This is a preprint of: "Volume entropy for minimal presentations of surface groups in all ranks", Lluís Alsedà, David Juher, Jérôme Los, Francesc Mañosas, *Geom. Dedicata*, 2015.
DOI: [10.1007/s10711-015-0103-7]

VOLUME ENTROPY FOR MINIMAL PRESENTATIONS OF SURFACE GROUPS IN ALL RANKS

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Abstract. We study the volume entropy of a class of presentations (including the classical ones) for all surface groups, called *minimal geometric presentations*. We rediscover a formula first obtained by Cannon and Wagreich [6] with the computation in a non published manuscript by Cannon [5]. The result is surprising: an explicit polynomial of degree n, the rank of the group, encodes the volume entropy of all classical presentations of surface groups. The approach we use is completely different. It is based on a dynamical system construction following an idea due to Bowen and Series [3] and extended to all geometric presentations in [15]. The result is an explicit formula for the volume entropy of minimal presentations for all surface groups, showing a polynomial dependence in the rank $n > 2$. We prove that for a surface group G_n of rank n with a classical presentation P_n the volume entropy is $\log(\lambda_n)$, where λ_n is the unique real root larger than one of the polynomial

$$x^n - 2(n-1)\sum_{j=1}^{n-1} x^j + 1.$$

1. Introduction

In the beginning of the 80's several breakthroughs occurred in group theory. The main one was the development of *large scale geometry* for groups, largely due to M. Gromov with, for instance, the classification of groups with polynomial growth function [14] or the introduction of the now standard notion of *hyperbolic groups* [13]. At about the same time R. Grigorchuck [12] found a class of groups with *intermediate growth function*. In all these classes of groups the *growth function* plays a central role. The growth function depends on the generating set X or on the presentation $P = \langle X/R \rangle$ of the group G. It is defined as the map $\mathbb{N} \mapsto \mathbb{N}$ such that

$$n \mapsto f_{G,P}(n) = \text{Card}\left\{g \in G : \text{length}_X(g) \le n\right\}.$$

From the growth function $f_{G,P}$ several asymptotic functions are defined such as the *volume entropy* or the *growth series* also called the *Poincaré series*.

Date: June 12, 2014.
1991 *Mathematics Subject Classification.* Primary: 57M07, 57M05. Secondary: 37E10, 37B40, 37B10.
Key words and phrases. Surface groups, Bowen-Series Markov maps, topological entropy, volume entropy.
The authors have been partially supported by MINECO grant numbers MTM2008-01486 and MTM2011-26995-C02-01. This work has been carried out thanks to the support of the ARCHIMEDE Labex (ANR-11-LABX- 0033).

The computational issues appeared also at about the same period. An idea due to J. Cannon [7] allows an inductive way to describe geodesics in the Cayley graph $\mathrm{Cay}^1(G, P)$ via the notion of *cone types*. This notion has been intensively used later on by Epstein, Cannon, Levy, Holt, Patterson, Thurston [9] with the introduction of a very large class of groups, called *automatic*, that contains the hyperbolic groups of Gromov. The computation of the growth function or the growth series becomes possible in principle from a geodesic automatic structure, when it exists. This is the case for hyperbolic groups. This computation, as it was noticed in [6], can also be obtained using the Floyd-Plotnick method [10].

In practice, finding an explicit geodesic automatic structure from the presentation is not so simple. For free groups with the free presentation all the computations are easy and, for instance, the volume entropy is simply $\log(2n-1)$, for the free group of rank n (see for instance [8]). The next simple case is the class of surface groups. For the classical presentations of surface groups, the growth series appeared in a paper by Cannon and Wagreich [6] without the explicit computation, leading to those series that were earlier obtained in a non published manuscript of Cannon [5]. For hyperbolic groups, the existence of a geodesic automatic structure for each presentation implies that the growth series is a rational function (see [9, 7]). In this case the volume entropy (sometimes called the critical exponent) is related to the largest pole of the growth series, i.e. the largest root of the denominator of the growth series (see for instance [4]). The result of Cannon and Wagreich for the classical presentations of surface groups shows that the denominator Q_n of the growth series is an explicit polynomial depending on the rank $n \geq 3$ of the surface group:

$$Q_n(x) := x^n - 2(n-1)\sum_{j=1}^{n-1} x^j + 1. \tag{1}$$

The fact that a single, explicit polynomial could encode the volume entropy for all surface groups is mysterious a priori, specially since the original computations of Cannon did not appear in published form.

In this paper we rediscover the polynomial $Q_n(x)$ from a completely different point of view and we hope that a part of the mystery will disappear. In our approach we compute the volume entropy of the group presentations from a dynamical system argument based on an idea due to R. Bowen and C. Series [3] and generalized in [15].

The original idea of Bowen and Series was to associate a specific map $\Phi_{B-S}\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ to a particular action of the group $G = \pi_1(S)$ on the hyperbolic plane $\mathbb{H}^2$, where $\mathbb{S}^1$ is seen as the space at infinity of $\mathbb{H}^2$. In [15], $\mathbb{S}^1$ is considered as the Gromov boundary of the group ∂G and a map $\Phi_P\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is constructed for each presentation P in a class, called *geometric*, characterized by the fact that the two dimensional Cayley complex $\mathrm{Cay}^2(G, P)$ is planar. The maps Φ_P are called *Bowen-Series-Like* and they satisfy several interesting properties, in particular the volume entropy of the presentation P equals the topological entropy of the map Φ_P. In addition the map Φ_P admits a finite Markov partition and the computation of the topological entropy for such maps is standard.

For any surface S, the classical presentation of the corresponding surface group $\Gamma = \pi_1(S)$ is geometric. These classical presentations are given by the minimal number of generators n and one relation of length $2n$. For orientable surfaces, n is even and equals $2g$, where g is the genus of the surface. In this case, the classical relation is a product of g commutators. In the non-orientable case, there is no restriction on the parity of n and the relation is given by the product of the squares of all generators (see for instance [18]). A presentation with the minimal number of generators is called *minimal*. The rank 2 cases (torus and Klein bottle) are, as usual, special: they are not hyperbolic, the growth function is quadratic and thus the volume entropy is 0. For $n > 2$ all minimal geometric presentations are proved to have the minimal volume entropy, among geometric presentations. It is conjectured that this minimum should be an absolute minimum in [15].

We rediscover here the surprising explicit polynomial $Q_n(x)$, $r \geq 3$. We will see that $Q_n(x)$ has a unique real root larger than one denoted λ_n. More precisely we prove:

Theorem 1.1. *For $n > 2$, let Γ be a surface group of rank n with a minimal geometric presentation P. Then, the volume entropy of Γ with respect to the presentation P is $\log(\lambda_n)$. Moreover, for $n \geq 4$, λ_n satisfies:*

$$2n - 1 - \frac{1}{(2n-1)^{n-2}} < \lambda_n < 2n - 1.$$

The above inequalities show that the difference between the volume entropy for the surface group and for the free group of the same rank is explicitly very small.

The genesis of this rediscovery is interesting. The dynamical system approach discussed above allows to compute the volume entropy of any geometric presentation P from an explicit Bowen-Series-Like map: $\Phi_P : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$. We developed an algorithm to compute the entropy of such maps, for the classical presentations of orientable surfaces, via the well known kneading invariant technique of Milnor and Thurston [16]. The polynomial $Q_n(x)$ appears that way in the computation for all orientable surfaces of genus $g \leq 43$. To obtain the theorem we needed to compute the determinant of a matrix whose size grows either linearly in n with polynomial entries using the Milnor-Thurston method, or quadratically in n with integer entries using the Markov matrix method. The computation leading to the proof of the theorem became possible by a succession of two surprises. First, by a particular choice of a minimal presentation P_n^+ the corresponding BSL map $\Phi_{P_n^+}$ admits an explicit symmetry of order $2n$. By a quotient process, the Markov matrix is reduced to an integer matrix whose size grows linearly in n. Then a method, developed in [2] under the nice name of the "Rome technique", was directly applicable to our case and reduced the computation to a 2×2 matrix with polynomial entries and no computer was necessary.

The paper is organized as follows. In Section 2 we recall the necessary ingredients for the construction of the map Φ_P in some particular geometric presentations. The map is then given explicitly for the particular minimal geometric presentations with a symmetry property, together with its Markov partition. In Section 3 we obtain a first formula for the volume entropy in the orientable case, in terms of the Markov matrix of the Bowen-Series-Like map. We exploit the symmetry of the presentation to obtain a formula for the volume entropy in terms of the spectral radius of a simpler matrix called the *compacted matrix of rank n*. In Section 4 we extend these results to the non-orientable case by showing that the volume entropy in this case is also the logarithm of the spectral radius of the compacted matrix of rank n. The computation of the spectral radius of this new matrix is still somewhat difficult. In Section 5 we obtain a simpler matrix with the same spectral radius. Finally, in Section 6, the "Rome method" is explained and applied to compute this spectral radius, and proving Theorem 1.1.

2. Bowen-Series-Like maps for geometric presentations

In this section we review the necessary ingredients for the construction of the Bowen-Series-Like maps defined in [15].

2.1. Geometric presentations.

Let $P = \langle X/R \rangle = \langle x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1} / R_1, \dots, R_k \rangle$ be a presentation of a group Γ. Recall that the Cayley graph $\mathrm{Cay}^1(\Gamma, P)$ is a metric space and let B_m be the ball of radius m centred at the identity. We denote the cardinality of any finite set A by $|A|$. The *volume entropy* of Γ with respect to the presentation P is denoted by $h_{\mathrm{vol}}(\Gamma, P)$ and defined as:

$$\lim_{m\to\infty} \frac{1}{m} \log|B_m|.$$

A presentation P of a surface group $\Gamma = \pi_1(S)$ is called *geometric* if the Cayley 2-complex $\mathrm{Cay}^2(\Gamma, P)$ is a plane. In particular the Cayley graph $\mathrm{Cay}^1(\Gamma, P)$ is a planar graph. A geometric presentation P is called *minimal* if the number of generators is minimal. For a group of an orientable surface of genus g it is well known that the minimal number of generators is $2g$ (see [18] for instance) and, in this case, there is a presentation with a single relation of length $4g$. The standard classical presentation in this case is the following:

$$\left\langle x_1^{\pm 1}, y_1^{\pm 1}, x_2^{\pm 1}, y_2^{\pm 1}, \dots, x_g^{\pm 1}, y_g^{\pm 1} / \prod_{i=1}^{g} [x_i, y_i] \right\rangle,$$

where $[x_i, y_i] = x_i \cdot y_i \cdot x_i^{-1} \cdot y_i^{-1}$ is a commutator.

For a rank n group of a non-orientable surface there is also a classical presentation with a single relation of length $2n$:

$$\left\langle x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1} / \prod_{i=1}^{n} x_i^2 \right\rangle.$$

It is easy to check that such classical presentations are geometric (see below).

Geometric presentations satisfy very simple combinatorial properties:

Lemma 2.1 (Floyd and Plotnick [10]). *If* $P = \langle x_1^{\pm 1}, \dots, x_n^{\pm 1} / R_1, \dots, R_k \rangle$ *is a geometric presentation of a surface group* Γ *then* P *satisfies the following properties:*

(a) *The set* $\{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$ *admits a cyclic ordering that is preserved by the* Γ*-action.*
(b) *Each generator appears exactly twice (with plus or minus exponent) in the set* $R = \{R_1, \ldots, R_k\}$ *of relations.*
(c) *Each pair of adjacent generators, according to the cyclic ordering (a), appears exactly once in* R *and defines uniquely a relation* $R_i \in R$.

The following statement is the main ingredient to compute the volume entropy of a geometric presentation. The statement also contains the main result about minimal geometric presentations. In what follows $\mathbb{S}^1$ will denote a (topological) circle. Recall that any surface group Γ is Gromov-hyperbolic [13] and its boundary is: $\partial\Gamma \simeq \mathbb{S}^1$.

Let us introduce the notion of a Markov partition. Let W be a finite set of $\mathbb{S}^1$. An interval of $\mathbb{S}^1$ will be called W*-basic* if it is the closure of a connected component of $\mathbb{S}^1 \setminus W$. Observe that two different W-basic intervals have pairwise disjoint interiors. Let $\phi \colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ and let $W \subset \mathbb{S}^1$ be finite. We say that W is a *Markov partition of* ϕ if W is $\phi-$invariant (i.e., $\phi(W) \subset W$) and the image by ϕ of every basic interval is a union of basic intervals.

Theorem 2.2 (Los [15])**.** *Let* P *be a geometric presentation of a surface group* Γ*. Then there exists a map* $\Phi_P \colon \partial\Gamma = \mathbb{S}^1 \longrightarrow \partial\Gamma = \mathbb{S}^1$ *with the following properties:*

(a) *The map* Φ_P *is Markov, i.e. it admits a finite Markov partition.*
(b) *The topological entropy of* Φ_P*,* $h_{top}(\Phi_P)$*, is equal to the volume entropy* $h_{vol}(\Gamma, P)$*.*

In addition, the volume entropy is minimal, among geometric presentations, for all minimal geometric presentations.

The map Φ_P satisfies more properties that are not needed here. Property (a) is specially interesting for computations since, for Markov maps, it is classical that the topological entropy is nothing but the logarithm of the spectral radius of a finite integer matrix, the Markov transition matrix (see [17] or [1] for instance). The goal of the next sections is to make such a Markov partition explicit in the particular cases of minimal geometric presentations.

2.2. **Construction of the Bowen-Series-Like map.**

In this subsection we review the definition and the necessary properties of the BSL maps, in the particular case of minimal geometric presentations.

2.2.1. *Bigons.* As we have seen, a presentation $P = \langle X/R \rangle$ defines the Cayley graph $\mathrm{Cay}^1(\Gamma, P)$ and the Cayley 2-complex $\mathrm{Cay}^2(\Gamma, P)$. A *bigon* in $\mathrm{Cay}^1(\Gamma, P)$ is a pair of distinct geodesics $\{\gamma_1, \gamma_2\}$ connecting two vertices $\{v, v'\} \in \mathrm{Cay}^1(\Gamma, P)$. We denote by $B_v(x, y)$ the set of bigons $\{\gamma_1, \gamma_2\}$ whose initial vertex is v and so that the geodesic γ_1 starts at v by the edge labelled x and γ_2 starts at v by the edge labelled y, with $x \neq y$. By the Γ-action we can fix the initial vertex v to be the identity and we denote $B_{\mathrm{id}}(x, y)$ by $B(x, y)$.

For geometric presentations of surface groups the set of bigons is particularly simple.

Lemma 2.3. *If $P = \langle X/R \rangle$ is a geometric presentation of a surface group Γ then the set of bigons $B(x, y)$ is non empty if and only if (x, y) is an adjacent pair of generators, according to the cyclic ordering of Lemma 2.1(a). In addition, if (x, y) is an adjacent pair of generators there is a unique bigon $\beta(x, y) \in B(x, y)$ of finite minimal length, called* minimal bigon.

This lemma is proved in [15, Lemmas 2.6 and 2.12]. Observe that each minimal bigon is particularly simple for a geometric presentation where all relations have even length. Indeed, each pair of adjacent generators (x, y) defines a unique relation by Lemma 2.1(c). The relation can be written, up to cyclic permutation and inversion ($R_i \to (R_i)^{-1}$), as: $R_i = \gamma_1 \cdot (\gamma_2)^{-1}$, where γ_1 is a word (or a path) starting by the letter x, γ_2 starts by the letter y and $l(\gamma_1) = l(\gamma_2)$. The observation is that for geometric presentations the two paths γ_1 and γ_2 start at the identity and end at the same vertex (since R_i is a relation) and are geodesics. In other words the pair $\{\gamma_1, \gamma_2\}$ is a bigon. It is minimal and unique by Lemma 2.1(c).

2.2.2. *Bigon-Rays.* We describe a canonical way to define a point on the boundary $\partial\Gamma$ associated to an adjacent pair of generators (x, y). Recall that a surface group is hyperbolic in the sense of Gromov [13] and its boundary $\partial\Gamma$ is the circle $\mathbb{S}^1$. By definition of $\partial\Gamma$, a point $\xi \in \partial\Gamma$ is the limit of geodesic rays, for instance starting at the identity, modulo the equivalence relation among rays that two rays are equivalent if they stay at a uniform bounded distance from each others (c.f. [13]). If $\xi \in \partial\Gamma$ is a point on the boundary we denote by $\{\xi\}$ a geodesic ray starting at identity and converging to ξ.

In what follows, given two integers k and l we will denote $k \pmod{l}$ by $[k]_l$. Also, we choose $1, 2, \ldots, l$ as the representatives of the classes modulo l; that is, $[0]_l = [l]_l = l$. However, unless necessary we omit the modulo part in the notations.

Notation 2.4. In what follows we denote the n generators (and their inverses) by $y_1, y_2, \ldots, y_{2n}$ in such a way that $y_{[i\pm1]_{2n}}$ are the elements adjacent to y_i with respect to the cyclic ordering from Lemma 2.1(a). We denote an adjacent pair by $(y_i, y_{[i+1]_{2n}})$ where, by convention, the edges denoted y_i and $y_{[i+1]_{2n}}$ are adjacent and oriented from the vertex. We also adopt the convention that y_i is on the *left* of $y_{[i+1]_{2n}}$ (see Figure 1). This convention defines an orientation of the plane $\mathrm{Cay}^2(\Gamma, P)$.

The parity of the number of adjacent pairs at each vertex implies that $(y_i, y_{[i+1]_{2n}})$ defines an *opposite* pair, with respect to the cyclic ordering of Lemma 2.1(a), defined by:

$$(y_i, y_{[i+1]_{2n}})^{\mathrm{opp}} := (y_{[i+n]_{2n}}, y_{[i+n+1]_{2n}})$$

(see Figure 2).

We construct a unique infinite sequence of adjacent pairs, bigons and vertices from any given pair $(y_i, y_{[i+1]_{2n}})$ by the following process:

Step 1. Each adjacent pair, at the identity, defines a unique minimal bigon $\beta(y_i, y_{i+1})$ by Lemma 2.3. The bigon $\beta(y_i, y_{i+1})$ is a pair of geodesics $\{\gamma_l, \gamma_r\}$, where the indices l, r stand for left and right, with respect to an orientation of the plane $\mathrm{Cay}^2(\Gamma, P)$. The geodesics $\{\gamma_l, \gamma_r\}$ connect the identity to a vertex $v_1 = v_1[\beta(y_i, y_{i+1})]$.

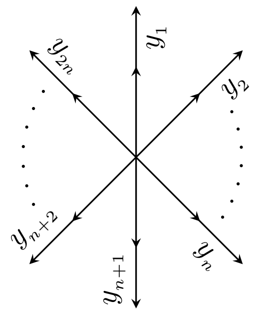

FIGURE 1. The labelling of the generators (and the cyclic ordering) fixed in Notation 2.4.

Step 2. The two geodesics $\{\gamma_l, \gamma_r\}$ end at v_1 by two generators that are adjacent by Lemma 2.3. Therefore the bigon $\beta(y_i, y_{i+1})$ defines a unique adjacent pair at v_1, called a *top pair* of $\beta(y_i, y_{i+1})$, which is denoted: $\text{topp}[\beta(y_i, y_{i+1})]$, based at $v_1 = v_1[\beta(y_i, y_{i+1})]$ and is uniquely defined by (y_i, y_{i+1}).

Step 3. The pair $\text{topp}[\beta(y_i, y_{i+1})]$ defines an opposite pair at v_1, denoted by:

$$\big(\text{topp}[\beta(y_i, y_{i+1})]\big)^{\text{opp}} := (y_i, y_{i+1})^{(1)}.$$

Step 4. We consider then the unique minimal bigon, at v_1, defined by the pair $(y_i, y_{i+1})^{(1)}$ by Lemma 2.3:

$$\beta^{(1)}(y_i, y_{i+1}) := \beta_{v_1}[(y_i, y_{i+1})^{(1)}].$$

Step 5. The bigon $\beta^{(1)}(y_i, y_{i+1})$ defines a new top pair $\text{topp}[\beta^{(1)}(y_i, y_{i+1})]$, at the vertex v_2.

The Steps 1–5 define, by induction, a unique infinite sequence of vertices and bigons (see Figure 2):

$$\begin{aligned} &\text{id}, v_1, v_2, \cdots \\ &\beta(y_i, y_{i+1}), \beta^{(1)}(y_i, y_{i+1}), \beta^{(2)}(y_i, y_{i+1}) \cdots \ . \end{aligned} \tag{2}$$

Each bigon in the infinite sequence $\{\beta^{(k)}(y_i, y_{i+1}) : k \in \mathbb{N}\}$ is a pair of geodesics $\left\{\gamma_l^{(k)}, \gamma_r^{(k)}\right\}$ with $k \in \mathbb{N}$ connecting the vertices v_k and v_{k+1}.

By definition, the terminal vertex v_{k+1} of $\beta^{(k)}$ is the initial vertex of the next bigon $\beta^{(k+1)}$ in the sequence. Therefore a *finite concatenation* of bigons $\beta^{(0)}(y_i, y_{i+1})\beta^{(1)}(y_i, y_{i+1}) \cdots \beta^{(k)}(y_i, y_{i+1})$ makes sense. It is defined by the finite collection of paths:

$$\left\{\gamma^{(0)}_{\epsilon(0)} \cdot \gamma^{(1)}_{\epsilon(1)} \cdots \gamma^{(k)}_{\epsilon(k)} : \epsilon(j) \in \{l, r\},\ j \in \{0, 1, \ldots, k\}\right\}.$$

We denote the infinite concatenation of all these paths as:

$$\beta^{\infty}(y_i, y_{i+1}) := \lim_{k\to\infty} \beta^{(0)}(y_i, y_{i+1})\beta^{(1)}(y_i, y_{i+1}) \cdots \beta^{(k)}(y_i, y_{i+1}).$$

Lemma 2.5 (Los [15, Lemma 3.1])**.** *With the above notation the following statements hold.*

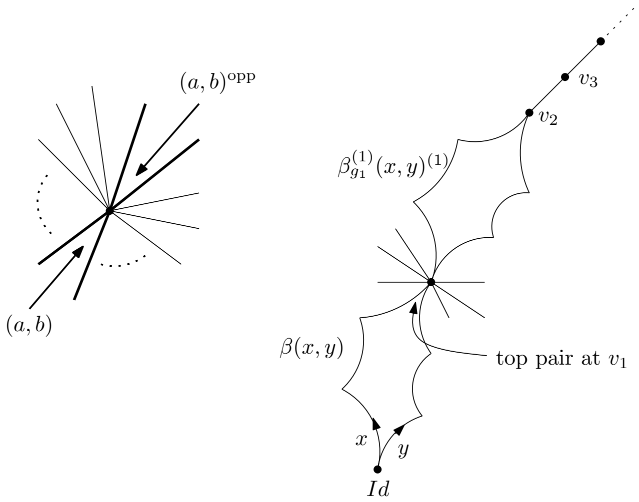

FIGURE 2. Opposite pair and bigon rays.

(a) *Each path in the collection:* $\beta^{(0)}(y_i,y_{i+1})\beta^{(1)}(y_i,y_{i+1})\cdots\beta^{(k)}(y_i,y_{i+1})$ *is a geodesic segment, for all* $k\in\mathbb{N}$*.*
(b) *Two geodesic segments in (a) stay at a uniform distance from each other for any* $k\in\mathbb{N}$*.*

In consequence, the infinite concatenation $\beta^\infty(y_i,y_{i+1})$ defines infinitely many geodesic rays with a unique limit point in $\partial\Gamma$. It will be denoted by $(y_i,y_{i+1})^\infty$.

2.2.3. *Cylinders, definition of the BSL map.* We define the *cylinder* of length one as the subset of the boundary:

$$\mathcal{C}_x := \{\xi\in\partial\Gamma : \text{there is a geodesic ray } \{\xi\} \text{ starting at id by } x\in X\}\,.$$

Lemma 2.6. *Let* $P=\langle X/R\rangle$ *be a geometric presentation of* Γ*. The boundary* $\partial\Gamma=\mathbb{S}^1$ *is covered by the cylinder sets* $\mathcal{C}_x,\ x\in X$ *and:*
(a) *Two cylinders have non-empty intersection:* $\mathcal{C}_x\bigcap\mathcal{C}_y\neq\emptyset$ *if and only if* (x,y) *is an adjacent pair of generators.*
(b) *Each cylinder* $\mathcal{C}_x,\ x\in X$ *is a non trivial connected interval of* $\partial\Gamma$*.*

This lemma is proved in [15, Lemmas 2.13 and 2.14]. Observe that the point $(y_i,y_{i+1})^\infty$ of Lemma 2.5 belongs, by definition, to the intersection $\mathcal{C}_{y_i}\bigcap\mathcal{C}_{y_{i+1}}$.

In what follows we consider the points in the circle ordered *clockwise.* That is, if r,s,t are pairwise different points of $\mathbb{S}^1$ we will write $r<s<t$ if s belongs to the clockwise arc starting at r and ending at t. The notation $r\le s\le t$ will also be used in the natural way. Then the interval $[r,t]$ is defined as the set $\left\{s\in\mathbb{S}^1 : r\le s\le t\right\}$. Also, if I,J,K are closed connected subsets of $\mathbb{S}^1$ with pairwise disjoint interiors we will write $I<J<K$ whenever $r\le s\le t$ for every $r\in I,\ s\in J$ and $t\in K$.

Definition 2.7. If $P=\langle X/R\rangle$ is a geometric presentation of a hyperbolic surface group Γ, then we denote by I_{y_i} the interval $[(y_{i-1},y_i)^\infty,(y_i,y_{i+1})^\infty]$. Clearly I_{y_i} is a subset of $\mathcal{C}_{y_i}$ for every $y_i\in X$.

We define the Bowen-Series-Like map $\Phi_P\colon \partial\Gamma \longrightarrow \partial\Gamma$ by

$$\Phi_P(\xi) = x^{-1}(\xi) \quad \text{if } \xi \in I_x,$$

where $x^{-1}(\xi)$ is the action, by homeomorphism, on $\partial\Gamma$ by the group element x^{-1}.

The map Φ_P satisfies the following elementary properties:

(i) It depends explicitly on the presentation P (the exact dependence will be explained below).

(ii) Since $I_x \subset \mathcal{C}_x$, each $\xi \in I_x$ has a writing, as a limit of a ray, as $\{\xi\} = x \cdot \omega$. The image under Φ_P is given by:

$$\{\Phi_P(\xi)\} = \{x^{-1}(x \cdot \omega)\} = \{\omega\}.$$

In other words, the map Φ_P is a shift map, on this particular writing as a ray.

2.3. **Markov partition for minimal geometric presentations.**

Theorem 2.2 states that the map Φ_P admits a Markov partition. In this subsection we will define a particular presentation, which will be called *symmetric*, and we will make the Markov partition explicit for this presentation.

The first step is to define subdivision points in each interval I_x, $x \in X$. Let us recall that the extreme points $(y,x)^\infty$ and $(x,z)^\infty$ of the intervals I_x are limit points of bigon rays $\beta^\infty(y,x)$ and $\beta^\infty(x,z)$. Let us focus on $(y,x)^\infty$. Let $\beta^\infty_v(y,x)$ be the bigon ray starting at the vertex $v \in \mathrm{Cay}^1(\Gamma,P)$. Observe that with this definition we can write:

$$\beta^\infty(y,x) = \beta(y,x) \cdot \beta^\infty_{v_1}[(y,x)^{(1)}], \tag{3}$$

with the notations of Subsection 2.2.2.

The particular property of a minimal geometric presentation that is useful here is that there is only one relation R of even length $2n$, when Γ is a surface group of rank n. In this case, any bigon $\beta(y,x)$ has the form $\{\gamma_l,\gamma_r\}$ with $\gamma_l \cdot (\gamma_r)^{-1}$ being one of the words representing the relation R, up to cyclic permutation and inversion. This word starts with the letter y and terminates with the letter x^{-1}.

Since the relation R has length $2n$, let us write the two paths $\{\gamma_l,\gamma_r\}$ as:

$$\{y \cdot x'_{i_2} \cdots x'_{i_n}, x \cdot x_{i_2} \cdots x_{i_n}\}. \tag{4}$$

We focus on the "x" side of Equations (3),(4), i.e. on the infinite collection of rays:

$$x \cdot x_{i_2} \cdots x_{i_n} \cdot \beta^{(\infty)}_v[(y,x)^{(1)}], \tag{5}$$

where v is the group element written: $v = x \cdot x_{i_2} \cdots x_{i_n}$. The vertices $v^1 = x$ and $v^j = x \cdot x_{i_2} \cdots x_{i_j}$, for $j = 2,3,\ldots,n-1$ of $\mathrm{Cay}^1(\Gamma,P)$ belong to γ_r and are ordered along γ_r (this notation is consistent with $v = v_n$).

The following pairs of consecutive edges:

$$\left\{(\overline{x}, x_{i_2}), (\overline{x_{i_2}}, x_{i_3}), \ldots, (\overline{x_{i_{n-1}}}, x_{i_n})\right\} \tag{6}$$

at the vertices $\left\{v^1,\ldots,v^{n-1}\right\}$, are *crossed* by the path γ_r, where the notation $\overline{x_{i_j}}$ means the edge x_{i_j} with the opposite orientation. Observe that each pair of consecutive letters along the paths γ_r are adjacent generators.

Lemma 2.8. *If the relation defining $\beta(y,x)$ has even length $2n$ then the collection:*

$$\mathcal{R}_L^x := \left\{ x \cdot x_{i_2} \cdots x_{i_j} \cdot \beta_{v^j}^{(\infty)}[(\overline{x_{i_j}}, x_{i_{j+1}})^{opp}] : j = 1, \dots, n-1 \right\}, \tag{7}$$

(see Figure 3) is called the left *(with respect to x)* subdivision rays*. They satisfy the following properties:*

(a) Each path in the infinite collection $\mathcal{R}_L^x$ is a ray starting at the identity.

(b) For a given $j \in \{1, 2, \dots, n-1\}$, all the rays in

$$\mathcal{R}_L^{(x,j)} = x \cdot x_{i_2} \cdots x_{i_j} \cdot \beta_{v^j}^{(\infty)}[(\overline{x_{i_j}}, x_{i_{j+1}})^{opp}]$$

converge to the same point $\lambda_x^j \in \partial\Gamma$.

(c) For any $j \neq p$, the rays in $\mathcal{R}_L^{(x,j)}$ and in $\mathcal{R}_L^{(x,p)}$ have a common beginning: $x \cdot x_{i_2} \cdots x_{i_\nu}$ where $\nu := \min\{j, p\}$ and are otherwise disjoint.

(d) Each λ_x^j, $j \in \{1, 2, \dots, n-1\}$ belongs to the interior of the interval I_x of Definition 2.7.

(e) The limit points λ_x^j are inversely ordered with respect to the index $j \in \{1, 2, \dots, n-1\}$ along $\partial\Gamma$ (that is, $\lambda_x^{n-1} < \lambda_x^{n-2} < \cdots < \lambda_x^2 < \lambda_x^1$).

This lemma is proved in [15, Lemma 4.1].

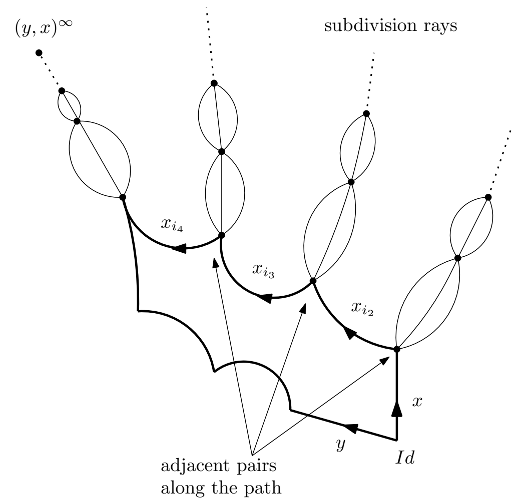

FIGURE 3. Subdivision rays.

We denote $\mathcal{L}_x = \{\lambda_x^1, \dots, \lambda_x^{n-1}\}$ this set of *left* (with respect to x) limit points. By the same analysis the adjacent pair (x, z) defines the set of *right* (with respect to x) limit points $\mathcal{R}_x = \{\rho_x^1, \dots, \rho_x^{n-1}\}$, which are ordered with respect to the superindex. Observe that we use here the fact that a minimal geometric presentation has only one relation (of length $2n$). Consider now the set of all such points:

$$\mathcal{S} = \bigcup_{x \in X} (\mathcal{R}_x \cup \mathcal{L}_x \cup \partial I_x), \tag{8}$$

called the *subdivision points*.

Lemma 2.9. *If P is a geometric presentation of a hyperbolic surface group Γ so that all relations have even length, then the set of subdivision points $\mathcal{S}$ is invariant under the map Φ_P of Definition 2.7 and defines a finite Markov partition of $\partial\Gamma$.*

This statement is a particular case of [15, Theorem 4.3]. For a minimal geometric presentation there is only one relation of length $2n$ for a surface group of rank n. In this case the partition of each interval I_x above is given by the points $\mathcal{R}_x \cup \mathcal{L}_x \cup \partial I_x$ which are ordered in the following way:

$$\lambda_x^n := (y,x)^\infty < \lambda_x^{n-1} < \cdots < \lambda_x^2 < \rho_x^1 < \lambda_x^1 < \rho_x^2 < \cdots < \rho_x^n := (x,z)^\infty .$$

We also observe here that the intervals I_x are ordered, along $\mathbb{S}^1$, by the (cyclic) ordering of the generators at the identity. Then, we can define a partition of each of the intervals I_x consisting on the following subintervals:

$$\begin{aligned} & L_x^j = \left[\lambda_x^j, \lambda_x^{j-1}\right] \text{ and } R_x^j = \left[\rho_x^{j-1}, \rho_x^j\right], \text{ for } j \in \{3,4,\ldots,n\}, \\ & C_x^L = \left[\lambda_x^2, \rho_x^1\right] \text{ and } C_x^R = \left[\lambda_x^1, \rho_x^2\right], \text{ and} \\ & C_x = \left[\rho_x^1, \lambda_x^1\right]. \end{aligned} \tag{9}$$

Recall that a subdivision point (left or right) has the following writing: $\{\lambda_x^j\} = x \cdot x_{i_2} \cdots x_{i_j} \cdot \beta_{v^j}^\infty[(\overline{x_{i_j}}, x_{i_{j+1}})^{\text{opp}}]$, for $j \in \{1,2,\ldots,n\}$.

Since the map Φ_P acts, on each interval I_x, on the ray writing as a shift map we obtain:

$$\begin{aligned} & \{\Phi_P(\lambda_x^1)\} = \beta^\infty[(\overline{x}, x_{i_2})^{\text{opp}}], \text{and} \\ & \{\Phi_P(\lambda_x^j)\} = x_{i_2} \cdots x_{i_j} \cdot \beta^\infty_{x_{i_2}\cdots x_{i_j}}[(\overline{x_{i_j}}, x_{i_{j+1}})^{\text{opp}}] \text{ for } j \in \{2,3,\ldots,n\} \end{aligned} \tag{10}$$

and there is a similar writing for the points ρ_x^j.

Lemma 2.10. *If P is a geometric presentation of a surface group with all relations of even length then the image of the* central *interval $C_x = [\rho_x^1, \lambda_x^1]$ under Φ_P is a single interval I_u, $u \in X$, where u is the generator that is opposite to x^{-1} for the cyclic ordering of Lemma 2.1(a) at the vertex x.*

Proof. By (10) we observe that $\{\Phi_P(\lambda_x^1)\} = \beta^\infty[(\overline{x}, x_{i_2})^{\text{opp}}]$ and similarly $\{\Phi_P(\rho_x^1)\} = \beta^\infty[(x'_{i_2}, \overline{x})^{\text{opp}}]$. Since the two adjacent pairs $(\overline{x}, x_{i_2})$ and $(x'_{i_2}, \overline{x})$ are adjacent at the vertex $v_1 = x$ then the two opposite pairs $(\overline{x}, x_{i_2})^{\text{opp}}$ and $(x'_{i_2}, \overline{x})^{\text{opp}}$ are also adjacent. That means that they share one edge u. This edge is just the one that is opposite to x^{-1} at the vertex x (see Figure 4). □

Next we define a particular presentation, which we call *symmetric*, for the rank n group $\pi_1(S)$, where S is an orientable surface. Recall that $n = 2g$, where g is the genus of S.

Definition 2.11. Given a surface group $\pi_1(S_g)$ of rank $n = 2g$, where S_g is orientable of genus g, the presentation

$$\left\langle x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1} \,/ x_1 x_2 \cdots x_n x_1^{-1} x_2^{-1} \cdots x_n^{-1} \right\rangle$$

will be called *symmetric* and denoted by P_n^+.

Proposition 2.12. *The symmetric presentation P_n^+ is minimal and geometric.*

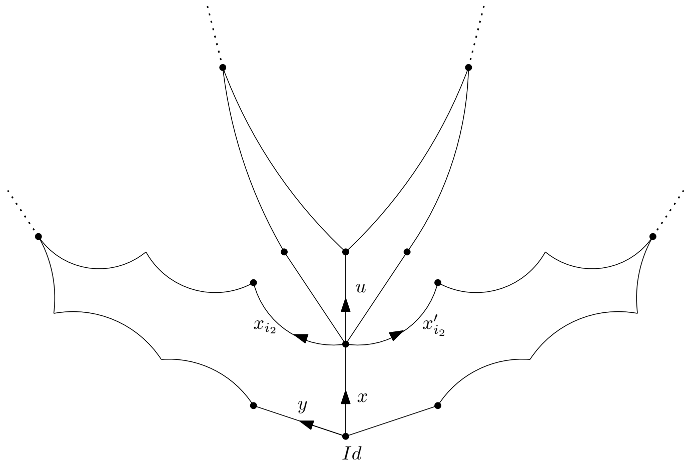

FIGURE 4. Central Interval.

Proof. Consider the polygon Δ_n with $2n$ sides, labelled by the elements of $\{x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}\}$ in the ordering:

$$x_1 \cdot x_2 \cdots x_n \cdot x_1^{-1} \cdot x_2^{-1} \cdots x_n^{-1}.$$

The identification of the side labelled x_i with the one labelled x_i^{-1} defines an orientable surface of genus g. The identification is an equivalence relation $\sim$ and $\Delta_n/\sim$ is the surface of genus g. The presentation P_n^+ is minimal since it has n generators and it is geometric because the universal cover of the surface $\Delta_n/\sim$ is nothing but the Cayley 2-complex $\mathrm{Cay}^2(\Gamma, P_n^+)$ that is a plane. □

Lemma 2.1 says that for geometric presentations the generators have a cyclic ordering at each vertex. For the presentation P_n^+ the cyclic ordering is

$$x_1 < x_2^{-1} < x_3 < x_4^{-1} < \cdots < x_{n-1} < x_n^{-1} < x_1^{-1} < x_2 < \cdots < x_{n-1}^{-1} < x_n.$$

3. The topological entropy of the map $\Phi_{P_n^+}$

The aim of this section is to start the computation of the topological entropy of the Bowen-Series-Like map $\Phi_{P_n^+}$ for the symmetric presentation $P_n^+ = \langle X/R \rangle$ of the orientable surface group of rank n.

Since the surface is orientable and the presentation is geometric and minimal, then all generators $x \in X$ act on $\partial\Gamma$ as an orientation preserving homeomorphism. By Definition 2.7(ii), $\Phi_P\big|_{I_x}$ is an orientation preserving homeomorphism for every $x \in X$ and from Lemma 2.9 the set $\mathcal{S}$ defines a Markov partition of Φ_P. Since $\partial I_{x_i} \subset \mathcal{S}$ we also have that Φ_P is a homeomorphism on every $\mathcal{S}-$basic interval.

In this situation the topological entropy can be easily computed as the logarithm of the spectral radius of the associated *Markov matrix.* Let us recall such result.

Let W be a Markov partition of a map $\phi\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$, and let $U_1, U_2, \dots,$ $U_{|W|}$ be a labelling of the $W-$basic intervals. The *Markov matrix of W* is

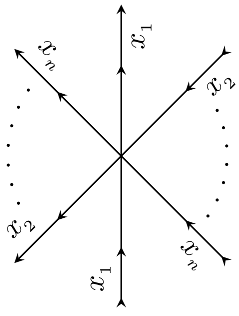

FIGURE 5. The cyclic ordering of the symmetric presentation.

defined as the $|W| \times |W|$ $(0,1)-$matrix $M = (m_{ij})_{i,j=1}^{|W|}$ such that $m_{ij} = 1$ if and only if $\phi(U_i) \supset U_j$.

For any square matrix M, we will denote its *spectral radius* by $\rho(M)$.

It is well known (see for instance [2] or [1, Theorem 4.4.5]), that if ϕ is monotone on each basic interval then

$$h_{\text{top}}(\phi) = \log \max\{\rho(M), 1\}. \tag{11}$$

We will use (11) to compute $h_{\text{top}}(\Phi_{P_n^+})$. To this end we first have to compute the Markov matrix of $\mathcal{S}$ that, in what follows, will be denoted by M_n^+. As we will see, a direct computation of $\rho(M_n^+)$ is infeasible at a practical level because the size of the matrix grows quadratically with n. So, the computation of $\rho(M_n^+)$ will be done in two steps by using spectral radius preserving transformations of the matrix M_n^+. In this section we will compute the Markov matrix M_n^+ for a symmetric presentation in the orientable case.

To do this, we need to specify completely the map $\Phi_{P_n^+}$ and then compute its Markov matrix. Recall that, for the symmetric presentation P_n^+ (see comments after Proposition 2.12), the cyclic ordering of Lemma 2.1 at any vertex is given by:

$$x_1 < x_2^{-1} < \cdots < x_{n-1} < x_n^{-1} < x_1^{-1} < x_2 < \cdots < x_{n-1}^{-1} < x_n < x_1$$

(see Figure 5). The main property of this cyclic ordering which makes the symmetric presentation very special and useful is that the edge that is opposite to x at any vertex is simply the edge x^{-1}.

The above cyclic ordering of the generators induces the following ordering of the intervals I_x along the boundary $\partial\Gamma = \mathbb{S}^1$:

$$I_{x_1} < I_{x_2^{-1}} < \cdots < I_{x_n^{-1}} < I_{x_1^{-1}} < I_{x_2} < \cdots < I_{x_n} < I_{x_1}.$$

The fact that the symmetric presentation has associated the above cyclic ordering gives the following immediate corollary of Lemma 2.10:

Corollary 3.1. *Let P_n^+ be the symmetric presentation of an orientable surface group of rank n. Then, $\Phi_{P_n^+}(C_x) = I_x$ for each generator x.*

For notational reasons we denote the ordered generators as

$$y_1 < y_2 < \cdots < y_{2n} < y_1$$

and the corresponding intervals as

$$I_{y_1} < I_{y_2} < \cdots < I_{y_{2n}} < I_{y_1},$$

where $y_i = x_i^{(-1)^{i+1}}$ for $1 \le i \le n$, and $y_i = x_{i-n}^{(-1)^i}$ for $n+1 \le i \le 2n$. Also, the fact that the edge that is opposite to x at any vertex is the edge x^{-1} now gives

$$y_i^{-1} = y_{[i+n]_{2n}}. \tag{12}$$

Observe from (9) that each of the $2n$ intervals I_{y_i} is divided into $2n-1$ intervals

$$L^n_{y_i} < \cdots < L^3_{y_i} < C^L_{y_i} < C_{y_i} < C^R_{y_i} < R^3_{y_i} < \cdots < R^n_{y_i}. \tag{13}$$

Hence, $|\mathcal{S}| = 2n(2n-1)$ and thus, the matrix M_n^+ is $2n(2n-1) \times 2n(2n-1)$.

Equations (10), (12) and Corollary 3.1 give the following images of the partition intervals defined in (9) (see Figure 6):

$$\begin{aligned}
&\Phi_{P_n^+}(L^j_{y_i}) = L^{j-1}_{y_{[i+n+1]_{2n}}} \text{ for } j \in \{4,5,\dots,n\},\\
&\Phi_{P_n^+}(L^3_{y_i}) = C^L_{y_{[i+n+1]_{2n}}} \cup C_{y_{[i+n+1]_{2n}}},\\
&\Phi_{P_n^+}(C^L_{y_i}) = C^R_{y_{[i+n+1]_{2n}}} \cup \left(\bigcup_{j=2}^{n} R^j_{y_{[i+n+1]_{2n}}}\right) \cup \left(\bigcup_{k=[i+n+2]_{2n}}^{[i-1]_{2n}} I_{y_k}\right),\\
&\Phi_{P_n^+}(C_{y_i}) = I_{y_i},\\
&\Phi_{P_n^+}(C^R_{y_i}) = C^L_{y_{[i+n-1]_{2n}}} \cup \left(\bigcup_{j=2}^{n} L^j_{y_{[i+n-1]_{2n}}}\right) \cup \left(\bigcup_{k=[i+1]_{2n}}^{[i+n-2]_{2n}} I_{y_k}\right),\\
&\Phi_{P_n^+}(R^3_{y_i}) = C_{y_{[i+n-1]_{2n}}} \cup C^R_{y_{[i+n-1]_{2n}}},\\
&\Phi_{P_n^+}(R^j_{y_i}) = R^{j-1}_{y_{[i+n-1]_{2n}}} \text{ for } j \in \{4,5,\dots,n\}.
\end{aligned} \tag{14}$$

From the formulae (14) it follows that the Markov matrix M_n^+ has a structure in blocks, all of size $(2n-1) \times (2n-1)$. So, it is convenient to write the matrix M_n^+ as

$$\begin{pmatrix}
M_{11} & M_{12} & \dots & M_{1,2n}\\
M_{21} & M_{22} & \dots & M_{2,2n}\\
\cdots & \cdots & \cdots & \cdots\\
M_{n1} & M_{n2} & \dots & M_{n,2n}\\
\cdots & \cdots & \cdots & \cdots\\
M_{2n,1} & M_{2n,2} & \dots & M_{2n,2n}
\end{pmatrix} \tag{15}$$

where each of the matrices $M_{lt} = (m^{lt}_{ij})_{i,j=1}^{2n-1}$ is of size $(2n-1) \times (2n-1)$.

Accordingly, we will label the basic intervals contained in I_{y_i} as U^i_j in such a way that they preserve the ordering given in (13). So, for $i = 1, 2, \dots, 2n$

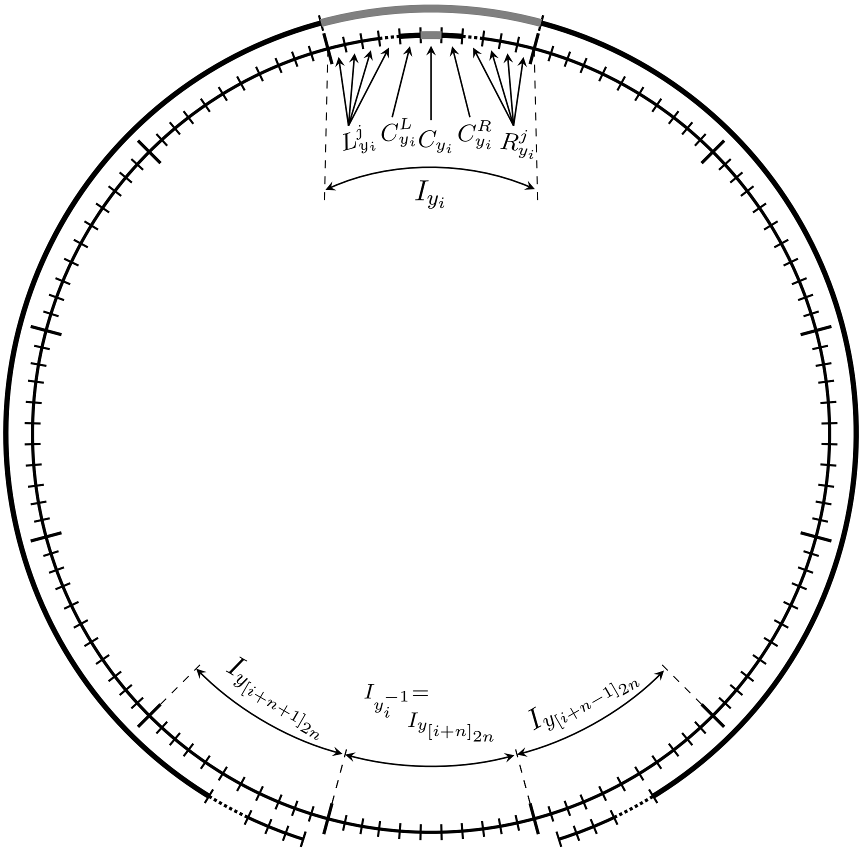

FIGURE 6. The intervals I_{y_i} in the circle together with the interior intervals. The outer curve is the image $\Phi_{P_n^+}\big|_{I_{y_i}}$ (which is order preserving). The intervals $L^j_{y_i}$, $R^j_{y_i}$ and their images are drawn with a continuous black line, $L^3_{y_i}$, $R^3_{y_i}$ and their images are drawn with a dotted line, $C^L_{y_i}$, $C^R_{y_i}$ and their images are drawn with a continuous thick black line and finally, C_{y_i} and its image are drawn with a continuous thick grey line.

and $j = 1, 2, \ldots, 2n-1$ it follows that

$$U^i_j = \begin{cases} L^{(n+1)-j}_{y_i} & \text{for } j = 1, 2, \ldots, n-2, \\ C^L_{y_i} & \text{for } j = n-1, \\ C_{y_i} & \text{for } j = n, \\ C^R_{y_i} & \text{for } j = n+1, \\ R^{j-(n-1)}_{y_i} & \text{for } j = n+2, n+3, \ldots, 2n-1. \end{cases} \tag{16}$$

With this labelling we define the matrix M_n^+ so that $m^{lt}_{ij} = 1$ if and only if $\phi(U^l_i) \supset U^t_j$.

The next theorem is a first reduction in the effective computation of $h_{\text{top}}(\Phi_{P_n^+})$.

$$
\left(\begin{array}{c|c|c|c|c|c|c|c}
0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0100000 & 0000000 & 0000000 \\
0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0011000 & 0000000 & 0000000 \\
0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000111 & 1111111 & 1111111 \\
1111111 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 \\
0000000 & 1111111 & 1111111 & 1110000 & 0000000 & 0000000 & 0000000 & 0000000 \\
0000000 & 0000000 & 0000000 & 0001100 & 0000000 & 0000000 & 0000000 & 0000000 \\
0000000 & 0000000 & 0000000 & 0000010 & 0000000 & 0000000 & 0000000 & 0000000 \\
\hline
0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0100000 & 0000000 \\
0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0011000 & 0000000 \\
1111111 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000111 & 1111111 \\
0000000 & 1111111 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 \\
0000000 & 0000000 & 1111111 & 1111111 & 1110000 & 0000000 & 0000000 & 0000000 \\
0000000 & 0000000 & 0000000 & 0000000 & 0001100 & 0000000 & 0000000 & 0000000 \\
0000000 & 0000000 & 0000000 & 0000000 & 0000010 & 0000000 & 0000000 & 0000000 \\
\hline
0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0100000 \\
0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0011000 \\
1111111 & 1111111 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000111 \\
0000000 & 0000000 & 1111111 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 \\
0000000 & 0000000 & 0000000 & 1111111 & 1111111 & 1110000 & 0000000 & 0000000 \\
0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0001100 & 0000000 & 0000000 \\
0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000010 & 0000000 & 0000000
\end{array}\right)
$$

FIGURE 7. The first three (of the total of eight) block rows of the Markov matrix $M_{P_4^+}$ corresponding to the symmetric presentation of an orientable surface group of rank 4.

Theorem 3.2.

$$h_{top}(\Phi_{P_n^+}) = \log \max \left\{ \rho\big(M_n^+\big), 1 \right\} = \log \max \left\{ \rho \left(\sum_{k=1}^{2n} M_{1k} \right), 1 \right\}.$$

An $(r,s)-$*block circulant matrix* is a matrix of the form

$$\begin{pmatrix} A_1 & A_2 & A_3 & \dots & A_r \\ A_r & A_1 & A_2 & \dots & A_{r-1} \\ A_{r-1} & A_r & A_1 & \dots & A_{r-2} \\ \dots & \dots & \dots & \dots & \dots \\ A_2 & A_3 & A_4 & \dots & A_1 \end{pmatrix}$$

where each A_i is an $s \times s$ matrix. Notice that a circulant matrix is completely determined by its first block row $(A_1\ A_2\ A_3\ \dots\ A_r)$.

The next lemma will be crucial in effectively computing the spectral radius of M_n^+ (see Figure 7 for an example in the case of rank 4).

Lemma 3.3. *The Markov matrix* M_n^+ *is a* $(2n, 2n-1)-$*block circulant matrix.*

Proof. From the formulae (14) it follows that $\Phi_{P_n^+}(U_i^l) \supset U_j^t$ if and only if $\Phi_{P_n^+}(U_i^{[l+1]_{2n}}) \supset U_j^{[t+1]_{2n}}$. In terms of the Markov matrix this amounts to $m_{ij}^{lt} = m_{ij}^{[l+1]_{2n},[t+1]_{2n}}$ for every $l, t \in \{1, 2, \dots, 2n\}$ and $i, j \in \{1, 2, \dots, 2n-1\}$. This implies that $M_{lt} = M_{[l+1]_{2n},[t+1]_{2n}}$. □

The next technical lemma provides a nice and useful result about the spectral radius of block circulant matrices.

Lemma 3.4. *Let*

$$A = \begin{pmatrix} A_1 & A_2 & A_3 & \dots & A_r \\ A_r & A_1 & A_2 & \dots & A_{r-1} \\ A_{r-1} & A_r & A_1 & \dots & A_{r-2} \\ \dots & \dots & \dots & \dots & \dots \\ A_2 & A_3 & A_4 & \dots & A_1 \end{pmatrix}$$

be a non-negative block circulant matrix. Then

$$\rho(A) = \rho\left(\sum_{i=1}^{r} A_i\right).$$

Proof. Since A is a block matrix, for every $m \geq 1$, A^m is a block matrix

$$\begin{pmatrix} A_{11}^{(m)} & A_{12}^{(m)} & \dots & A_{1r}^{(m)} \\ A_{21}^{(m)} & A_{22}^{(m)} & \dots & A_{2r}^{(m)} \\ \dots & \dots & \dots & \dots \\ A_{r1}^{(m)} & A_{r2}^{(m)} & \dots & A_{rr}^{(m)} \end{pmatrix}$$

where each block has size $s \times s$ and is a sum of r^{m-1} non-commutative products of m matrices among the blocks $A_1, A_2, A_3, \dots, A_r$. That is, each $A_{ij}^{(m)}$ is the sum of r^{m-1} matricial products of the form $A_{r_1} A_{r_2} \cdots A_{r_m}$ with $r_1, r_2, \dots, r_m \in \{1, 2, \dots, r\}$.

Moreover, since every block A_i appears exactly once in every block row and every block column it can be proved by induction that every product of the form $A_{r_1} A_{r_2} \cdots A_{r_m}$ appears exactly once in every block row and every block column of A^m. Therefore, for every $q \in \{1, 2, \dots, r\}$, $\sum_{i=1}^{r} A_{qi}^{(m)}$ and $\sum_{i=1}^{r} A_{iq}^{(m)}$ are the sum of r^m matricial products, and every product $A_{r_1} A_{r_2} \cdots A_{r_m}$ appears exactly once in each of these expressions. Hence,

$$\sum_{i=1}^{r} A_{qi}^{(m)} = \sum_{i=1}^{r} A_{iq}^{(m)} = \left(\sum_{i=1}^{r} A_i\right)^m.$$

Consequently, the classical matrix norms satisfy:

$$\|A^m\|_\infty = \|(\textstyle\sum_{i=1}^{r} A_i)^m\|_\infty \text{ and } \|A^m\|_1 = \|(\textstyle\sum_{i=1}^{r} A_i)^m\|_1.$$

Since the spectral radius is defined as: $\rho(A) = \lim_{m\to\infty} \|A^m\|^{1/m}$ for any matrix norm, it follows that $\rho(A) = \rho\left(\sum_{i=1}^{r} A_i\right)$. □

Remark 3.5. In fact the above lemma holds for every matrix for which a given block appears exactly once in every block row and every block column.

Now we are endowed with the necessary ingredients to prove Theorem 3.2.

Proof of Theorem 3.2. It follows from (11) and Lemmas 3.3 and 3.4. □

Next, to complete the first reduction in the computation of $h_{\text{top}}(\Phi_{P_n^+})$, we will give an explicit formula for the matrix $\sum_{k=1}^{2n} M_{1k}$.

The *compacted matrix of rank* n is the $(2n-1)\times(2n-1)$ matrix $\mathsf{C}_n = (c_{ij})$ defined by

$$
c_{ij} = \begin{cases}
1 & j = i+1 \text{ and } i \in \{1,2,\dots,n-3\},\\
1 & \text{if } j \in \{n-1,n\} \text{ and } i = n-2,\\
n-2 & \text{if } j \in \{1,2,\dots,n\} \text{ and } i = n-1,\\
n-1 & \text{if } j \in \{n+1,n+2,\dots,2n-1\} \text{ and } i = n-1,\\
1 & \text{if } i = n,\\
n-1 & \text{if } j \in \{1,2,\dots,n-1\} \text{ and } i = n+1,\\
n-2 & \text{if } j \in \{n,n+1,\dots,2n-1\} \text{ and } i = n+1,\\
1 & \text{if } j \in \{n,n+1\} \text{ and } i = n+2,\\
1 & j = i-1 \text{ and } i \in \{n+3,n+4,\dots,2n-1\}, \text{ and}\\
0 & \text{otherwise.}
\end{cases} \tag{17}
$$

In matrix form, C_n is

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0\\
n-2 & n-2 & n-2 & \cdots & n-2 & n-2 & n-2 & n-1 & n-1 & \cdots & n-1 & n-1 & n-1\\
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1\\
n-1 & n-1 & n-1 & \cdots & n-1 & n-1 & n-2 & n-2 & n-2 & \cdots & n-2 & n-2 & n-2\\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0
\end{pmatrix}
$$

Lemma 3.6.

$$\sum_{k=1}^{2n} M_{1k} = \mathsf{C}_n.$$

To prove this lemma we need to explicitly describe the matrix M_n^+. To this end we introduce the following notation. The zero matrix of size $k \times k$ will be denoted by $\mathbf{0}_k$, and $\mathbf{J}_k$ will denote the $k \times k$ $(0,1)-$matrix with ones in the anti-diagonal. Also, if $i \in \{1,2,\dots,k\}$, $\mathbf{U}_k^i$ will denote the $k \times k$ matrix such that all entries in the $i-$th row are 1 and all other entries are 0. Finally, for $k \geq 5$ odd, $\mathbf{T}_k = (t_{ij})$ is the $k \times k$ $(0,1)-$matrix such that $t_{ij} = 1$ if and only if (see Figure 8 for examples):

- $j = i+1$ and $i \in \{1,2,\dots,\widetilde{k}-3\}$, or
- $j \in \{\widetilde{k}-1, \widetilde{k}\}$ and $i = \widetilde{k}-2$ or
- $\widetilde{k}+1 \leq j \leq k$ and $i = \widetilde{k}-1$,

where $\widetilde{k} = \frac{k+1}{2}$. Observe that (see again Figure 8) $\mathbf{J}_k\mathbf{T}_k\mathbf{J}_k$ is the matrix obtained from $\mathbf{T}_k$ by a symmetry with respect to the central coordinate $t_{\widetilde{k},\widetilde{k}}$.

Proof of Lemma 3.6. From formulae (14) and taking into account the labelling of the basic intervals (16) it follows that $m_{ij}^{lt} = 1$ if and only if (see

$$\mathbf{U}_7^3 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \qquad \mathbf{T}_7 = \begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&1&1&0&0&0\\ 0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

$$\mathbf{J}_7 = \begin{pmatrix} 0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0 \end{pmatrix} \qquad \mathbf{J}_7\mathbf{T}_7\mathbf{J}_7 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 1&1&1&0&0&0&0\\ 0&0&0&1&1&0&0\\ 0&0&0&0&0&1&0 \end{pmatrix}$$

FIGURE 8. Examples of the matrices $\mathbf{U}_k^i$, $\mathbf{T}_k$, $\mathbf{J}_k$ and $\mathbf{J}_k\mathbf{T}_k\mathbf{J}_k$ with $k = 7$.

Figure 7 for an example in the case of rank 4):

$$\begin{cases} j = i+1,\ t = [l+n+1]_{2n} & \text{for } i = 1,2,\dots,n-3,\\ j \in \{n-1,n\},\ t = [l+n+1]_{2n} & \text{for } i = n-2,\\ \left.\begin{array}{l} n+1 \le j \le 2n-1,\ t = [l+n+1]_{2n}, \text{ and}\\ j \in \{1,2,\dots,2n-1\},\ [l+n+2]_{2n} \le t \le [l-1]_{2n} \end{array}\right\} & \text{for } i = n-1,\\ j \in \{1,2,\dots,2n-1\},\ t = l & \text{for } i = n,\\ \left.\begin{array}{l} j \in \{1,2,\dots,n-1\},\ t = [l+n-1]_{2n}, \text{ and}\\ j \in \{1,2,\dots,2n-1\},\ [l+1]_{2n} \le t \le [l+n-2]_{2n} \end{array}\right\} & \text{for } i = n+1,\\ j \in \{n,n+1\},\ t = [l+n-1]_{2n} & \text{for } i = n+2,\\ j = i-1,\ t = [l+n-1]_{2n} & \text{for } i = n+3,\dots,2n-1. \end{cases}$$

In matrix block form the above formulae become (see again Figure 7):

$$\begin{aligned} M_{l,[l+n+1]_{2n}} &= \mathbf{T}_{2n-1}\\ M_{lt} &= \mathbf{U}_{2n-1}^{n-1} \text{ for } [l+n+2]_{2n} \le t \le [l-1]_{2n}\\ M_{ll} &= \mathbf{U}_{2n-1}^{n}\\ M_{lt} &= \mathbf{U}_{2n-1}^{n+1} \text{ for } [l+1]_{2n} \le t \le [l+n-2]_{2n}\\ M_{l,[l+n-1]_{2n}} &= \mathbf{J}_{2n-1}\mathbf{T}_{2n-1}\mathbf{J}_{2n-1}\\ M_{l,[l+n]_{2n}} &= \mathbf{0}_{2n-1} \end{aligned}$$

for $l \in \{1,2,\dots,2n\}$. Consequently,

$$\begin{aligned} \sum_{t=1}^{n} M_{1t} &= \mathbf{T}_{2n-1} + \mathbf{J}_{2n-1}\mathbf{T}_{2n-1}\mathbf{J}_{2n-1} + \mathbf{U}_{2n-1}^{n} + (n-2)\left(\mathbf{U}_{2n-1}^{n-1} + \mathbf{U}_{2n-1}^{n+1}\right)\\ &= \mathsf{C}_n. \end{aligned}$$

□

The next corollary gives an explicit formula for the entropy in the orientable case in terms of the spectral radius of a $(2n-1)\times(2n-1)$ matrix which is a "compacted" version of the Markov matrix M_n^+.

Corollary 3.7.

$$h_{top}(\Phi_{P_n^+}) = \log\max\left\{\rho\left(\mathsf{C}_n\right), 1\right\}.$$

Proof. It follows from Theorem 3.2 and Lemma 3.6. □

Remark 3.8. Note that the map $\Phi_{P_n^+}$ commutes with a rigid rotation R of period $2n$. The quotient space obtained by identifying each orbit of R to a point is a circle. The map induced by $\Phi_{P_n^+}$ on this quotient space is also a Markov map. The matrix C_n is nothing but the Markov matrix of this induced map (see (14) and Figure 6).

4. The non-orientable case

We start this section by extending the definition of *symmetric presentation* (Definition 2.11) to non orientable surface groups.

Definition 4.1. Given a surface group $\Gamma = \pi_1(S)$ of rank n, where S is a non orientable surface, the following presentation of Γ will be called *symmetric* and denoted by P_n^-. Its definition depends on the parity of n as follows. For n odd, we define P_n^- as

$$\left\langle x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1} \,/ x_1 x_2 \cdots x_n x_{n-1} x_{n-2} \cdots x_1 x_n \right\rangle$$

while, for n even, P_n^- is defined as

$$\left\langle x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1} \,/ x_1 x_2 \cdots x_n x_{n-1} x_{n-2} \cdots x_1 x_n^{-1} \right\rangle.$$

Similar arguments to the ones used in the proof of Proposition 2.12 yield that the symmetric presentation P_n^- is minimal and geometric.

As in the orientable case, the nomenclature *symmetric* for the presentation P_n^- accounts for the fact that, at each vertex, the cyclic ordering of the generators (Lemma 2.1) exhibits the useful property that the edge opposite to x at any vertex is simply the edge x^{-1}. Indeed, one can check that the ordering of the generators at any vertex is

$$x_1 < x_2^{-1} < x_3 < \cdots < x_{n-1} < x_n^{-1} < x_1^{-1} < x_2 < x_3^{-1} < \cdots < x_{n-1}^{-1} < x_n$$

when n is even, and

$$x_1 < x_2^{-1} < x_3 < \cdots < x_{n-1}^{-1} < x_n < x_1^{-1} < x_2 < x_3^{-1} < \cdots < x_{n-1} < x_n^{-1}$$

when n is odd.

The fact that the symmetric presentation has associated the above cyclic ordering implies that Corollary 3.1 also holds for the non-orientable case:

Corollary 4.2. *Let P_n^- be the symmetric presentation of a non-orientable surface group of rank n. Then, $\Phi_{P_n^-}(C_x) = I_x$ for each generator x.*

Notice that map $\Phi_{P_n^+}$ and the Markov matrix M_n^+ are only defined for n even since the group corresponds to an orientable surface. However, all associated formulae extend to the case n odd. In this sense below we will

$$
\left(\begin{array}{ccccc|ccccc|ccccc|ccccc|ccccc|ccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1\\
1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right)
$$

FIGURE 9. The first three (of the total of six) block rows of the Markov matrix $M_{P_3^-}$ corresponding to the symmetric presentation of a non-orientable surface group of rank 3.

compare the maps $\Phi_{P_n^+}$ and $\Phi_{P_n^-}$ and the associated Markov matrices M_n^+ and M_n^-, independently on the parity of n.

Using the notations introduced in the previous sections one can check that the Markov map $\Phi_{P_n^-}$ behaves essentially as $\Phi_{P_n^+}$ in all intervals I_{y_i} except when $i \in \{n, 2n\}$. In these two intervals the map reverses orientation. So, when $i \notin \{n, 2n\}$ the equation (14) holds with $\Phi_{P_n^-}$ instead of $\Phi_{P_n^+}$. When $i = n$,

$$
\begin{aligned}
&\Phi_{P_n^-}(L^j_{y_n}) = R^{j-1}_{y_{2n-1}} \text{for } j \in \{4, 5, \dots, n\},\\
&\Phi_{P_n^-}(L^3_{y_n}) = C_{y_{2n-1}} \cup C^R_{y_{2n-1}},\\
&\Phi_{P_n^-}(C^L_{y_n}) = C^L_{y_{2n-1}} \cup \left(\bigcup_{j=2}^{n} L^j_{y_{2n-1}}\right) \cup \left(\bigcup_{k=n+1}^{2n-2} I_{y_k}\right),\\
&\Phi_{P_n^-}(C_{y_n}) = I_{y_n},\\
&\Phi_{P_n^-}(C^R_{y_n}) = C^R_{y_1} \cup \left(\bigcup_{j=2}^{n} R^j_{y_1}\right) \cup \left(\bigcup_{k=2}^{n-1} I_{y_k}\right),\\
&\Phi_{P_n^-}(R^3_{y_n}) = C^L_{y_1} \cup C_{y_1},\\
&\Phi_{P_n^-}(R^j_{y_n}) = L^{j-1}_{y_1} \text{for } j \in \{4, 5, \dots, n\},
\end{aligned}
\tag{18}
$$

and analogous formulae hold for the interval $I_{y_{2n}}$. Hence, in a similar way to the previous section it follows that the Markov matrix M_n^- of $\Phi_{P_n^-}$ is of the form (15) with $M_{i,j}$ replaced by $\mathbf{J}_{2n-1} M_{i,j}$ for $i \in \{n, 2n\}$ and $j = 1, 2, \dots, 2n$ (see Figure 9).

Next we want to prove Lemma 4.3 which is an analogue of Lemma 3.4 for this case. This will allow us to simplify the computation of the spectral radius of M_n^-. As expected, a further consequence of Lemmas 4.3 and 3.4 will be that the orientation-reversing character of $\Phi_{P_n^-}$ in the intervals I_{y_n} and $I_{y_{2n}}$ has no effects in the entropy. To do this it is convenient to introduce the notion of *disoriented block circulant matrix* as follows. An $(r,s)-$*disoriented block circulant matrix* is a matrix of the form

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1r} \\ A_{21} & A_{22} & \dots & A_{2r} \\ \dots & \dots & \dots & \dots \\ A_{r1} & A_{r2} & \dots & A_{rr} \end{pmatrix}$$

where each A_{ij} is an $s \times s$ matrix for which there exists an $(r,s)-$block circulant matrix

$$\widetilde{A} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1r} \\ \widetilde{A}_{21} & \widetilde{A}_{22} & \dots & \widetilde{A}_{2r} \\ \dots & \dots & \dots & \dots \\ \widetilde{A}_{r1} & \widetilde{A}_{r2} & \dots & \widetilde{A}_{rr} \end{pmatrix}$$

such that given $i \in \{2,\dots,r\}$, either $A_{ij} = \widetilde{A}_{ij}$ for every $j = 1,2,\dots,r$ or $A_{ij} = \mathbf{J}_s\widetilde{A}_{ij}$ for every $j = 1,2,\dots,r$. That is, every block row of A coincides with the corresponding block row of $\widetilde{A}$ or is obtained from the corresponding block row of $\widetilde{A}$ by pre-multiplying each block by $\mathbf{J}_s$. Observe that this last operation permutes the individual rows of the block row symmetrically with respect to the central horizontal axis. The matrix $\widetilde{A}$ will be called the *parallelization of* A. Observe that the assumption that the first block row of A and $\widetilde{A}$ coincide implies that the parallelization of A is unique.

Lemma 4.3. *Let*

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1r} \\ A_{21} & A_{22} & \dots & A_{2r} \\ \dots & \dots & \dots & \dots \\ A_{r1} & A_{r2} & \dots & A_{rr} \end{pmatrix}$$

be a non-negative disoriented $(r,s)-$*block circulant matrix such that*

$$\left(\sum_{j=1}^{r} A_{1j}\right)\mathbf{J}_s = \mathbf{J}_s\left(\sum_{j=1}^{r} A_{1j}\right) . \tag{19}$$

Then

$$\rho(A) = \rho\left(\sum_{j=1}^{r} A_{1j}\right).$$

Proof. Let

$$\widetilde{A} = \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} & \dots & \widetilde{A}_{1r} \\ \widetilde{A}_{21} & \widetilde{A}_{22} & \dots & \widetilde{A}_{2r} \\ \dots & \dots & \dots & \dots \\ \widetilde{A}_{r1} & \widetilde{A}_{r2} & \dots & \widetilde{A}_{rr} \end{pmatrix}$$

be the unique parallelization of A. We will prove that $\|A^m\|_\infty = \|\widetilde{A}^m\|_\infty$ for every $m \geq 0$. Then,

$$\rho(A) = \lim_{m\to\infty} \|A^m\|_\infty^{1/m} = \lim_{m\to\infty} \|\widetilde{A}^m\|_\infty^{1/m} = \rho(\widetilde{A}) ,$$

and the result follows from Lemma 3.4.

For every $m \in \mathbb{N}$ we will write A^m and $\widetilde{A}^m$ as

$$\begin{pmatrix} A_{11}^{(m)} & A_{12}^{(m)} & \dots & A_{1r}^{(m)} \\ A_{21}^{(m)} & A_{22}^{(m)} & \dots & A_{2r}^{(m)} \\ \dots & \dots & \dots & \dots \\ A_{r1}^{(m)} & A_{r2}^{(m)} & \dots & A_{rr}^{(m)} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \widetilde{A}_{11}^{(m)} & \widetilde{A}_{12}^{(m)} & \dots & \widetilde{A}_{1r}^{(m)} \\ \widetilde{A}_{21}^{(m)} & \widetilde{A}_{22}^{(m)} & \dots & \widetilde{A}_{2r}^{(m)} \\ \dots & \dots & \dots & \dots \\ \widetilde{A}_{r1}^{(m)} & \widetilde{A}_{r2}^{(m)} & \dots & \widetilde{A}_{rr}^{(m)} \end{pmatrix},$$

respectively. Then,

$$\|A^m\|_\infty = \max_{i=1,2,\dots,r} \Big\| \sum_{j=1}^{r} A_{ij}^{(m)} \Big\|_\infty = \max_{i=1,2,\dots,r} \left(\sum_{j=1}^{r} A_{ij}^{(m)} \right) \mathbf{u}_s \text{ and, similarly,}$$

$$\|\widetilde{A}^m\|_\infty = \max_{i=1,2,\dots,r} \left(\sum_{j=1}^{r} \widetilde{A}_{ij}^{(m)} \right) \mathbf{u}_s \ ,$$

where $\mathbf{u}_s$ denotes the (column) vector of size s with all the entries equal to 1. So, to prove the lemma it is enough to show that

$$\left(\sum_{j=1}^{r} A_{ij}^{(m)} \right) \mathbf{u}_s = \left(\sum_{j=1}^{r} \widetilde{A}_{ij}^{(m)} \right) \mathbf{u}_s \tag{20}$$

for every $i = 1, 2, \dots, r$ and $m \in \mathbb{N}$.

Before proving this claim we will prove two necessary technical results on the block rows of the matrix A that are stronger versions of assumption (19). The first one is the following:

$$\left(\sum_{j=1}^{r} A_{ij} \right) \mathbf{J}_s = \mathbf{J}_s \left(\sum_{j=1}^{r} A_{ij} \right) \text{ and } \left(\sum_{j=1}^{r} A_{ij} \right) \mathbf{w}_s = \left(\sum_{j=1}^{r} A_{rj} \right) \mathbf{w}_s \tag{21}$$

for every non-negative vector $\mathbf{w}_s$ of size s such that $\mathbf{J}_s \mathbf{w}_s = \mathbf{w}_s$ and every $i, r \in \{1, 2, \dots, r\}$.

From the definitions of disoriented block circulant matrix, parallelization and block circulant matrix we get:

$$\sum_{j=1}^{r} A_{1j} = \sum_{j=1}^{r} \widetilde{A}_{1j} = \sum_{j=1}^{r} \widetilde{A}_{ij} \tag{22}$$

and $\sum_{j=1}^{r} A_{ij}$ is either $\sum_{j=1}^{r} \widetilde{A}_{ij}$ or $\mathbf{J}_s \left(\sum_{j=1}^{r} \widetilde{A}_{ij} \right)$. Thus, from (22) and (19) we get, respectively,

$$\sum_{j=1}^{r} A_{ij} = \begin{cases} \sum_{j=1}^{r} \widetilde{A}_{ij} = \sum_{j=1}^{r} A_{1j} \\ \mathbf{J}_s \left(\sum_{j=1}^{r} \widetilde{A}_{ij} \right) = \mathbf{J}_s \left(\sum_{j=1}^{r} A_{1j} \right) = \left(\sum_{j=1}^{r} A_{1j} \right) \mathbf{J}_s. \end{cases}$$

Consequently, if $\mathbf{w}_s$ is a non-negative vector of size s such that $\mathbf{J}_s \mathbf{w}_s = \mathbf{w}_s$, we have $\left(\sum_{j=1}^{r} A_{ij} \right) \mathbf{w}_s = \left(\sum_{j=1}^{r} A_{1j} \right) \mathbf{w}_s$ for every $i \in \{1, 2, \dots, r\}$. This proves the second equality of (21). To prove the first one we use the above expression for $\sum_{j=1}^{r} A_{ij}$ and (19). In the first case we have

$$\left(\sum_{j=1}^{r} A_{ij} \right) \mathbf{J}_s = \left(\sum_{j=1}^{r} A_{1j} \right) \mathbf{J}_s = \mathbf{J}_s \left(\sum_{j=1}^{r} A_{1j} \right) = \mathbf{J}_s \left(\sum_{j=1}^{r} A_{ij} \right),$$

and in the second one,

$$\left(\sum_{j=1}^r A_{ij}\right) \mathbf{J}_s = \mathbf{J}_s \left(\sum_{j=1}^r A_{1j}\right) \mathbf{J}_s = \mathbf{J}_s \mathbf{J}_s \left(\sum_{j=1}^r A_{1j}\right) = \mathbf{J}_s \left(\sum_{j=1}^r A_{ij}\right).$$

This ends the proof of (21).

The second technical result that we need is a weaker version of (21) but that holds for all powers of A. More precisely, for every $i, r \in \{1, 2, \ldots, r\}$ and $m \in \mathbb{N}$,

$$\mathbf{J}_s \left(\sum_{j=1}^r A_{ij}^{(m)}\right) \mathbf{u}_s = \left(\sum_{j=1}^r A_{rj}^{(m)}\right) \mathbf{u}_s \ . \tag{23}$$

In fact this implies that $\left(\sum_{j=1}^r A_{\ell j}^{(m)}\right) \mathbf{u}_s$ is a vector independent on ℓ, call it $\mathbf{w}_s^m$, such that $\mathbf{J}_s \mathbf{w}_s^m = \mathbf{w}_s^m$.

For $m = 1$, (23) follows directly from (21) and the fact that $\mathbf{J}_s \mathbf{u}_s = \mathbf{u}_s$. Now we assume that (23) holds for $m \geq 1$ and prove it for $m + 1$. Clearly,

$$\sum_{j=1}^r A_{ij}^{(m+1)} = \sum_{j=1}^r \left(\sum_{\ell=1}^r A_{i\ell} A_{\ell j}^{(m)}\right) = \sum_{\ell=1}^r A_{i\ell} \left(\sum_{j=1}^r A_{\ell j}^{(m)}\right) \tag{24}$$

for every $i \in \{1, 2, \ldots, r\}$. Thus, from the induction hypothesis and (21),

$$\begin{aligned} \mathbf{J}_s \left(\sum_{j=1}^r A_{ij}^{(m+1)}\right) \mathbf{u}_s &= \mathbf{J}_s \sum_{\ell=1}^r A_{i\ell} \left(\sum_{j=1}^r A_{\ell j}^{(m)}\right) \mathbf{u}_s = \mathbf{J}_s \big(\sum_{\ell=1}^r A_{i\ell}\big) \mathbf{w}_s^m \\ &= \big(\sum_{\ell=1}^r A_{i\ell}\big) \mathbf{J}_s \mathbf{w}_s^m = \big(\sum_{\ell=1}^r A_{i\ell}\big) \mathbf{w}_s^m \\ &= \big(\sum_{\ell=1}^r A_{r\ell}\big) \mathbf{w}_s^m = \sum_{\ell=1}^r A_{r\ell} \left(\sum_{j=1}^r A_{\ell j}^{(m)}\right) \mathbf{u}_s \\ &= \left(\sum_{j=1}^r A_{rj}^{(m+1)}\right) \mathbf{u}_s. \end{aligned}$$

This completes the induction step and, thus, (23) is proved.

Now we will prove formula (20) by induction on m for a fixed but arbitrary $i \in \{1, 2, \ldots, r\}$. Assume that $m = 1$. If $A_{ij} = \widetilde{A}_{ij}$ for $j = 1, 2, \ldots, r$ then the equality is trivially true. Otherwise, $A_{ij} = \mathbf{J}_s \widetilde{A}_{ij}$ for $j = 1, 2, \ldots, r$. Hence, from (21),

$$\begin{aligned} \left(\sum_{j=1}^r A_{ij}\right) \mathbf{u}_s &= \left(\sum_{j=1}^r A_{ij}\right) (\mathbf{J}_s \mathbf{u}_s) = \mathbf{J}_s \left(\sum_{j=1}^r A_{ij}\right) \mathbf{u}_s \\ &= \mathbf{J}_s \left(\sum_{j=1}^r \mathbf{J}_s \widetilde{A}_{ij}\right) \mathbf{u}_s = \left(\sum_{j=1}^r \widetilde{A}_{ij}\right) \mathbf{u}_s, \end{aligned}$$

because $\mathbf{J}_s$ is an involution (i.e. $\mathbf{J}_s^2$ is the identity of size s).

Assume that (20) holds for $m \geq 1$. As above, we will consider two cases. If $A_{i\ell} = \widetilde{A}_{i\ell}$ for $\ell = 1, 2, \ldots, r$, then from (24), the analogous formula for $\widetilde{A}$ and the induction assumption we get

$$\begin{aligned} \left(\sum_{j=1}^r A_{ij}^{(m+1)}\right) \mathbf{u}_s &= \sum_{\ell=1}^r A_{i\ell} \left(\sum_{j=1}^r A_{\ell j}^{(m)}\right) \mathbf{u}_s \\ &= \sum_{\ell=1}^r \widetilde{A}_{i\ell} \left(\sum_{j=1}^r \widetilde{A}_{\ell j}^{(m)}\right) \mathbf{u}_s = \left(\sum_{j=1}^r \widetilde{A}_{ij}^{(m+1)}\right) \mathbf{u}_s. \end{aligned}$$

So, we are left with the case $A_{i\ell} = \mathbf{J}_s \widetilde{A}_{i\ell}$ for $\ell = 1, 2, \dots, r$. In a similar way to the previous case we get

$$\begin{aligned}\left(\sum_{j=1}^r A_{ij}^{(m+1)}\right) \mathbf{u}_s &= \mathbf{J}_s \sum_{\ell=1}^r \widetilde{A}_{i\ell} \left(\sum_{j=1}^r \widetilde{A}_{\ell j}^{(m)}\right) \mathbf{u}_s \\ &= \mathbf{J}_s \left(\sum_{j=1}^r \widetilde{A}_{ij}^{(m+1)}\right) \mathbf{u}_s.\end{aligned}$$

Consequently, from (23) we get:

$$\left(\sum_{j=1}^r A_{ij}^{(m+1)}\right) \mathbf{u}_s = \mathbf{J}_s \left(\sum_{j=1}^r A_{ij}^{(m+1)}\right) \mathbf{u}_s = \left(\sum_{j=1}^r \widetilde{A}_{ij}^{(m+1)}\right) \mathbf{u}_s.$$

This ends the proof of the lemma. □

With the help of Lemma 4.3, as in the orientable case we obtain:

Corollary 4.4.

$$h_{top}(\Phi_{P_n^-}) = \log \max \{\rho(\mathsf{C}_n), 1\}.$$

5. Computation of the topological entropy — second reduction: Super compacting the matrix C_n

The *super compacted matrix of rank* n is the $n \times n$ matrix $\mathsf{SC}_n = (s_{ij})$ defined as follows:

$$s_{ij} = \begin{cases} 1 & \text{if } i \le n-2 \text{ and } j = i+1 \text{ or } i = n, \\ 2 & \text{if } i = n-2 \text{ and } j = n, \\ 2n-3 & \text{if } i = n-1 \text{ and } j < n, \\ 2n-4 & \text{if } i = n-1 \text{ and } j = n, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

In matrix form we have:

$$\mathsf{SC}_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 \\ 2n-3 & 2n-3 & 2n-3 & 2n-3 & \cdots & 2n-3 & 2n-3 & 2n-4 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{pmatrix}$$

In this section we prove

Proposition 5.1. *For every* $n \ge 3$,

$$\max \{\rho(\mathsf{C}_n), 1\} = \max \{\rho(\mathsf{SC}_n), 1\}.$$

To prove the above result we need another intermediate matrix which we obtain from C_n. We introduce the *divided compacted matrix of rank* n of size

$2n \times 2n$, denoted by $\mathsf{DC}_n = (d_{ij})$, which we define as follows:

$$
d_{ij} = \begin{cases} c_{ij} & \text{if } i < n \text{ and } j \le n, \\ c_{i,j-1} & \text{if } i < n \text{ and } j \ge n+1, \\ c_{i-1,j} & \text{if } i > n+1 \text{ and } j \le n, \\ c_{i-1,j-1} & \text{if } i > n+1 \text{ and } j \ge n+1, \\ 1 & \text{if } i = n \text{ and } j \le n \text{ or } i = n+1 \text{ and } j \ge n+1, \\ 0 & \text{if } i = n \text{ and } j \ge n+1 \text{ or } i = n \text{ and } j \le n, \end{cases} \tag{26}
$$

where $\mathsf{C}_n = (c_{ij})$. In matrix form, DC_n is (compare with the definition of the matrix C_n in Page 18):

$$
\left(\begin{array}{cccccc|cc|cccccc}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
n-2 & n-2 & n-2 & \cdots & n-2 & n-2 & n-2 & n-2 & n-1 & n-1 & \cdots & n-1 & n-1 & n-1 \\
\hline
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\
\hline
n-1 & n-1 & n-1 & \cdots & n-1 & n-1 & n-2 & n-2 & n-2 & n-2 & \cdots & n-2 & n-2 & n-2 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0
\end{array}\right)
$$

Notice that the matrix DC_n is indeed the Markov matrix of a topological model obtained by subdividing the central interval of the topological model from Remark 3.8 at a fixed point (that exists because the central interval covers itself).

Proof of Proposition 5.1. First we will prove that

$$\operatorname{Spec}(\mathsf{DC}_n) = \operatorname{Spec}(\mathsf{C}_n) \cup \{1\},$$

where $\operatorname{Spec}(\cdot)$ denotes the set of eigenvalues of a matrix. To do this observe that 1 is an eigenvalue of DC_n with eigenvector $(0, 0, \dots, 0, 1, -1, 0 \dots, 0, 0)$, where the two non-zero elements of this vector are at the n and $n+1$ entries. Also, if μ is an eigenvalue of C_n with eigenvector $(v_1, v_2, \dots, v_{2n-1})$, then, it follows directly from the definition of the matrix DC_n that μ is also an eigenvalue of DC_n with eigenvector

$$\left(v_1, v_2, \dots, v_{n-1}, \tfrac{v_n}{2}, \tfrac{v_n}{2}, v_{n+1}, \dots, v_{2n-1}\right).$$

Conversely, if μ is an eigenvalue of DC_n with eigenvector $(v_1, v_2, \dots, v_{2n})$, then, again from the definition of the matrix DC_n, it follows that μ is also an eigenvalue of C_n with eigenvector

$$(v_1, v_2, \dots, v_{n-1}, v_n + v_{n+1}, v_{n+2}, \dots, v_{2n}).$$

This proves the statement.

The second step of the proof will be to show that $\rho(\mathsf{DC}_n) = \rho(\mathsf{SC}_n)$. To do this it is convenient to write the matrix DC_n in block form, with blocks of size $n \times n$:

$$\mathsf{DC}_n = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}.$$

Observe that DC_n is symmetric with respect to the central point. So, $D_{21} = \mathbf{J}_n D_{12} \mathbf{J}_n$ and $D_{22} = \mathbf{J}_n D_{11} \mathbf{J}_n$, which amounts to:

$$\mathsf{DC}_n = \begin{pmatrix} D_{11} & D_{12} \\ \mathbf{J}_n D_{12} \mathbf{J}_n & \mathbf{J}_n D_{11} \mathbf{J}_n \end{pmatrix}.$$

Now let us consider the block matrix of size $2n \times 2n$ defined by

$$\mathbf{Z} := \begin{pmatrix} \mathbf{I}_n & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{J}_n \end{pmatrix},$$

where $\mathbf{I}_n$ denotes the identity matrix of size $n \times n$. Clearly $\mathbf{Z}$ is non-singular and $\mathbf{Z}^{-1} = \mathbf{Z}$. Hence,

$$\mathbf{Z}\mathsf{DC}_n\mathbf{Z}^{-1} = \begin{pmatrix} D_{11} & D_{12}\mathbf{J}_n \\ D_{12}\mathbf{J}_n & D_{11} \end{pmatrix}$$

because $\mathbf{J}_n\mathbf{J}_n = \mathbf{I}_n$. Observe that $\mathbf{Z}\mathsf{DC}_n\mathbf{Z}^{-1}$ is a non-negative block circulant matrix. Thus,

$$\rho\left(\mathsf{DC}_n\right) = \rho\left(\mathbf{Z}\mathsf{DC}_n\mathbf{Z}^{-1}\right) = \rho\left(D_{11} + D_{12}\mathbf{J}_n\right)$$

by Lemma 3.4. Moreover, by direct inspection it follows that $D_{11} + D_{12}\mathbf{J}_n = \mathsf{SC}_n$.

Summarizing, we have proved

$$\max\left\{\rho\left(\mathsf{C}_n\right), 1\right\} = \max\left\{\rho\left(\mathsf{DC}_n\right), 1\right\} = \max\left\{\rho\left(\mathsf{SC}_n\right), 1\right\}.$$

□

Remark 5.2. The topological model in Remark 3.8 has a fixed point and commutes with the symmetry of degree -1 with respect to this fixed point. The quotient space obtained by identifying each orbit of the symmetry to a point is a closed interval, and the induced map on this quotient space is also a Markov map. The matrix SC_n is in fact the Markov matrix of this quotient map.

6. The spectral radius of SC_n and proof of Theorem 1.1

To compute the spectral radius of SC_n we will use the *rome method* proposed in [2]. To this end we have to introduce some notation.

Let $M = (m_{ij})$ be a $k \times k$ matrix. Given a sequence $p = (p_j)_{j=0}^{\ell}$ of elements of $\{1, 2, \dots, k\}$ we define the *width of p,* denoted by $w(p)$, as the number $\prod_{j=1}^{\ell} m_{p_{j-1}p_j}$. If $w(p) \neq 0$ then p is called a *path of length* ℓ. The length of a path p will be denoted by $\ell(p)$. A loop is a path such that $p_\ell = p_0$ i.e. that begins and ends at the same index.

A subset R of $\{1, 2, \dots, k\}$ is called a *rome* if there is no loop outside R, i.e. there is no path $(p_j)_{j=0}^{\ell}$ such that $p_\ell = p_0$ and $\{p_j : 0 \le j \le \ell\}$ is disjoint from R. For a rome R a path $(p_j)_{j=0}^{\ell}$ is called *simple* if $p_i \in R$ for $i = 0, \ell$ and $p_i \notin R$ for $i = 1, 2, \dots, \ell - 1$.

If $R = \{r_1, r_2, \dots, r_\ell\}$ is a rome of a matrix M then we define an $\ell \times \ell$ matrix-valued real function $M_R(x)$ by setting $M_R(x) = (a_{ij}(x))$, where $a_{ij}(x) = \sum_p w(p) \cdot x^{-\ell(p)}$, where the summation is over all simple paths originating at r_i and terminating at r_j.

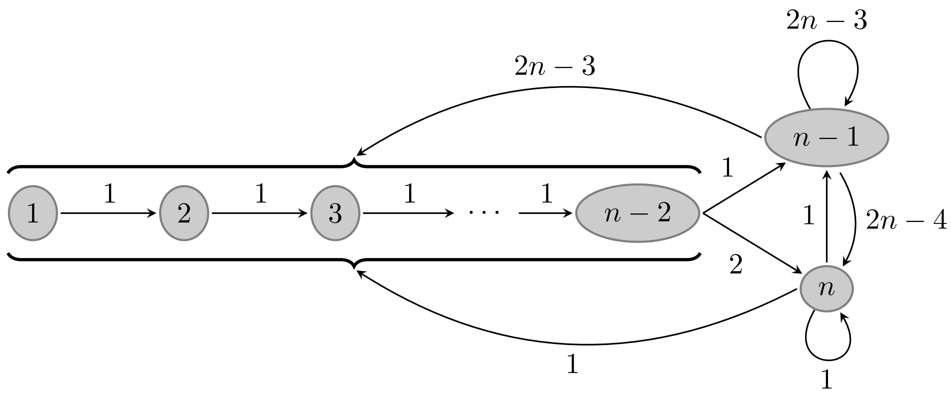

FIGURE 10. The combinatorial graph associated to SC_n. The arrows ending at braces indicate multiple arrows with the same weight, each one directed towards a node under the brace.

Theorem 6.1 (Theorem 1.7 of [2])**.** *If R is a rome of cardinality ℓ of a $k \times k$ matrix M then the characteristic polynomial of M is equal to*

$$(-1)^{k-\ell} x^k \det(M_R(x) - \mathbf{I}_\ell).$$

To use Theorem 6.1 it is helpful to represent the matrix M in form of a combinatorial graph which amounts to draw all paths of length 1 associated to M. To do this we introduce the following notation. A path (i, j) of length 1 will be written as $i \xrightarrow{w} j$, where w denotes the width of the path. For the matrix M the width w of the path $i \xrightarrow{w} j$, is just the entry $(M)_{i,j} \neq 0$. Observe that, with this notation, a path $p = (p_j)_{j=0}^{k}$ is written as

$$p_0 \xrightarrow{w_0} p_1 \xrightarrow{w_1} \cdots p_{k-1} \xrightarrow{w_{k-1}} p_k$$

and $w(p) = \prod_{i=0}^{k-1} w_i$.

We will compute the spectral radius of SC_n in Proposition 6.3 by using Theorem 6.1. In Figure 10 we show the combinatorial graph associated to SC_n.

Remark 6.2. The combinatorial graph associated to SC_n shown in Figure 10 is in fact the generalized Markov graph of the topological model obtained in Remark 5.2.

Proposition 6.3. *The spectral radius of SC_n is the largest root of the polynomial $Q_n(x)$.*

Proof. By direct inspection of the graph of Figure 10 it follows that SC_n is an irreducible non-negative integer matrix. Hence, by the Perron-Frobenius Theorem (see [11]), we get that the spectral radius of SC_n is the largest eigenvalue of the characteristic polynomial of SC_n. It is larger than 1 and simple. So, to prove the theorem, we have to show that the characteristic polynomial of SC_n is $Q_n(x)$.

Clearly $R = \{n-1, n\}$ is a rome of SC_n (see Figure 10). Hence,

$$M_R(x) = \begin{pmatrix} \beta\big(x^{-1} + z(x)\big) & (\beta-1)x^{-1} + 2\beta z(x) \\ x^{-1} + z(x) & x^{-1} + 2z(x) \end{pmatrix}$$

where $\beta = 2n-3$, $z(x) := \sum_{\ell=2}^{n-1} x^{-\ell}$.

By Theorem 6.1, the characteristic polynomial of SC_n is

$$\begin{aligned}
(-1)^{n-2}x^n &\begin{vmatrix} \beta\big(x^{-1} + z(x)\big) - 1 & 2\beta\big(x^{-1} + z(x)\big) - (\beta+1)x^{-1} \\ x^{-1} + z(x) & 2\big(x^{-1} + z(x)\big) - x^{-1} - 1 \end{vmatrix} \\
&= x^n\Big(\big(x^{-1} - \beta - 2\big)\big(x^{-1} + z(x)\big) + x^{-1} + 1\Big) \\
&= x^n\big(x^{-1} - (2n-1)\big)\left(\sum_{\ell=1}^{n-1} x^{-\ell}\right) + x^{n-1} + x^n \\
&= x^n - (2n-2)\sum_{j=1}^{n-1} x^j + 1.
\end{aligned}$$

This ends the proof of the proposition. □

Next we prove a technical lemma that studies the polynomial (1) and gives the bounds for λ_n.

Lemma 6.4. *For every $n \geq 3$, $Q_n(x)$ has a unique real root λ_n larger than one. Moreover, for $n \geq 4$,*

$$2n - 1 - \frac{1}{(2n-1)^{n-2}} < \lambda_n < 2n - 1.$$

Proof. Observe that $Q_n(0) = 1$, $Q_n(1) = -2n(n-2) < 0$ and

$$Q_n'(x) = n\left(x^{n-1} - \frac{2(n-1)}{n}\sum_{j=1}^{n-1} jx^{j-1}\right) \leq n\left(x^{n-1} - \frac{2(n-1)}{n}\right).$$

Since $x^{n-1} - \frac{2(n-1)}{n}$ is negative for every for $n \geq 3$ and $x \in [0,1]$, it follows that Q_n has a unique root in $(0,1)$.

On the other hand, it is easy to see that $Q_n(x) = x^n Q_n(x^{-1})$ (that is, Q_n is a *reciprocal polynomial*) and, hence, z is a root of Q_n if and only if z^{-1} is a root of Q_n. Consequently, $Q_n(x)$ has a unique real root larger than one.

Also,

$$Q_n(2n-1) = (2n-1)^n - 2(n-1)\frac{(2n-1)^n - (2n-1)}{2(n-1)} + 1 = 2n.$$

So, $\lambda_n < 2n-1$.

To end the proof of the lemma it is enough to show that

$$Q_n\left(2n - 1 - \tfrac{1}{(2n-1)^{n-2}}\right) < 0$$

for $n \geq 4$. We have

$$\begin{aligned}
Q_n(z) &= z^n - 2(n-1)s\frac{z^n - z}{2(n-1)s - 1} + 1 \\
&= -\frac{z^n - 4n(n-1)s + 2n - 1}{2(n-1)s - 1},
\end{aligned}$$

where $z := 2n - 1 - \frac{1}{s}$ and $s := (2n-1)^{n-2}$.

Since $2(n-1)s - 1 > 2n - 1 > 0$, $z^n - 4n(n-1)s > 0$ implies that $Q_n(z) < 0$.

An exercise shows that $z^n > (2n-1)^n - n(2n-1)$. Hence,

$$z^n - 4n(n-1)s > (2n-1)^{n-2}\left(1 - \frac{n}{(2n-1)^{n-3}}\right)$$

and $z^n - 4n(n-1)s > 0$ for $n \geq 4$. So, $Q_n(z) < 0$ and the lemma is proved. □

Proof of Theorem 1.1. It follows from Theorem 2.2, Corollaries 3.7 and 4.4, Propositions 5.1 and 6.3, and Lemma 6.4.

□

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