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Medium amplitude limit cycles of some classes of generalized Liénard systems

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1. Introduction and statement of the results

The bifurcation of limit cycles by perturbing a planar system which has a continuous family of *cycles*, i.e. periodic orbits, has been an intensively studied phenomenon; see for instance [13, 16, 2] and references therein. The simplest planar system having a continuous family of cycles is the linear center, and a special family of its perturbations is given by the generalized polynomial Liénard systems:

$$\dot{x} = y + \sum_{i=1}^{\mu} \varepsilon^i F_i(x), \quad \dot{y} = -x + \sum_{i=1}^{\nu} \varepsilon^i g_i(x), \tag{1_ε}$$

where $\mu, \nu \in \mathbb{N}$, $g_i(x)$ and $F_i(x)$ are polynomials for $i \geq 1$, and ε is a small parameter.

The classical and generalized Liénard systems appear very often in several branches of science and engineering, as biology, chemistry, mechanics, electronics, etc., see for instance [20] and references therein. In particular Liénard systems are frequent specially in physiological processes, see for instance [10]. Further, some planar systems can be transformed into (generalized) Liénard systems, see for example [5, 15]. In addition, the generalized polynomial Liénard systems is one of the most considered families in the study of limit cycles, see [18].

We assume $F_\mu(x) \not\equiv 0$ and $g_\nu(x) \not\equiv 0$, then we define

$$m = \max_{1 \leq i \leq \mu} \{\deg F_i(x)\}$$

and

$$n = \max_{1 \leq i \leq \nu} \{\deg g_i(x)\}.$$

For a small enough ε, let $\mathcal{H}_\nu^\mu(m,n)$ be the maximum number of limit cycles of (1_ε) that bifurcate from cycles of the *linear center* (1_0), i.e. the maximum number of *medium amplitude limit cycles* which can bifurcate from (1_0) under the perturbation (1_ε), in short

$$\mathcal{H}_\nu^\mu(m,n) = \left\{ \begin{array}{c} \text{Maximum number of medium} \\ \text{amplitude limit cycles of } (1_\varepsilon) \end{array} \right\}.$$

The main problem concerning $\mathcal{H}_\nu^\mu(m,n)$ is finding its exact value.

We are interested in $\nu \geq 1$ because when $g_i(x) \equiv 0$ for all $i \geq 1$ the maximum number of medium amplitude limit cycles of (1_ε) is well-known. Indeed, if we denote by $\mathcal{H}_0^\mu(m)$ the maximum number of medium amplitude limit cycles of (1_ε) in such a case, then we know from [17] that $\mathcal{H}_0^1(m) \geq [(m-1)/2]$, where $[\cdot]$ denotes the integer part function. Moreover, by following [7, Theorem 3.1] we can prove that $\mathcal{H}_0^\mu(m) = [(m-1)/2]$ for $\mu \geq 1$. An explicit proof of this statement is provided in [1, Section 3.2.2] because it is known that the cyclicity of a non-degenerated center (as in our case) coincide with the cyclicity of the open period annulus surrounding it. See for instance [9]. Theorem 1 (below) is a generalization of this result, and it also improves the results of Section 4.3.2 in [1] which prove that $\mathcal{H}_\nu^\mu(m,n) = [(m-1)/2]$ for some families of generalized Liénard systems. A rewiev about the results conserning small and medium amplitude limit cycles of (1_ε) is given in [19], where is also proved that

$$\begin{aligned}
\mathcal{H}_1^1(m,n) &\geq \left[\frac{m-1}{2}\right],\\
\mathcal{H}_2^2(m,n) &\geq \max\left\{\left[\frac{m-1}{2}\right], \left[\frac{m}{2}\right] + \left[\frac{n}{2}\right] - 1\right\},\\
\mathcal{H}_3^3(m,n) &\geq \left[\frac{m+n-1}{2}\right].
\end{aligned}$$

However, the exact values of $\mathcal{H}_1^1(m,n)$, $\mathcal{H}_2^2(m,n)$, and $\mathcal{H}_3^3(m,n)$ were not reported there.

In this paper we give the exact value of $\mathcal{H}_\nu^\mu(m,n)$ for two subfamilies of (1_ε). More precisely, we consider the families:

$$\mathcal{GL}1 := \left\{ \begin{array}{c} \text{Systems } (1_\varepsilon) \text{ assuming that} \\ g_i(x) \text{ is odd for } 1 \leq i \leq \nu \end{array} \right\}$$

and

$$\mathcal{GL}2 := \left\{ \begin{array}{c} \text{Systems } (1_\varepsilon) \text{ assuming that} \\ F_i(x) \text{ is even for } \mu_0 < i \leq \mu \end{array} \right\},$$

where μ_0 is the smallest integer with $1 \leq \mu_0 \leq \mu$ such that $F_{\mu_0}(x) \not\equiv 0$.

We will give the exact values of $\tilde{\mathcal{H}}_\nu^\mu(m,n)$ and $\bar{\mathcal{H}}_\nu^\mu(m,n)$ the maximum number of medium amplitude limit cycles of systems in $\mathcal{GL}1$ and $\mathcal{GL}2$, respectively. We note that if $\mu_0 = \mu$, then $\bar{\mathcal{H}}_\nu^\mu(m,n) = \mathcal{H}_\nu^\mu(m,n)$.

Our main result is the following.

Theorem 1. *The following statements hold.*

(a) *The exact value of* $\tilde{\mathcal{H}}_\nu^\mu(m,n)$ *is* $\left[\frac{m-1}{2}\right]$. *Moreover, for each* s *with* $0 \leq s \leq \left[\frac{m-1}{2}\right]$ *there exist systems in* $\mathcal{GL}1$ *having exactly* s *hyperbolic limit cycles.*

(b) *The exact value of* $\bar{\mathcal{H}}_\nu^\mu(m,n)$ *is either* $\left[\frac{m-1}{2}\right]$ *if* m *is odd or* $\left[\frac{m}{2}\right] + \left[\frac{n}{2}\right] - 1$ *if* m *is even. Moreover, for each* s *with* $0 \leq s \leq \left[\frac{m}{2}\right] + \left[\frac{n}{2}\right] - 1$ *there exist systems in* $\mathcal{GL}2$ *having exactly* s *hyperbolic limit cycles.*

The assumptions on $g_i(x)$ and $F_i(x)$ in definitions of $\mathcal{GL}1$ and $\mathcal{GL}2$, respectively, are necessary. Otherwise, we can construct systems (1_ε) having more medium amplitude limit cycles, see Remark 1 in Section 3.

Theorem 1 is a generalization of Theorem 1.1 in [22], where the case $\mu = \nu = 1$ was considered. We note that in such a case $\bar{\mathcal{H}}_1^1(m,n) = \mathcal{H}_1^1(m,n)$. Hence Theorem 1.$(b)$ gives the exact value of $\mathcal{H}_1^1(m,n)$.

The proof of Theorem 1 is based on computing the maximum number of isolated zeros of the first non-vanishing Poincaré–Pontryagin–Melnikov function of the displacement function of (1_ε), by taking into account the restrictions: $g_i(x)$ odd for $1 \leq i \leq \nu$ and $F_i(x)$ even for $\mu_0 < i \leq \mu$, respectively.

The paper is organized as follows. In Section 2 we recall the definition of the displacement function of (1_ε), as well as the algorithm to compute the Poincaré–Pontryagin–Melnikov functions. Preliminary results that allow us to provide elementary proofs of the main result are given in Section 3. Finally, in Section 4 we will prove Theorem 1.

2. Poincaré–Pontryagin–Melnikov functions

The linear center (1_0) is the Hamiltonian system associated to the polynomial $H = (x^2 + y^2)/2$; hence its cycles are the circles $\gamma_c = \{H - c = 0\}$ with $c > 0$. By using c as a parameter, the first return map of (1_ε) can be expressed in terms of ε and c: $\mathcal{P}(\varepsilon, c)$. Therefore the corresponding *displacement function* $L(\varepsilon, c) = \mathcal{P}(\varepsilon, c) - c$ is analytic for small enough ε and can be written as the power series in ε

$$L(\varepsilon, c) = \varepsilon L_1(c) + \varepsilon^2 L_2(c) + O(\varepsilon^3), \tag{2}$$

where $L_i(c)$ with $i \geq 1$ is the *Poincaré–Pontryagin–Melnikov function* of order i, which is defined for $c \geq 0$.

Let $L_k(c)$ with $k \geq 1$ be the first non-vanishing coefficient in (2). The zeros of $L_k(c)$ are important in the study of medium amplitude limit cycles of (1_ε) because of the *Poincaré–Pontryagin–Andronov criterion*: The maximum number of isolated zeros, counting multiplicities, of $L_k(c)$ is an upper bound for $\mathcal{H}^\mu_\nu(m, n)$. Furthermore each simple zero c_0 of $L_k(c)$ corresponds to one and only one limit cycle of (1_ε) with ε small enough bifurcating from the cycle γ_{c_0}.

Now, we will recall the algorithm to compute the functions $L_i(c)$. System (1_ε) can be written as

$$\dot{x} = y, \qquad \dot{y} = -x + \sum_{i \geq 1} \varepsilon^i \left(g_i(x) + f_i(x) y\right)$$

where $f_i(x) = F_i'(x)$, or equivalently as

$$dH - \varepsilon \omega_1 - \varepsilon^2 \omega_2 - \cdots = 0 \tag{3_ε}$$

with $\omega_i = (g_i(x) + f_i(x)y)\, dx$ and $\omega_i \equiv 0$ for $i > \max\{\mu, \nu\}$.

As we know, $L_1(c)$ is given by the classical Poincaré–Pontryagin formula $L_1(c) = \int_{\gamma_c} \omega_1$. A construction to compute the second order Poincaré–Pontryagin–Melnikov function of a perturbed system of the form $dH - \varepsilon\omega_1$ with ω_1 an arbitrary polynomial 1-form was given by Yakovenko [1995]. After, Françoise [1996] gave the algorithm to know the Poincaré–Pontryagin–Melnikov function of any order of $dH - \varepsilon\omega_1$. Finally, Iliev [1999] gave the result for computing the higher order Poincaré–Pontryagin–Melnikov functions of a perturbed system of the form $dH - \varepsilon\omega_1 - \varepsilon^2\omega_2 - \cdots = 0$, where ω_i for $i \geq 1$ are arbitrary polynomial 1-forms. His result is the following.

Theorem 2. [11]. *If $k \geq 2$ and $L_1(c) \equiv \cdots \equiv L_{k-1}(c) \equiv 0$, then there are polynomials $q_1, \ldots, q_{k-1}$ and $Q_1, \ldots, Q_{k-1}$ such that $\Omega_1 = dQ_1 + q_1 dH, \ldots, \Omega_{k-1} = dQ_{k-1} + q_{k-1} dH$, and*

$$L_k(c) = \int_{\gamma_c} \Omega_k,$$

where

$$\Omega_1 = \omega_1,\ \Omega_l = \omega_l + \sum_{i+j=l} q_i \omega_j, \text{ and } 2 \leq l \leq k. \tag{4}$$

The proof of this result easily follows from the Poincaré–Pontryagin formula, and the Ilyashenko–Gavrilov theorem ([12], [8]): If $\int_{\gamma_c} \omega = 0$ for all $c \geq 0$, then $\omega = dQ + qdH$, where Q and q are polynomials, and by applying an induction argument. For a detailed proof, see for instance [11], [14].

On the other hand, we know from [11] that $L_k(c)$ has at most $[k(\max\{n, m\} - 1)/2]$ positive zeros, counting multiplicities. However, this result does not give the value of $\mathcal{H}_\nu^\mu(m, n)$ because the upper bound for k depending on μ, ν, m, and n is unknown. In addition, the number of isolated zeros of the first non-vanishing Poincaré–Pontryagin–Melnikov function does not provide the number of limit cycles of (1_ε) with ε small enough as shows next example.

Example 1. *Consider the Liénard system*

$$\dot{x} = y + \varepsilon x - \varepsilon^2 x^3, \qquad \dot{y} = -x, \tag{5_ε}$$

or equivalently $dH - \varepsilon\omega_1 - \varepsilon^2\omega_2 = 0$ *with* $\omega_1 = ydx$ *and* $\omega_2 = 3x^2ydx$, *where* ε *is a small parameter.*

A simple computation gives $L_1(c) = \int_{\gamma_c} \omega_1 = -2\pi c$. *Hence system* (5_ε) *does not have limit cycles bifurcating from the cycles of the linear center. However, for any* $\varepsilon > 0$ *small enough the system* (5_ε) *has a limit cycle bifurcating from the infinity; more precisely, if we consider* (5_ε) *on the Poincaré sphere* $\mathbb{S}^2$, *then the limit cycle bifurcates from the equator of* $\mathbb{S}^2$ *which is known as "the circle at infinity" or "points at infinity" of* $\mathbb{R}^2$ *[21]. Indeed, by using the function*

$$V_\varepsilon(x, y) = 4y^2 + 4\varepsilon x\left(1 - \varepsilon x^2\right)y + 4x^2 - \frac{3}{\varepsilon^2}$$

it is not difficult to prove that the $1/V_\varepsilon(x, y)$ *is a Dulac function for* (5_ε) *in* $\mathbb{R}^2 \setminus \{V_\varepsilon(x, y) = 0\}$*; moreover, it is easy to see that for* $\varepsilon \in (0, 1)$ *the curve* $\{V_\varepsilon(x, y) = 0\}$ *has an oval surrounding the origin (the unique singularity of* (5_ε)*). Hence,* $\mathbb{R}^2 \setminus \{V_\varepsilon(x, y) = 0\}$ *has a 1-connected component* $\widetilde{U}_\varepsilon$, *then the generalized Bendixon–Dulac theorem [6] ensures that* (5_ε) *has a hyperbolic limit cycle* Γ_ε *in* $\widetilde{U}_\varepsilon$ *for each* $\varepsilon \in (0, 1)$. *Thus,* Γ_ε *contains the oval of* $\{V_\varepsilon(x, y) = 0\}$. *See Section 4 of [4] for more details. Finally, a straightforward computation shows that the circle* $x^2 + y^2 = 1/(2\varepsilon)^2$ *is contained in the bounded region limited by the oval of* $\{V_\varepsilon(x, y) = 0\}$. *This implies that* Γ_ε *bifurcates from the "the circle at infinity" of* $\mathbb{R}^2$.

In next section we will give some properties on ω_i which will allow us to simplify the computation of the Poincaré–Pontryagin–Melnikov functions

3. Preliminary results

For computing $L_k(c)$ for (1_ε) we will use the following two elementary lemmas whose proof is omitted.

Lemma 3. *Let* P *be a polynomial in the ring* $\mathbb{R}\left[x^2, H\right]$. *We define* $\deg_2 P$ *to be the degree of* P *in* $\mathbb{R}[x^2, H]$.

(*a*) *For* $i, j \geq 0$ *there are homogeneous polynomials* Q_{ij}, $q_{ij} \in \mathbb{R}\left[x^2, H\right]$ *with* $\deg_2 Q_{ij} = i + j$ *and* $\deg_2 q_{ij} = i + j - 1$, *such that*

$$H^i x^{2j} dx = d\left(xQ_{ij}\right) + \left(xq_{ij}\right)dH$$

or

$$H^i x^{2j+1} dx = d\left(x^2 Q_{ij}\right) + \left(x^2 q_{ij}\right)dH.$$

If $i = 0$, *then* $q_{ij} \equiv 0$.

(*b*) *For* $i, j \geq 0$ *there are homogeneous polynomials* Q_{ij}, $q_{ij} \in \mathbb{R}[x^2, H]$ *with* $\deg_2 Q_{ij} = i + j + 1$ *and* $\deg_2 q_{ij} = i + j$, *such that*

$$H^i x^{2j+1} y dx = d\left(yQ_{ij}\right) + \left(yq_{ij}\right) dH.$$

(*c*) *For* $i, j \geq 0$ *we have*

$$\int_{\gamma_c} H^i x^{2j} y dx = \frac{-\pi c}{2^j(2j+1)} \binom{2(j+1)}{j+1} c^{i+j}.$$

Lemma 4. *If* $\omega, \eta \in \mathcal{A}$ *and* $q \in \mathcal{S}$ *where*

$$\mathcal{A} := \left\{(xA + xyB)\, dx \,\middle|\, A, B \in \mathbb{R}\left[x^2, H\right]\right\}$$

and

$$\mathcal{S} := \left\{x^2 q_1 + yq_2 \,\middle|\, q_1, q_2 \in \mathbb{R}\left[x^2, H\right]\right\},$$

then $\omega + \eta \in \mathcal{A}$ *and* $q\omega \in \mathcal{A}$.

The next two results are straightforward consequences of these two previous lemmas.

Corollary 5. *If* $\omega \in \mathcal{A}$, *then* $\int_{\gamma_c} \omega \equiv 0$, $\omega = dQ + qdH$ *with* $q \in \mathcal{S}$, *and* $q\omega \in \mathcal{A}$.

Corollary 6. *If* $P\left(x^2\right) = \sum_{r=0}^{d} p_r x^{2r} \in \mathbb{R}\left[x^2\right]$, *then*

$$\int_{\gamma_c} P\left(x^2\right) y dx = -\pi c \sum_{r=0}^{d} \binom{2(r+1)}{r+1} \frac{p_r}{2^r(2r+1)} c^r.$$

We now will prove two lemmas which will be useful in the proof of Theorem 1.

Lemma 7. *Suppose* $k \geq 2$. *Then the following statements hold.*

(*a*) $\omega_l \in \mathcal{A}$ *for* $1 \leq l \leq k-1$ *if and only if* $\Omega_l \in \mathcal{A}$ *for* $1 \leq l \leq k-1$, *where* Ω_l *is defined as in* (4).

(*b*) *If* $\Omega_l \in \mathcal{A}$ *for* $1 \leq l \leq k-1$, *then* $L_1(c) \equiv \cdots \equiv L_{k-1}(c) \equiv 0$ *and* $L_k(c) = \int_{\gamma_c} \omega_k$.

Proof. (*a*) We proceed by induction on k. If $k = 2$, then statement (*a*) is true because $\Omega_1 = \omega_1 \in \mathcal{A}$. We now assume that the statement is true for $k - 1$, and we will prove it for k.

From the induction hypothesis it follows that $\omega_l, \Omega_l \in \mathcal{A}$ for $1 \leq l \leq k-2$. Thus, by Corollary 5, $\Omega_l = dQ_l + q_l dH$ with $q_l \in \mathcal{S}$ for all $1 \leq l \leq k-2$, and by using Lemma 4 we conclude that $\overline{\Omega}_{k-1} := \sum_{i+j=l} q_i \omega_j \in \mathcal{A}$. Hence, since $\Omega_{k-1} = \omega_{k-1} + \overline{\Omega}_{k-1}$, Lemma 4 implies that $\omega_{k-1} \in \mathcal{A}$ if and only if $\Omega_{k-1} \in \mathcal{A}$.

(*b*) By Corollary 5, $\Omega_l = dQ_l + q_l dH$ with $q_l \in \mathcal{S}$ for all $1 \leq l \leq k-1$, and $L_1(c) \equiv \cdots \equiv L_{k-1}(c) \equiv 0$. In addition, by the statement (*a*), $\omega_l \in \mathcal{A}$ for $1 \leq l \leq k-1$. Thus, $\overline{\Omega}_k := \sum_{i+j=k} q_i \omega_j \in \mathcal{A}$ because of Lemma 4, which implies that $\int_{\gamma_c} \overline{\Omega}_k \equiv 0$ by Corollary 5. Finally, from Theorem 2 we have $L_k(c) = \int_{\gamma_c} \omega_k + \int_{\gamma_c} \overline{\Omega}_k$. Therefore $L_k(c) = \int_{\gamma_c} \omega_k$. □

Before announce next lemma, we note that each polynomial $h(x) = \sum_{r=0}^{m-1} a_r x^r$ of degree $m - 1$ can be written as

$$h(x) = \hat{h}\left(x^2\right) + x\tilde{h}\left(x^2\right) \tag{6}$$

where

$$\hat{h}(x^2) = \sum_{r=0}^{\left[\frac{m-1}{2}\right]} a_{2r+1}x^{2r}, \quad \tilde{h}(x^2) = \sum_{r=0}^{\left[\frac{m-2}{2}\right]} a_{2r+2}x^{2r}. \tag{7}$$

Lemma 8. *Let* $\omega = (g(x) + f(x)y)\,dx$*, where* $f(x) = \sum_{r=0}^{m-1} a_r x^r$ *and* $g(x) = \sum_{s=0}^{n} b_s x^s$*.*

(a) $\displaystyle\int_{\gamma_c} \omega = \int_{\gamma_c} \hat{f}(x^2)ydx$ *and its value is*

$$-\pi c \sum_{r=0}^{\left[\frac{m-1}{2}\right]} \binom{2(r+1)}{r+1} \frac{a_{2r+1}}{2^r(2r+1)} c^r.$$

(b) If $\int_{\gamma_c}\omega \equiv 0$*, then* $\omega = dQ + (y\bar{q})dH$ *with* $\bar{q} \in \mathbb{R}\left[x^2, H\right]$ *of degree* $\deg_2 \bar{q} = [(m-2)/2]$*, and* $\displaystyle\int_{\gamma_c} (y\bar{q})\,\omega = \int_{\gamma_c} \bar{q}\hat{g}(x^2)ydx$ *whose value is*

$$-\pi c \sum_{s=0}^{\left[\frac{n}{2}\right]} \sum_{r=0}^{\left[\frac{m-2}{2}\right]} \binom{2(s+r+1)}{s+r+1} \frac{(b_{2s})(a_{2r+2})}{2^{s+r}(2s+1)} c^{s+r}.$$

(c) $\int_{\gamma_c} (y\bar{q})\,\omega \equiv 0$ *if and only if* $\bar{q} \equiv 0$ *or* $\hat{g}(x^2) \equiv 0$*.*

Proof. (a) By statements (a) and (b) of Lemma 3, $\int_{\gamma_c}\omega = \int_{\gamma_c}\hat{f}(x^2)ydx$, and the value of this integral follows from Corollary 6.

(b) If $\int_{\gamma_c}\omega \equiv 0$, then $\hat{f}(x^2) \equiv 0$ by (a). This implies that $\omega = g(x)dx + x\tilde{f}(x^2)ydx$ and by (7) we have

$$\omega = d\left(\int g(x)dx\right) + \sum_{r=0}^{\left[\frac{m-2}{2}\right]} a_{2r+2}x^{2r+1}ydx.$$

From Lemma 3.(b) we obtain $x^{2r+1}ydx = d\left(y\overline{Q}_r\right) + \left(y\overline{q}_r\right)dH$ for some homogeneous polynomials $\overline{Q}_r, \overline{q}_r \in \mathbb{R}\left[x^2, H\right]$ with $\deg_2 \overline{Q}_r = r+1$ and $\deg_2 \overline{q}_r = r$, respectively. Hence

$$\omega = dQ + (y\overline{q})\,dH$$

with

$$Q = \int g(x)dx + y \sum_{r=0}^{\left[\frac{m-2}{2}\right]} a_{2r+2}\overline{Q}_r \quad \text{and} \quad \overline{q} = \sum_{r=0}^{\left[\frac{m-2}{2}\right]} a_{2r+2}\overline{q}_r.$$

Thus $\overline{q} \in \mathbb{R}\left[x^2, H\right]$ is homogeneous and $\deg_2 \overline{q} = \left[\frac{m-2}{2}\right]$. Moreover, a simple computation shows that

$$\overline{q}_r = 2\sum_{i=0}^{r} \binom{r+1}{i} \left(\frac{r+1-i}{2i+1}\right) (2H)^{r-i} \left(x^2 - 2H\right)^i. \tag{8}$$

As $(y\overline{q})\,\omega = \overline{q}\hat{g}\big(x^2\big)ydx + \overline{q}\tilde{g}\big(x^2\big)xydx + \overline{q}\tilde{f}\big(x^2\big)xy^2dx$ and $\overline{q}\tilde{f}\big(x^2\big)xy^2dx = \overline{q}\tilde{f}\big(x^2\big)x\left(2H - x^2\right)dx$, it follows that $(y\overline{q})\,\omega = \overline{q}\hat{g}\big(x^2\big)ydx + dQ_2 + q_2dH$ because of statements (a) and (b) of Lemma 3. Hence we obtain $\int_{\gamma_c}(y\bar{q})\,\omega = \int_{\gamma_c}\overline{q}\hat{g}\big(x^2\big)ydx$. That is,

$$\int_{\gamma_c}(y\overline{q})\,\omega = \int_{\gamma_c}\left(\sum_{r=0}^{[\frac{m}{2}]-1} a_{2r+2}\overline{q}_r\right)\left(\sum_{s=0}^{[\frac{n}{2}]} b_{2s}x^{2s}\right)ydx.$$

By using expression (8) of $\overline{q}_r$, a straightforward computation, and Lemma 3(c) we obtain the formula given in the statement. Finally, statement (c) follows from the formula given in statement (b). $\square$

Remark 1. System (1_ε) with $\mu = \nu = 1$, $F_1(x) = -x^2$, and $g_1(x) = 1 - x^2$ does not satisfy the hypothesis in definition of $\mathcal{GL}1$ because $g_1(x)$ is not an odd function. Here $m = n = 2$ and from Theorem 1.(a) it follows that $\tilde{\mathcal{H}}_1^1(2,2) = 0$; however, for ε small enough, this system has one medium amplitude limit cycle. Indeed, we need only to prove that the first non-vanishing coefficient of the displacement function (2), associated to this system, has a simple positive zero. For that we write system in the form (3_ε) as $dH - \varepsilon\omega = 0$ with $\omega = (1 - x^2 - 2xy)dx$. By Lemma 8.$(a)$, $L_1(c) \equiv 0$, and by Theorem 2 and Lemma 8.(b), $L_2(c) = -\pi c(4 - 2c)$.

Now, system (1_ε) with $\mu = \nu = 2$, $F_1(x) = -3x^2$, $F_2(x) = -2x^3$, $g_1(x) = x^2 + x^3$, and $g_2(x) = \left(-5 + 25x^2\right)/6$ does not satisfy the hypothesis in definition of $\mathcal{GL}2$ because $F_2(x)$ is not an even function. In this case $m = n = 3$ and by Theorem 1.(b), $\tilde{\mathcal{H}}_2^2(3,3) = 1$; however, for ε small enough, the resulting system has two medium amplitude limit cycles. Indeed, following previous ideas, and using Theorem 2 and Lemma 8 it is easy to see that $L_1(c) \equiv 0$, $L_2(c) \equiv 0$, and $L_3(c) = -\pi c(c-1)(c-2)$.

4. Proof of the Theorem 1

We can assume, after a linear change of variables if necessary, that $F_i(0) = 0$ for all $1 \le i \le \mu$. Suppose that $F_i(x) = \sum_{r=1}^{m}(a_{i(r-1)}/r)x^r$ and $g_i(x) = \sum_{s=0}^{n} b_{is}x^s$. Thus, $f_i(x) = F_i'(x) = \sum_{r=0}^{m-1} a_{ir}x^r$ and $g_i(x)$ can be written as $f_i(x) = \hat{f}_i\big(x^2\big) + x\tilde{f}_i\big(x^2\big)$ and $g_i(x) = \hat{g}_i\big(x^2\big) + x\tilde{g}_i\big(x^2\big)$, respectively, according to (6).

Proof of Theorem 1. (a) By hypothesis, $g_i(x)$ is odd for $1 \le i \le \nu$, which means that $g_i(x) = x\tilde{g}_i\big(x^2\big)$ for $1 \le i \le \nu$. Let $L_k(c)$ be the first non-vanishing Poincaré–Pontryagin–Melnikov function in (2). For proving the statement we will prove first that $L_k(c)$ has at most $[(m-1)/2]$ positive zeros, and then that for each s with $0 \le s \le [(m-1)/2]$ we can choose systems in $\mathcal{GL}1$ in such a way that $L_k(c)$ has exactly s simple positive zeros.

If $k = 1$, then the assertion is true. Indeed, we have

$$L_1(c) = \int_{\gamma_c} x\tilde{g}_i\big(x^2\big)dx + \hat{f}_1\big(x^2\big)ydx + \tilde{f}_1\big(x^2\big)xydx.$$

Thus, as $\int_{\gamma_c} x\tilde{g}_i\big(x^2\big)dx \equiv 0$ and $\int_{\gamma_c}\tilde{f}_1\big(x^2\big)xydx \equiv 0$ by Corollary 5, we obtain $L_1(c) = \int_{\gamma_c}\hat{f}_1\big(x^2\big)ydx$. From (7) we know that $\deg_2 \hat{f}_1\big(x^2\big) = [(m-1)/2]$, which implies that $L_1(c)$ has at most $[(m-1)/2]$ positive zeros because of Corollary 6. In addition, since $\hat{f}_1\big(x^2\big)$ has $[(m-1)/2] + 1$ independent coefficients, for each s with $0 \le s \le [(m-1)/2]$ we can choose suitable coefficients of $\hat{f}_1\big(x^2\big)$ in such a way that $L_k(c)$ has exactly s simple positive zeros. Therefore, by applying the Poincaré–Pontryagin–Andronov criterion it follows that the corresponding system (1_ε), which belongs to $\mathcal{GL}1$, has exactly s hyperbolic limit cycles. In particular we have proved that $\tilde{\mathcal{H}}_\nu^\mu(m,n) = [(m-1)/2]$.

Suppose then that $k \geq 2$. If we prove that $\Omega_l \in \mathcal{A}$ for $1 \leq l \leq k-1$, then $L_k(c) = \int_{\gamma_c} \omega_k$ by Lemma 7, and by applying the same idea as in previous paragraph the assertion follows. Therefore, it remains to show that $\Omega_l \in \mathcal{A}$ for $1 \leq l \leq k-1$.

We proceed by induction on k. If $k = 2$, then $L_1(c) \equiv 0$, which implies that

$$\Omega_1 = \Big(x\tilde{g}_1\big(x^2\big) + xy\tilde{f}_1\big(x^2\big)\Big)\, dx \in \mathcal{A}.$$

Hence the assertion is true for $k = 2$.

We now assume that the assertion is true for $k-2$, and we will prove it for $k-1$. By induction hypothesis, $\Omega_i \in \mathcal{A}$ for $1 \leq i \leq k-2$, which implies that $\Omega_i = dQ_i + q_i dH$ with $q_i \in \mathcal{S}$ for $1 \leq i \leq k-2$ by Corollary 5. Furthermore, by Lemma 7, $\omega_j \in \mathcal{A}$ for $1 \leq j \leq k-2$. Hence $\overline{\Omega}_{k-1} := \sum_{i+j=k-1} q_i\omega_j$ (with $1 \leq i,\, j \leq k-2$) is an element of $\mathcal{A}$ because of Lemma 4. Since $\Omega_{k-1} = \omega_{k-1} + \overline{\Omega}_{k-1}$,

$$L_{k-1}(c) = \int_{\gamma_c} \Omega_{k-1} = \int_{\gamma_c} \omega_{k-1} = \int_{\gamma_c} \hat{f}_{k-1}\big(x^2\big)ydx,$$

which vanishes identically. Therefore, $\omega_{k-1} = \Big(x\tilde{g}_{k-1}\big(x^2\big) + xy\tilde{f}_{k-1}\big(x^2\big)\Big)\, dx \in \mathcal{A}$. Thus $\Omega_{k-1} \in \mathcal{A}$.

(b) Firstly we will show two properties concerning ω_i and $\int_{\gamma_c} \omega_i$ which we will use along the proof of the statement. Then, we will split the proof into two cases: m odd and m even.

For $1 \leq i < \mu_0$ the 1-form ω_i has the form $g_i(x)dx$ that is exact: $\omega_i = dQ_i + q_i dH$ with $q_i \equiv 0$. Hence, from (4) it follows that $\Omega_i = \omega_i$ for $1 \leq i \leq \mu_0$. Thus, $L_i(c) = \int_{\gamma_c} \Omega_i \equiv 0$ for $1 \leq i < \mu_0$ and $L_{\mu_0}(c) = \int_{\gamma_c} \Omega_{\mu_0} = \int_{\gamma_c} \omega_{\mu_0}$. On the other hand, since $F_i(x)$ is even for $\mu_0 < i \leq \mu$, $f_i(x) = x\tilde{f}_i\big(x^2\big)$ for $\mu_0 < i \leq \mu$. Thus, for $i > \mu_0$ we have $\omega_i = d\left(\int g_i(x)dx\right) + x\tilde{f}_i\big(x^2\big)ydx$. Moreover, $x^{2r+1}ydx = d\left(y\overline{Q}_{0r}\right) + \left(y\overline{q}_{0r}\right) dH$ because of Lemma 3.(b). Hence,

$$\omega_i = d\left(\bar{Q}_i\right) + \left(y\bar{q}_i\right) dH; \tag{9}$$

of course $\bar{q}_i \equiv 0$ for $i > \mu$. Therefore we obtain

$$\int_{\gamma_c} \omega_i \equiv 0 \quad \text{for } i > \mu_0. \tag{10}$$

Case m odd. In this case $\deg F_{\mu_0}(x) = m$ because $F_i(x)$ is an even polynomial for $\mu_0 < i \leq \mu$. Since $F'_{\mu_0}(x) = f_{\mu_0}(x) = \hat{f}_{\mu_0}\big(x^2\big) + x\tilde{f}_{\mu_0}\big(x^2\big)$ has an even degree, $\hat{f}_{\mu_0}\big(x^2\big) \not\equiv 0$. Hence, from Lemma 8.($a$) it follows that $L_{\mu_0}(c) = \int_{\gamma_c} \omega_{\mu_0} = \int_{\gamma_c} \hat{f}_{\mu_0}\big(x^2\big)ydx \not\equiv 0$, and it has at most $[(m-1)/2]$ positive zeros, counting multiplicities; moreover, we can choose suitable coefficients of $F_{\mu_0}(x)$ in such a way that $L_{\mu_0}(c)$ has exactly $[(m-1)/2]$ simple positive zeros. Therefore by the Poincaré–Pontryagin–Andronov criterion, $\tilde{\mathcal{H}}_\nu^\mu(m,n) = [(m-1)/2]$.

Case m even. Let $L_k(c)$ be the first non-vanishing Poincaré–Pontryagin–Melnikov function of (2). If $k = \mu_0$, then $L_{\mu_0}(c) = \int_{\gamma_c} \omega_{\mu_0}$ has at most $[(m-1)/2]$ positive zeros, counting multiplicities, because of Lemma 8.(a). Since m is even, $[(m-1)/2] \leq [m/2] + [n/2] - 1$. Hence $L_{\mu_0}(c)$ has at most $[m/2]+[n/2]-1$ positive zeros, counting multiplicities.

We claim that if $k \geq \mu_0 + 1$, then $\omega_1, \ldots, \omega_{k-1-\mu_0} \in \mathcal{A}$, $\Omega_i = dQ_i + q_i dH$ with $q_i \in \mathcal{S}$ for $\mu_0 \leq i \leq k-1$, and $L_k(c) = \int_{\gamma_c} \left(y\bar{q}_{\mu_0}\right) \omega_{k-\mu_0} = \int_{\gamma_c} \bar{q}_{\mu_0}\hat{g}_{k-\mu_0}\big(x^2\big)ydx$. By assuming that this assertion is true and by applying Lemma 8.(b) we conclude that $L_k(c)$ has at most $[m/2]+[n/2]-1$ positive zeros, counting multiplicities; moreover, for each s with $0 \leq s \leq [m/2]+[n/2]-1$ we can choose suitable coefficients of $\bar{q}_{\mu_0}$ and $\hat{g}_{k-\mu_0}(x^2)$ in such a way that $L_k(c)$ has exactly s simple positive zeros. Thus, by the Poincaré–Pontryagin–Andronov criterion the corresponding system (1_ε) has exactly s hyperbolic limit cycles; in particular we have $\bar{\mathcal{H}}^{\mu}_{\nu}(m,n)=[m/2]+[n/2]-1$. Therefore, to finish the proof of statement (b) we need only to confirm our assertion, which we prove next by proceeding by induction on k.

If $k=\mu_0+1$, then we will prove that $\Omega_{\mu_0}=dQ_{\mu_0}+q_{\mu_0}dH$ with $q_{\mu_0}\in\mathcal{S}$, and that $L_{\mu_0+1}(c)=\int_{\gamma_c}\bar{q}_{\mu_0}\hat{g}_1(x^2)ydx$. We know that $\Omega_{\mu_0}=\omega_{\mu_0}$ and since $k=\mu_0+1$, $L_{\mu_0}(c)=\int_{\gamma_c}\Omega_{\mu_0}\equiv 0$. Thus, from Lemma 8.$(b)$ it follows that $\Omega_{\mu_0}=\omega_{\mu_0}=dQ_{\mu_0}+q_{\mu_0}dH$, where $q_{\mu_0}=y\bar{q}_{\mu_0}\not\equiv 0\in\mathcal{S}$. On the other hand, by Theorem 2, $L_{\mu_0+1}(c)=\int_{\gamma_c}\Omega_{\mu_0+1}$, where $\Omega_{\mu_0+1}=\omega_{\mu_0+1}+\sum_{i+j=\mu_0+1}q_i\omega_j$. Since $q_i\equiv 0$ for $1\leq i<\mu_0$, $\Omega_{\mu_0+1}=\omega_{\mu_0+1}+q_{\mu_0}\omega_1$. Hence, by (10) we obtain $L_{\mu_0+1}(c)=\int_{\gamma_c}(y\bar{q}_{\mu_0})\,\omega_1=\int_{\gamma_c}\bar{q}_{\mu_0}\hat{g}_1(x^2)ydx$.

If $k=\mu_0+2$, then $\Omega_{\mu_0}=\omega_{\mu_0}=dQ_{\mu_0}+q_{\mu_0}dH$, where $q_{\mu_0}=y\bar{q}_{\mu_0}\not\equiv 0\in\mathcal{S}$ and $L_{\mu_0+1}(c)\equiv 0$. Since $\bar{q}_{\mu_0}\not\equiv 0$, $\hat{g}_1(x^2)\equiv 0$ by Lemma 8.(c). Thus, $\Omega_1=\omega_1\in\mathcal{A}$, and Corollary 5 implies that $\Omega_1=dQ_1+q_1dH$ with $q_1\in\mathcal{S}$. Moreover, from (9) $\omega_{\mu_0+1}=d\left(\bar{Q}_{\mu_0+1}\right)+(y\bar{q}_{\mu_0+1})\,dH$, whence

$$\Omega_{\mu_0+1}=\omega_{\mu_0+1}+q_{\mu_0}\omega_1=dQ_{\mu_0+1}+q_{\mu_0+1}dH$$

with $q_{\mu_0+1}\in\mathcal{S}$ because of Corollary 5.

On the other hand, from Theorem 2 we have

$$L_{\mu_0+2}(c)=\int_{\gamma_c}\omega_{\mu_0+2}+\sum_{i+j=\mu_0+2}\int_{\gamma_c}q_i\omega_j.$$

As $\omega_1\in\mathcal{A}$ and $q_{\mu_0+1}\in\mathcal{S}$, then $q_{\mu_0+1}\omega_1\in\mathcal{A}$ following Lemma 4 and $\int_{\gamma_c}q_{\mu_0+1}\omega_1\equiv 0$ by Corollary 5. In addition, we know that $q_i\equiv 0$ for $1\leq i<\mu_0$ and $\int_{\gamma_c}\omega_{\mu_0+2}\equiv 0$ by (10). Hence $L_{\mu_0+2}(c)=\int_{\gamma_c}(y\bar{q}_{\mu_0})\,\omega_2=\int_{\gamma_c}\bar{q}_{\mu_0}\hat{g}_2(x^2)ydx$.

We now assume that the assertion holds for $k-1$, and we will prove it for k. By Theorem 2, $L_k(c)=\int_{\gamma_c}\Omega_k$, where

$$\Omega_k=\omega_k+q_1\omega_{k-1}+\cdots+q_{\mu_0-1}\omega_{k+1-\mu_0}+q_{\mu_0}\omega_{k-\mu_0}+q_{\mu_0+1}\omega_{k-1-\mu_0}+\cdots+q_{k-2}\omega_2+q_{k-1}\omega_1.$$

Since $q_i\equiv 0$ for $1\leq i<\mu_0$, $\Omega_k=\omega_k+q_{\mu_0}\omega_{k-\mu_0}+q_{\mu_0+1}\omega_{k-1-\mu_0}+\cdots+q_{k-2}\omega_2+q_{k-1}\omega_1$.

On the other hand, from the induction hypothesis it follows that $\omega_1,\ldots,\omega_{k-2-\mu_0}\in\mathcal{A}$, $\Omega_i=dQ_i+q_idH$ with $q_i\in\mathcal{S}$ for $\mu_0\leq i\leq k-2$, and $L_{k-1}(c)=\int_{\gamma_c}\bar{q}_{\mu_0}\hat{g}_{k-1-\mu_0}(x^2)ydx$. Since $L_{k-1}(c)\equiv 0$, $\hat{g}_{k-1-\mu_0}(x^2)\equiv 0$ because of Lemma 8.(c), which implies that $\omega_{k-1-\mu_0}\in\mathcal{A}$. Therefore, $q_{\mu_0}\omega_{k-1-\mu_0}+\cdots+q_{k-3}\omega_2+q_{k-2}\omega_1\in\mathcal{A}$ by Lemma 4. Moreover, from (10) $\omega_{k-1}=d\left(\bar{Q}_{k-1}\right)+(y\bar{q}_{k-1})\,dH$, and by applying Corollary 5 we obtain

$$\Omega_{k-1}=dQ_{k-1}+q_{k-1}dH,\quad \text{with } q_{k-1}\in\mathcal{S}.$$

Hence $q_{\mu_0+1}\omega_{k-1-\mu_0}+\cdots+q_{k-2}\omega_2+q_{k-1}\omega_1\in\mathcal{A}$ by Lemma 4. In addition, $\omega_k=d\left(\bar{Q}_k\right)+(y\bar{q}_k)\,dH$ by (10). Thus, we obtain $L_k(c)=\int_{\gamma_c}q_{\mu_0}\omega_{k-\mu_0}=\int_{\gamma_c}\bar{q}_{\mu_0}\hat{g}_{k-\mu_0}(x^2)ydx$. □

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