that in complex notation can be written as

$$\dot{z} = (\lambda + i)z + (z\bar{z})^{\frac{d-5}{2}}(Az^5 + Bz^4\bar{z} + Cz^3\bar{z}^2 + Dz^2\bar{z}^3 + Ez\bar{z}^4 + F\bar{z}^5),$$

where $\lambda \in \mathbb{R}$ and $A, B, C, D, E, F \in \mathbb{C}$. Note that if $d = 5$ we obtain the full class of polynomial differential systems of the form a linear system with homogeneous polynomial nonlinearities of degree 5. Our study uses algorithms of computational algebra based on the Groebner basis theory and modular arithmetics for simplifying the computations.

1. Introduction and statement of the main results

The center-focus problem is one of the main problems in the qualitative theory of real planar polynomial systems. For nondegenerate singular points this problem is equivalent to have an analytic first integral in a neighborhood of the singular point, see [26, 27] and [3, 14, 15].

We recall that a singular point is a *center* if it has a neighborhood such that all the orbits, with the exception of the singular point, in this neighborhood are periodic, and that a singular point is a *focus* if it has neighborhood such that all the orbits, with the exception of the singular point, spiral either in forward or in backward time to the singular point.

In this paper we study the center-focus problem for a class of polynomial systems which generalizes the class of polynomial systems with homogeneous nonlinearities of degree 5. The characterization of the centers of the polynomial differential systems begun with the quadratic ones and the class of cubic polynomial systems with only homogeneous nonlinearities, see [2, 25, 32, 34]. See [30, 31] for an update on these cases. Actually we are very far from obtaining a complete classification of the centers for the class of all polynomial systems of degree 3. However some subclasses of cubic systems with centers are know, see for instance [35, 36] and references therein. The centers of polynomial systems of the form a linear center with homogeneous polynomial nonlinearities of degree $k > 3$ are not classified, but there are partial results for $k = 4, 5, 6, 7$ see [4, 5, 13, 18, 19, 20, 21, 22, 23, 24]. However the huge amount of computations which usually are necessary becomes the center problem in general computationally intractable, see [17] and references therein.

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In this paper we consider the real polynomial differential systems in the plane that has a singular point at the origin with eigenvalues $\lambda \pm i$ and that can be written in complex notation as

$$\dot{z} = (\lambda + i)z + (z\bar{z})^{\frac{d-5}{2}}(Az^5 + Bz^4\bar{z} + Cz^3\bar{z}^2 + Dz^2\bar{z}^3 + Ez\bar{z}^4 + F\bar{z}^5), \tag{1}$$

where $z = x + iy$, $d \geq 5$ is an arbitrary odd integer, $\lambda \in \mathbb{R}$ and $A, B, C, D, E, F \in \mathbb{C}$. For some particular systems first we want to determine the conditions that ensure that the origin is a center. Of course, these systems for $d = 5$ coincide with the class of polynomial differential systems of the form a linear system with homogeneous polynomial nonlinearities of degree 5. So the class of polynomial differential systems (1) of odd degree $d \geq 5$ generalizes the linear systems with homogeneous polynomial nonlinearities of degree 5. We remark that there are very few results about the centers for classes of polynomial differential systems of arbitrary degree. The resolution of this problem implies the effective computation of the Poincaré-Liapunov constants. Indeed, setting

$$\begin{array}{rlll} A & = a_1 + ia_2, & B = a_3 + ia_4, & C = a_5 + ia_6, \\ D & = a_7 + ia_8, & E = a_9 + ia_{10}, & F = a_{11} + ia_{12}, \end{array}$$

and writing (1) in polar coordinates, i.e., doing the change of variables $r^2 = z\bar{z}$ and $\theta = \arctan(\operatorname{Im} z / \operatorname{Re} z)$, system (1) becomes

$$\dot{r} = \lambda r + F(\theta)\, r^d, \qquad \dot{\theta} = 1 + G(\theta)\, r^{d-1}, \tag{2}$$

where $F(\theta)$ and $G(\theta)$ are the homogeneous trigonometric polynomials

$$\begin{array}{rcl} F(\theta) & = & a_5 + (a_3 + a_7)\cos(2\theta) + (a_8 - a_4)\sin(2\theta) + (a_1 + a_9)\cos(4\theta) \\ & & +(a_{10} - a_2)\sin(4\theta) + a_{11}\cos(6\theta) + a_{12}\sin(6\theta), \\ G(\theta) & = & a_6 + (a_4 + a_8)\cos(2\theta) + (a_3 - a_7)\sin(2\theta) + (a_{10} + a_2)\cos(4\theta) \\ & & +(a_1 - a_9)\sin(4\theta) + a_{12}\cos(6\theta) - a_{11}\sin(6\theta). \end{array}$$

In order to determine the necessary conditions to have a center we work with the Poincaré series

$$H(r, \theta) = \sum_{n=2}^{\infty} H_n(\theta) r^n,$$

where $H_2(\theta) = 1/2$ and $H_n(\theta)$ are homogeneous trigonometric polynomials respect to θ of degree n. Imposing that this power series is a formal first integral of system (2) we obtain

$$\dot{H}(r, \theta) = \sum_{k=2}^{\infty} V_{2k} r^{2k} = 0.$$

where V_{2k} are the *Poincaré-Lyapunov constants* that depend on the parameters of system (1). Indeed, it is easy to see from the recursive equations that generate the V_{2k}, that these V_{2k} are polynomials in the parameters of system (1) see [16]. As system (1) is polynomial, due to the Hilbert Basis theorem, the ideal $J = < V_2, V_4, ... >$ generated by the Poincaré-Liapunov constants is finitely generated, i.e. there exist $W_1, W_2, ..., W_k$ in J such that $J = < W_1, W_2, ..., W_k >$. Such a set of generators is called a *basis* of J and is a finite set of necessary conditions to have a center for system (1). The set of coefficients for which all the Poincaré-Liapunov constants V_{2k} vanish is called the *center variety* of the family of polynomial differential systems and also it is an algebraic set. The determination of the first Poincaré-Liapunov constants has been made using Mathematica and it took several hours of computation.

In practice we determine a number of Poincaré-Liapunov constants assuming that inside these number there is the set of generators of all Poincaré-Liapunov constants. From this set the much harder problem is to decompose this algebraic set into its irreducible components. For simple cases this can be done by hand, see [4, 5, 18]. However for more difficult systems the use of a computer algebra system is essential. The computational tool which we use is the routine `minAssGTZ` [10] of the computer algebra system SINGULAR [12] which is based on the Gianni-Trager-Zacharias algorithm [11]. Since computations are too laborious they cannot be completed in the field of rational numbers. Therefore we choose the approach based on making use of modular computations [29]. We have chosen the prime $p = 32003$. To perform the rational reconstruction we used the MATHEMATICA code presented in [29]. We follow the algorithm described in [29]. The last step of this algorithm has not been verified because computations cannot be overcome. This step ensures that all the points of the center variety have been found. That is, we know that all the encountered points belong to the decomposition of the center variety but we do not know whether the given decomposition is complete. We remark that, nevertheless, it is practically sure that the given list is complete, see also [29]. Therefore in the following we provide necessary conditions for having a center (a complete list of them with very high probability), and later on we shall prove that they sufficient. We remark that since the computations were performed using modular arithmetics we cannot guarantee that we obtain all the centers, but the probability that this not be the case is very small.

From system (2) we can obtain the associated equation

$$\frac{dr}{d\theta} = \frac{\lambda r + F(\theta)\, r^d}{1 + G(\theta)\, r^{d-1}}, \tag{3}$$

It is clear that equation (3) is well defined in a sufficient small neighborhood of the origin. Hence if system (2) has a center at the origin, then equation (3) when $\dot{\theta} > 0$ also has a center at the origin. The transformation $(r, \theta) \to (\rho, \theta)$ introduced by Cherkas [8] defined by

$$\rho = \frac{r^{d-1}}{1 + G(\theta) r^{d-1}}, \quad \text{whose inverse is} \quad r = \frac{\rho^{1/(d-1)}}{(1 - \rho G(\theta))^{1/(d-1)}}, \tag{4}$$

is a diffeomorphism from the region $\dot{\theta} > 0$ into its image. If we transform equation (3) using the transformation (4), we obtain the following Abel equation

$$\begin{aligned} \frac{d\rho}{d\theta} &= (d-1)G(\theta)[\lambda G(\theta) - F(\theta)]\rho^3 \\ &\quad +[(d-1)(F(\theta) - 2\lambda G(\theta)) - G'(\theta)]\rho^2 + (d-1)\lambda\rho \\ &= A(\theta)\rho^3 + B(\theta)\rho^2 + C\rho. \end{aligned} \tag{5}$$

The solution $\rho(\theta, \rho_0)$ of (5) satisfying that $\rho(0, \rho_0) = \rho_0$ can be expanded in a convergent series of $\rho_0 \geq 0$ sufficiently small of the form

$$\rho(\theta, \rho_0) = \rho_1(\theta)\rho_0 + \rho_2(\theta)\rho_0^2 + \rho_3(\theta)\rho_0^3 + \cdots \tag{6}$$

with $\rho_1(\theta) = 1$ and $\rho_k(0) = 0$ for $k \geq 2$. Let $P : [0, \tilde{\rho}_0] \to \mathbb{R}$ be the Poincaré return map defined by $P(\tilde{\rho}_0) = \rho(2\pi, \tilde{\rho}_0)$ for a convenient $\tilde{\rho}_0$. System (1) has a center at the origin if and only if $\rho_k(2\pi) = 0$ for every $k \geq 2$. If we assume that $\rho_2(2\pi) = \cdots = \rho_{m-1}(2\pi) = 0$ we say that $v_m = \rho_m(2\pi)$ is the *m-th Poincaré-Liapunov-Abel* constant of system (1). Of course the set of coefficients for which all the Poincaré-Liapunov-Abel constants v_m vanish is the same that the set for which all the Poincaré-Liapunov constants V_{2k} vanish. This set, as we mentioned, is the center variety of system (1).

Once we have determined the center variety of system (1) we also can to determine which of these centers are isochronous. A center of system (1) is *isochronous* if the period of all periodic orbits in a neighborhood of the origin is constant. In fact the isochronous centers of the linear centers perturbed by fifth degree polynomials are known, see [28]. The contribution of this work is the classification of the isochronous centers for the generalized families studied.

From the second equation of (2) we have

$$T = \int_0^{2\pi} \frac{d\theta}{\dot{\theta}} = \int_0^{2\pi} \frac{1}{1 + G(\theta) r(\theta)^{d-1}} d\theta. \tag{7}$$

Using the change (4) the previous integral becomes

$$T = \int_0^{2\pi} (1 - G(\theta)\rho(\theta)) d\theta = 2\pi - \int_0^{2\pi} G(\theta)\rho(\theta) d\theta,$$

where $\rho(\theta) = \sum_{j \geq 1} \rho_j(\theta) \rho_0^j$ is the solution given in (6). System (1) has an isochronous center at the origin if it is a center and satisfies

$$\int_0^{2\pi} G(\theta)\rho(\theta) d\theta = \sum_{j \geq 1} \left(\int_0^{2\pi} G(\theta) \rho_j(\theta) d\theta \right) \rho_0^j = 0.$$

That is $T = \displaystyle\int_0^{2\pi} d\theta/\dot{\theta} = 2\pi - \sum_{j \geq 1} T_j \rho_0^j = 2\pi$, where

$$T_j = \int_0^{2\pi} G(\theta) \rho_j(\theta) d\theta,$$

are called the *period Abel constants*.

The next result proved in [33] gives the relationship between the existence of a transversal commuting and the existence of an isochronous center.

Theorem 1. *Let (S) and (S_T) be transversal plane differential systems of class C^2. Assume that the local flows defined by the solutions of (S) and (S_T) commute (in the sense of the Lie bracket). Then any center of (S) is isochronous.*

We note that the space of systems (1) with a center at the origin is invariant with respect to the action group $\mathbb{C}^*$ of change of variables $z \to \xi z$:

$$\begin{array}{ll} A \to \xi^{(d-7)/2} \bar{\xi}^{(d-5)/2} \xi^5 A, & B \to \xi^{(d-7)/2} \bar{\xi}^{(d-5)/2} \xi^4 \bar{\xi} B. \\ C \to \xi^{(d-7)/2} \bar{\xi}^{(d-5)/2} \xi^3 \bar{\xi}^2 C, & D \to \xi^{(d-7)/2} \bar{\xi}^{(d-5)/2} \xi^2 \bar{\xi}^3 D. \\ E \to \xi^{(d-7)/2} \bar{\xi}^{(d-5)/2} \xi \bar{\xi}^4 E, & F \to \xi^{(d-7)/2} \bar{\xi}^{(d-5)/2} \bar{\xi}^5 F. \end{array} \tag{8}$$

The next result will be used to check when system (1) is reversible with respect to a straight line through the origin. Indeed system (1) is invariant with respect to a straight line through the origin if it is invariant under the change of variables $w = e^{i\gamma} z$, $\tau = -t$ for some real γ. The next result proved in [9].

Lemma 2. *System* (1) *is reversible if and only if*

$$\begin{array}{lll} A = -\bar{A}e^{-4i\gamma}, & B = -\bar{B}e^{-2i\gamma}, & C = -\bar{C}, \\ D = -\bar{D}e^{2i\gamma}, & E = -\bar{E}e^{4i\gamma}, & F = -\bar{F}e^{6i\gamma}, \end{array}$$

for some $\gamma \in \mathbb{R}$. *Furthermore, in this situation the origin of system* (1) *has a center at the origin.*

The main results of this paper are Theorem 3 where we classify the centers of the polynomial differential systems (1) for two particular subclasses, namely

$$(c.1)\ A = \mathrm{Im}(C) = 0, \qquad (c.2)\ E = \mathrm{Im}(C) = 0,$$

determining the conditions on the parameters λ, A, B, C, D, E and F in order that the origin of the polynomial differential system (1) of degree $d \geq 5$ odd be a center, and Theorem 4 where we classify the isochronous centers for such families.

Theorem 3. *For* $d \geq 5$ *odd the following statement hold.*

(a) *System* (1) *satisfying conditions* (c.1) *has a center at the origin if one of the following conditions hold:*
- (a.1) $\lambda = C = E = 2B + \bar{D} = 0$,
- (a.2) $\lambda = C = F = 3B + \bar{D} = 0$,
- (a.3) $\lambda = C = E = F = (d-3)B + (d+1)\bar{D} = 0$,
- (a.4) $\lambda = C = \mathrm{Im}(BD) = \mathrm{Re}(DE\bar{F}) = \mathrm{Re}(\bar{B}E\bar{F}) = \mathrm{Re}(D^2\bar{E}) = \mathrm{Re}(B\bar{D}E) = \mathrm{Re}(B^2E) = \mathrm{Im}(D\bar{E}^2F) = \mathrm{Im}(DE^2\bar{F}) = \mathrm{Im}(D^3\bar{F}) = \mathrm{Im}(D^2\bar{B}\bar{F}) = \mathrm{Im}(\bar{B}^2D\bar{F}) = \mathrm{Im}(B^3F) = \mathrm{Re}(E^3\bar{F}^2) = 0$.

(b) *System* (1) *satisfying conditions* (c.2) *has a center at the origin if one of the following conditions hold:*
- (b.1) $\lambda = C = F = B + 3\bar{D} = 0$ *for* $d = 5$,
- (b.2) $\lambda = A = C = F = (d-3)B + (d+1)\bar{D} = 0$,
- (b.3) $\lambda = A = C = D + 2\bar{B} = 0$,
- (b.4) $\lambda = B = C = 0$ *and conditions* (16) *(see below) for* $d = 5$,
- (b.5) $\lambda = C = \mathrm{Im}(BD) = \mathrm{Re}(D\bar{A}\bar{F}) = \mathrm{Re}(\bar{B}\bar{A}\bar{F}) = \mathrm{Re}(D^2A) = \mathrm{Re}(BAE) = \mathrm{Re}(B^2\bar{A}) = \mathrm{Im}(DA^2F) = \mathrm{Im}(D\bar{A}^2\bar{F}) = \mathrm{Im}(D^3\bar{F}) = \mathrm{Im}(D^2\bar{B}\bar{F}) = \mathrm{Im}(\bar{B}^2D\bar{F}) = \mathrm{Im}(B^3F) = \mathrm{Re}(\bar{A}^3\bar{F}^2) = 0$.

The proof of statements (a) and (b) of Theorem 3 are given in sections 2 and 3, respectively.

Theorem 4. *For* $d \geq 5$ *odd the following statement hold.*

(a) *System* (1) *satisfying conditions* (c.1) *has an isochronous center at the origin if one of the following conditions hold:*
- (a.1) $\lambda = C = E = F = (d-3)B + (d+1)\bar{D} = 0$,
- (a.2) $\lambda = C = E = F = D - \bar{B} = 0$,
- (a.3) $\lambda = C = D = E = 0$, $F = -\bar{B}^2/B$ *with* $B \neq 0$ *and* $d = 7$,
- (a.4) $\lambda = C = D = E = 0$, $B = -F$, $\mathrm{Im}(F) = 0$ *and* $d = 7$,
- (a.5) $\lambda = C = D = E = 0$, $B = e^{\pm i\pi/3}$, $\mathrm{Im}(F) = 0$ *and* $d = 7$.

(b) *System* (1) *satisfying conditions* (c.2) *has a center at the origin if one of the following conditions hold:*
- (b.1) $\lambda = C = A = F = (d-3)B + (d+1)\bar{D} = 0$,

(b.2) $\lambda = C = A = F = D - \bar{B} = 0$,
(b.3) $\lambda = C = D = A = 0$, $F = -\bar{B}^2/B$ *with* $B \neq 0$ *and* $d = 7$,
(b.4) $\lambda = C = D = A = 0$, $B = -F$, $\mathrm{Im}(F) = 0$ *and* $d = 7$,
(b.5) $\lambda = C = D = A = 0$, $B = e^{\pm i\pi/3}$, $\mathrm{Im}(F) = 0$ *and* $d = 7$,
(b.6) $\lambda = C = F = \mathrm{Im}(D^2 A) = 3D + \bar{B} = 0$ *and* $d = 5$,
(b.7) $\lambda = C = B = D = F = 0$ *and* $d = 5$.

The proof of statements (a) and (b) of Theorem 4 are given in sections 4 and 5, respectively.

2. Proof of Theorem 3 (a)

Proof of statement (a.1). The conditions (a.1) written in real parameters are $\lambda = a_5 = a_9 = a_{10} = 2a_4 - a_8 = 2a_3 + a_7 = 0$. In this case system (1) takes the form

$$\dot{z} = iz + (z\bar{z})^{(d-5)/2}(Bz^4\bar{z} - 2\bar{B}z^2\bar{z}^3 + F\bar{z}^5). \tag{9}$$

Now we rescale system (9) by $(z\bar{z})^{(d-5)/2} = |z|^{d-5}$ and system (9) becomes

$$\dot{z} = iz|z|^{5-d} + Bz^4\bar{z} - 2\bar{B}z^2\bar{z}^3 + F\bar{z}^5 = i\frac{\partial H}{\partial \bar{z}}, \tag{10}$$

where for $d \geq 5$ odd with $d \neq 7$ we have

$$H = \frac{2}{7-d}|z|^{7-d} - \frac{i}{2}Bz^4\bar{z}^2 + \frac{i}{2}\bar{B}z^2\bar{z}^4 - \frac{i}{6}F\bar{z}^6 + \frac{i}{6}\bar{F}z^6,$$

and for $d = 7$ we have

$$H = \log|z|^2 - \frac{i}{2}Bz^4\bar{z}^2 + \frac{i}{2}\bar{B}z^2\bar{z}^4 - \frac{i}{6}F\bar{z}^6 + \frac{i}{6}\bar{F}z^6.$$

Note that the first integrals $\exp(H)$ for $d = 7$, and H for $d \geq 5$ odd with $d \neq 7$, are real functions well defined at the origin. Therefore in this case the origin is a Hamiltonian center. □

Proof of statement (a.2). The conditions (a.2) expressed in real parameters are $\lambda = a_5 = a_{11} = a_{12} = 3a_4 - a_8 = 3a_3 + a_7 = 0$. System (1) can be written into the form

$$\begin{aligned} \dot{z} &= iz + (z\bar{z})^{(d-5)/2}(Bz^4\bar{z} - 3\bar{B}z^2\bar{z}^3 + Ez\bar{z}^4) \\ &= iz + (z\bar{z})^{(d-3)/2}(Bz^3 - 3\bar{B}z^2 + E\bar{z}^3). \end{aligned} \tag{11}$$

If we rescale system (11) by $|z|^{d-3}$ we get

$$\dot{z} = iz|z|^{3-d} + Bz^3 - 3\bar{B}z^2 + E\bar{z}^3 = i\frac{\partial H}{\partial \bar{z}},$$

where for $d = 5$

$$H = \log|z|^2 - i(Bz^3\bar{z} - \bar{B}z\bar{z}^3) - \frac{i}{4}(E\bar{z}^4 - \bar{E}z^4),$$

and for $d \geq 7$ odd we have

$$H = \frac{2}{5-d}|z|^{5-d} - i(Bz^3\bar{z} - \bar{B}z\bar{z}^3) - \frac{i}{4}(E\bar{z}^4 - \bar{E}z^4).$$

Note that the first integrals $\exp(H)$ for $d = 5$ and H for $d \geq 7$ odd are real functions well defined at the origin. Therefore the origin is a center. □

Proof of statement $(a.3)$. The conditions (a.3) written in real parameters are $\lambda = a_5 = (d-3)a_4 - (d+1)a_8 = (d-3)a_3 + (d+1)a_7 = a_9 = a_{10} = a_{11} = a_{12} = 0$. Since in conditions $(c.1)$ we have $A = 0$ and in conditions $(c.2)$ we have $E = 0$, it follows that case $(a.3)$ is a subcase of $(b.5)$. □

Proof of statement $(a.4)$. The conditions of this case expressed in the real parameters are

$$
\begin{aligned}
&p_1 = a_5 = 0,\\
&p_2 = a_4a_7 + a_3a_8 = 0,\\
&p_3 = a_7a_9a_{11} - a_8a_{10}a_{11} + a_8a_9a_{12} + a_7a_{10}a_{12} = 0,\\
&p_4 = a_3a_9a_{11} + a_4a_{10}a_{11} - a_4a_9a_{12} + a_3a_{10}a_{12} = 0,\\
&p_5 = a_7^2a_9 - a_8^2a_9 + 2a_7a_8a_{10} = 0,\\
&p_6 = a_3a_7a_9 + a_4a_8a_9 + 2a_3a_8a_{10} = 0,\\
&p_7 = a_3^2a_9 - a_4^2a_9 - 2a_3a_4a_{10} = 0,\\
&p_8 = a_8a_9^2a_{11} - 3a_8a_{10}^2a_{11} + a_7a_9^2a_{12} + 4a_8a_9a_{10}a_{12} + a_7a_{10}^2a_{12} = 0,\\
&p_9 = a_4a_9^2a_{11} - 3a_4a_{10}^2a_{11} - a_3a_9^2a_{12} + 4a_4a_9a_{10}a_{12} - a_3a_{10}^2a_{12} = 0,\\
&p_{10} = 3a_7^2a_8a_{11} - a_8^3a_{11} - a_7^3a_{12} + 3a_7a_8^2a_{12} = 0,\\
&p_{11} = 3a_3a_7a_8a_{11} + a_4a_8^2a_{11} - a_3a_7^2a_{12} + 3a_3a_8^2a_{12} = 0,\\
&p_{12} = 3a_3^2a_8a_{11} - a_4^2a_8a_{11} - a_3^2a_7a_{12} - 3a_3a_4a_8a_{12} = 0,\\
&p_{13} = 3a_3^2a_4a_{11} - a_4^3a_{11} + a_3^3a_{12} - 3a_3a_4^2a_{12} = 0,\\
&p_{14} = a_9^3a_{11}^2 - 3a_9a_{10}^2a_{11}^2 + 6a_9^2a_{10}a_{11}a_{12} - 2a_{10}^3a_{11}a_{12} - a_9^3a_{12}^2 + 3a_9a_{10}^2a_{12}^2 = 0.
\end{aligned}
$$

We now rewrite each of the conditions p_j for $j = 1, \ldots, 14$ in terms of complex parameters of system (1). We obtain that $p_1 = \mathrm{Re}(C)$, $p_2 = \mathrm{Im}(BD)$, $p_3 = \mathrm{Re}(DE\bar{F})$, $p_4 = \mathrm{Re}(\bar{B}E\bar{F})$, $p_5 = \mathrm{Re}(D^2\bar{E})$, $p_6 = \mathrm{Re}(B\bar{D}E)$, $p_7 = \mathrm{Re}(B^2E)$, where to obtain p_6 we have used the condition that $p_2 = 0$. Moreover, using that $p_3 = 0$ we get $p_8 = \mathrm{Im}(D\bar{E}^2F)$, and using that $p_4 = 0$ we get $p_9 = \mathrm{Im}(DE^2\bar{F})$. Furthermore, $p_{10} = \mathrm{Im}(D^3\bar{F})$, $p_{11} = \mathrm{Im}(D^2\bar{B}\bar{F})$, $p_{12} = \mathrm{Im}(\bar{B}^2D\bar{F})$, $p_{13} = \mathrm{Im}(B^3F)$ and $p_{14} = \mathrm{Re}(E^3\bar{F}^2)$, where in p_{11} we have used that $p_2 = 0$. In summary we have the conditions of statement (a.3). Therefore we have

$$
\begin{aligned}
&C = -\bar{C}, \quad \frac{\bar{B}}{B} = \frac{D}{\bar{D}}, \quad \frac{D}{\bar{D}} = -\Big(\frac{\bar{E}F}{E\bar{F}}\Big), \quad \Big(\frac{D}{\bar{D}}\Big)^2 = -\Big(\frac{E}{\bar{E}}\Big),\\
&\frac{D}{\bar{D}} = \frac{E^2\bar{F}}{\bar{E}^2F}, \quad \Big(\frac{D}{\bar{D}}\Big)^3 = \frac{F}{\bar{F}}, \quad \Big(\frac{E}{\bar{E}}\Big)^3 = -\Big(\frac{F}{\bar{F}}\Big)^2.
\end{aligned} \tag{12}
$$

Now let $\theta_1, \theta_2, \theta_3, \theta_4$ be such that

$$e^{i\theta_1} = -\frac{\bar{B}}{B}, \quad e^{i\theta_2} = -\frac{\bar{D}}{D}, \quad e^{i\theta_3} = -\frac{\bar{E}}{E}, \quad e^{i\theta_4} = -\frac{\bar{F}}{F}.$$

From conditions (12) we have that

$$\theta_1 = -\theta_2 (\mathrm{mod}.2\pi), \ \theta_3 = 2\theta_2 (\mathrm{mod}.2\pi), \ \theta_4 = 3\theta_2 (\mathrm{mod}.2\pi). \tag{13}$$

Now taking $\gamma = \theta_1/2$ and using (13) we obtain

$$e^{2i\gamma} = e^{i\theta_1} = -\frac{\bar{B}}{B}, \quad e^{-2i\gamma} = e^{-i\theta_1} = e^{i\theta_2} = -\frac{\bar{D}}{D},$$

and

$$e^{-4i\gamma} = e^{-2i\theta_1} = e^{2i\theta_2} = e^{\theta_3} = -\frac{\bar{E}}{E}, \quad e^{-6i\gamma} = e^{-3i\theta_1} = e^{3i\theta_2} = e^{\theta_4} = -\frac{\bar{F}}{F}.$$

Hence by Lemma 2 system (1) under the conditions of statement (a.3) is reversible and consequently it has a center at the origin. □

3. Proof of Theorem 3 (b)

Proof of statement (*b*.1). The conditions (b.1) written in real parameters are $\lambda = a_5 = a_4 - 3a_8 = a_3 + 3a_7 = a_{11} = a_{12} = 0$ for $d = 5$. System (1) under these conditions becomes

$$\begin{aligned} \dot{x} &= -y + a_1x^5 - 2a_7x^5 - 5a_2x^4y - 8a_8x^4y - 10a_1x^3y^2 + 8a_7x^3y^2 \\ &\quad + 10a_2x^2y^3 - 4a_8x^2y^3 + 5a1xy^4 + 10a_7xy^4 - a_2y^5 + 4a_8y^5, \\ \dot{y} &= x + a_2x^5 + 4a_8x^5 + 5a_1x^4y - 10a_7x^4y - 10a_2x^3y^2 - 4a_8x^3y^2 \\ &\quad - 10a_1x^2y^3 - 8a_7x^2y^3 + 5a_2xy^4 - 8a_8xy^4 + a_1y^5 + 2a_7y^5. \end{aligned} \tag{14}$$

System (14) has the invariant algebraic curve

$$\begin{aligned} f &= 1 + 2(a_2 + 4a_8)x^4 + 8(a_1 - 2a_7)x^3y - 12a_2x^2y^2 - 8(a_1 + 2a_7)xy^3 \\ &\quad + 2(a_2 - 4a_8)y^4, \end{aligned}$$

Moreover $V = f_1^{5/4}$ is an inverse integrating factor of system (14) which gives an analytic first integral in a neighborhood of the origin. □

Proof of statement (*b*.2). The conditions (b.2) expressed in real parameters are $\lambda = a_1 = a_2 = a_5 = (d-3)a_4 - (d+1)a_8 = (d-3)a_3 + (d+1)a_7 = a_{11} = a_{12} = 0$. Since in conditions (*c*.1) we have $A = 0$ and in conditions (*c*.2) we have $E = 0$, it is clear that case (*b*.2) is a subcase of (*a*.3). □

Proof of statement (*b*.3). The conditions (b.3) written in real parameters are $\lambda = a_5 = a_2 = a_1 = 2a_4 - a_8 = 2a_3 + a_7 = 0$. System (1) can be written into the form

$$\dot{z} = iz + (z\bar{z})^{(d-5)/2}(Bz^4\bar{z} - 2\bar{B}z^2\bar{z}^3 + F\bar{z}^5). \tag{15}$$

If we rescale system (15) by $|z|^{d-5}$ we get

$$\dot{z} = iz|z|^{5-d} + Bz^4\bar{z} - 2\bar{B}z^2\bar{z}^3 + F\bar{z}^5 = i\frac{\partial H}{\partial \bar{z}},$$

where for $d \geq 5$ odd with $d \neq 7$ we have

$$H = \frac{2}{7-d}|z|^{7-d} - \frac{i}{2}(Bz^4\bar{z}^2 - \bar{B}z^2\bar{z}^4) - \frac{i}{6}(F\bar{z}^6 - \bar{F}z^6),$$

and for $d = 7$ we have

$$H = \log|z|^2 - \frac{i}{2}(Bz^4\bar{z}^2 - \bar{B}z^2\bar{z}^4) - \frac{i}{6}(F\bar{z}^6 - \bar{F}z^6).$$

Note that the first integrals $\exp(H)$ for $d = 7$ and H for $d \geq 5$ odd with $d \neq 7$ are real functions well defined at the origin. Therefore the origin is a Hamiltonian center. □

Proof of statement (b.4). The conditions (b.4) in real parameters are $a_5 = a_4 = a_3 = 0$ and

$$
\begin{array}{ll}
s_1 = & 2a_8^2 + a_7a_{11} - a_{11}^2 + a_8a_{12} - a_{12}^2 = 0,\\
s_2 = & 2a_7a_8 + a_8a_{11} - a_7a_{12} = 0,\\
s_3 = & 2a_7^2 - a_7a_{11} - a_{11}^2 - a_8a_{12} - a_{12}^2 = 0,\\
s_4 = & a_2a_7 + a_1a_8 + a_2a_{11} + a_1a_{12} = 0,\\
s_5 = & a_1a_7 - a_2a_8 - a_1a_{11} + a_2a_{12} = 0,\\
s_6 = & 2a_1a_8a_{11} + a_2a_{11}^2 - 2a_2a_8a_{12} + a_2a_{12}^2 = 0,\\
s_7 = & a_1^2a_8 + a_2^2a_8 + 2a_1a_2a_{11} + a_1^2a_{12} - a_2^2a_{12} = 0,\\
s_8 = & 2a_2^2a_8a_{11} + 3a_1a_2a_{11}^2 + 2a_1a_2a_8a_{12} + 2a_1^2a_{11}a_{12}\\
 & -2a_2^2a_{11}a_{12} - a_1a_2a_{12}^2 = 0,\\
s_9 = & 3a_1^2a_2a_{11}^2 - a_2^3a_{11}^2 + 2a_1^3a_{11}a_{12} - 6a_1a_2^2a_{11}a_{12} - 3a_1^2a_2a_{12}^2\\
 & +a_2^3a_{12}^2 = 0.
\end{array}
\tag{16}
$$

If we sum the polynomials $s_1 + s_3$ we obtain $a_7^2 + a_8^2 - (a_{11}^2 + a_{12}^2) = 0$. Now we introduce the reparametrization $a_{11} = k_1 \sin t_1$, $a_{12} = k_1 \cos t_1$, $a_7 = k_2 \sin t_2$ and $a_8 = k_2 \cos t_2$. With this reparametrization $s_1 + s_3 = (k_2 - k_1)(k_2 + k_1)$.

In the case $k_1 = k_2$ the conditions s_1, s_2 and s_3 take the form

$$s_1 = -s_3 = k_2^2(\cos(t_1 - t_2) + \cos(2t_2)), \quad s_2 = k_2^2(\sin(t_1 - t_2) + \sin(2t_2)),$$

The case $k_2 = 0$ implies $a_7 = a_8 = a_{11} = a_{12} = 0$ which is a subcase of (b.1). The case $\sin(t_1-t_2)+\sin 2t_2 = 0$ and $\sin(t_1-t_2)+\sin(2t_2) = 0$ implies $t_1 = \pi+3t_2+2k\pi$ or $t_1 = 3t_2 - \pi + 2k\pi$ with $k \in \mathbb{Z}$. In both cases the conditions s_4 and s_5 become

$$
\begin{array}{ll}
s_4 & = 2k_2 \sin t_2(-a_2 \cos(2t_2) + a_1 \sin(2t_2)),\\
s_5 & = -2k_2 \cos t_2(a_2 \cos(2t_2) - a_1 \sin(2t_2)).
\end{array}
$$

The annulation of these conditions implies $a_1 = a_2 \cot(2t_2)$ and all the rest of conditions $s_6, \dots, s_9$ are zero. Undoing the parametrization, substituting $\cos t_2 \to a_8/k_2$ and $\sin t_2 \to a_7/k_2$ and $k_2 = \sqrt{a_7^2 + a_8^2}$, system (1) takes the form

$$
\begin{array}{ll}
\dot{x} = & -y + \dfrac{(a_7 - a_8)(a_7 + a_8)(-a_2a_7^2 + 4a_7^2a_8 - a_2a_8^2)x^5}{2a_7a_8(a_7^2 + a_8^2)}\\
& -\dfrac{(5a_2a_7^2 - 16a_7^2a_8 + 5a_2a_8^2 + 4a_8^3)x^4y}{a_7^2 + a_8^2}\\
& -\dfrac{(-5a_2a_7^4 + 8a_7^4a_8 - 32a_7^2a_8^3 + 5a_2a_8^4)x^3y^2}{a_7a_8(a_7^2 + a_8^2)}\\
& +\dfrac{2(5a_2a_7^2 - 14a_7^2a_8 + 5a_2a_8^2 + 6a_8^3)x^2y^3}{a_7^2 + a_8^2}\\
& +\dfrac{(-5a_2a_7^4 + 12a_7^4a_8 - 28a_7^2a_8^3 + 5a_2a_8^4)xy^4}{2a_7a_8(a_7^2 + a_8^2)}\\
& -\dfrac{(a_2a_7^2 - 4a_7^2a_8 + a_2a_8^2)y^5}{a_7^2 + a_8^2},
\end{array}
\tag{17}
$$

$$
\begin{aligned}
\dot y = \quad & x+\frac{(a_2a_7^2+4a_7^2a_8+a_2a_8^2)x^5}{a_7^2+a_8^2}\\
&-\frac{(5a_2a_7^4+12a_7^4a_8-28a_7^2a_8^3-5a_2a_8^4)x^4y}{2a_7a_8(a_7^2+a_8^2)}\\
&-\frac{2(5a_2a_7^2+14a_7^2a_8+5a_2a_8^2-6a_8^3)x^3y^2}{a_7^2+a_8^2}\\
&+\frac{(5a_2a_7^4+8a_7^4a_8-32a_7^2a_8^3-5a_2a_8^4)x^2y^3}{a_7a_8(a_7^2+a_8^2)}\\
&+\frac{(5a_2a_7^2+16a_7^2a_8+5a_2a_8^2-4a_8^3)xy^4}{a_7^2+a_8^2}\\
&-\frac{(a_7-a_8)(a_7+a_8)(a_2a_7^2+4a_7^2a_8+a_2a_8^2)y^5}{2a_7a_8(a_7^2+a_8^2)}.
\end{aligned}
$$

This system has the invariant algebraic curve

$$
f_1=1+2a_2x^4-\frac{4a_2(a_7^2-a_8^2)}{a_7a_8}x^3y-12a_2x^2y^2+\frac{4a_2(a_7^2-a_8^2)}{a_7a_8}xy^3+2a_2y^4,
$$

However we have not been able to find any additional invariant algebraic curve and only with the curve f_1 is not possible to find any integrating factor or a first integral for system (17). Hence we consider the initial differential equation (1) with $\lambda=0$

$$
\dot z=iz+(Az^5+Bz^4\bar z+Cz^3\bar z^2+Dz^2\bar z^3+Ez\bar z^4+F\bar z^5),
$$

and its associated complex conjugated equation

$$
\dot{\bar z}=-i\bar z+(\bar A\bar z^5+\bar B\bar z^4z+\bar C\bar z^3z^2+\bar D\bar z^2z^3+\bar E\bar zz^4+\bar Fz^5),
$$

Both equations define a differential system in $\mathbb{C}^2$ with a complex saddle at the origin that rewriting using the variables (x,y) and also applying the change of time $t\to it$ becomes

$$
\begin{aligned}
\dot x &= x-a_{40}x^5-a_{31}x^4y-a_{22}x^3y^2-a_{13}x^2y^3-a_{04}xy^4-a_{15}y^5,\\
\dot y &= -y+b_{51}x^5+b_{40}x^4y+b_{31}x^3y^2+b_{22}x^2y^3+b_{13}xy^4+b_{04}y^5,
\end{aligned}
$$

where $a_{40}=iA$, $a_{31}=iB$, $a_{22}=iC$, $a_{13}=iD$, $a_{04}=iE$, $a_{15}=iF$, and $b_{ij}=\bar a_{ji}$. Substituting these values in the system (17) we obtain

$$
\begin{aligned}
\dot x &= x+\frac{ia_2(a_7-ia_8)^2x^5}{2a_7a_8}+(a_8-ia_7)x^2y^3+\frac{(ia_7-a_8)^3y^5}{a_7^2+a_8^2},\\
\dot y &= -y+\frac{(a_7-ia_8)^2x^5}{ia_7-a_8}-(a_8+ia_7)x^3y^2+\frac{ia_2(a_7+ia_8)^2y^5}{2a_7a_8}.
\end{aligned}
\tag{18}
$$

System (18) has the invariant algebraic curve

$$
f_2=1+\frac{a_2((a_8+ia_7)x^2+(ia_7-a_8)y^2)((a_7-ia_8)x^2-(a_7-ia_8)y^2)}{2a_7a_8}
$$

which is the transformation of the curve f_1 to these new variables. Now we reparameterize system (18) of the form

$$\text{(19)}\qquad \begin{aligned} \dot{x} &= x - a_{40}x^5 + \frac{b_{31}^2 x^2 y^3}{b_{51}} + \frac{b_{31}^3 y^5}{b_{51}^2}, \\ \dot{y} &= -y + b_{51}x^5 + b_{31}x^3y^2 - \frac{a_{40}b_{31}^2 y^5}{b_{51}^2}. \end{aligned}$$

where $a_{40} = -ia_2(a_7 - ia_8)^2/(2a_7a_8)$, $b_{31} = -i(a_7 - ia_8)$, $b_{51} = (a_7 - ia_8)^2/(ia_7 - a_8)$, and the curve f_2 takes the simple form

$$f_2(x, y) = 1 - a_{40}x^4 + \frac{a_{40}b_{31}^2 y^4}{b_{51}^2}.$$

Let $b_{31} = r_1 e^{i\phi_1}$ and $b_{51} = r_2 e^{i\phi_2}$ where r_1 and r_2 are the moduli and ϕ_1 and ϕ_2 are the arguments, respectively.

We are going to prove that system (19) has an integrable saddle at the origin and consequently system (**??**) has a real center. Note that $f_2^{1/2}$ has two roots. We fix one of them and we call if $h_0(x, y)$. Now we consider the following change of variables given by

$$\text{(20)}\qquad u = u(x, y) = k_2 y/f_2^{\frac{1}{4}}(x, y), \qquad v = v = (x, y) = k_1 x/f_2^{\frac{1}{4}}(x, y),$$

where $k_1 = -i\sqrt{r_2/r_1}e^{i(\phi_2 - \phi_1)/2}$, $k_2 = i\sqrt{r_1/r_2}e^{i(\phi_1 - \phi_2)/2}$ and we take for both u and v the same fourth root of $f_2(x, y)$, that we call $h_1(x, y)$, such that $h_1^2(x, y) = h_0(x, y)$. The inverse change is given by

$$x = x(u, v) = \frac{\sqrt{r_2}e^{i\phi_2/2}k_2 v}{H_1(u, v)}, \quad y = y(u, v) = \frac{\sqrt{r_2}e^{i\phi_2/2}k_1 u}{H_1(u, v)},$$

where $H_1(u, v) = h_1(x(u, v), y(u, v))$ and

$$H_1^4 = b_{51}^2 k_2^4 (k_1^4 + a_{40}u^4) - a_{40}b_{31}^2 k_1^4 v^4.$$

Let

$$g_2(u, v) = f_2(x(u, v), y(u, v)) = \frac{r_2^2 e^{2i\phi_2}}{H_1^4} = \frac{r_2^2 e^{2i\phi_2}}{H_0^2},$$

where $H_0(u, v) = h_0(x(u, v), y(u, v))$. Then

$$xy = uv\frac{r_2 e^{i\phi_2}}{H_1^2} = uv\frac{r_2 e^{i\phi_2}}{H_0} = uvg_2(u, v)^{1/2}.$$

In these variables (u, v) system (19) becomes

$$\frac{du}{dt} = -\mathcal{X}(u, v)g_2(u, v), \qquad \frac{dv}{dt} = -\mathcal{Y}(u, v)g_2(u, v).$$

where $\mathcal{X}(u, v)$ and $\mathcal{Y}(u, v)$ is exactly the same system (19) in the variables (u, v) that is

$$\begin{aligned} \mathcal{X}(u, v) &= u - a_{40}u^5 + \frac{b_{31}^2 u^2 v^3}{b_{51}} + \frac{b_{31}^3 v^5}{b_{51}^2}, \\ \mathcal{Y}(u, v) &= -v + b_{51}u^5 + b_{31}u^3v^2 - \frac{a_{40}b_{31}^2 v^5}{b_{51}^2}. \end{aligned}$$

Suppose that the origin of system (19) is not integrable, then there exists a formal power series $F(x,y) = xy + h.o.t$, such that

$$\tag{21} \begin{aligned} \left.\frac{dF(x,y)}{dt}\right|_{(18)} &= \frac{\partial F(x,y)}{\partial x}\mathcal{X}(x,y) + \frac{\partial F(x,y)}{\partial y}\mathcal{Y}(x,y) \\ &= \lambda_m (xy)^{4m+1} + h.o.t., \end{aligned}$$

where h.o.t. indicates higher order terms, m is a positive integer and $\lambda_m \neq 0$ is the focus quantity. Using the inverse change of (20) we define the function G as

$$G(u,v) = F(x(u,v), y(u,v)) = uv + h.o.t.,$$

then we have that

$$dG(u,v)/dt = -g_2(u,v)\left[\frac{\partial G(u,v)}{\partial u}\mathcal{X}(u,v) + \frac{\partial G(u,v)}{\partial v}\mathcal{Y}(u,v)\right].$$

Doing the change of variables (20) the previous expression becomes

$$\tag{22} \begin{aligned} &- f_2(x,y)\left[\frac{\partial F(x,y)}{\partial x}\mathcal{X}(x,y) + \frac{\partial F(x,y)}{\partial y}\mathcal{Y}(x,y)\right] + h.o.t. \\ &= -(1 + h.o.t.)[\lambda_m (xy)^{4m+1} + h.o.t.] \\ &= -\lambda_m (xy)^{4m+1} + h.o.t.\big|_{(20)} = -\lambda_m (uv)^{4m+1} + h.o.t.. \end{aligned}$$

From (21) and (22) we have that $\lambda_m = 0$. This implies that system (19) is integrable and therefore it has a complex center at the origin.

In the case $k_1 = -k_2$ the conditions s_1, s_2 and s_3 take the form

$$s_1 = -s_3 = k_2^2(-\cos(t_1 - t_2) + \cos(2t_2)), \quad s_2 = k_2^2(-\sin(t_1 - t_2) + \sin(2t_2)).$$

From $-\sin(t_1 - t_2) + \sin(2t_2) = 0$ we obtain $t_1 = \pi - t_2$ and substituting this value we have $s_1 = -s_3 = k_2^2 \cos(2t_2)$ which implies $t_2 = \pi/4 + k\pi$ with $k \in \mathbb{Z}$. The annulation of rest of conditions $s_4, \ldots, s_9$ implies $a_1 = 0$. Hence is a particular case of previous one with $t_2 = \pi/4 + k\pi$ with $k \in \mathbb{Z}$, i.e. we get system (17) with $a_7^2 = a_8^2$. □

Proof of statement (b.5). The conditions (b.5) expressed in real parameters are

$$\begin{aligned}
q_1 &= a_5 = 0,\\
q_2 &= a_4a_7 + a_3a_8 = 0,\\
q_3 &= a_1a_7a_{11} + a_2a_8a_11 - a_2a_7a_{12} + a_1a_8a_{12} = 0,\\
q_4 &= a_1a_3a_{11} - a_2a_4a_{11} - a_2a_3a_{12} - a_1a_4a_{12} = 0,\\
q_5 &= a_1a_7^2 - 2a_2a_7a_8 - a_1a_8^2 = 0,\\
q_6 &= a_1a_3a_7 - 2a_2a_3a_8 + a_1a_4a_8 = 0,\\
q_7 &= a_1a_3^2 + 2a_2a_3a_4 - a_1a_4^2 = 0,\\
q_8 &= 3a_7^2a_8a_{11} - a_8^3a_{11} - a_7^3a_{12} + 3a_7a_8^2a_{12} = 0,\\
q_9 &= 3a_3a_7a_8a_{11} + a_4a_8^2a_{11} - a_3a_7^2a_{12} + 3a_3a_8^2a_{12} = 0,\\
q_{10} &= 3a_3^2a_8a_{11} - a_4^2a_8a_{11} - a_3^2a_7a_{12} - 3a_3a_4a_8a_{12} = 0,\\
q_{11} &= a_1^2a_8a_11 - 3a_2^2a_8a_{11} + a_1^2a_7a_{12} + a_2^2a_7a_{12} - 4a_1a_2a_8a_{12} = 0,\\
q_{12} &= 3a_3^2a_4a_{11} - a_4^3a_{11} + a_3^3 * a_{12} - 3a_3a_4^2a_{12} = 0,\\
q_{13} &= a_1^2a_4a_{11} - 3a_2^2a_4a_{11} - a_1^2a_3a_{12} - a_2^2a_3a_{12} - 4a_1a_2a_4a_{12} = 0,\\
q_{14} &= a_1^3a_{11}^2 - 3a_1a_2^2a_{11}^2 - 6a_1^2a_2a_{11}a_{12} + 2a_2^3a_{11}a_{12} - a_1^3a_{12}^2 + 3a_1a_2^2a_{12}^2 = 0.
\end{aligned}$$

We now rewrite each of the conditions q_j for $j = 1, \dots, 14$ in terms of complex parameters of system (1). We obtain that $q_1 = \mathrm{Re}(C)$, $q_2 = \mathrm{Im}(BD)$, $q_3 = \mathrm{Re}(D\bar{A}\bar{F})$, $q_4 = \mathrm{Re}(\bar{B}\bar{A}\bar{F})$, $q_5 = \mathrm{Re}(D^2A)$, $q_6 = \mathrm{Re}(B\bar{D}\bar{A})$, $q_7 = \mathrm{Re}(B^2\bar{A})$, where to obtain q_6 we have used the condition that $q_2 = 0$. Moreover we have that $q_8 = \mathrm{Im}(D^3\bar{F})$ and $q_9 = \mathrm{Im}(D^2\bar{B}\bar{F})$. Furthermore $q_{10} = \mathrm{Im}(\bar{B}^2D\bar{F})$. Now using $q_3 = 0$ we get $q_{11} = \mathrm{Im}(DA^2F)$ and $q_{12} = \mathrm{Im}(B^3F)$. Finally using $q_4 = 0$ we get $q_{13} = \mathrm{Im}(D\bar{A}^2\bar{F})$, and using $q_2 = 0$ we have $q_{14} = \mathrm{Re}(\bar{A}^3\bar{F}^2)$. In summary, we have the conditions of statement (b.5). Therefore we have

$$\begin{aligned}
&C = -\bar{C}, \quad \frac{\bar{B}}{B} = \frac{D}{\bar{D}}, \quad \frac{D}{\bar{D}} = -\Big(\frac{AF}{\bar{A}\bar{F}}\Big), \quad \Big(\frac{D}{\bar{D}}\Big)^2 = -\Big(\frac{\bar{A}}{A}\Big),\\
&\frac{D}{\bar{D}} = \frac{\bar{A}^2\bar{F}}{A^2F}, \quad \Big(\frac{D}{\bar{D}}\Big)^3 = \frac{F}{\bar{F}}, \quad \Big(\frac{\bar{A}}{A}\Big)^3 = -\Big(\frac{F}{\bar{F}}\Big)^2.
\end{aligned} \tag{23}$$

Now let $\theta_1, \theta_2, \theta_3, \theta_4$ be such that

$$e^{i\theta_1} = -\frac{\bar{B}}{B}, \quad e^{i\theta_2} = -\frac{\bar{D}}{D}, \quad e^{i\theta_3} = -\frac{A}{\bar{A}}, \quad e^{i\theta_4} = -\frac{\bar{F}}{F}.$$

From conditions (23) we have that

$$\theta_1 = -\theta_2(\mathrm{mod}(2\pi)), \ \theta_3 = 2\theta_2(\mathrm{mod}(2\pi)), \ \theta_4 = 3\theta_2(\mathrm{mod}(2\pi)). \tag{24}$$

Now taking $\gamma = \theta_1/2$ and using (24) we have

$$e^{2i\gamma} = e^{i\theta_1} = -\frac{\bar{B}}{B}, \quad e^{-2i\gamma} = e^{-i\theta_1} = e^{i\theta_2} = -\frac{\bar{D}}{D}$$

and

$$e^{-4i\gamma} = e^{-2i\theta_1} = e^{2i\theta_2} = e^{\theta_3} = -\frac{A}{\bar{A}}, \quad e^{-6i\gamma} = e^{-3i\theta_1} = e^{3i\theta_2} = e^{\theta_4} = -\frac{\bar{F}}{F}.$$

Hence by Lemma 2 system (1) under the conditions of statement (b.5) is reversible and consequently has a center at the origin. □

4. Proof of Theorem 4 (a)

Proof of statement (a.1). The conditions (a.1) written in real parameters are $\lambda = a_5 = a_9 = a_{10} = a_{11} = a_{12} = (d-3)a_4 - (d+1)a_8 = (d-3)a_3 + (d+1)a_7 = 0$. Then system (1) can be written as

$$\dot{z} = iz + (z\bar{z})^{(d-5)/2}\Big(Bz^4\bar{z} - \frac{d-3}{d+1}\bar{B}z^2\bar{z}^3\Big). \tag{25}$$

Doing the change of variables

$$w = \xi z, \text{ where } \xi = \Big(\frac{B^{d+1}}{\bar{B}^{d-3}}\Big)^{1/(4(d-1)}, \tag{26}$$

system (25) becomes

$$\dot{w} = iw + (w\bar{w})^{(d-5)/2}\Big(w^4\bar{w} - \frac{d-3}{d+1}w^2\bar{w}^3\Big).$$

In polar coordinates this differential equation writes

$$\dot{r} = \frac{4r^d}{d+1}\cos(2\theta), \qquad \dot{\theta} = 1 + \frac{2(d-1)}{d+1}r^{d-1}\sin(2\theta).$$

Therefore we have the following associated equation for the trajectories

$$\frac{dr}{d\theta} = \frac{4r^d\cos(2\theta)}{d+1+2(d-1)r^{d-1}\sin(2\theta)} \quad \text{with} \quad r(0) = r_0.$$

Now integrating it and taking into account that $r(\theta) \geq 0$ for any θ we get

$$r(\theta)^{1-d} = \frac{-2(d-1)\sin(2\theta) + \sqrt{(d+1)^2 r_0^{2-2d} + 4(d-1)^2\sin^2(2\theta)}}{d+1}. \tag{27}$$

Note that

$$\sqrt{(d+1)^2 r_0^{2-2d} + 4(d-1)^2\sin^2(2\theta)} > |2(d-1)\sin(2\theta)|,$$

and thus $r(\theta)$ given in (27) is positive. Therefore introducing (27) into (7) we have that

$$\int_0^{2\pi}\frac{d\theta}{\dot{\theta}} = \int_0^{2\pi}\left(1 - \frac{2(d-1)\sin(2\theta)}{\sqrt{4(d-1)^2\sin^2(2\theta) + (d+1)^2 r_0^{2-2d}}}\right)d\theta = 2\pi,$$

because the function $2(d-1)\sin(2\theta)/\sqrt{4(d-1)^2\sin^2(2\theta) + (d+1)^2 r_0^{2-2d}}$ is odd in θ. □

Proof of statement (a.2). The conditions (a.2) expressed in real parameters are $\lambda = a_5 = a_9 = a_{10} = a_{11} = a_{12} = a_3 - a_7 = a_4 + a_8 = 0$. Then system (1) becomes

$$\dot{z} = iz + (z\bar{z})^{(d-5)/2}\Big(Bz^4\bar{z} + \bar{B}z^2\bar{z}^3\Big). \tag{28}$$

We do the change of variables (26), and in these new variables system (25) becomes

$$\dot{w} = iw + (w\bar{w})^{(d-5)/2}\Big(w^4\bar{w} + w^2\bar{w}^3\Big).$$

This system in polar coordinates writes

$$\dot{r} = 2r^d\cos 2\theta, \qquad \dot{\theta} = 1,$$

which is clearly isochronous. □

Proof of statement (a.3). The conditions (a.3) written in real parameters are $\lambda = a_5 = a_7 = a_8 = a_9 = 0$, $a_{11} = -(a_3^3 - 3a_3a_4^2)/(a_3^2 + a_4^2)$, $a_{12} = (3a_3^2a_4 - a_4^3)/(a_3^2 + a_4^2)$ and $d = 7$. Therefore system (1) becomes

$$\dot{z} = iz + (z\bar{z})\bar{z}(z^4 - \bar{z}^4). \tag{29}$$

In polar coordinates equation (29) has the form

$$\dot{r} = r^7(\cos(2\theta) - \cos(6\theta)), \quad \dot{\theta} = 1 + r^6(\sin(2\theta) + \sin(6\theta)). \tag{30}$$

Note that system (30) has the invariant (in the sense that dI/dt is zero on the solutions of system (30))

$$I = I(t, r, \theta) = 6\theta + 4r^6 \cos^3(2\theta) - 6t.$$

Using this invariant we can express the time in function of the variables (r, θ, I) as follows

$$t = \theta + \frac{2}{3}r^6 \cos^3(2\theta) - \frac{I}{6}.$$

Let $r(\theta)$ be a solution of system (30) sufficiently close to the origin. Since the origin is a center, we get that $r(2\pi) = r(0)$ in a certain neighborhood of the origin. Thus

$$T = t(2\pi) - t(0) = \left[2\pi + \frac{2}{3}r(2\pi)^6 - \frac{I}{6}\right] - \left[\frac{2}{3}r(0)^6 - \frac{I}{6}\right] = 2\pi.$$

See [6] where as far as we know for first time it was used the invariants to determine isochronicity. □

Proof of statement (a.4). The conditions (a.4) expressed in real parameters are $\lambda = a_4 = a_5 = a_7 = a_8 = a_9 = a_{10} = a_{12} = 0$, $a_3 = -a_{11}$ and $d = 7$. So system (1) becomes

$$\dot{z} = iz + (z\bar{z})(-Fz^4\bar{z} + F\bar{z}^5), \quad \text{with } F = \bar{F}. \tag{31}$$

Doing the change of variables

$$w = \xi z, \text{ where } \xi = \left(\frac{1}{F}\right)^{1/6}, \tag{32}$$

system (33) becomes

$$\dot{w} = iw + (w\bar{w})\bar{w}(-w^4 + \bar{w}^4).$$

In polar coordinates it has the form

$$\dot{r} = r^7(-\cos(2\theta) + \cos(6\theta)), \qquad \dot{\theta} = 1 - r^6(\sin(2\theta) + \sin(6\theta)).$$

Note that this system has an invariant of the form

$$I = I(t, r, \theta) = -6\theta - 4r^6 \cos^3(2\theta) + 6t.$$

Using this invariant we can prove the isochronicity in this case as in the previous one. □

Proof of statement (a.5). The conditions (a.5) written in real parameters are $\lambda = a_5 = a_7 = a_8 = a_9 = a_{10} = a_{12} = 0$, $a_3 = a_{11}/2$, $a_4 = \pm\sqrt{3}a_{11}/2$ and $d = 7$. Then system (1) becomes

$$\dot{z} = iz + (z\bar{z})(e^{\pm i\pi/3}Fz^4\bar{z} + F\bar{z}^5), \quad \text{with } F = \bar{F}. \tag{33}$$

Doing the change of variables (32) system (33) becomes

$$\dot{w} = iw + (w\bar{w})\bar{w}\Big(e^{\pm i\pi/3}w^4 + \bar{w}^4\Big).$$

In polar coordinates it writes

$$\dot r = r^7\Big(\frac{1}{2}\cos(2\theta)+\cos(6\theta)\mp\frac{\sqrt{3}}{2}\sin(2\theta)\Big),$$
$$\dot\theta = 1+r^6\Big(\pm\frac{\sqrt{3}}{2}\cos(2\theta)+\frac{1}{2}\sin(2\theta)-\sin(6\theta)\Big).$$

Note that this system has the invariant

$$I = I(t,r,\theta) = -6\theta + 4r^6\cos^3(2(\pm\pi/3-\theta))+6t.$$

Using this invariant we can prove the isochronicity as before. □

5. Proof of Theorem 4 (b)

Since in conditions (c.1) we have $A=0$ and in conditions (c.2) we have $E=0$ it follows that the conditions (b.1), (b.2), (b.3), (b.4) and (b.5) are the same than (a.1), (a.2), (a.3), (a.4) and (a.5), respectively.

Proof of statement (b.6). The conditions (b.6) expressed in real parameters are $\lambda = a_5 = a_{11} = a_{12} = a_4 - 3a_8 = a_3 + 3a_7 = 0$, $a_2a_7^2 + 2a_1a_7a_8 - a_2a_9^2 = 0$ and $d=5$. Hence system (1) becomes

$$\dot z = iz + Az^5 + Bz^4\bar z - \frac{1}{3}\bar B z^2\bar z^3, \quad \text{with } \operatorname{Im}(D^2A)=0. \tag{34}$$

Doing the change of variables (26) with $d=5$ system (34) becomes

$$\dot w = iw + a_1w^5 + w^4\bar w - \frac{1}{3}w^2\bar w^3, \tag{35}$$

This system in polar coordinates takes the form

$$\begin{aligned}\dot r &= \frac{r^5}{3}(2\cos(2\theta)+3a_1\cos(4\theta)),\\ \dot\theta &= 1+\frac{4}{3}r^4\sin(2\theta)+a_1r^4\sin(4\theta).\end{aligned} \tag{36}$$

Following [7] system (41) has the first integral

$$H = \frac{r^8}{1+2r^4Q(\theta)}, \quad \text{where} \quad Q(\theta) = \frac{4}{3}\sin(2\theta)+a_1\sin(4\theta).$$

From $H(r,\theta)=K$, being K an arbitrary real constant different from zero, we can express

$$r^4 = \frac{Q(\theta)\pm\sqrt{Q(\theta)^2+K}}{K}. \tag{37}$$

From the differential equation $\dot\theta = 1 + r^4Q(\theta)$ and using (37) we get

$$T = \int_0^{2\pi}\Big(1\pm\frac{Q(\theta)}{\sqrt{Q(\theta)+K}}\Big)d\theta = 2\pi \pm \int_0^{2\pi}\frac{Q(\theta)}{\sqrt{Q(\theta)+K}}d\theta = 2\pi,$$

where to see that the last integral is zero we do the change of variables $\varphi = -\theta$ and we use the periodicity of $Q(\theta)$. □

Proof of statement (*b*.7). The conditions (b.7) expressed in real parameters are $\lambda = a_3 = a_4 = a_5 = a_7 = a_8 = a_{11} = a_{12} = 0$ and $d = 5$. Therefore system (1) becomes

$$\dot{z} = iz + Az^5. \tag{38}$$

Doing the change of variables

$$w = \xi z, \text{ where } \xi = \left(\frac{i}{A}\right)^{1/4}, \tag{39}$$

system (38) becomes

$$\dot{w} = iw + iw^5. \tag{40}$$

This system in polar coordinates takes the form

$$\begin{aligned} \dot{r} &= -r^5 \sin(4\theta), \\ \dot{\theta} &= 1 + r^4 \cos(4\theta). \end{aligned} \tag{41}$$

and in cartesian coordinates becomes

$$\begin{aligned} \dot{x} &= -y + y(-5x^4 + 10x^2y^2 - y^4), \\ \dot{y} &= x + x(x^4 - 10x^2y^2 + 5y^4). \end{aligned}$$

This system has the transversal commuting system

$$\begin{aligned} \dot{x} &= x + x(x^4 - 10x^2y^2 + 5y^4), \\ \dot{y} &= y + y(5x^4 - 10x^2y^2 + y^4). \end{aligned}$$

Applying Theorem 1 (see also [7]), system (41) has an isochronous center at the origin. □

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References

[1] A.A. Andronov, E.A. Leontovich, I.I. Gordon, A.G. Maier, *Qualitative theory of second-order dynamic systems.* John Wiley & Sons, New York-Toronto, Israel Program for Scientific Translations, Jerusalem-London, 1973.

[2] N.N. Bautin, *On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type*, Mat. Sb. **30 (72)** (1952), 181–196; Amer. Math. Soc. Transl. **100** (1954), 397–413.

[3] J. Chavarriga, H. Giacomini, J. Giné, J. Llibre, *On the integrability of two–dimensional flows*, J. Differential Equations **157** (1999), 163–182.

[4] J. Chavarriga, J. Giné, *Integrability of a linear center perturbed by a fourth degree homogeneous polynomial*, Publ. Mat. **40** (1996), 21–39.

[5] J. Chavarriga, J. Giné, *Integrability of a linear center perturbed by a fifth degree homogeneous polynomial*, Publ. Mat. **41** (1997), 335–356.

[6] J. Chavarriga, J. Giné, I.A. García, *Isochronous centers of a linear center perturbed by fourth degree homogeneous polynomial*, Bull. Sci. Math. 123 (1999), no. 2, 77–96.

[7] J. Chavarriga, J. Giné, I.A. García, *Isochronous centers of a linear center perturbed by fifth degree homogeneous polynomials*, J. Comput. Appl. Math. 126 (2000), no. 1-2, 351–368.

[8] L.A. Cherkas, *The number of limit cycles of a certain second order autonumous system* (Russian), Differencial'nye Uravnenija **12** (1976), no. 5, 944–946.
[9] A. Cima, A. Gasull, V. Mañosa, F. Mañosas, *Algebraic properties of the Liapunov and period constants*, Rocky Mountain J. Math. **27** (1997), 471–501.
[10] W. Decker, S.Laplagne, G. Pfister, and H.A. Schonemann, SINGULAR *3-1 library for computing the prime decomposition and radical of ideals, primdec.lib*, 2010.
[11] P. Gianni, B. Trager, and G. Zacharias, *Gröbner bases and primary decompositions of polynomials*, J. Symbolic Comput. **6** (1988), 146–167.
[12] G.M. Greuel, G. Pfister, and H.A. Schönemann, `SINGULAR 3.0` *A Computer Algebra System for Polynomial Computations*, Centre for Computer Algebra, University of Kaiserlautern (2005). `http://www.singular.uni-kl.de.`
[13] J. Giné, *The center problem for a linear center perturbed by homogeneous polynomials*, Acta Math. Sin. (Engl. Ser.) **22** (2006), no. 6, 1613–1620.
[14] J. Giné, *On the centers of planar analytic differential systems*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **17** (2007), 3061–3070.
[15] J. Giné, *On the first integrals in the center problem*, Bull. Sci. Math. **137** (2013), no. 4, 457–465.
[16] J. Giné, X. Santallusia, *On the Poincaré-Lyapunov constants and the Poincaré series*, Appl. Math. (Warsaw) 28 (2001), no. 1, 1730.
[17] J. Giné, X. Santallusia, *Implementation of a new algorithm of computation of the Poincaré-Liapunov constants*, J. Comput. Appl. Math. **166** (2004), 465–476.
[18] J. Llibre, C. Valls, *Classification of the centers, of their cyclicity and isochronicity for two classes of generalized quintic polynomial differential systems*, NoDEA Nonlinear Differential Equations Appl. **16** (2009), no. 5, 657–679.
[19] J. Llibre, C. Valls, *Centers for polynomial vector fields of arbitrary degree*, Com. Pure and Applied Analysis **8** (2009), 725–742.
[20] J. Llibre, C. Valls, *Classification of the centers, their cyclicity and isochronicity for a class of polynomial differential systems generalizing the linear systems with cubic homogeneous nonlinearities*, J. Differential Equations **246** (2009), 2192–2204.
[21] J. Llibre, C. Valls, *Classification of the centers and their cyclicity for the generalized quadratic polynomial differential systems*, J. Math. Anal. and Appl. **357** (2009), 427–437.
[22] J. Llibre, C. Valls, *Classification of the centers and of the isochronous centers for a class of quartic–like systems*, Nonlinear Analysis: Theory and Methods and Applications **71** (2009), 3119–3128.
[23] J. Llibre, C. Valls, *Centers and isochronous centers for two classes of generalized seventh and ninth systems*, J. Dynamics and Differential Equations **22** (2010), 657–675.
[24] J. Llibre, C. Valls, *Classification of the centers and their isochronicity for a class of polynomial differential systems of arbitrary degree*, Advances in Mathematics **227** (2011), 472–493.
[25] V.A. Lunkevich, K.S. Sibirskii, *Integrals of a system with homogeneous nonlinearities of the third degree in cases of a center*, (Russian) Differentsial'nye Uravneniya **20** (1984), no. 8, 1360–1365.
[26] M.A. Lyapunov, *Problème général de la stabilité du mouvement*, Ann. of Math. Stud. **17**, Princeton University Press, 1947.
[27] H. Poincaré, *Sur l'intégration des équations différentielles du premier ordre et du premier degré I and II*, Rendiconti del circolo matematico di Palermo **5** (1891), 161–191; **11** (1897), 193–239.
[28] V.G. Romanovski, Xingwu Chen, Zhaoping Hu, *Linearizability of linear systems perturbed by fifth degree homogeneous polynomials*, J. Phys. A, Math. Theor. **40** (2007), no. 22, 5905–5919.
[29] V.G. Romanovski, M. Prešern, *An approach to solving systems of polynomials via modular arithmetics with applications*, J. Comput. Appl. Math. **236** (2011), no. 2, 196–208.
[30] D. Schlomiuk, *Algebraic and geometric aspects of the theory of polynomial vector fields*, in: Bifurcations and Periodic Orbits of Vector Fields, Montreal, PQ, 1992, in: NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 408, Kluwer Acad. Publ., Dordrecht, 1993, pp. 429–467.
[31] D. Schlomiuk, *Algebraic particular integrals, integrability and the problem of the center*, Trans. Amer. Math. Soc. 338 (1993), 799–841.

[32] K.S. Sibirskii, *On the number of limit cycles in the neighborhood of a singular point* (Russian) Differencial'nye Uravnenija **1** (1965) 5366; Differential Equations **1** (1965), 36-47.
[33] M. Villarini, *Regularity properties of the period function near a center of a planar vector field*, Nonlinear Anal. **19** (1992), no. 8, 787–803.
[34] N.I. Vulpe, K.S. Sibirskii, *Centro-affine invariant conditions for the existence of a center of a differential system with cubic nonlinearities* (Russian) Dokl. Akad. Nauk SSSR **301** (1988), no. 6, 1297–1301; translation in Soviet Math. Dokl. **38** (1989), no. 1, 198–201.
[35] H. Żołądek, *The classification of reversible cubic systems with center*, Topol. Methods Nonlinear Anal. **4** (1994), 79–136.
[36] H. Żołądek, *Remarks on: The classification of reversible cubic systems with center [Topol. Methods Nonlinear Anal.* **4** *(1994), 79–136]*, Topol. Methods Nonlinear Anal. **8** (1996), 335–342.

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