PHASE PORTRAITS OF UNIFORM ISOCHRONOUS QUARTIC CENTERS

JACKSON ITIKAWA¹ AND JAUME LLIBRE¹

Abstract. In this paper we classify the global phase portraits in the Poincaré disc of all quartic polynomial differential systems with a uniform isochronous center at the origin.

1. Introduction and statement of the main results

Christian Huygens is credited with being one of the first scholars to study isochronous systems in the XVII century, even before the development of the differential calculus. Huygens investigated the cycloidal pendulum, which has isochronous oscillations in opposition to the monotonicity of the period of the usual pendulum. It is probably the first example of a nonlinear isochrone. For more details see [7].

Isochronicity appears in a wide variety of Physics phenomena and it is also closely related to the uniqueness and existence of solutions for some boundary value, perturbation, or bifurcation problems. Moreover it is important in stability theory, since a periodic solution of the central region is Liapunov stable if and only if the neighboring periodic solutions have the same period. For more details on these topics see [4].

In the last decades the study of isochronous systems has been increased due to the proliferation of powerful methods of computerized research, and special attention has been dedicated to polynomial differential systems, see [2, 5] and the bibliography therein.

In this paper we classify the global phase portraits of all quartic polynomial differential systems with a uniform isochronous center at the origin.

Let \( p \in \mathbb{R}^2 \) be a center of a differential polynomial system in \( \mathbb{R}^2 \), without loss of generality we can assume that \( p \) is the origin of coordinates. We say that \( p \) is an isochronous center if it is a center having a neighborhood such that all the periodic orbits in this neighborhood have the same period. We say that \( p \) is a uniform

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isochronous center if the system, in polar coordinates \( x = r \cos \theta, \ y = r \sin \theta \), takes the form \( \dot{r} = G(\theta, r), \ \dot{\theta} = k, \ k \in \mathbb{R} \setminus \{0\} \), for more details see Conti [5]. The next result is well-known.

**Proposition 1.** Assume that a planar differential polynomial system 
\[
\dot{x} = P(x, y), \ \dot{y} = Q(x, y)
\]
of degree \( n \) has a center at the origin of coordinates. Then, this center is uniform isochronous if and only if by doing a linear change of variables and a rescaling of time it can be written into the form
\[
\dot{x} = -y + x f(x, y), \ \dot{y} = x + y f(x, y),
\]
where \( f(x, y) \) is a polynomial in \( x \) and \( y \) of degree \( n - 1 \), and \( f(0, 0) = 0 \).


**Theorem 2.** Consider \( f(x, y) = \sum_{i=1}^{3} f_i(x, y) \) where \( f_i(x, y) \) for \( i = 1, 2, 3 \) are homogeneous polynomials of degree \( i \), \( f_1^2 + f_2^2 \neq 0 \) and \( f_3 \neq 0 \) such that (1) be a quartic polynomial differential system. Then the only case of local analytic integrability in an open neighborhood of the origin of system (1) is given, modulo a rotation, by the time-reversible system.
\[
\dot{x} = -y + x(A_1 x + B_2 xy + C_1 x^3 + C_3 xy^2), \ 
\dot{y} = x + y(A_1 x + B_2 xy + C_1 x^3 + C_3 xy^2),
\]
where \( A_1, B_2, C_1, C_3 \in \mathbb{R} \).

Due to the following classical result due to Poincaré [13] and Liapunov [11] Theorem 2 characterizes the quartic uniform isochronous centers, except the ones for which the polynomial \( f(x, y) \) is a homogeneous polynomial of degree 3, and the characterization of such systems has already been done by Conti in [5].

**Theorem 3.** An analytic differential system \( \dot{x} = -y + F_1(x, y), \ \dot{y} = x + F_2(x, y) \), with \( F_1(x, y) \) and \( F_2(x, y) \) real analytic functions without constant and linear terms defined in a neighborhood of the origin, has a center at the origin if and only if there exists a local analytic first integral of the form \( H = x^2 + y^2 + G(x, y) \) defined in a neighborhood of the origin, where \( G \) starts with terms of order higher than two.

Algaba et al [1] provided the phase portraits of systems (2) in the particular case \( C_1 = 0 \). In such case systems (2) have a polynomial commutator, allowing to get the bifurcation diagram of the systems. In Theorem 4 presented below, we provide all the global phase portraits of systems (2).
Theorem 4. Consider a quartic polynomial differential system $X : \mathbb{R}^2 \to \mathbb{R}^2$ and assume that $X$ has a uniform isochronous center at the origin. Then the global phase portrait of $X$ is topologically equivalent to one of the 14 phase portraits of Figure 1.

More precisely, since $X$ can always be written as system (2), the global phase portrait of $X$ is topologically equivalent to the phase portrait (a) of Figure 1 if either $C_1C_3 > 0$, or $C_3 = 0$, $B_2 < 0$;
(b) of Figure 1 if $C_1 = 0$, $C_3 \neq 0$ and if either $r_1, r_2, r_3 > 0$, or $r_1, r_2, r_3 < 0$;

(c) of Figure 1 if $C_1 = 0$, $C_3 \neq 0$ and if either $r_1 < 0, r_2, r_3 > 0$, or $r_1, r_2 < 0, r_3 > 0$;

(d) of Figure 1 if $C_1 = 0$, $C_3 \neq 0$ and if either $r_1 r_2 > 0, r_3 = r_2$, or $r_2 = r_1, r_1 < 0, r_3 > 0$;

(e) of Figure 1 if $C_1 = 0$, $C_3 \neq 0$ and if either $r_1 < 0, r_2 > 0, r_3 = r_2$, or $r_2 = r_1, r_1 < 0, r_3 > 0$;

(f) of Figure 1 if $C_1 = 0$, $C_3 \neq 0$ and if either $r_3 = r_2 = r_1, \forall r_1, r_2, r_3 \in \mathbb{R}$, or $r_1 \neq 0$ and $r_{2,3} = a \pm bi$, $\forall r_1, b \in \mathbb{R}$, $a \in \mathbb{R}$;

(g) of Figure 1 if $C_3 = 0$, $C_1 \neq 0$, $B_2 > 0$, $C_1 \neq -A_1 B_2$;

(h) of Figure 1 if either $C_3 = 0$, $C_1 \neq 0$, $B_2 > 0$, $C_1 = -A_1 B_2$, or $B_3 = C_3 = 0$;

(i) or (j) or (k) of Figure 1 if $C_1 C_3 < 0$, $B_2 = 0$;

(l) or (m) or (n) of Figure 1 if $C_1 C_3 < 0$, $B_2 \neq 0$;

where in the cases with $C_1 = 0$, we have that $r_1, r_2, r_3$ are the roots of the polynomial $-C_3 - B_2 x - A_1 x^2 - x^3$ and we assume that $r_1 \leq r_2 \leq r_3$ when these roots are real.

Our results have been checked with the software $P_4$, see for more details on this software the Chapters 9 and 10 of [6].

The rest of the paper is organized as follows. In section 2 we present some results and technical propositions used in our study. In section 3 we prove Theorem 4.

2. Preliminary results

In this section we present some results necessary to our study.

**Poincaré compactification.** Let $\mathcal{X}$ be a planar vector field of degree $n$. The Poincaré compactified vector field $p(\mathcal{X})$ corresponding to $\mathcal{X}$ is an analytic vector field induced on $\mathbb{S}^2$ as follows (see, for instance [8], or Chapter 5 of [6]). Let $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ (the Poincaré sphere) and $T_y\mathbb{S}^2$ be the tangent space to $\mathbb{S}^2$ at point $y$. Consider the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \rightarrow \mathbb{S}^2$. This map defines two copies of $\mathcal{X}$, one in the northern hemisphere and the other in the southern hemisphere. Denote by $\mathcal{X}'$ the vector field $Df \circ \mathcal{X}$ defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. Clearly $\mathbb{S}^1$ is identified to the infinity of $\mathbb{R}^2$. In order to extend $\mathcal{X}'$ to a vector field on $\mathbb{S}^2$ (including $\mathbb{S}^1$) it is necessary that $\mathcal{X}$ satisfies suitable conditions. In
the case that $X$ is a planar vector field of degree $n$ then $p(X)$ is the only analytic extension of $y_0^{n-1}X'$ to $S^2$. On $S^2 \setminus S^1$ there are two symmetric copies of $X$, and knowing the behaviour of $p(X)$ around $S^1$, we know the behaviour of $X$ at infinity. The projection of the closed northern hemisphere of $S^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \mapsto (y_1, y_2)$ is called the Poincaré disc, and it is denoted by $D^2$. The Poincaré compactification has the property that $S^1$ is invariant under the flow of $p(X)$.

As $S^2$ is a differentiable manifold, for computing the expression for $p(X)$, we consider the six local charts $U_i = \{ y \in S^2 : y_i > 0 \}$, and $V_i = \{ y \in S^2 : y_i < 0 \}$ where $i = 1, 2, 3$; and the diffeomorphisms $F_i : U_i \to \mathbb{R}^2$ and $G_i : V_i \to \mathbb{R}^2$ for $i = 1, 2, 3$ are the inverses of the central projections from the planes tangent at the points $(1, 0, 0), (-1, 0, 0)$, $(0, 1, 0), (0, -1, 0), (0, 0, 1)$ and $(0, 0, -1)$ respectively. We denote by $(u, v)$ the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$ (so $(u, v)$ represents different things according to the local charts under consideration).

The expression for $p(X)$ in the local chart $(U_1, F_1)$ is given by

$$\dot{u} = v^n \left[ -uP \left( \frac{1}{v}, \frac{u}{v} \right) + Q \left( \frac{1}{v}, \frac{u}{v} \right) \right], \quad \dot{v} = -v^{n+1} P \left( \frac{1}{v}, \frac{u}{v} \right).$$

The expression for $p(X)$ in local chart $(U_2, F_2)$ is

$$\dot{u} = v^n \left[ P \left( \frac{u}{v}, \frac{1}{v} \right) - uQ \left( \frac{u}{v}, \frac{1}{v} \right) \right], \quad \dot{v} = -v^{n+1} Q \left( \frac{u}{v}, \frac{1}{v} \right),$$

and for $(U_3, F_3)$ is

$$\dot{u} = P(u, v), \quad \dot{v} = Q(u, v).$$

The expression for $p(X)$ in the chart $(V_i, G_i)$ is the same than in the chart $(U_i, F_i)$ multiplied by $(-1)^{n-1}$ for $i = 1, 2, 3$. The points of $S^3$ in any chart have $v = 0$. Thus we obtain a polynomial vector field in each local chart.

Since the unique singular points at infinity which cannot be contained into the charts $U_1 \cup V_1$ are the origins of $U_2$ and $V_2$, when we study the infinite singular points on the charts $U_2 \cup V_2$, we only need to verify if the origin of these charts are singular points.

**Topological equivalence.** We say that two polynomial vector fields $X$ and $Y$ on $\mathbb{R}^2$ are *topologically equivalent* if there exists a homeomorphism on $S^2$ preserving the infinity $S^1$ carrying orbits of the flow induced by $p(X)$ into orbits of the flow induced by $p(Y)$, preserving or reversing simultaneously the sense of all orbits.
A separatrix of $p(X)$ is an orbit which is either a singular point, or a limit cycle, or a trajectory which lies in the boundary of a hyperbolic sector at a singular point, finite or infinity.

We denote by $\text{Sep}(p(X))$ the set formed by all separatrices of $p(X)$. Neumann [12] proved that the set $\text{Sep}(p(X))$ is closed. Each open connected component of $S^2 \setminus \text{Sep}(p(X))$ is called a canonical region of $p(X)$. A separatrix configuration is defined as a union of $\text{Sep}(p(X))$ plus one representative solution chosen from each canonical region. We say that $\text{Sep}(p(X))$ and $\text{Sep}(p(Y))$ are equivalent if there exists a homeomorphism in $S^2$ preserving the infinity $S^1$ carrying orbits of $\text{Sep}(p(X))$ into orbits of $\text{Sep}(p(Y))$, preserving or reversing simultaneously the sense of all orbits.

The next two theorems, due to Neumann [12], states the characterization of two topologically equivalent Poincaré compactified vector fields.

**Theorem 5.** Suppose that $p(X)$ and $p(Y)$ are two continuous flows in $S^2$ with finitely many singular points. Then $p(X)$ and $p(Y)$ are topologically equivalent if and only if their separatrix configurations are equivalent.

Theorem 5 implies that, in order to obtain the global phase portrait of a polynomial vector field $p(X)$ with finitely many singular points, we essentially need to determine the $\alpha-$ and the $\omega-$limit sets of all separatrices of $p(X)$.

**Theorem 6.** If $p(X)$ and $p(Y)$ have the infinity filled of singular points and finitely many finite singular points, we shall work with $X$ and $Y$. Then $X$ and $Y$ are topologically equivalent if and only if their separatrix configurations are equivalent.

Theorem 6 implies that, in order to obtain the global phase portrait of a polynomial vector field $X$ with the infinity filled of singular points and finitely many finite singular points, we essentially need to determine the $\alpha-$ and the $\omega-$limit sets of all separatrices of $X$.

### 3. Proof of Theorem 4

For providing all the possible phase portraits in the Poincaré disc for the planar quartic polynomial differential systems with a uniform isochronous center at the origin, we shall start studying all the finite and infinite singular points of such systems. We remark that in this work we never consider the quartic polynomial differential systems (1)
with a uniform isochronous center such that \( f(x, y) \) is a homogeneous polynomial.

By Theorem 2 a planar quartic polynomial differential system with a uniform isochronous center at the origin always can be written as a system (2). These systems are invariant under the transformation \((x, y, t) \mapsto (-x, y, -t)\), and therefore, all their phase portraits are symmetric respect to the \( y \)-axis.

3.1. Finite singular points. In polar coordinates \((r, \theta)\) defined by \((x, y) = (r \cos \theta, r \sin \theta)\) a planar differential system with a uniform isochronous center at the origin (2) always can be written as \( \dot{r} = P(r, \theta), \ \dot{\theta} = 1 \). Hence such systems have no finite singular points except the origin.

The period annulus of a center is the biggest punctured neighborhood only foliated by periodic orbits. Compactifying \( \mathbb{R}^2 \) to the Poincaré disc, the boundary of the period annulus of a center has two connected components: the center itself and a graphic. Since the origin is the only finite singular point in the systems we are interested to study, the graphic of the period annulus of uniform isochronous center cannot present finite singularities. Hence, such a graphic shall be formed either by infinite singular points and separatrices of these infinite singular points, or by the periodic solution at infinity when at infinity there are no singular points.

3.2. Infinite singular points. In the chart \( U_1 \) the differential system (2) becomes

\[
\dot{u} = (1 + u^2)v^3, \\
\dot{v} = (-C_1 - C_3u^2 - B_2uv^2 - A_1v^2 + uv^3)v,
\]

and therefore all the points \((u, 0)\) for all \( u \in \mathbb{R} \) are infinite singular points of the differential system (2) in \( U_1 \). In order to obtain the local phase portraits at these points, we perform a change of coordinates of the form \( ds = vdt \), and system (3) becomes

\[
\dot{u'} = (1 + u^2)v^2, \\
\dot{v'} = -C_1 - C_3u^2 + v(-B_2u - A_1v + uv^2),
\]

where the prime denotes derivative with respect to \( s \).

In chart \( U_2 \), system (2) becomes

\[
\dot{u} = -(1 + u^2)v^3, \\
\dot{v} = (-C_3u - C_1u^3 - B_2uv + A_1uv^2 - uv^3)v.
\]
We only need to study the point \((0,0)\) of \(U_2\). We perform a change of coordinates of the form \(ds = vdt\), obtaining the system

\[
\begin{align*}
    u' &= -(1 + u^2)v^2, \\
    v' &= -C_3u - C_1u^3 - B_2uv - A_1uw^2 - uv^3.
\end{align*}
\]

We shall apply the well known results for the hyperbolic, semi-hyperbolic and nilpotent singular points, see for instance Theorems 2.15, 2.19 and 3.15 of [6], for the characterization of the local phase portraits at each singular point of systems (4) and (6).

**Case I: \(C_1 = 0\).** We remark that if \(C_1 = C_3 = 0\) system (2) degenerates to a cubic polynomial differential system, whose global phase portraits are well known, see for instance [9]. Therefore in Case I we assume \(C_3 \neq 0\).

We first analyze the chart \(U_2\). We denote by \(O_{U_2}\) the origin of the chart \(U_2\). The corresponding linear part of system (6) at \(O_{U_2}\) is

\[
\begin{pmatrix}
0 & 0 \\
-C_3 & 0
\end{pmatrix}.
\]

Therefore \(O_{U_2}\) is a nilpotent singularity and applying Theorem 3.5 of [6] we conclude that it is a cusp, whose behavior depends on the sign of the coefficient \(C_3\). Hence, the local phase portrait at the origin for system (6) might be one of the two shown in Figure 2.

![Figure 2. Local phase portrait at the origin of system (6).](image)

We now perform the study for the chart \(U_1\). Clearly the only singular point at infinity in the chart \(U_1\) is the origin, which we denote by \(O_{U_1}\).

The corresponding linear part of system (4) at \(O_{U_1}\) is identically zero. So it is necessary to apply a directional blow up \((u,v) \mapsto (u,w)\) where \(v = uw\), and we obtain the system

\[
\begin{align*}
    u' &= (1 + u^2)u^2w^2, \\
    w' &= u(-C_3 - B_2w - A_1w^2 - w^3).
\end{align*}
\]


Performing a change of the independent variable of the form \(dT = u\, ds\) in system (7), we get the system:

\[
\begin{align*}
u' &= (1 + u^2)uw^2, \\
w' &= -C_3 - B_2w - A_1w^2 - w^3,
\end{align*}
\]

where the prime now denotes derivative with respect to \(T\). The singular points of system (8) are of the form \((0, r_i), \ i = 1, 2, 3\), where \(r_1, r_2, r_3\) are the roots of the polynomial \(-C_3 - B_2w - A_1w^2 - w^3\), that is, 

\[
A_1 = -(r_1 + r_2 + r_3), \quad B_2 = r_1r_2 + r_1r_3 + r_2r_3, \quad C_3 = -r_1r_2r_3.
\]

We observe that, since we are assuming \(C_3 \neq 0\), we have \(r_1r_2r_3 \neq 0\). Hence, we have the following cases. Of course, of these roots we only need to take into account the real ones.

**Subcase I.1:** Three simple real roots. Without loss of generality we assume that \(r_1 < r_2 < r_3\). The singular points at the infinity are \(p_1 = (0, r_1), p_2 = (0, r_2), \) and \(p_3 = (0, r_3)\). The corresponding linear part of system (8) at each of these points are respectively

\[
\begin{pmatrix}
r_1^2 & 0 \\
0 & -(r_1 - r_2)(r_1 - r_3)
\end{pmatrix}, \quad \begin{pmatrix}
r_2^2 & 0 \\
0 & (r_1 - r_2)(r_2 - r_3)
\end{pmatrix}, \quad \begin{pmatrix}
r_3^2 & 0 \\
0 & -(r_1 - r_3)(r_2 - r_3)
\end{pmatrix}.
\]

Applying Theorem 2.15 of [6] and the hypotheses \(r_1 < r_2 < r_3, \ r_1r_2r_3 \neq 0\) in the above expressions we conclude that \(p_1\) and \(p_3\) are saddles, and \(p_2\) is an unstable node. The resulting singularity obtained from the blow down of \(p_1, p_2\) and \(p_3\) depends on the position of these singular points with respect to the origin of the \(u-\)axis. Hence we have the following subcases.

**Subcase I.1.1:** \(0 < r_1 < r_2 < r_3\). The local phase portraits at the singularities \(p_i, i = 1, 2, 3\) for system (8) and system (7) are shown in Figures 3 and 4, respectively.

Going back through the blow up we get the local phase portrait at the origin of system (4), see Figure 5. Finally, taking into account the rescaling of time \(ds = v\, dt\), we obtain that the phase portrait at the origin of system (3) is topologically equivalent to the one of Figure 6.

For the chart \(U_2\), since \(r_1, r_2, r_3 > 0\) then \(C_3 = -r_1r_2r_3 < 0\), and we obtain a local phase portrait as the one in Figure 2(\(C_3 < 0\)).

In short, the global phase portrait in this case is obtained taking into account all the local phase portraits of the finite and infinite singular points, the existence and uniqueness theorem for the solutions of a differential system, the fact that all the phase portraits of planar
quartic polynomial differential systems with a uniform isochronous center at the origin are symmetric with respect to the $y$-axis, and that the graphic at the boundary of the period annulus of the uniform isochronous center at the origin is formed by separatrices of infinite singular points. Hence we obtain that the global phase portrait for Subcase I.1.1 is topologically equivalent to the one of Figure 1(b) of Theorem 4.

**Subcase I.1.2:** $r_1 < 0 < r_2 < r_3$. The resulting local phase portrait at the origin of system (3) is given in Figure 7. This local phase portrait is obtained proceeding in a similar way to Case I.1.1.

For the chart $U_2$, since $r_1 < 0$ and $r_2, r_3 > 0$ then $C_3 = -r_1r_2r_3 > 0$ and we have a local phase portrait topologically equivalent to the one
Subcase I.1.3: $r_1 < r_2 < 0 < r_3$. The phase portrait at the origin of system (3) is given in Figure 8. This local phase portrait is obtained proceeding in a similar way to Case I.1.1.

For the chart $U_2$, since $C_3 = -r_1r_2r_3 < 0$, we have a local phase portrait topologically equivalent to the one of Figure 2($C_3 < 0$). Then the global phase portrait for Subcase I.1.3 is shown in Figure 1(c) of Theorem 4.

Subcase I.1.4: $r_1 < r_2 < r_3 < 0$. The resulting phase portrait at the origin of system (3) is given in Figure 9, obtained as in case I.1.1.

For the chart $U_2$, since $C_3 = -r_1r_2r_3 > 0$, we have a local phase portrait topologically equivalent to the one of Figure 2($C_3 > 0$). So the global phase portrait for Subcase I.1.3 is shown in Figure 1(b) of Theorem 4.

Subcase I.2: One simple real root and one double real root.

Without loss of generality we consider two distinct cases depending on the relative position of the simple and the double real roots: $r_1 < r_2 = r_3$ and $r_1 = r_2 < r_3$. We start with the first case. The singular points
at infinity are \( p_1 = (0, r_1) \) and \( p_2 = (0, r_2) \). The corresponding linear part of system (8) at each of these points are respectively

\[
\begin{pmatrix}
  r_1^2 & 0 \\
  0 & -(r_1 - r_2)^2
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
  r_2^2 & 0 \\
  0 & 0
\end{pmatrix}.
\]

Now we assume \( r_1 < r_2 = r_3 \), and \( r_1, r_2 \neq 0 \) in the above expressions and applying Theorems 2.15 and 2.19 of [6], we conclude that \( p_1 \) is a saddle and \( p_2 \) is a saddle-node. The resulting singularity obtained from the blow down of \( p_1 \) and \( p_2 \) depends on the position of such singular points with respect to the horizontal axis. Hence we have the following cases.

**Subcase I.2.1:** \( 0 < r_1 < r_2 \). In this case the local phase portrait at the origin of system (3) is given in Figure 10, obtained as in case I.1.1.

For the chart \( U_2 \), since \( C_3 = -r_1 r_2^2 < 0 \), we have a local phase portrait similar to the one in Figure 2\((C_3 < 0)\). Consequently the global phase portrait for Subcase I.2.1 is shown in Figure 1(d) of Theorem 4.

**Subcase I.2.2:** \( r_1 < 0 < r_2 \). The local phase portrait at the origin of system (3) is given in Figure 11, obtained as in case I.1.1.

For the chart \( U_2 \), since \( C_3 = -r_1 r_2^2 > 0 \), we have a local phase portrait topologically equivalent to the one of Figure 2\((C_3 > 0)\), and the global phase portrait for Subcase I.2.2 is shown in Figure 1(e) of Theorem 4.

**Subcase I.2.3:** \( r_1 < r_2 < 0 \). The resulting phase portrait at the origin of system (3) is given in Figure 12. This local phase portrait is obtained proceeding in a similar way to the case I.1.1.

For the chart \( U_2 \), since \( C_3 = -r_1 r_2^2 > 0 \), we have a local phase portrait topologically equivalent to the one of Figure 2\((C_3 > 0)\).

The resulting global phase portrait for Subcase I.2.3 is shown in Figure 1(d) of Theorem 4.
Now we analyze the case $r_1 = r_2 < r_3$. The singular points at the
infinity are $p_1 = (0, r_1), p_2 = (0, r_3)$. The corresponding linear part of
system (8) at each of these points are respectively
\[
\begin{pmatrix}
  r_1^2 & 0 \\
  0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
  r_3^2 & 0 \\
  0 & -(r_1 - r_3)^2
\end{pmatrix},
\]

Considering that we assume $r_1 < r_3$, and $r_1, r_3 \neq 0$ in the above
expressions and applying Theorems 2.15 and 2.19 of [6], we conclude
that $p_1$ is a saddle-node, and $p_2$ is a saddle. The resulting singularity
obtained from the blow down of $p_1$, and $p_2$ depends on the position
of such singular points to the horizontal axis. Hence, we have the
following cases.

**Subcase I.2.4:** $0 < r_1 < r_3$. In this case the local phase portrait at
the origin of system (3) is given in Figure 13, obtained as in case I.1.1.

We remark that the dynamics in the above phase portrait is almost
topologically equivalent to the one of Case I.2.1, except between the
two separatrices.

For the chart $U_2$, since $C_3 = -r_1^2 r_3 < 0$, we have a local phase
portrait topologically equivalent to the one of Figure 2($C_3 < 0$). Then
the global phase portrait for Subcase I.2.4 is shown in Figure 1(d) of
Theorem 4.

**Subcase I.2.5:** $r_1 < 0 < r_3$. In this case the local phase portrait at
the origin of system (3) is given in Figure 14.

For the chart $U_2$, since $C_3 = -r_1^2 r_3 < 0$, we have a local phase
portrait topologically equivalent to the one of Figure 2($C_3 < 0$). Hence
the global phase portrait for Subcase I.2.5 is shown in Figure 1(e) of
Theorem 4.

**Subcase I.2.6:** $r_1 < r_3 < 0$. In this case the local phase portrait at
the origin of system (3) is given in Figure 15.
We remark that the dynamics in the above phase portrait is almost topologically equivalent to the one of Case I.2.1, except between the two separatrices.

For the chart $U_2$, since $C_3 = -r_2^2 r_3 > 0$, we have a local phase portrait topologically equivalent to the one of Figure 2($C_3 > 0$). Therefore the global phase portrait for Subcase I.2.6 is shown in Figure 1(d) of Theorem 4.

**Subcase I.3: One triple real root.** In this case we have $r_1 = r_2 = r_3$. Hence the only singular point at infinity is $p_1 = (0, r_1)$. The corresponding linear part of system (8) at $p_1$ is

\[
\begin{pmatrix}
  r_1^2 & 0 \\
  0 & 0
\end{pmatrix}.
\]

Since $r_1 \neq 0$ from Theorem 2.19 of [6], it follows that $p_1$ is a saddle. Then the resulting singularity obtained from the blow down of $p_1$ depends on the position of such singular point with respect to the horizontal axis. So we distinguish the following cases.

**Subcase I.3.1: $r_1 > 0$.** In this case the local phase portrait at the origin of system (3) is given in Figure 16.

For the chart $U_2$, since $C_3 = -r_3^3 < 0$, we have a local phase portrait topologically equivalent to the one of Figure 2($C_3 < 0$). So the global phase portrait for Subcase I.3.1 is shown in Figure 1(f) of Theorem 4.

**Subcase I.3.2: $r_1 < 0$.** In this case the local phase portrait at the origin of system (3) is given in Figure 17.

For the chart $U_2$, since $C_3 = -r_1^3 > 0$, we have a local phase portrait topologically equivalent to the one of Figure 2($C_3 > 0$). Then the global phase portrait for Subcase I.3.2 is shown in Figure 1(f) of Theorem 4.

**Subcase I.4: One simple real root and two complex conjugate roots.** We denote the real root as $r_1$ and the two complex conjugate
roots as $r_{2,3} = a \pm ib$. Note that, if at least one of the roots $r_i, i = 1, 2, 3$ is zero, we have $C_3 = -r_1(a^2 + b^2) = 0$ and as already commented, the case $C_1 = C_3 = 0$ leads to a cubic polynomial differential system, which has already been studied. Since we are only interested in analyzing the quartic systems, we assume $C_3 \neq 0$, which leads to $r_1, b \neq 0$.

The unique real singular point at infinity is $p_1 = (0, r_1)$. The linear part of system (8) at $p_1$ is

$$
\begin{pmatrix}
r_1^2 & 0 \\
0 & -(b^2 + (a - r_1)^2)
\end{pmatrix}.
$$

Since $r_1, b \neq 0$ by Theorem 2.15 of [6], we get that $p_1$ is a saddle. Then the singularity obtained from the blow down of $p_1$ depends on the position of such singular point with respect to the horizontal axis. So we have the following cases.

**Subcase I.4.1:** $r_1 > 0$. In this case the local phase portrait at the origin of system (3) is given in Figure 18.

For the chart $U_2$, since $C_3 = -r_1(a^2 + b^2) < 0$, we have a local phase portrait topologically equivalent to the one of Figure 2 ($C_3 < 0$). Therefore the resulting global phase portrait for Subcase I.4.1 is shown in Figure 1(f) of Theorem 4.

**Subcase I.4.2:** $r_1 < 0$. In this case the local phase portrait at the origin of system (3) is given in Figure 19.

For the chart $U_2$, since $C_3 = -r_1(a^2 + b^2) > 0$, we have a local phase portrait topologically equivalent to the one of Figure 2 ($C_3 > 0$), and the global phase portrait for Subcase I.4.2 is shown in Figure 1(f) of Theorem 4.

**Case II:** $C_3 = 0$. We assume $C_1 \neq 0$ since otherwise system (2) becomes a cubic differential system.

![Figure 16. Phase portrait of system (3) for $0 < r_1 = r_2 = r_3$.](image1)

![Figure 17. Phase portrait of system (3) for $r_3 = r_2 = r_1 < 0$.](image2)
We first study the chart $U_1$. Then system (4) in this chart becomes
\begin{align*}
u' &= (1 + u^2)v^2, \\
v' &= -C_1 - B_2uv - A_1v^2 + uv^3.
\end{align*}
Analyzing system (9) we obtain that there is no singular point at infinity in the chart $U_1$.

For the chart $U_2$ we only need to study the origin $O_{U_2}$. The system (6) in that chart writes
\begin{align*}
u' &= -(1 + u^2)v^2, \\
v' &= -C_1u^3 - B_2uv - A_1uv^2 - uv^3,
\end{align*}
The linear part of system (10) at $O_{U_2}$ is identically zero. Thus it is necessary to apply a directional blow up $v = uw$ to it, resulting the following system
\begin{align*}
u' &= -(1 + u^2)uw^2, \\
v' &= u(-C_1u - B_2w - A_1uw^2 + w^3).
\end{align*}
We perform a change of coordinates of the form $dT = uds$ in system (11) and we get
\begin{align*}
u' &= -(1 + u^2)uw^2, \\
v' &= -C_1u - B_2w - A_1uw^2 + w^3,
\end{align*}
where the prime now denotes derivative with respect to $T$. The singular points of system (12) are $p_1 = (0, 0), p_2 = (0, -\sqrt{B_2})$ and $p_3 = (0, \sqrt{B_2})$. Hence we consider the following cases.

**Subcase II.1: $B_2 > 0$.** We have $p_1, p_2, p_3$ as three distinct real singular points. The corresponding linear part of system (12) at $p_1$ is
\[
\begin{pmatrix}
0 & 0 \\
-C_1 & -B_2
\end{pmatrix}.
\]
Applying Theorem 2.19 of [6] we conclude that $p_1$ is an unstable node.

The linear parts of system (12) at $p_2$ and $p_3$ are identical, namely
\[
\begin{pmatrix}
-B_2 & 0 \\
-A_1B_2 - C_1 & 2B_2
\end{pmatrix}.
\]

Applying Theorem 2.15 of [6] it follows that $p_2$ and $p_3$ are saddles. Going back with the blow down we get the local phase portrait at the origin of system (5) topologically equivalent to the one of Figure 20.

![Figure 20. Phase portrait of system (5) for $B_2 > 0$.](image)

We remark that, as it was mentioned, all global phase portraits of planar quartic polynomial differential systems with a uniform isochronous center at the origin are symmetric with respect to the $y$–axis. Moreover, the graphic at the boundary of the period annulus of the uniform isochronous center at the origin is formed by separatrices of infinite singular points. Considering these results and the above calculations above, we shall have two distinct global phase portraits for Subcase II.1.

**Subcase II.1.1: $C_1 = -A_1B_2$.** Under this hypothesis we have the following result.

**Lemma 7** (Invariant straight lines). *If $B_2 > 0$, $C_1 = -A_1B_2$ and $C_3 = 0$ in the quartic polynomial differential system (2) of Theorem 2, then the system has the two invariant straight lines $x = \pm \sqrt{1/B_2}$."

*Proof.* If $B_2 > 0$, $C_1 = -A_1B_2$ and $C_3 = 0$ in system (2), then it writes
\[
\dot{x} = (B_2x^2 - 1)(-A_1x^2 + y), \quad \dot{y} = x(1 + A_1y - A_1B_2x^2y + B_2y^2).
\]
Hence $x = \pm \sqrt{1/B_2}$ are invariant. \qed

Using this information we obtain easily the global phase portrait for Subcase II.1.1 shown in Figure 1(h) of Theorem 4.

**Subcase II.1.2: $C_1 \neq -A_1B_2$.** Then the global phase portrait is shown in Figure 1(g) of Theorem 4.
**Subcase II.2: \( B_2 < 0 \).** The only real singular point is the origin, \( p_1 = (0,0) \). The linear part of system (12) at \( p_1 \) is

\[
\begin{pmatrix}
0 & 0 \\
-C_1 & -B_2
\end{pmatrix}.
\]

Applying Theorem 2.19 of [6] we conclude that \( p_1 \) is a saddle.

The local phase portrait at the origin of system (5) depends on the sign of the coefficient \( C_1 \) as shown in Figure 21, obtained as in case I.1.1.

![Figure 21. Phase portrait of system (5) for \( B_2 < 0 \).](image)

Although the system might present two distinct local phase portraits at the origin, the corresponding global phase portraits are topologically equivalent, and it is shown in Figure 1(a) of Theorem 4.

**Subcase II.3: \( B_2 = 0 \).** The only singular point is the origin, \( p_1 = (0,0) \). The linear part of system (12) at \( p_1 \) is

\[
\begin{pmatrix}
0 & 0 \\
-C_1 & 0
\end{pmatrix}.
\]

Therefore \( p_1 \) is a nilpotent singular point. Applying Theorem 3.5 of [6], we conclude that \( p_1 \) is a saddle, similar to the one illustrated in Figure 22.

![Figure 22. Saddle of a nilpotent singularity.](image)

The local phase portrait at the origin of system (5) in this case is given in Figure 23. Then the global phase portrait is shown in Figure 1(h) of Theorem 4.
Case III: $C_1 C_3 > 0$. There are only two possible singular points in the chart $U_1$, $(-\sqrt{-C_1/C_3}, 0)$ and $(\sqrt{-C_1/C_3}, 0)$. Since $C_1 C_3 > 0$ system (4) in $U_1$ has no real singular points.

In the chart $U_2$ the origin, which we denote by $O_{U_2}$ is the only real singular point of system (6). Its linear part is

$$
\begin{pmatrix}
0 & 0 \\
-C_3 & 0
\end{pmatrix}.
$$

Therefore $O_{U_2}$ is a nilpotent singularity and by Theorem 3.5 of [6] it is a cusp, whose behavior depends on the sign of the coefficient $C_3$. Hence the local phase portrait at the origin for system (5) might be one of the two shown in Figure 2. Then the global phase portrait is shown in Figure 1(a) of Theorem 4.

Case IV: $B_2 = 0, C_1 C_3 < 0$. The expression (4) for the system in the local chart $U_1$ is

$$
\begin{align*}
 u' &= (1 + u^2)v^2, \\
 v' &= -C_1 - C_3 u^2 - A_1 v^2 + u v^3.
\end{align*}
$$

So there are two singular points at infinity in $U_1$: $p_1 = (\sqrt{-C_1/C_3}, 0)$ and $p_2 = (-\sqrt{-C_1/C_3}, 0)$. Similarly in the chart $U_2$ the origin $O_{U_2}$ is a singularity, because the system in that chart is

$$
\begin{align*}
 u' &= -(1 + u^2)v^2, \\
 v' &= -C_3 u - C_1 u^3 - A_1 u v^2 - u v^3.
\end{align*}
$$

The linear parts of system (13) at $p_1$ and $p_2$, and of system (14) at $O_{U_2}$ are

$$
\begin{pmatrix}
0 & 0 \\
-2C_3\sqrt{-C_1/C_3} & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 \\
2C_3\sqrt{-C_1/C_3} & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 \\
-C_3 & 0
\end{pmatrix},
$$

respectively.

The point $O_{U_2}$ in the chart $U_2$ is a nilpotent singularity and by Theorem 3.5 of [6] it is a cusp, whose local phase portrait depends
on the sign of the coefficient \( C_3 \). Hence the local phase portrait at the origin for system (5) might be one of the two shown in Figure 2.

Since both \( p_1 \) and \( p_2 \) are also nilpotent singularities we apply the same theorem to determine that \( p_1 \) and \( p_2 \) are cusps, whose behavior also depend on the sign of \( C_3 \), but are slightly distinct from that one of \( O_{U_2} \). For \( C_3 < 0 \) the local phase portraits at \( p_1 \) and \( p_2 \) are topologically equivalent to Figure 24(I) and (II) respectively, whereas for \( C_3 > 0 \), the local phase portraits at \( p_1 \) and \( p_2 \) are topologically equivalent to Figure 24(II) and (I), respectively.

![Figure 24. Phase portrait of system (5) for \( B_2 = 0 \).](image)

Using similar arguments as in the previous cases, we shall have three possible configurations for the global phase portraits, they are shown in Figures 1(i), (j) and (k) of Theorem 4.

**Case V**: \( B_2 \neq 0, C_1 C_3 < 0 \). The expression (4) for the system in the local chart \( U_1 \) is

\[
\begin{align*}
    u' &= (1 + u^2)v^2, \\
    v' &= -C_1 - C_3 u^2 - B_2 u v^2 - A_1 v^2 + u v^3,
\end{align*}
\]

So there are two singular points at infinity, \( p_1 = (\sqrt{-C_1/C_3}, 0) \) and \( p_2 = (-\sqrt{-C_1/C_3}, 0) \). Similarly in the chart \( U_2 \) the origin \( O_{U_2} \) is a singularity, because this system writes

\[
\begin{align*}
    u' &= -(1 + u^2)v^2, \\
    v' &= -C_3 u - C_1 u^3 - B_2 u v - A_1 u v^2 - u v^3.
\end{align*}
\]

The linear part of system (15) at \( p_1 \) and \( p_2 \), and of system (16) at \( O_{U_2} \) are respectively

\[
\begin{align*}
    \left( \begin{array}{cc} 0 & 0 \\
    -2C_3 \sqrt{-C_1/C_3} & -B_2 \sqrt{-C_1/C_3} \end{array} \right), \\
    \left( \begin{array}{cc} 0 & 0 \\
    2C_3 \sqrt{-C_1/C_3} & B_2 \sqrt{-C_1/C_3} \end{array} \right), \\
    \left( \begin{array}{cc} 0 & 0 \\
    -C_3 & 0 \end{array} \right).
\end{align*}
\]
The singular points $p_1$ and $p_2$ are semi-hyperbolic singularities. By Theorem 2.19 of [6], $p_1$ and $p_2$ are saddle-nodes. We remark that in Case IV, which only differs from the present one by the vanishing of the coefficient $B_2$, the two singular points $p_1$ and $p_2$ in the chart $U_1$ are cusps. Both cusps and saddle-nodes are singular points of index 0. The cusps in the previous case bifurcate to saddle-nodes by changing $B_2$ from zero to non-zero values.

The local phase portraits at $p_1$ and $p_2$ depend on the values of the coefficients $A_1, B_2, C_1$ and $C_3$. The possible local phase portraits are shown in figures 25 and 26. These local phase portraits are obtained as in case I.1.1.

![Phase portrait of system (5) for $A_1, C_1 > 0$, $B_2, C_3 < 0$.](image1)

![Phase portrait of system (5) for $C_1 > 0$, $A_1, B_2, C_3 < 0$.](image2)

The point $O_{U_2}$ in the chart $U_2$ is a nilpotent singular point. Applying Theorem 3.5 of [6] we see that it is a cusp, whose behavior depends on the sign of the coefficient $C_3$. Hence the local phase portrait at the origin for system (5) might be one of the two shown in Figure 2.

Using similar arguments as in the previous cases, we shall have three possible configurations for the global phase portraits, they are shown in Figures 1(l), (m) and (n) of Theorem 4. We remark that all three configurations are possible, setting $B_2 = \varepsilon > 0$ in the examples presented in Case IV.
Figure 27. Phase portrait of system (5) for $A_1, C_3 > 0$, $B_2, C_1 < 0$.

Figure 28. Phase portrait of system (5) for $C_3 > 0$, $A_1, B_2, C_1 < 0$.

Figure 29. Phase portrait of system (5) for $A_1 \in \mathbb{R}$, $B_2, C_1 > 0$, $C_3 < 0$.

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