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POLYNOMIAL DIFFERENTIAL SYSTEMS IN $\mathbb{R}^3$ HAVING AN INVARIANT QUADRIC AND A DARBOUX INVARIANT

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Abstract. We give the normal forms of all polynomial differential systems in $\mathbb{R}^3$ which have a non–degenerate or degenerate quadric as an invariant algebraic surface. We also characterize among these systems those which have a Darboux invariant constructed uniquely using the invariant quadric, giving explicitly their expressions. As an example we apply the obtained results in the determination of the Darboux invariants for the Chen system with an invariant quadric.

1. Introduction and statement of the main results

Let $\mathbb{K}[x, y, z]$ be the ring of the polynomials in the variables x, y and z with coefficients in $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. We consider the polynomial differential system in $\mathbb{R}^3$ defined by

$$\dot{x} = P(x, y, z), \qquad \dot{y} = Q(x, y, z), \qquad \dot{z} = R(x, y, z), \tag{1}$$

where $P, Q, R \in \mathbb{R}[x, y, z]$ are relatively prime polynomials. The dot denotes derivative with respect to the independent variable t usually called the *time*. In what follows we shall write system (1) simply as $\dot{x} = P$, $\dot{y} = Q$, $\dot{z} = R$.

We say that $m = \max\{\deg P, \deg Q, \deg R\}$ is the *degree* of system (1) and we can naturally associate to this system the vector field

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}.$$

Systems like (1) appear frequently in the literature due to their theoretical importance as well as to their use in applied mathematics, since polynomial systems are usually used to model natural phenomena, arising in Physics, Biology, Chemistry and other sciences, see for instance [23, 24] and references therein. In this way many books and hundreds of papers have been published aiming to describe the dynamics of the solutions of system (1). However this dynamics is far from being completely understood, even in the quadratic case, i.e. when system (1) has degree $m = 2$. Indeed the dynamics generated by the flow of system (1) with degree $m \geq 2$ is, in general, very complex and difficult to be studied. Beyond singular points, periodic, homoclinic and heteroclinic orbits, which is commonly encountered in the phase space of planar polynomial vector fields, 3–dimensional systems

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like (1) may present more complex dynamical behavior, as the occurrence of invariant tori, quasiperiodic orbits, strange or chaotic attractors and several other phenomena [7, 22, 25]. One of the tools used to study the dynamics of system (1) is the determination of 2–dimensional surfaces embedded in $\mathbb{R}^3$ which are invariant under the flow of this system. When system (1) has an invariant algebraic surface, this helps strongly to understand the dynamics of the system, see for instance [10, 11, 12, 13, 14].

In this paper we give the normal forms of all polynomial differential systems in $\mathbb{R}^3$ which have a non–degenerate or a degenerate quadric as an invariant algebraic surface. We also characterize among these systems those which have a Darboux invariant constructed uniquely using the invariant quadric, giving explicitly their expressions. The precise definition of invariant algebraic surface and Darboux invariant as well as the statement of the main results obtained are given ahead. In order to do this we need some definitions of the Darboux theory of integrability. This theory for planar polynomial vector fields can be found in Chapter 2 of [4] and in Chapter 8 of [6]. It can be extended in a natural way to polynomial vector fields in $\mathbb{R}^n$ or $\mathbb{C}^n$, see for instance [8, 19, 20, 21].

1.1. **Preliminaries.** Let U be an open subset of $\mathbb{R}^3$. If there exists a non–locally constant analytic function $H : U \to \mathbb{R}$, which is constant on all solution curves $(x(t), y(t), z(t))$ of system (1) contained in U, then H is called a *first integral* of X in U. Clearly H is a first integral of system (1) if and only if $X(H) \equiv 0$ on U, i.e.

$$X(H) = \frac{\partial H}{\partial x}P + \frac{\partial H}{\partial y}Q + \frac{\partial H}{\partial z}R = \frac{dH}{dt} = 0 \quad \text{on the orbits of } X \text{ contained in } U.$$

An *invariant* of system (1) on the open subset $U \subset \mathbb{R}^3$ is a non–locally constant analytic function I in the variables x, y, z and t such that I is constant on all solution curves $(x(t), y(t), z(t))$ of system (1) contained in U, i.e.

$$\frac{dI}{dt} = \frac{\partial I}{\partial x}P + \frac{\partial I}{\partial y}Q + \frac{\partial I}{\partial z}R + \frac{\partial I}{\partial t} = 0 \quad \text{on the orbits of } X \text{ contained in } U.$$

In some sense an invariant I is a first integral of system (1) which depends on the time t.

Let $f \in \mathbb{C}[x, y, z] \setminus \mathbb{R}$. The surface $f(x, y, z) = 0$ is an *invariant algebraic surface* of system (1) if for some polynomial $K \in \mathbb{C}[x, y, z]$ we have

$$X(f) = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} + R\frac{\partial f}{\partial z} = Kf.$$

The polynomial K is called the *cofactor* of the invariant algebraic surface $f = 0$ and if m is the degree of the polynomial vector field (1), then the degree of K is at most $m - 1$. Note that when $K = 0$, then f is a polynomial first integral. We remark that in the definition of invariant algebraic surface $f = 0$ we always allow that this surface to be complex, even in the case that the polynomial differential system is real. This is due to the fact that sometimes for real polynomial systems the existence of a real first integral can be forced by the existence of complex invariant algebraic surfaces, for more details see Chapter 8 of [6].

Let $g, h \in \mathbb{C}[x, y, z] \setminus \{0\}$ and assume that g and h are relatively prime in the ring $\mathbb{C}[x, y, z]$, or that $h = 1$. Then the function $F = \exp(g/h)$ is called an *exponential factor* of system (1) if for some polynomial $L \in \mathbb{C}[x, y, z]$ of degree at most $m - 1$ we have that

$$X(F) = P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} + R\frac{\partial F}{\partial z} = LF.$$

We say that an invariant I of X is of *Darboux type*, or simply a *Darboux invariant* if it can be written as

$$I(x, y, z, t) = f_1^{\lambda_1} ... f_p^{\lambda_p} F_1^{\mu_1} ... F_q^{\mu_q} e^{st}, \tag{2}$$

where $f_i = 0$ are invariant algebraic surfaces of X for $i = 1, ..., p$; F_j are exponential factors of X for $j = 1, ..., q$; $\lambda_i, \mu_j \in \mathbb{C}$ and $s \in \mathbb{R} \setminus \{0\}$.

Let $\varphi_p(t)$ be the solution of system (1) passing through the point $p \in \mathbb{R}^3$, defined on its maximal interval (α_p, ω_p) such that $\varphi_p(0) = p$. If $\omega_p = \infty$ we define the set

$$\omega(p) = \{q \in \mathbb{R}^3 : \text{there exist } \{t_n\} \text{ with } t_n \to \infty \text{ and } \varphi_p(t_n) \to q \text{ when } n \to \infty\}.$$

In the same way, if $\alpha_p = -\infty$ we define the set

$$\alpha(p) = \{q \in \mathbb{R}^3 : \text{there exist } \{t_n\} \text{ with } t_n \to -\infty \text{ and } \varphi_p(t_n) \to q \text{ when } n \to \infty\}.$$

The sets $\omega(p)$ and $\alpha(p)$ are called the *ω–limit set* and the *α–limit set* of p, respectively.

While the knowledge of a first integral of the differential system (1) in $\mathbb{R}^3$ allows to reduce the study of this system in one dimension, the knowledge of a Darboux invariant provides information about the ω– and α–limit sets of all orbits of system (1). More precisely, there is the following result proved in [15] for planar polynomial differential systems, but it can be easily extended to $\mathbb{R}^n$.

Proposition 1. *Let $I(x, y, z, t) = f(x, y, z)e^{st}$ be a Darboux invariant of system (1). Let $p \in \mathbb{R}^3$ and $\varphi_p(t)$ be the solution of system (1) with maximal interval (α_p, ω_p) such that $\varphi_p(0) = p$. The following statements hold.*

(a) *If $\omega_p = \infty$ then $\omega(p) \subset \overline{\{f(x, y, z) = 0\}} \cup \mathbb{S}^2$, where $\mathbb{S}^2$ is the boundary of the Poincaré ball.*

(b) *If $\alpha_p = -\infty$ then $\alpha(p) \subset \overline{\{f(x, y, z) = 0\}} \cup \mathbb{S}^2$, where $\mathbb{S}^2$ is the boundary of the Poincaré ball.*

For a definition of Poincaré ball see for instance [3]. Note that in Proposition 1 the function f is of the form $f = f_1^{\lambda_1} ... f_p^{\lambda_p} F_1^{\mu_1} ... F_q^{\mu_q}$, as in (2).

A surface $\mathcal{G} = \mathcal{G}(x, y, z) = 0$ in $\mathbb{R}^3$ implicitly defined by an algebraic equation of degree two is called a *quadric*. The quadric $\mathcal{G} = 0$ is *non–degenerate* if the polynomial $\mathcal{G}$ is irreducible in $\mathbb{C}[x, y, z]$. The non–degenerate quadrics are classified as ellipsoid or sphere (S), parabolic cylinder (PC), hyperbolic cylinder (HC), elliptic cylinder (EC), elliptic paraboloid (EP), hyperbolic paraboloid (HP), hyperboloid of one sheet (HOS), hyperboloid of two sheets (HTS) and cone (C). The non–degenerate quadrics and their normal forms are shown in Figure 1. A *degenerate quadric* $\mathcal{G} = 0$ is a quadric that is not non–degenerate. The degenerate quadrics are classified as two real parallel planes (RPP), two complex parallel planes (CPP), one double real plane (DP), two real planes intersecting in a straight line (r), two complex planes intersecting in a real straight line (s), and a real point (P).

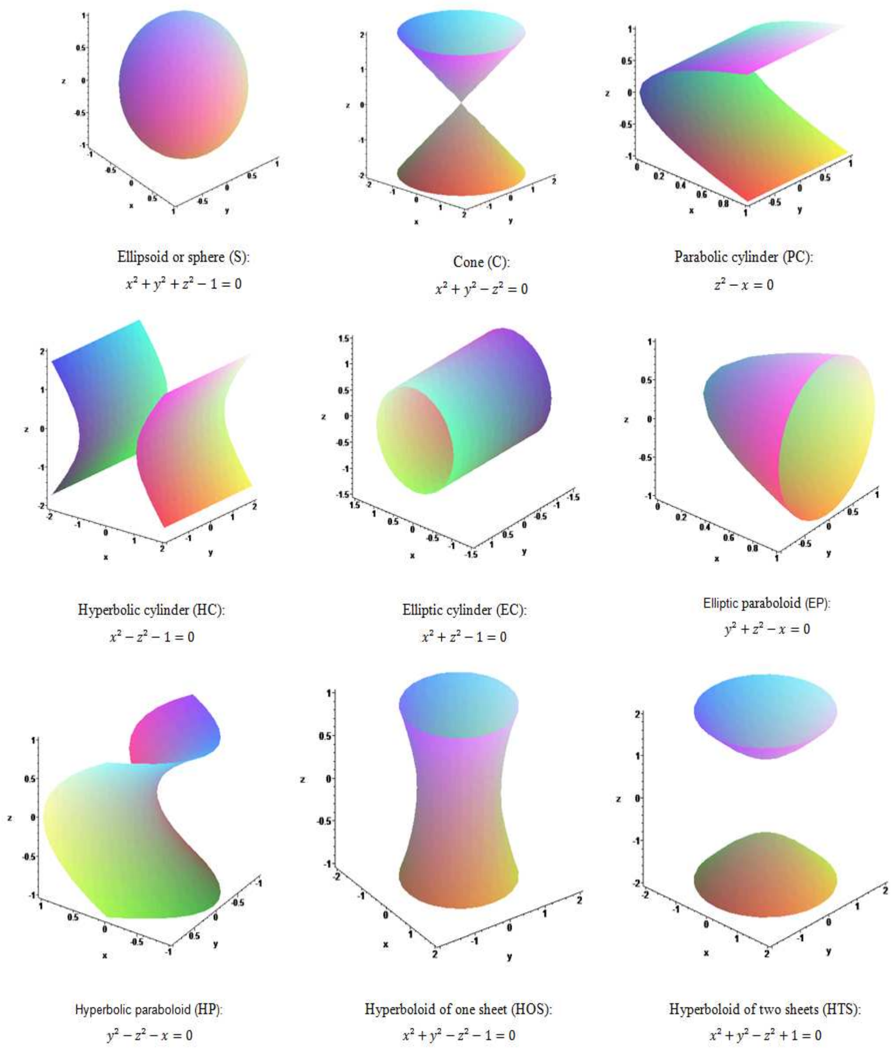

FIGURE 1. Non-degenerate quadrics and their (normalized) equations.

1.2. **Statement of the main results.** Now we give the normal form of all real polynomial differential systems in $\mathbb{R}^3$ having one of the quadrics described above as an invariant set.

Theorem 2. *Assume that a non–degenerate quadric $\mathcal{G} = 0$ is an invariant algebraic surface of the polynomial differential system* (1)*. Then after an affine change of coordinates system* (1) *and the quadric $\mathcal{G} = 0$ can be written in one of the following eight normal forms, where A, B, C, D, E, F, G and Q are arbitrary polynomials of $\mathbb{R}[x, y, z]$.*

(S) *If $\mathcal{G}$ is of type* (S)*, then system* (1) *can be written as*
$\dot{x} = \mathcal{G}A - 2yD + 2zE, \quad \dot{y} = \mathcal{G}B + 2xD - 2zF, \quad \dot{z} = \mathcal{G}C - 2xE + 2yF,$
with $\mathcal{G} = x^2 + y^2 + z^2 - 1$.

(PC) *If $\mathcal{G}$ is of type* (PC)*, then system* (1) *can be written as*
$\dot{x} = \mathcal{G}A + 2zE, \quad \dot{y} = Q, \quad \dot{z} = \mathcal{G}C + E,$ *with* $\mathcal{G} = z^2 - x$.

(HC) *If $\mathcal{G}$ is of type* (HC)*, then system* (1) *can be written as*
$\dot{x} = \mathcal{G}A - 2zE, \quad \dot{y} = Q, \quad \dot{z} = \mathcal{G}C - 2xE,$ *with* $\mathcal{G} = x^2 - z^2 - 1$.

(EC), *If $\mathcal{G}$ is of type* (EC)*, then system* (1) *can be written as*
$\dot{x} = \mathcal{G}A + 2zE, \quad \dot{y} = Q, \quad \dot{z} = \mathcal{G}C - 2xE,$ *with* $\mathcal{G} = x^2 + z^2 - 1$.

(EP) *If $\mathcal{G}$ is of type* (EP)*, then system* (1) *can be written as*
$\dot{x} = \mathcal{G}A - 2yD + 2zE, \quad \dot{y} = \mathcal{G}B - 2zF - D, \quad \dot{z} = \mathcal{G}C + 2yF + E,$
with $\mathcal{G} = y^2 + z^2 - x$.

(HP) *If $\mathcal{G}$ is of type* (HP)*, then system* (1) *can be written as*
$\dot{x} = \mathcal{G}A - 2yD - 2zE, \quad \dot{y} = \mathcal{G}B + 2zF - D, \quad \dot{z} = \mathcal{G}C + 2yF + E,$
with $\mathcal{G} = y^2 - z^2 - x$.

(HOS) *If $\mathcal{G}$ is of type* (HOS)*, then system* (1) *can be written as*
$\dot{x} = \mathcal{G}A - 2yD - 2zE, \quad \dot{y} = \mathcal{G}B + 2xD + 2zF, \quad \dot{z} = \mathcal{G}C - 2xE + 2yF,$
with $\mathcal{G} = x^2 + y^2 - z^2 - 1$.

(HTS) *If $\mathcal{G}$ is of type* (HTS)*, then system* (1) *can be written as the same system of statement* (HOS)*, with* $\mathcal{G} = x^2 + y^2 - z^2 + 1$.

(C) *If $\mathcal{G}$ is of type* (C)*, then system* (1) *can be written as*
$\dot{x} = \mathcal{G}A - 2yD - 2zE + xG, \quad \dot{y} = \mathcal{G}B + 2xD + 2zF + yG,$
$\dot{z} = \mathcal{G}C - 2xE + 2yF + zG,$ *with* $\mathcal{G} = x^2 + y^2 - z^2$.

Theorem 2 is proved in section 3.

Theorem 3. *Assume that a degenerate quadric $\mathcal{G} = 0$ is an invariant algebraic surface of the polynomial differential system* (1)*. Then after an affine change of coordinates system* (1) *and the quadric $\mathcal{G} = 0$ can be written in one of the following four normal forms, where A, B, C, D, E, F, G, Q and R are arbitrary polynomials of $\mathbb{R}[x, y, z]$.*

(x) *If $\mathcal{G}$ is of type* (RPP)*, then we can write $\mathcal{G} = x^2 - 1$; if $\mathcal{G}$ is of type* (CPP)*, then we can write $\mathcal{G} = x^2 + 1$; and if $\mathcal{G}$ is of type* (DP)*, then we can write $\mathcal{G} = x^2$. In these three cases system* (1) *can be written as*

$\dot{x} = \mathcal{G}M, \quad \dot{y} = Q, \quad \dot{z} = R,$

with M a polynomial on $\mathbb{R}[x, y, z]$ such that $2xM = K$, where K is the cofactor of the invariant quadric $\mathcal{G} = 0$.

(r) *If $\mathcal{G}$ is of type (r), then we can write $\mathcal{G}=xy$ and system (1) as*

$$\dot x = xK_1, \quad \dot y = yK_2, \quad \dot z = R,$$

where K_1 and K_2 are the cofactors of the invariant algebraic planes $x=0$ and $y=0$, respectively.

(s) *If $\mathcal{G}$ is of type (s), then we can write $\mathcal{G}=x^2+y^2$ and system (1) as*

$$\dot x = \mathcal{G}A - 2yD + xG, \quad \dot y = \mathcal{G}B + 2xD + yG, \quad \dot z = R.$$

(P) *If $\mathcal{G}$ is of type (P), then we can write $\mathcal{G}=x^2+y^2+z^2$ and system (1) as*

$$\dot x = \mathcal{G}A - 2yD + 2zE + xG, \quad \dot y = \mathcal{G}B + 2xD - 2zF + yG,$$
$$\dot z = \mathcal{G}C - 2xE + 2yF + zG.$$

Theorem 3 is also proved in section 3.

See the definition of a double real invariant plane or an invariant plane of multiplicity 2 in section 2.

In what follows when we have a polynomial differential system having the invariant quadric $\mathcal{G}=0$ we shall say that this system is of *type* $(\mathcal{G})$. For instance a system of type (PC) will be a system having an invariant parabolic cylinder.

Finally in the next two theorems we present the characterization of all polynomial differential systems in $\mathbb{R}^3$ having a quadric as an invariant algebraic surface and a Darboux invariant *constructed uniquely using the invariant quadric*, we do not repeat this assumption in every statement of the next two theorems. In these two theorems and also later on *when we say system (W) this will mean the system of statement (W) of Theorems 2 or 3.*

Theorem 4. *In the statement of this theorem A, B, C, D, E, F and Q are arbitrary polynomials of $\mathbb{R}[x,y,z]$ and $a,b,c\in\mathbb{R}$ with $ac\neq 0$.*

$(\overline{PC})$ *Systems (PC) having a Darboux invariant I can be written as*

$$\dot x = \mathcal{G}(a+2zC)+2zE, \quad \dot y = Q, \quad \dot z = \mathcal{G}C+E,$$
and $I=(z^2-x)^{1/a}e^t$.

$(\overline{EP})$ *Systems (EP) having a Darboux invariant can be written as*

$$\dot x = \mathcal{G}(a+2yB+2zC)-2yD+2zE, \ \dot y = \mathcal{G}B-2zF-D, \ \dot z = \mathcal{G}C+2yF+E,$$
and $I=(y^2+z^2-x)^{1/a}e^t$.

$(\overline{HP})$ *Systems (HP) having a Darboux invariant can be written as*

$$\dot x = \mathcal{G}(a+2yB-2zC)-2yD-2zE, \ \dot y = \mathcal{G}B+2zF-D, \ \dot z = \mathcal{G}C+2yF+E,$$
and $I=(y^2-z^2-x^2)^{1/a}e^t$.

$(\overline{C})$ *Systems (C) having a Darboux invariant I can be written as*

$$\dot x = (y^2-z^2)A-(2D+xB)y+(-2E+xC)z-ax/2,$$
$$\dot y = (x^2-z^2)B+(2D-yA)x+(2F+yC)z-ay/2,$$
$$\dot z = (x^2+y^2)C-(2E+zA)x+(2F-zB)y-az/2,$$
and $I=(x^2+y^2-z^2)^{1/a}e^t$.

$(\overline{N})$ *Systems (S), (HC), (EC), (HOS) and (HTS) have no Darboux invariants.*

Theorem 4 is proved in section 4.

Theorem 5. *In the statement of this theorem A, B, C, D, E, F, Q and R are arbitrary polynomials of $\mathbb{R}[x,y,z]$ and $a,b,c \in \mathbb{R}$ with $ac \neq 0$.*

($\overline{x}$) *Systems* (x) *having a Darboux invariant I can be written as*
$\dot{x} = \mathcal{G}, \quad \dot{y} = Q, \quad \dot{z} = R,$
where either $\mathcal{G} = x^2 - 1$ and $I = \sqrt{(x+1)/(x-1)}\, e^t$; or $\mathcal{G} = x^2 + 1$ and $I = e^{t+\arctan(1/x)}$; or $\mathcal{G} = x^2$ and $I = e^{t+1/x}$, respectively.

($\overline{r}$) *Systems* (r) *having a Darboux invariant I can be written as*
$\dot{x} = xA, \quad \dot{y} = y(a+bA), \quad \dot{z} = R,$ *and* $I = y/(x^b e^{at})$.

($\overline{s}$) *Systems* (s) *having a Darboux invariant can be written either as*
$\dot{x} = (x-by)A - ay, \quad \dot{y} = (bx+y)A + ax, \quad \dot{z} = R,$
and $I = (x^2+y^2)^b e^{-2\arctan(y/x)+2at}$*; or as*
$\dot{x} = -yB + cx, \quad \dot{y} = xB + cy, \quad \dot{z} = R,$
and $I = (x^2+y^2)e^{-2ct}$.

($\overline{P}$) *Systems* (P) *having a Darboux invariant I can be written as*
$$\dot{x} = (y^2+z^2)A - (2D+xB)y + (2E-xC)z - ax/2,$$
$$\dot{y} = (x^2+z^2)B + (2D-yA)x - (2F+yC)z - ay/2,$$
$$\dot{z} = (x^2+y^2)C - (2E+zA)x + (2F-zB)y - az/2,$$
where $I = (x^2+y^2+z^2)^{1/a}e^t$.

Theorem 5 is proved in section 4.

Note that by Theorem 5 all the polynomial differential systems having a degenerate quadric as an invariant algebraic surface always have a Darboux invariant, while this is not the case for the non–degenerate quadrics, as stated in case ($\overline{N}$) of Theorem 4.

The results presented here provide an extension to $\mathbb{R}^3$ of a similar study developed in [9] concerning polynomial differential systems in $\mathbb{R}^2$ having a conic as an invariant algebraic curve and a Darboux invariant forced uniquely by their conic.

In section 2 we present some results of the Darboux theory of integrability that will be used for proving Theorems 2 to 5. In section 5 we apply the obtained results in the determination of the Darboux invariants for the Chen system with invariant quadrics.

2. Some results of the Darboux theory of integrability

The Darboux theory of integrability for polynomial differential systems in $\mathbb{R}^3$ allows to compute a first integral if the systems have convenient invariant algebraic surfaces. The next result explains how to find Darboux invariants. Its proof for polynomial differential systems in $\mathbb{R}^2$ can be found in [6] (see statement (vi) of Theorem 8.7), but it can be trivially extended to $\mathbb{R}^n$.

Proposition 6. *Suppose that a polynomial differential system* (1) *of degree m admits p invariant algebraic surfaces $f_i = 0$ with cofactors K_i for $i = 1, ..., p$, and q*

exponential factors $F_j = \exp(g_j/h_j)$ *with cofactors* L_j *for* $j = 1, ..., q$. *There exist* λ_i, $\mu_j \in \mathbb{C}$ *not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -s, \tag{3}$$

for some $s \in \mathbb{R} \setminus \{0\}$, *if and only if the real (multi-valued) function*

$$f_1^{\lambda_1} ... f_p^{\lambda_p} F_1^{\mu_1} ... F_q^{\mu_q} e^{st}$$

is a Darboux invariant of system (1).

The following proposition is easy to prove.

Proposition 7. *For the real polynomial differential system* (1), $f = 0$ *is an invariant algebraic surface with cofactor* K *if and only if* $\bar{f} = 0$ *is an invariant algebraic surface with cofactor* $\bar{K}$, *where* $\bar{f}$ *and* $\bar{K}$ *denote the conjugates of* f *and* K, *respectively.*

Observe that if among the invariant algebraic surfaces of system (1) a complex conjugate pair $f = 0$ and $\bar{f} = 0$ occurs and they appear in the expression of the Darboux invariant, then the Darboux invariant has a real factor of the form $f^\lambda \bar{f}^{\bar{\lambda}}$, which is the multi–valued real function

$$[(\mathrm{Re}(f))^2 + (\mathrm{Im}(f))^2]^{\mathrm{Re}(\lambda)} e^{-2\,\mathrm{Im}(\lambda) \arctan(\mathrm{Im}(f)/\,\mathrm{Re}(f))}. \tag{4}$$

So if system (1) is real then the Darboux invariant is also real, independently of having complex invariant algebraic surfaces. A similar result occurs if in the expression of the Darboux invariant appear complex exponential factors.

The next result for planar polynomial differential systems is proved in Proposition 8.4 of [6], but its proof for polynomial differential systems in $\mathbb{R}^n$ is the same.

Proposition 8. *Suppose that* $f \in \mathbb{C}[x, y, z]$ *and let* $f = f_1^{n_1} ... f_r^{n_r}$ *be its factorization into irreducible factors in* $\mathbb{C}[x, y, z]$. *Then for a polynomial differential system* (1), $f = 0$ *is an invariant algebraic surface with cofactor* K_f *if and only if* $f_i = 0$ *is an invariant algebraic surface for each* $i = 1, ..., r$ *with cofactor* K_{f_i}. *Moreover* $K_f = n_1 K_{f_1} + ... + n_r K_{f_r}$.

About exponential factors it is known the following result (the proof for planar polynomial differential systems is given in Proposition 8.6 of [6]). Again the result can be easily extended to $\mathbb{R}^n$.

Proposition 9. *If* $F = \exp(g/h)$ *is an exponential factor for the polynomial differential system* (1) *and* h *is not a constant, then* $h = 0$ *is an invariant algebraic surface and* g *satisfies the equation* $Xg = gK_h + hK_F$, *where* K_h *and* K_F *are the cofactors of* h *and* F, *respectively.*

It is known that if a vector field X associated to system (1) has the invariant plane $ax + by + cz + d = 0$, then the polynomial $ax + by + cz + d$ divides the polynomial

$$R(x,y,z) = \begin{vmatrix} 1 & x & y & z \\ 0 & X(x) & X(y) & X(z) \\ 0 & X^2(x) & X^2(y) & X^2(z) \\ 0 & X^3(x) & X^3(y) & X^3(z) \end{vmatrix}, \tag{5}$$

where $X(v) = \dot{v}$, $X^2(v) = X(X(v))$ and $X^3(v) = X(X^2(v))$, with v running on the variables x, y and z. Moreover, if $(ax+by+cz+d)^k | R(x,y,z)$ and $(ax+by+cz+d)^{k+1} \nmid R(x,y,z)$, then we say that the invariant plane $ax+by+cz+d=0$ has *multiplicity* k. For more details on the multiplicity of invariant planes see [5, 20].

Let f_1,f_2 and f_3 be real functions defined in an open subset U of $\mathbb{R}^3$. As usual the *Jacobian matrix* of the functions f_1, f_2 and f_3 is defined by

$$J = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial z} \end{pmatrix}. \tag{6}$$

The *Jacobian of* J is the determinant of the matrix J, and here it is denoted by

$$\{f_1, f_2, f_3\} = \det(J).$$

The next result is a particular case of Theorem 1 of [16] and it characterizes all polynomial differential systems (1) having a given invariant algebraic surface. Since the proof of the more general Theorem 1 of [16] is more tedious than the proof of the result presented here we shall prove it.

Theorem 10. *Let f_1 be a polynomial on $\mathbb{R}[x,y,z]$. Then any polynomial differential system in $\mathbb{R}^3$ which admits $f_1 = 0$ as an invariant algebraic surface is of the form*

$$\dot{v} = \lambda_1\{v, f_2, f_3\} + \lambda_2\{f_1, v, f_3\} + \lambda_3\{f_1, f_2, v\}, \tag{7}$$

where v runs over the variables x, y and z, $\lambda_1 = \varphi f_1$ and φ, λ_2, λ_3 are rational functions, f_2 and f_3 are arbitrary polynomials on $\mathbb{R}[x,y,z]$ which must be chosen in such a way that the Jacobian $\{f_1, f_2, f_3\} \not\equiv 0$.

Proof. Define the vector field

$$\{*, f_2, f_3\} = \det \begin{pmatrix} \dfrac{\partial *}{\partial x} & \dfrac{\partial *}{\partial y} & \dfrac{\partial *}{\partial z} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial z} \end{pmatrix},$$

and similarly the vector fields $\{f_1, *, f_3\}$ and $\{f_1, f_2, *\}$. Then we can consider the polynomial vector field (7) of the statement of the theorem written in the form

$$X(*) = \lambda_1\{*\,, f_2, f_3\} + \lambda_2\{f_1,\,*\,, f_3\} + \lambda_3\{f_1, f_2,\,*\,\}. \tag{8}$$

By using this representation we get

$$X(f_1) = \lambda_1\{f_1, f_2, f_3\} + \lambda_2\{f_1, f_1, f_3\} + \lambda_3\{f_1, f_2, f_1\} = K f_1.$$

Therefore, $f_1 = 0$ is an invariant algebraic surface of the polynomial vector field X with cofactor $K = \varphi\{f_1, f_2, f_3\}$.

Now we shall prove that the vector field X is the most general polynomial vector field which admits $f_1 = 0$ as an invariant algebraic surface. Indeed let

$$Y = \big(Y_1(x, y, z), Y_2(x, y, z), Y_3(x, y, z)\big)$$

be a polynomial vector field having $f_1 = 0$ as an invariant algebraic surface. Then taking

$$\lambda_j = \frac{Y(f_j)}{\{f_1, f_2, f_3\}}, \quad \text{for } j = 1, 2, 3,$$

and substituting these lambdas into the definition (8) of the polynomial vector field X we get for an arbitrary polynomial F that

$$X(F) = Y(f_1)\frac{\{F, f_2, f_3\}}{\{f_1, f_2, f_3\}} + Y(f_2)\frac{\{f_1, F, f_3\}}{\{f_1, f_2, f_3\}} + Y(f_3)\frac{\{f_1, f_2, F\}}{\{f_1, f_2, f_3\}}.$$

Substituting

$$Y(f_j) = Y_1\frac{\partial f_j}{\partial x} + Y_2\frac{\partial f_j}{\partial y} + Y_3\frac{\partial f_j}{\partial z},$$

in $X(F)$ we obtain after an easy, but tedious computation, that $X(F) = Y(F)$. In view of the arbitrariness of F the theorem is proved. □

3. Normal forms for polynomial differential systems in $\mathbb{R}^3$ having a quadric as invariant algebraic surface

In this section we will prove Theorems 2 and 3 stated in section 1, which provide the normal forms of the polynomial differential systems in $\mathbb{R}^3$ having one non–degenerate or degenerate quadric as an invariant algebraic surface.

Proof of Theorem 2. Suppose that the polynomial differential system (1) is of type (S), i.e. it has an ellipsoid or a sphere as an invariant algebraic surface. After an affine change of coordinates, we can assume that $\mathcal{G} = x^2 + y^2 + z^2 - 1 = 0$ is the invariant sphere (or ellipsoid) of system (1).

By Theorem 10 any polynomial differential system having the invariant algebraic surface $f_1 = \mathcal{G} = 0$ can be written as system (7), i.e. it is given by

$$\begin{aligned}
\dot{x} &= \varphi\, f_1\,(f_{2y}f_{3z} - f_{2z}f_{3y}) + 2y(\lambda_3 f_{2z} - \lambda_2 f_{3z}) + 2z(\lambda_2 f_{3y} - \lambda_3 f_{2y}),\\
\dot{y} &= \varphi\, f_1\,(f_{3x}f_{2z} - f_{2x}f_{3z}) + 2x(\lambda_2 f_{3z} - \lambda_3 f_{2z}) + 2z(\lambda_3 f_{2x} - \lambda_2 f_{3x}),\\
\dot{z} &= \varphi\, f_1\,(f_{2x}f_{3y} - f_{3x}f_{2y}) + 2x(\lambda_3 f_{2y} - \lambda_2 f_{3y}) + 2y(\lambda_2 f_{3x} - \lambda_3 f_{2x}),
\end{aligned} \tag{9}$$

where $f_{iv} = \partial f_i/\partial v$, with v running on the variables x, y and z, for $i = 2, 3$, while φ, λ_2, λ_3 are rational functions and $f_1 = \mathcal{G}, f_2$, f_3 are polynomials on $\mathbb{R}[x, y, z]$ satisfying the condition $\{f_1, f_2, f_3\} \not\equiv 0$.

Defining

$$J_1 = f_{2y}f_{3z} - f_{2z}f_{3y},$$
$$J_2 = f_{3x}f_{2z} - f_{2x}f_{3z}, \tag{10}$$
$$J_3 = f_{2x}f_{3y} - f_{3x}f_{2y},$$

we can write system (9) as

$$\begin{aligned}
\dot{x} &= \varphi\, f_1\, J_1 + \lambda_2(2zf_{3y} - 2yf_{3z}) + \lambda_3(2yf_{2z} - 2zf_{2y}),\\
\dot{y} &= \varphi\, f_1\, J_2 + \lambda_2(2xf_{3z} - 2zf_{3x}) + \lambda_3(2zf_{2x} - 2xf_{2z}),\\
\dot{z} &= \varphi\, f_1\, J_3 + \lambda_2(2yf_{3x} - 2xf_{3y}) + \lambda_3(2xf_{2y} - 2yf_{2x}).
\end{aligned} \tag{11}$$

Then taking $f_1 = x^2 + y^2 + z^2 - 1$ and

$$\varphi = \frac{xA + yB + zC}{xJ_1 + yJ_2 + zJ_3},$$

$$\lambda_2 = \frac{(xf_{2y} - yf_{2x})2D + (zf_{2x} - xf_{2z})2E + (yf_{2z} - zf_{2y})2F + (f_{2x}A + f_{2y}B + f_{2z}C)f_1}{2xJ_1 + 2yJ_2 + 2zJ_3},$$

$$\lambda_3 = \frac{(xf_{3y} - yf_{3x})2D + (zf_{3x} - xf_{3z})2E + (yf_{3z} - zf_{3y})2F + (f_{2x}A + f_{2y}B + f_{2z}C)f_1}{2xJ_1 + 2yJ_2 + 2zJ_3},$$

in system (11), where A, B, C, D, E and F are arbitrary polynomials in $\mathbb{R}[x, y, z]$, we obtain the system of statement (S) of Theorem 2, which has $\mathcal{G} = x^2+y^2+z^2-1 = 0$ as an invariant algebraic surface with cofactor $K = 2(xA + yB + zC)$. Indeed we have

$$\begin{aligned}
&X(\mathcal{G}) = 2xP + 2yQ + 2zR =\\
&= 2x(\mathcal{G}A - 2yD - 2zE) + 2y(\mathcal{G}B + 2xD - 2zF) + 2z(\mathcal{G}C - 2xE + 2yF) =\\
&= 2x\mathcal{G}A + 2y\mathcal{G}B + 2z\mathcal{G}C = 2(xA + yB + zC)\,\mathcal{G}.
\end{aligned}$$

Therefore statement (S) of Theorem 2 is proved.

In a similar way if the polynomial differential system (1) is of type (PC), (HC), (EC), (EP), (HP), (HOS), and (HTS), then after an affine change of coordinates we can take $\mathcal{G} = z^2 - x = 0$, $\mathcal{G} = x^2 - z^2 - 1 = 0$, $\mathcal{G} = x^2 + z^2 - 1 = 0$, $\mathcal{G} = y^2 + z^2 - x = 0$, $\mathcal{G} = y^2 - z^2 - x = 0$, $\mathcal{G} = x^2 + y^2 - z^2 - 1 = 0$ and $\mathcal{G} = x^2 + y^2 - z^2 + 1 = 0$ as the invariant quadric of system (1), respectively. Then repeating the same arguments of the proof of statement (S) we can prove statements (PC), (HC), (EC), (EP), (HP), (HOS) and (HTS) of Theorem 2. In all of these cases we can determine φ, λ_2 and λ_3 in system (7) of Theorem 10 to obtain the respective normal forms. Moreover in each one of these cases $\mathcal{G}$ is an invariant algebraic surface with cofactor $K = f_{1x}A + f_{1y}B + f_{1z}C$, where $f_1 = \mathcal{G}$ is the invariant quadric.

It only remains to prove statement (C) of Theorem 2. In this case, after an affine change of coordinates, we can assume without loss of generality that $\mathcal{G} = x^2 +$

$y^2 - z^2 = 0$ is the invariant cone of system (1). Since this quadric is given by a homogeneous function, the expression for φ is slightly different from the previous cases. Note that in this case the additional terms xG, yG and zG appear in the equations of system (C). From Theorem 10 if we take the invariant algebraic surface $f_1 = \mathcal{G} = 0$, system (7) is written as

$$\begin{aligned} \dot{x} &= \varphi\, f_1\, (f_{2y} f_{3z} - f_{2z} f_{3y}) + 2y(\lambda_3 f_{2z} - \lambda_2 f_{3z}) + 2z(-\lambda_2 f_{3y} + \lambda_3 f_{2y}),\\ \dot{y} &= \varphi\, f_1\, (f_{3x} f_{2z} - f_{2x} f_{3z}) + 2x(\lambda_2 f_{3z} - \lambda_3 f_{2z}) + 2z(-\lambda_3 f_{2x} + \lambda_2 f_{3x}),\\ \dot{z} &= \varphi\, f_1\, (f_{2x} f_{3y} - f_{3x} f_{2y}) + 2x(\lambda_3 f_{2y} - \lambda_2 f_{3y}) + 2y(\lambda_2 f_{3x} - \lambda_3 f_{2x}), \end{aligned} \tag{12}$$

where again $f_{iv} = \partial f_i / \partial v$, with v running on the variables x, y and z, for $i = 2, 3$, φ, λ_2, λ_3 are rational functions and $f_1 = \mathcal{G}, f_2$, f_3 are polynomials on $\mathbb{R}[x, y, z]$ satisfying the condition $\{f_1, f_2, f_3\} \not\equiv 0$. Then taking J_1, J_2 and J_3 as in (10) system (12) can be written as

$$\begin{aligned} \dot{x} &= \varphi\, f_1\, J_1 - \lambda_2(2z f_{3y} + 2y f_{3z}) + \lambda_3(2y f_{2z} + 2z F_{2y}),\\ \dot{y} &= \varphi\, f_1\, J_2 + \lambda_2(2x f_{3z} + 2z f_{3x}) - \lambda_3(2x f_{2z} - 2z f_{2x}),\\ \dot{z} &= \varphi\, f_1\, J_3 + \lambda_3(2x f_{2y} - 2y f_{2x}) \;+\; \lambda_3(2x f_{2y} - 2y f_{2x}). \end{aligned} \tag{13}$$

Hence taking $f_1 = x^2 + y^2 - z^2$ and

$$\begin{aligned} \varphi &= \frac{xA + yB - zC + G}{-xJ_1 - yJ_2 + zJ_3},\\ \lambda_2 &= \frac{(x f_{2y} - y f_{2x})2D \;-\; (x f_{2z} + z f_{2x})2E \;+\; (z f_{2y} + y f_{2z})2F}{2xJ_1 + 2yJ_2 - 2zJ_3} +\\ &\quad + \frac{(f_{2x}A + f_{2y}B + f_{2z}C) f_1 + (x f_{2x} + y f_{2y} + z f_{2z})G}{2xJ_1 + 2yJ_2 - 2zJ_3},\\ \lambda_3 &= \frac{(x f_{3y} - y f_{3x})2D \;-\; (x f_{3z} + z f_{3x})2E \;+\; (z f_{3y} + y f_{3z})2F}{2xJ_1 + 2yJ_2 - 2zJ_3} +\\ &\quad + \frac{(f_{3x}A + f_{3y}B + f_{3z}C) f_1 + (x f_{3x} + y f_{3y} + z f_{3z})G}{2xJ_1 + 2yJ_2 - 2zJ_3}, \end{aligned}$$

in system (13), where A, B, C, D, E, F and G are arbitrary polynomials in $\mathbb{R}[x, y, z]$, we obtain the system of statement (C) of Theorem 2, which has $\mathcal{G} = x^2 + y^2 - z^2 = 0$ as an invariant algebraic surface with cofactor $K = 2(xA + yB - zC + G)$. Indeed we have

$$\begin{aligned} X(\mathcal{G}) &= 2xP + 2yQ + 2zR \;=\; 2x(\mathcal{G}A - 2yD - 2zE + xG) +\\ &+ 2y(\mathcal{G}B + 2xD + 2zF + yG) - \; 2z(\mathcal{G}C - 2xE + 2yF + zG) =\\ &= 2x\mathcal{G}A \;+\; 2y\mathcal{G}B - 2z\mathcal{G}C + 2G(x^2 + y^2 - z^2) \;=\\ &= (2xA + 2yB - 2zC)\mathcal{G} \;+\; 2G\mathcal{G} \;=\; 2(xA + yB - zC + G)\mathcal{G}. \end{aligned}$$

Therefore Theorem 2 is proved. □

Proof of Theorem 3. In order to prove cases (x) and (r) of Theorem 3 we use Proposition 8 of section 2 and the decomposition of the quadrics (RPP), (CPP), (DP) and (r), then we apply the definition of invariant algebraic surfaces to obtain the normal forms.

Suppose that the polynomial differential system (1) is of type (RPP), i.e. it has two real parallel planes as invariant algebraic surface. After an affine change of coordinates, we can take $\mathcal{G} = x^2 - 1 = 0$ as the invariant quadric of system (1). In this case $\mathcal{G} = g_1 g_2$ where $g_1 = x - 1$ and $g_2 = x + 1$, hence by Proposition 8 we have that $g_1 = x - 1 = 0$ and $g_2 = x + 1 = 0$ are two invariant parallel planes of system (1) with cofactors K_1 and K_2, respectively. So, from the definition of invariant algebraic surface, we have that

$$P\frac{\partial g_1}{\partial x} + Q\frac{\partial g_1}{\partial y} + R\frac{\partial g_1}{\partial z} = K_1 g_1 \quad \Rightarrow \quad P = K_1(x-1).$$

Analogously

$$P\frac{\partial g_2}{\partial x} + Q\frac{\partial g_2}{\partial y} + R\frac{\partial g_2}{\partial z} = K_2 g_2 \quad \Rightarrow \quad P = K_2(x+1).$$

Then $P = \mathcal{G}M$, where M is a polynomial in $\mathbb{R}[x, y, z]$ such that $K = 2xm$ is the cofactor of the quadric $\mathcal{G}$. Therefore, we can write system (1) as the system of statement (x) of Theorem 3. Observe that as $\mathcal{G}$ depends only on the variable x, we have its partial derivatives with respect to y and z equal to zero, hence we can take Q and R as arbitrary polynomials in the normal form of statement (x). It will happen in another cases and will be clear along the text.

If system (1) is of type (CPP) then, after an affine change of coordinates, we can consider $\mathcal{G} = x^2 + 1 = 0$ as the invariant quadric of system (1), and again by Proposition 8 $g_1 = x - i = 0$ and $g_2 = x + i = 0$ are two invariant complex parallel planes of system (1). So, using the same arguments of the previous paragraph, we obtain $P = \mathcal{G}M$, where M is a polynomial in $\mathbb{R}[x, y, z]$ such that $K = 2xm$ is the cofactor of the quadric $\mathcal{G}$, then we can write system (1) as the system of statement (x) of Theorem 3.

Suppose that the polynomial differential system (1) is of type (DP), i.e. it has one double real plane as invariant algebraic surface. After an affine change of coordinates we can consider $\mathcal{G} = x^2 = 0$ as the invariant quadric, and $g_1 = x = 0$ as the double invariant plane of system (1). Given a small pertubation of system (1) we can bifurcate the double invariant plane $x = 0$ in two real and parallel planes $g_1 = x - \varepsilon = 0$ and $g_2 = x + \varepsilon = 0$ (see the different interpretations of the multiplicity of an invariant surface in [5], where it is given for invariant algebraic curves in $\mathbb{R}^2$, but these results extend in a natural way to higher dimensions, and in particular to invariant algebraic surfaces in $\mathbb{R}^3$, see also [19, 20]). Now from the case of systems of type (RPP), we can write $P = (x^2 - \varepsilon^2)M$, with M as above. Making $\varepsilon \to 0$ we obtain that $P = \mathcal{G}M$ and we can write system (1) as the system of statement (x) of Theorem 3. Finally it is easy to check, using the polynomial (5), that the plane $x = 0$ of the system of statement (x) of Theorem 3 with $\mathcal{G} = x^2$ has multiplicity two. This completes the statement (x) of Theorem 3.

In a similar way to the quadrics of type (RPP) and (CPP) if system (1) is of type (r) then, after an affine change of coordinates, we can take $\mathcal{G} = xy = 0$ as the invariant quadric of system (1), and consequently by Proposition 8 $g_1 = x = 0$ and $g_2 = y = 0$ are two real invariant planes of system (1). Then we obtain $P = xK_1$ and $Q = yK_2$, where K_1 and K_2 are the cofactors of $g_1 = 0$ and $g_2 = 0$, respectively. Therefore we can write system (1) as the system of statement (r) of Theorem 3, hence proving this statement.

Now in order to prove statements (s) and (P) of Theorem 3 we use Theorem 10 again. Suppose that the polynomial differential system (1) is of type (s), i.e. it has two complex planes intersecting at a real straight line as an invariant algebraic surface. After an affine change of coordinates, we can take $\mathcal{G} = x^2 + y^2 = 0$ as the invariant quadric. As the quadric $\mathcal{G}$ is given by a homogeneous function, we can follow the proof of statement (C) of Theorem 2. According to Theorem 10, we can write system (1) as system (7), with $f_1 = \mathcal{G}$. From system (7) we obtain

$$
\begin{aligned}
\dot{x} &= \varphi f_1\,(f_{2y}f_{3z} - f_{2z}f_{3y}) + 2y(\lambda_3 f_{2z} - \lambda_2 f_{3z}),\\
\dot{y} &= \varphi f_1\,(f_{3x}f_{2z} - f_{2x}f_{3z}) + 2x(\lambda_2 f_{3z} - \lambda_3 f_{2z}),\\
\dot{z} &= \varphi f_1\,(f_{2x}f_{3y} - f_{3x}f_{2y}) + 2x(\lambda_3 f_{2y} - \lambda_2 f_{3y}) + 2y(\lambda_2 f_{3x} - \lambda_3 f_{2x}),
\end{aligned} \tag{14}
$$

where φ, λ_2, λ_3 are rational functions, $f_1 = \mathcal{G}$ and f_2, f_3 are polynomials on $\mathbb{R}[x, y, z]$ satisfying the condition $\{f_1, f_2, f_3\} \not\equiv 0$. Then, considering that $f_{1z} = 0$ we can take $\dot{z} = R$ being R an arbitrary polynomial, and system (14) can be written as

$$
\begin{aligned}
\dot{x} &= \varphi f_1\, J_1 - 2\lambda_{2y} f_{3z} + 2\lambda_{3y} f_{2z},\\
\dot{y} &= \varphi f_1\, J_2 + 2\lambda_{2x} f_{3z} - 2\lambda_{3x} f_{2z},\\
\dot{z} &= R.
\end{aligned} \tag{15}
$$

Now taking

$$
\begin{aligned}
\varphi &= \frac{xA + yB + G}{xJ_1 + yJ_2},\\
\lambda_2 &= \frac{(2xJ_1 + 2yJ_2)D + (yJ_1 - xJ_2)G \;+\; (J_1B - J_2A)\,f_1}{(2xJ_1 + 2yJ_2)f_{3z}},\\
\lambda_3 &= 0,
\end{aligned}
$$

where A, B, D and G are arbitrary polynomials in $\mathbb{R}[x, y, z]$, J_1, J_2 as in (10). Therefore we can write system (15) as the system of statement (s) of Theorem 3. Moreover we have

$$
\begin{aligned}
X(\mathcal{G}) = 2xP + 2yQ \;&=\; 2x(\mathcal{G}A - 2yD + 2zE + xG) +\\
&+ 2y(\mathcal{G}B + 2xD - 2zF + yG) = 2x\mathcal{G}A \;+\; 2y\mathcal{G}B \;+ 2G(x^2 + y^2) =\\
&= (2xA + 2yB)\,\mathcal{G} \;+ 2G\mathcal{G} \;=\; 2(xA + yB + G)\,\mathcal{G}.
\end{aligned}
$$

Hence $\mathcal{G} = x^2 + y^2 = 0$ is an invariant algebraic surface of system (s) with cofactor $K = 2(xA + yB + G)$. So statement (s) of Theorem 3 is proved.

Lastly suppose that the polynomial differential system (1) is of type (P), i.e. it has a real point as an invariant algebraic surface. After an affine change of coordinates, we can take $\mathcal{G} = x^2 + y^2 + z^2 = 0$ as the invariant quadric. Working in an analogous way as in the proof of statement (s), we obtain the system (P) of Theorem 3. This completes the proof of Theorem 3. □

4. Characterization of polynomial differential systems in $\mathbb{R}^3$ having an invariant quadric and a Darboux invariant

In this section we prove Theorems 4 and 5 which give the characterization of all polynomial differential systems in $\mathbb{R}^3$ having one quadric as an invariant algebraic surface and a Darboux invariant.

Proof of Theorem 4. Suppose that the polynomial differential system (PC) of Theorem 2 has a Darboux invariant. Note that $\mathcal{G} = z^2 - x = 0$ is the invariant quadric of system (PC) with cofactor $K = 2zC - A$. As system (PC) has a Darboux invariant constructed uniquely with the invariant quadric $\mathcal{G} = 0$, by Proposition 6, there exists $\mu \in \mathbb{C}$ not zero such that, for some $s \in \mathbb{R} \setminus \{0\}$,

$$\mu K = -s \Leftrightarrow \mu(2zC - A) = -s \Leftrightarrow \mu A = 2\mu zC + s \Leftrightarrow A = 2zC + a$$

where $a = s/\mu \in \mathbb{R} \setminus \{0\}$. Substituting A into system (PC) we obtain system $(\overline{PC})$. Using Proposition 6 it follows that the Darboux invariant of system $(\overline{PC})$ is $I = \mathcal{G}^\mu e^{st}$. As $\mu = s/a$ and clearly it is not restrictive to choose $s = 1$, so $I = (z^2 - x)^{1/a} e^t$.

In a similar way it can be proved that if systems (EP) and (HP) have a Darboux invariant constructed uniquely using the invariant quadric, then we get systems of statements $(\overline{PC})$, $(\overline{EP})$ and $(\overline{HP})$, respectively.

Now we prove statement $(\overline{C})$ of Theorem 4. Suppose that the polynomial differential system (C) has a Darboux invariant. Then $\mathcal{G} = x^2 + y^2 - z^2 = 0$ is the invariant quadric of system (C) with cofactor $K = 2(xA + yB - zC + G)$. Since system (C) has a Darboux invariant constructed uniquely with the invariant quadric $\mathcal{G} = 0$, by Proposition 6, there exists $\mu \in \mathbb{C}$ not zero such that, for some $s \in \mathbb{R} \setminus \{0\}$,

$$\mu K = -s \Leftrightarrow 2\mu(xA + yB - zC + G) = -s \Leftrightarrow xA + yB - zC + G = -s/(2\mu)$$
$$\Leftrightarrow G = -xA - yB + zC - a/2$$

where $a = s/\mu \in \mathbb{R} \setminus \{0\}$. Substituting G, putting $\mathcal{G} = x^2 + y^2 - z^2$ and grouping the polynomials conveniently into system (C) we obtain system $(\overline{C})$. Using Proposition 6, in a similar way of case $(\overline{PC})$, it follows that the Darboux invariant of system $(\overline{C})$ is $I = (x^2 + y^2 - z^2)^{1/a} e^t$.

Lastly we prove statement $(\overline{N})$. Suppose that the polynomial differential system (S) has a Darboux invariant constructed uniquely using the quadric $\mathcal{G} = x^2 + y^2 + z^2 - 1 = 0$ with cofactor $K = 2(xA + yB + zC)$. Then equation (3) from Proposition 6 becomes $\mu K = -s$, for some $s \in \mathbb{R} \setminus \{0\}$. However the polynomial $\mu K = 2\mu(xA + yB + zC)$ has no independent term, so the equation $\mu K = -s$ never holds. Therefore, by Proposition 6 it follows that system (S) has no Darboux invariants. Analogously we can prove that the polynomial differential systems (HC), (EC), (HOS) and (HTS) have no Darboux invariants and statement $(\overline{N})$ is proved. □

Proof of Theorem 5. Suppose that a polynomial differential system (x) with the invariant quadratic (RPP) given by $\mathcal{G} = x^2 - 1 = 0$ has a Darboux invariant constructed uniquely from this quadric. By Proposition 8 this system (x) has two invariant parallel planes $g_1 = x - 1 = 0$ and $g_2 = x + 1 = 0$ with cofactors $K_1 = (x+1)M$ and $K_2 = (x-1)M$, respectively. By Proposition 6 there exist $\mu_1, \mu_2 \in \mathbb{C}$ not all zero satisfying equation (3), i.e. $\mu_1 K_1 + \mu_2 K_2 = -s$, for some $s \in \mathbb{R} \setminus \{0\}$. Then from (3), we have that

$$\mu_1(x+1)M + \mu_2(x-1)M = -s \quad \Leftrightarrow \quad [(\mu_1+\mu_2)x + \mu_1 - \mu_2]M = -s.$$

Hence $\mu_1 = -\mu_2$ and $M = s/(2\mu_2) \in \mathbb{R} \setminus \{0\}$ in order that the Darboux invariant exists. Without loss of generality we can take $s = 1$, $\mu_2 = 1/2$, then $\mu_1 = -1/2$ and $M = 1$. Then substituting M into system (x), it becomes the system of statement $(\overline{x})$, and its Darboux invariant, using Proposition 6, is $I = g_1^{\mu_1} g_2^{\mu_2} e^{st} = \sqrt{(x+1)/(x-1)}\, e^t$. Hence statement $(\overline{x})$ is proved for systems (RPP).

Suppose that a polynomial differential system (x) with the invariant quadratic (CPP) given by $\mathcal{G} = x^2 + 1 = 0$ has a Darboux invariant constructed uniquely from this quadric. By Proposition 8 this system (x) has the complex invariant parallel planes $g_1 = x - i = 0$ and $g_2 = x + i = 0$ with cofactors $K_1 = (x+i)M$ and $K_2 = (x-i)M$, respectively. By Proposition 6 there exist $\mu_1, \mu_2 \in \mathbb{C}$ not all zero satisfying equation (3), i.e. $\mu_1 K_1 + \mu_2 K_2 = -s$, for some $s \in \mathbb{R} \setminus \{0\}$. Then from (3), we have that

$$[\mu_1(x+i) + \mu_2(x-i)]M = -s \quad \Leftrightarrow \quad [(\mu_1+\mu_2)x + i(\mu_1 - \mu_2)]M = -s.$$

Hence $\mu_1 = -\mu_2$ and $M = s/(2\lambda) \in \mathbb{R} \setminus \{0\}$ in order that the Darboux invariant exists. Without loss of generality we can take $s = 1$, $\mu_2 = i/2$, then $\mu_1 = -i/2$ and $M = 1$. Then substituting M into system (x), it becomes the system of statement $(\overline{x})$. By Proposition 6 its Darboux invariant is $I = (x-i)^{(i/2)}(x+i)^{(-i/2)}e^t$. Then using (4) we obtain $I = e^{t+\arctan(1/x)}$. This ends the proof of statement $(\overline{x})$ for systems (CPP).

Suppose that a polynomial differential system (x) with the invariant quadratic (DP) given by $\mathcal{G} = x^2 = 0$ has a Darboux invariant constructed uniquely from this quadric. This system (x) has the invariant plane $g_1 = x = 0$. Due to the fact that the invariant plane $g_1 = 0$ has multiplicity two it is easy to check that it has the exponential factor $F_1 = e^{1/x}$ (for more details see [5, 20]). Moreover the cofactors of $g_1 = 0$ and of F_1 are $K_1 = xM$ and $L_1 = -M$, respectively. By Proposition 6 there exist $\mu_1, \mu_2 \in \mathbb{C}$ not all zero satisfying equation (3), i.e. $\mu_1 K_1 + \mu_2 L_1 = -s$, for some $s \in \mathbb{R} \setminus \{0\}$. Then from (3), we have that

$$(\mu_1 x - \mu_2)M = -s.$$

Hence $\mu_1 = 0$ and $M = s/\mu_2 \in \mathbb{R} \setminus \{0\}$. Taking $s = \mu_2 = 1$ we obtain $M = 1$. Consequently, by Proposition 6 its Darboux invariant is $I = e^{t+1/x}$. This shows statement $(\overline{x})$ for systems (DP). Hence statement $(\overline{x})$ is proved.

Assume that a polynomial differential system (r) with the invariant quadratic (r) given by $\mathcal{G} = xy = 0$ has a Darboux invariant constructed uniquely from this quadric. By Proposition 8 this system has the invariant planes $g_1 = x = 0$ and $g_2 = y = 0$ with cofactors K_1 and K_2, respectively. By Proposition 6 there exist $\mu_1, \mu_2 \in \mathbb{C}$ not all zero satisfying equation (3), i.e. $\mu_1 K_1 + \mu_2 K_2 = -s$, for some $s \in \mathbb{R} \setminus \{0\}$. In particular, without loss of generality we can consider $\mu_1, \mu_2 \in \mathbb{R}$ and $\mu_2 \neq 0$. Then $K_2 = -(s + \mu_1 K_1)/\mu_2$. Therefore, from system (r) we get the following system

$$\dot{x} = xK_1, \quad \dot{y} = -\frac{1}{\mu_2} y(s + \mu_1 K_1), \quad \dot{z} = R.$$

So, taking $a = -s/\mu_2 \in \mathbb{R}\setminus\{0\}$ and $b = -\mu_1/\mu_2$, we obtain the system of statement $(\overline{r})$. Moreover, it follows directly from Proposition 6 that the Darboux invariant of this system is given by $I = y/(x^b e^{at})$. This completes the proof of statement $(\overline{r})$.

We now prove statement (s) of Theorem 5. First we write the polynomials D and G of a polynomial differential system (s) in the form $D = d + D_1$ and $G = g + G_1$, where $d, g \in \mathbb{R}$ and $D_1, G_1 \in \mathbb{R}[x, y, z]$ satisfying $D_1(0,0,0) = G_1(0,0,0) = 0$. Second we assume that a polynomial differential system (s) with the invariant quadratic (s) given by $x^2 + y^2 = 0$ has a Darboux invariant constructed uniquely from this quadric. By Proposition 8 $f_1 = x + iy = 0$ and $f_2 = x - iy = 0$ are the complex invariant planes intersecting in a real straight line with cofactors $K_1 = (x-iy)(A+iB)+2i(d+D_1)+2(g+G_1)$ and $K_2 = \bar{K}_1$ (the conjugate of K_1), respectively. By Proposition 6 there exist $\mu, \lambda \in \mathbb{C}$ not all zero satisfying equation (3), i.e.

$$(\mu + \lambda)(xA + yB + 2(g + G_1)) + (\mu - \lambda)i(xB - yA + 2(d + D_1)) = -s. \tag{16}$$

Considering $\hat{A} = xA + yB + 2G_1$ and $\hat{B} = xB - yA + 2D_1$, we can rewrite system (s) into the form

$$\dot{x} = x\hat{A} - y\hat{B} + 2(gx - dy), \quad \dot{y} = y\hat{A} + x\hat{B} + 2(gy + dx), \quad \dot{z} = R, \tag{17}$$

and from equation (16) we get

$$(\mu + \lambda)(\hat{A} + 2g) + (\mu - \lambda)i(\hat{B} + 2d) = -s. \tag{18}$$

So $\lambda = \bar{\mu}$. We take $\mu = \mu_1 + i\mu_2$, where $\mu_1, \mu_2 \in \mathbb{R}$. Then from equation (18) we obtain

$$2\mu_1(\hat{A} + 2g) - 2\mu_2(\hat{B} + 2d) = -s,$$

which provides the following linear system

$$\mu_1\hat{A} - \mu_2\hat{B} = 0, \quad \mu_1 g - \mu_2 d = -\frac{s}{4}, \tag{19}$$

in the variables μ_1 and μ_2.

We first consider $\mu_2 \neq 0$. Then from equation (19) we have that $\hat{B} = (\mu_1/\mu_2)\hat{A}$ and $d = s/(4\mu_2) + g\mu_1/\mu_2$. We obtain the first system of statement $(\overline{s})$ replacing $\hat{B}$ and d in system (17) and taking $A = \hat{A} + 2g$, $a = \mu_1/\mu_2$ and $b = s/(2\mu_2)$. Moreover from Proposition 6 and equation (4) it follows that the Darboux invariant of this system is $I = (x^2 + y^2)^b e^{-2\arctan(y/x) + 2at}$.

Now we consider $\mu_2 = 0$. Obviously $\mu_1 \neq 0$. So from equation (19) we have that $\hat{A} = 0$ and $g = -s/(4\mu_1)$. We obtain the second system of statement $(\overline{s})$

replacing $\hat{A}$ and g in this system and taking $B = \hat{B} + 2d$ and $c = -s/(2\mu_1)$. Moreover from Proposition 6 it follows that the Darboux invariant of this system is $I = (x^2 + y^2)e^{-2ct}$. This completes the proof of statement $(\overline{s})$.

Finally suppose that a polynomial differential (P) with the invariant quadratic (P) given by $\mathcal{G} = x^2 + y^2 + z^2 = 0$ has a Darboux invariant constructed uniquely from this quadric. This invariant point has cofactor $K = xA + yB + zC + G$. Let $G = -a/2 + \hat{G}$, where $a \in \mathbb{R}$ and $\hat{G} \in \mathbb{R}[x, y, z]$ with $\hat{G}(0, 0, 0) = 0$. So by Proposition 6 there exists $\mu \in \mathbb{R} \setminus \{0\}$ such that $\mu K = -s$ with $s \in \mathbb{R} \setminus \{0\}$. Then $a = s/\mu \neq 0$ and $\hat{G} = -xA - yB - zC$. Replacing G into system (P) we obtain the system of statement $(\overline{P})$. From Proposition 6 and choosing $s = 1$ we obtain the Darboux invariant $I = (x^2 + y^2 + z^2)^{1/a}e^t$ of this system. This completes the proof of statement $(\overline{P})$.

The proof of Theorem 5 is finally completed. □

5. Darboux invariants of the Chen system

The Chen system is the three-parameter family of quadratic polynomial differential equations given by

$$\dot{u} = \alpha(v - u), \quad \dot{v} = (\gamma - \alpha)u - uw + \gamma v, \quad \dot{w} = uv - \beta w, \tag{20}$$

where $(u, v, w) \in \mathbb{R}^3$ and α, β and γ are real parameters, see [2]. For suitable choices of the values of parameters α, β and γ this system exhibit chaotic phenomena. In this way, some important properties of system (20) are similar to the properties of the well-known Lorenz system [22]. Indeed it was shown recently that system (20) is topologically equivalent to the Lorenz system, see [1]. Hence the dynamical properties proved for the Chen system hold also for the Lorenz system.

The following result about system (20) was proved in [18, 17] (see also [14] for the analysis of system (20) having invariant algebraic surfaces).

Proposition 11. *If $\alpha \neq 0$ then system* (20) *has the invariant algebraic surfaces $F_i = F_i(u, v, w) = 0$, for $i = 1, ..., 6$, given as follows.*
(a) If $\beta = 2\alpha$ then $F_1 = u^2 - 2\alpha w$.
(b) If $\alpha = -\beta = \gamma$ then $F_2 = v^2 + w^2$.
(c) If $\alpha = \beta = -\gamma$ then $F_3 = 2u^2 + v^2 + w^2$.
(d) If $3\alpha + \gamma = 0$ or $\beta = 0$ then $F_4 = u^4 + \frac{4}{3}\gamma u^2 w - \frac{4}{9}\gamma^2 v^2 - \frac{8}{9}\gamma^2 uv - \frac{16}{9}\gamma^2 u^2$.
(e) If $\alpha + \gamma = 0$ or $\beta = 4\alpha$ then $F_5 = u^4 + 4\gamma u^2 w - 4\gamma^2 v^2 + 8\gamma^2 uv + 8\gamma^2 u^2 + 48\gamma^3 w$.
(f) If $\beta = \gamma = 0$ then $F_6 = v^2 + w^2 + 2\alpha w$.

Note that in Proposition 11 the invariant algebraic surfaces F_1, F_2, F_3 and F_6 are quadrics. Now we use Theorems 2, 3, 4 and 5 proved in this paper to put the Chen system in its normal form in cases (a), (b), (c) and (f) of Proposition 11 and to obtain the Darboux invariant in each one of these cases.

(a) If we take $\beta = 2\alpha$ into system (20) then $F_1 = u^2 - 2\alpha w = 0$ is an invariant algebraic surface of system (20) with cofactor $K_1 = -2\alpha$. After the change of coordinates $x = 2\alpha w$, $y = v$ and $z = u$, we can write system (20) as

$$\dot{x} = 2\alpha(yz - x), \quad \dot{y} = (\gamma - \alpha)z - \frac{xz}{2\alpha} + \gamma y, \quad \dot{z} = \alpha(y - z), \tag{21}$$

and the parabolic cylinder $\mathcal{G} = z^2 - x = 0$ is an invariant algebraic surface of system (21) with cofactor $K = -2\alpha$. Note that we can write system (21) as system (PC) of Theorem 2 if we consider $A = 2\alpha$, $C = 0$, $E = \alpha(y - z)$ and $Q = (\gamma - \alpha)z - xz/2\alpha + \gamma y$. Furthermore, system (21) is of the form $(\overline{PC})$ of Theorem 4, considering A, C, E and Q as before and $a = 2\alpha$, hence it has the Darboux invariant given by $I = (z^2 - x)^{1/2\alpha} e^t$.

(b) If we take $\alpha = -\beta = \gamma$ into system (20) then $F_2 = v^2 + w^2 = 0$ is an invariant algebraic surfaceof system (20) with cofactor $K_2 = 2\alpha$. After the change of coordinates $x = w$, $y = v$ and $z = u$, we can write system (20) as

$$\dot{x} = yz + \alpha x, \quad \dot{y} = -xz + \alpha y, \quad \dot{z} = \alpha(y - z), \tag{22}$$

and $\mathcal{G} = x^2 + y^2 = 0$ is its invariant quadric of system (22) with cofactor $K = 2\alpha$. So system (22) is of type (s) of Theorem 3 hence it has two invariant complex planes $x + iy$ and $x - iy$ intersecting at a real straight line. Note that if we consider $A = B = 0$, $D = -z/2$, $G = \alpha$ and $R = \alpha(y - z)$ into system (s) of Theorem 3 we obtain system (22). Moreover, system (22) is of the second form presented in statement $(\overline{s})$ of Theorem 5 with $c = \alpha$ and $B = -z$. Therefore it has the Darboux invariant $I = (x^2 + y^2)e^{-2\alpha t}$.

(c) If we take $\alpha = \beta = -\gamma$ into system (20) then $F_3 = 2u^2 + v^2 + w^2 = 0$ is an invariant algebraic surface. Doing the change of coordinates $x = \sqrt{2}u$, $y = v$ and $z = w$, we can write system (20) as

$$\dot{x} = \sqrt{2}\alpha y - \alpha x, \quad \dot{y} = -\sqrt{2}\alpha x - \frac{\sqrt{2}}{2} xz - \alpha y, \quad \dot{z} = \frac{\sqrt{2}}{2} xy - \alpha z, \tag{23}$$

and $\mathcal{G} = x^2 + y^2 + z^2 = 0$ is its invariant algebraic surface with cofactor $K = -2\alpha$. So system (23) is of type (P) of Theorem 3. Note that if we consider $A = B = C = E = 0$, $D = -\sqrt{2}\alpha/2$, $F = \sqrt{2}x/4$ and $G = -\alpha$ into system (P) we obtain system (23). Furthermore, system (23) is of the form $(\overline{P})$ of Theorem (5) with $a = 2\alpha$. Therefore it has the Darboux invariant $I = (x^2 + y^2 + z^2)^{1/2\alpha} e^t$.

(f) If we take $\beta = \gamma = 0$ into system (20) then $F_6 = v^2 + w^2 + 2\alpha w = v^2 + (w - \alpha)^2 - \alpha^2 = 0$ is an invariant algebraic surface of system (20). Doing the change of coordinates $x = v$, $y = u$ and $z = w + \alpha$ and considering $\alpha = 1$ we can write system (20) as

$$\dot{x} = -yz, \quad \dot{y} = x - y, \quad \dot{z} = xy, \tag{24}$$

and the elliptic cylinder $\mathcal{G} = x^2 + z^2 - 1 = 0$ is its invariant algebraic surface with cofactor $K = 0$. Hence system (24) is of type (EC) of Theorem (2) and $\mathcal{G}$ is a polynomial first integral of system (24), generating a family of cylinders invariant by its flow. We obtain the same results if we consider $\alpha = -1$. The global dynamics of the Chen system in this case was studied in [14], where the authors have shown that this system has infinitely many heteroclinic connections, each one belonging to one of these invariant cylinders.

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