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ISOCHRONICITY AND LINEARIZABILITY OF PLANAR POLYNOMIAL HAMILTONIAN SYSTEMS

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Abstract. In this paper we study isochronicity and linearizability of planar polynomial Hamiltonian systems. First we prove a theorem which supports a negative answer to the following open question: Do there exist planar polynomial Hamiltonian systems of even degree having an isochronous center? stated by Jarque and Villadelprat in the J. Differential Equations **180** (2002), 334–373. Additionally we obtain some conditions for linearizability of complex cubic Hamiltonian systems.

1. Introduction and statement of the main results

Consider real planar polynomial Hamiltonian system, i.e. differential systems of the form

$$\begin{aligned} \dot{x} &= -H_y(x,y), \\ \dot{y} &= H_x(x,y), \end{aligned} \tag{1}$$

where H is a real polynomial in the variables x and y. The maximum degree of the polynomials $H_x(x,y)$ and $H_y(x,y)$ is the *degree* of the Hamiltonian system (1). Assume the origin is a center for the system. The period function of a center provides the period of each periodic orbit inside the period annulus of the center. When this period function is constant the center is called *isochronous*.

Several authors have studied the isochronicity of centers. But such isochronicity has been characterized for very few families of polynomial differential systems. The first important result is due to Loud [16], who classified the quadratic isochronous centers. After Pleshkan [20] classified the cubic polynomial differential systems with homogeneous nonlinearities and recently isochronous centers of polynomial systems with homogeneous nonlinearities of degree five have been classified in [21]. Several authors (see [4, 10, 25]) have shown that Hamiltonian systems have no isochronous centers if they have homogeneous nonlinearities. For the Hamiltonian polynomial systems with Hamiltonian of the form $H(x,y) = F(x) + G(y)$ the unique isochronous center is the linear one, see [5]. The cubic polynomial Hamiltonian isochronous centers were classified in [6]. Some other results on isochronicity can be found in [2, 18, 19, 23] and references inside.

From the work of Loud it follows easily that planar polynomial Hamiltonian differential systems of degree two have no isochronous centers. Jarque and Villadelprat [15] in 2002 proved that every center of a planar polynomial Hamiltonian differential system of degree four is nonisochronous, and they stated the following:

Open problem. *Do there exist planar polynomial Hamiltonian systems of even degree having an isochronous center?*

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The authors of the question said that an argument in support of a negative answer is the following result of Mañosas and Villadelprat [17], see also Ito [14]. The Hamiltonian differential system (1) has an isochronous center of period 2π at the origin if and only if

$$H(x,y) = \frac{1}{2}\left(X(x,y)^2 + Y(x,y)^2\right), \tag{2}$$

the map $(x,y) \to (X(x,y), Y(x,y))$ defined in a neighborhood of the origin is analytic, $X(0,0) = Y(0,0) = 0$, and its Jacobian is constant and equal to one.

Later on in 2008 Chen, Romanovski and Zhang in [3] provided more support to the negative answer proving that there is no planar polynomial Hamiltonian systems with only even degree nonlinearities having an isochronous center at the origin.

The following statement provides further support to the negative answer to the open question stated above.

Theorem 1. *Planar polynomial Hamiltonian systems* (1) *of even degree such that their corresponding analytic maps $X(x,y)$ and $Y(x,y)$ given in* (2) *are defined in the whole plane, have no isochronous centers.*

Theorem 1 is proved in Section 2.

Of course, if in the statement of Theorem 1 the functions $X(x,y)$ and $Y(x,y)$ are polynomials, they are defined in the whole plane. Also in this case the analytic map $(x,y) \to (X(x,y), Y(x,y))$ converges in the whole plane.

We want to mention that to know when a center of a Hamiltonian system is isochronous is important due to its connection with the Jacobian conjecture, see for more details the articles of Gavrilov [11] and Sabatini [24]. Indeed if we take a polynomial map $(x,y) \to (X(x,y), Y(x,y))$ with nonzero constant Jacobian such that $X(0,0) = Y(0,0) = 0$, then the polynomial Hamiltonian system with Hamiltonian given by (2) has an isochronous center at the origin. If we prove that the period annulus of any center constructed in this way is the whole plane then the Jacobian conjecture follows.

When $X(x,y)$ and $Y(x,y)$ in (2) are polynomials, then the isochronous center is called trivial [6]. It was shown in [6] that all isochronous centers of cubic Hamiltonian systems are trivial. More precisely, the result obtained in [6] is as follows.

Theorem 2. *A cubic Hamiltonian system has an isochronous center at the origin if and only if after a linear change of coordinates its Hamiltonian can be written as*

$$H(x,y) = (k_1 x)^2 + (k_2 y + P(x))^2, \tag{3}$$

where k_1 and k_2 are different from zero and $P(x) = k_3 x + k_4 x^2$.

Note that the linearizing transformation for system (1) with the Hamiltonian function (3) is

$$X = k_1 x, \quad Y = k_2 y + P(x).$$

However not all isochronous centers of polynomial Hamiltonian systems are trivial. An example of Hamiltonian system with nonlinearities of degree seven is given in [6].

Up to now we discussed isochronous centers of real systems. However it is known that isochronicity of a real center is equivalent to its linearizability. The problem of linearizability can be studied also for complex systems and, since after a complexification any real plane system can be embedded into a complex two-dimensional system (see e.g. [23, §3.2]) the problem of linearizability can be considered as a generalization of the problem of isochronicity. Thus it appears natural to study also the problem of linearizability of plane complex Hamiltonian systems

$$\dot{x} = -H_y(x,y), \quad \dot{y} = H_x(x,y), \tag{4}$$

where $H(x,y)$ is a complex function of the form

$$H = -xy + h.o.t. \tag{5}$$

and compare the results with those obtained for real systems. One of advantages of working with complex systems (4-5) is that computation of the linearizability quantities is much simpler for such systems than for real systems. The singular point at the origin of (4-5) is often called a complex center.

An analog of the open problem stated above for system (4) is as follows: *are there linearizable systems (4-5) of even degree?*

It is easy to see that the answer to this question is positive. More precesely for the quadratic system

$$\begin{aligned} \dot{x} =& x - a_{10}x^2 - a_{01}xy - a_{-12}y^2, \\ \dot{y} =& -y + b_{2,-1}x^2 + b_{10}xy + b_{01}y^2 \end{aligned} \tag{6}$$

we have the following theorem.

Theorem 3. *Hamiltonian system* (6)*, that is, system* (6) *with*

$$a_{01} = 2b_{01}, \quad b_{10} = 2a_{10} \tag{7}$$

is linearizable if and only if $b_{2,-1} = a_{10} = 0$ *or* $b_{01} = a_{-12} = 0$.

Theorem 3 is proven in section 3.

In this paper we also study the linearizability of the cubic Hamiltonian system

$$\begin{aligned} \dot{x} =& x - a_{10}x^2 - a_{20}x^3 - a_{01}xy - a_{11}x^2y - a_{-12}y^2 - a_{02}xy^2 - a_{-13}y^3 = P(x,y), \\ \dot{y} =& -y + b_{2,-1}x^2 + b_{3,-1}x^3 + b_{10}xy + b_{20}x^2y + b_{01}y^2 + b_{11}xy^2 + b_{02}y^3 = Q(x,y), \end{aligned} \tag{8}$$

where

$$b_{10} = 2a_{10}, \quad a_{01} = 2b_{01}, \quad a_{02} = 3b_{02}, \quad b_{20} = 3a_{20}, \quad b_{11} = a_{11}. \tag{9}$$

Clearly, system (8) with condition (9) represents the whole family of cubic Hamiltonian systems of the form (4-5). For this system we have obtained the following result.

Theorem 4. *The Hamiltonian system (8) (that is, system (8) with conditions (9)) is linearizable at the origin if one of the following conditions holds:*

1) $b_{3,-1} = b_{2,-1} = b_{02} = a_{20} = a_{11} = a_{10} = 0$,

2) $b_{02} = b_{01} = a_{20} = a_{-13} = a_{-12} = a_{11} = 0$,

3) $b_{3,-1} = b_{2,-1} = 2b_{01}^2 + 9b_{02} = a_{20} = 2a_{-12}b_{02} - a_{-13}b_{01} = 4a_{-12}b_{01} + 9a_{-13} = a_{11} = a_{10} = 0$,

4) $a_{-12} = a_{-13} = b_{02} = b_{01} = a_{11} = 2a_{10}^2 + 9a_{20} = 2b_{2,-1}a_{20} - b_{3,-1}a_{10} = 4b_{2,-1}a_{10} + 9b_{3,-1} = 0$,

5) $27b_{01}^3 + a_{-12}^2b_{2,-1} = 9a_{10}b_{01} - a_{-12}b_{2,-1} = a_{10}a_{-12} + 3b_{01}^2 = 3a_{10}^2 + b_{01}b_{2,-1} = -4/3a_{10}b_{2,-1} + b_{3,-1} = -4b_{01}^2 + b_{02} = 4/3b_{01}b_{2,-1} + a_{20} = -4/3a_{-12}b_{01} + a_{-13} = 4/3a_{-12}b_{2,-1} + a_{11} = 0$.

Note that the computational approach we employ to prove this theorem allows to find linearizable systems in any given parametric family of polynomial systems.

The paper is organized as follows. In the next section we prove Theorem 1. In Section 3 proofs of Theorems 3 and 4 are given, interrelation of isochronocity of real Hamiltonian systems and linearizability of complex Hamiltonian systems is discussed and an open problem is stated.

2. Proof of Theorem 1

From the mentioned result of Mañosas and Villadelprat we can suppose that the Hamiltonian of our Hamiltonian system (1) having an isochronous center at the origin is of the form (2) with $X(x,y)$ and $Y(x,y)$ analytic functions such that $X(0,0)=Y(0,0)=0$, and its Jacobian $X_x(x,y)Y_y(x,y)-X_y(x,y)Y_x(x,y)=1$.

Proof of Theorem 1. Since $H(x,y)=(X(x,y)^2+Y(x,y)^2)/2$ it follows that the function $H(x,y)$ is positive semidefinite, i.e. $H(x,y)\geq 0$ for all $(x,y)\in\mathbb{R}^2$. Let $H_k(x,y)$ be the homogeneous part of H of degree k. Assume that the polynomial $H(x,y)$ has degree m. Then, Theorem 1 will be proved if we show that m is even.

Indeed, assume that m is odd. We claim that near infinity the polynomial $H(x,y)$ changes sign. Indeed, write $g(t)=H(tx,ty)$ for a point (x,y) such that $H_m(x,y)$ is not zero. Note that $g(t)$ is a polynomial of odd degree, because $g(t)=H_0(x,y)+H_1(x,y)t+\ldots+H_m(x,y)t^m$. Then the sign of $g(t)$ when $t\to+\infty$ is opposite to the sign of $g(t)$ when $t\to-\infty$. Therefore the claim is proved. But this is in contradiction with the fact that the polynomial $H(x,y)$ is positive semidefinite. Hence m is even. This completes the proof of Theorem 1. □

Second proof of Theorem 1 when $X(x,y)$ and $Y(x,y)$ are polynomials. Indeed, let m be the maximum of the degrees of these two polynomials, and let $X_k(x,y)$ and $Y_k(x,y)$ be the homogeneous part of the polynomials $X(x,y)$ and $Y(x,y)$ of degree k for $k=0,1,\ldots,m$. From the definition of m we have that

$$X_m(x,y)^2+Y_m(x,y)^2\neq 0. \tag{10}$$

The Hamiltonian system with Hamiltonian (2) is

$$\begin{aligned}\dot{x}&=-X(x,y)X_y(x,y)-Y(x,y)Y_y(x,y),\\ \dot{y}&=X(x,y)X_x(x,y)+Y(x,y)Y_x(x,y).\end{aligned} \tag{11}$$

Since we assume that this Hamiltonian system has even degree we have that

$$\begin{aligned}X_m(x,y)X_{my}(x,y)+Y_m(x,y)Y_{my}(x,y)&=0,\\ X_m(x,y)X_{mx}(x,y)+Y_m(x,y)Y_{mx}(x,y)&=0,\end{aligned} \tag{12}$$

otherwise the Hamiltonian system (11) would have odd degree. Multiplying the first equation of (12) by y and the second by x and summing the two equations obtained we get

$$X_m(x,y)\big(xX_{mx}(x,y)+yX_{my}(x,y)\big)+Y_m(x,y)\big(xY_{mx}(x,y)+yY_{my}(x,y)\big)=0.$$

By the Euler theorem on homogeneous functions this equation becomes

$$m\big(X_m(x,y)^2+Y_m(x,y)^2\big)=0,$$

in contradiction with (10). This completes the proof for the case when $X(x,y)$ and $Y(x,y)$ are polynomials. $\square$

3. Complex Hamiltonian systems

In this section we prove Theorems 3 and 4 and discuss the relation of the results on linearizability of complex Hamiltonian systems and the results on isochronicity of real Hamiltonian systems obtained in [6].

Proof of Theorem 3. It is known that the linearizability variety of quadratic system (6) is defined by the first three pairs of the linearizability quantities (see e.g. Section 4 of [23] for the proof of this fact, the definition of linearizability quantities and a computational algorithm for computing the quantities).

When condition (7) is fulfilled the quantities are as follows

$$\begin{aligned} I_{11} = -J_{11} =& 6a_{10}b_{01} + 2/3a_{-12}b_{2,-1}, \\ I_{22} = -J_{22} =& 20a_{10}^3a_{-12} + 20b_{01}^3b_{2,-1} - 20/9a_{-12}^2b_{2,-1}^2, \\ I_{33} = -J_{33} =& 280/27a_{-12}^3b_{2,-1}^3, \end{aligned}$$

where I_{22}, J_{22} are reduced modulo $\langle I_{11}\rangle$ and I_{33}, J_{33} are reduced modulo $\langle I_{11}, I_{22}\rangle$. Obviously, the variety of the ideal

$$\mathcal{L} = \langle I_{11}, J_{11}, I_{22}, J_{22}, I_{33}, J_{33}\rangle$$

consists of two components (i) $b_{2,-1} = a_{10} = 0$ and (ii) $b_{01} = a_{-12} = 0$. Thus the conditions of Theorem 3 are necessary conditions for the linearization of system (6-7). They are also the sufficient ones. Indeed, in case (i) the system has the form

$$\dot{x} = x - 2b_{01}xy - a_{-12}y^2, \quad \dot{y} = -y + b_{01}y^2, \tag{13}$$

and it is linearizable by the substitution

$$X = \left(x - b_{01}xy - \frac{a_{-12}y^2}{3}\right)(1 - b_{01}y), \quad Y = \frac{y}{(1 - b_{01}y)}.$$

Case (ii) is dual to (i) under the involution

$$a_{ij} \leftrightarrow b_{ji}. \tag{14}$$

The theorem is proved. $\square$

One possibility to find all linearizable Hamiltonian systems in family (8) is using the similar way as in the proof of Theorem 3, that is, by computing a sufficient number of the linearizability quantities for the system[1], then finding the irreducible decomposition of the linearizability variety and then proving the linearizability of systems corresponding to each of components of the variety. For system (8-9) we computed the first 7 pair of linearizability quantities $I_{11}, J_{11}, \dots, I_{77}, J_{77}$ and have tried to find the decomposition of the variety of the ideal

$$\mathcal{L} = \langle I_{11}, J_{11}, \dots, I_{77}, J_{77}\rangle, \tag{15}$$

[1]however unlike for the case of quadratic system (6) it is unknown how many first pairs of linearizability quantities define the linearizability variety of system (8)

but we failed to find the decomposition of the variety $V(\mathcal{L})$ of $\mathcal{L}$ using our computational facility. So we employ another approach to obtain the result stated in Theorem 4.

Proof of Theorem 4. We look for a linearization of the first equation of system (8) in the form

$$\begin{aligned} X =& x + h_1x^2 + h_2xy + h_3y^2 + h_4x^3 + h_5x^2y+ \\ & h_6xy^2 + h_7y^3 + h_8x^4 + h_9x^3y + h_{10}x^2y^2 + h_{11}xy^3 + h_{12}y^4. \end{aligned} \tag{16}$$

Letting $\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ the vector field associated to (8), we see that substitution (16) linearizes the first equation of (8) if

$$\mathcal{X}X - X = 0. \tag{17}$$

By our assumption system (8) is Hamiltonian with the Hamiltonian function

$$H = -xy+a_{10}x^2y+\frac{1}{2}a_{11}x^2y^2+\frac{a_{-12}y^3}{3}+\frac{a_{-13}y^4}{4}+a_{20}x^3y+b_{01}xy^2+b_{02}xy^3+\frac{b_{2,-1}x^3}{3}+\frac{b_{3,-1}x^4}{4}.$$

If a polynomial (16) satisfies (17) then $X = 0$ is an invariant algebraic curve of (8) (see e.g. [23, Proposition 3.6.2]). Then by the analytic Nullstellenzatz [13] $g = (-H)/X$ is an analytic function of the form $g(x, y) = y + h.o.t.$ and the second equation of (8) is linearizable by the substitution $Y = g(x, y)$. Thus, it is sufficient to look only for the linearization of one equation of the system.

Equating coefficients of the polynomial on the left hand side of (17) to zero we obtain the following system:

$$\begin{aligned}
-a_{10} + h_1 = -2a_{-13}h_{10} + 4a_{11}h_{12} = -a_{-13}h_{11} + 4b_{02}h_{12} = -2b_{01} - h_2 = \\
-a_{-12} - 3h_3 = -a_{11} - 4b_{01}h_1 + a_{10}h_2 + 2b_{2,-1}h_3 = \\
-a_{20} - 2a_{10}h_1 + b_{2,-1}h_2 + 2h_4 = \\
-3b_{02} - 2a_{-12}h_1 - b_{01}h_2 + 4a_{10}h_3 - 2h_6 = -a_{-13} - a_{-12}h_2 + 2b_{01}h_3 - 4h_7 = \\
-2a_{-13}h_1 - 3h_{11} - 2b_{02}h_2 + 2a_{11}h_3 - 2a_{-12}h_5 + 6a_{10}h_7 = \\
-2a_{-12}h_{10} + b_{01}h_{11} + 8a_{10}h_{12} - 2a_{-13}h_5 - b_{02}h_6 + 3a_{11}h_7 = \\
-5h_{12} - a_{-13}h_2 + 2b_{02}h_3 - a_{-12}h_6 + 3b_{01}h_7 = \\
-a_{-12}h_{11} + 4b_{01}h_{12} - a_{-13}h_6 + 3b_{02}h_7 = \\
-6b_{02}h_1 - h_{10} + 6a_{20}h_3 - 3a_{-12}h_4 - 3b_{01}h_5 + 3a_{10}h_6 + 3b_{2,-1}h_7 = \\
-2a_{20}h_1 + b_{3,-1}h_2 - 3a_{10}h_4 + b_{2,-1}h_5 + 3h_8 = 2b_{3,-1}h_{10} - 4a_{11}h_8 = \\
-2a_{11}h_1 + 2a_{20}h_2 + 2b_{3,-1}h_3 - 6b_{01}h_4 + 2b_{2,-1}h_6 + h_9 = \\
2b_{2,-1}h_{10} - 3a_{11}h_4 + a_{20}h_5 + 2b_{3,-1}h_6 - 8b_{01}h_8 - a_{10}h_9 = \\
4a_{20}h_{10} + 3b_{3,-1}h_{11} - 12b_{02}h_8 - 2a_{11}h_9 = \\
5a_{10}h_{11} - 2b_{01}h_{10} + 4b_{2,-1}h_{12} - 3a_{-13}h_4 - 5b_{02}h_5 + a_{11}h_6 + 9a_{20}h_7 - 3a_{-12}h_9 = \\
-4b_{02}h_{10} + 2a_{11}h_{11} + 12a_{20}h_{12} - 3a_{-13}h_9 = \\
2a_{10}h_{10} + 3b_{2,-1}h_{11} - 9b_{02}h_4 - a_{11}h_5 + 5a_{20}h_6 + 3b_{3,-1}h_7 - 4a_{-12}h_8 - 5b_{01}h_9 = \\
8a_{20}h_{11} + 4b_{3,-1}h_{12} - 4a_{-13}h_8 - 8b_{02}h_9 = \\
-3a_{20}h_4 + b_{3,-1}h_5 - 4a_{10}h_8 + b_{2,-1}h_9 = \\
-4a_{20}h_8 + b_{3,-1}h_9 = 0.
\end{aligned} \tag{18}$$

Let I be the ideal generated by the polynomials written above. Then the variety $V(I)$ of I is the set of solutions of the above system. To find conditions for linearizability we have to eliminate from the ideal I variables $h_1, \dots, h_{12}$ and then compute minimal associate primes of the obtained ideal. However since both procedures are very time and memory consuming we were not able to complete calculations with our computational facilities. So we applied modular computations (see [1, 22]). Working in the ring of polynomials over the field of characteristic 32003 using the routine `eliminate` of the computer algebra system SINGULAR [8] we eliminate from (18) variables $h_1, \ldots, h_{12}$, that is, we compute the twelfth elimination ideal I_{12} of I in the ring

$$\mathbb{Z}_{32003}[h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9, h_{10}, h_{11}, h_{12} a_{10}, a_{11}, a_{-12}, a_{-13}, a_{20}, b_{01}, b_{02}, b_{2,-1}, b_{3,-1}]$$

(see e.g. [7, 23] for more details). Then with `minAssGTZ`[2] of `primdec` library [9] of Singular we compute the irreducible decomposition of $V(I_{12})$ and after the rational reconstruction with the algorithm of [26] obtain component 1), 2), 3), 5) given in the statement of the theorem. Looking similarly as above for a linearization of the second equation of system (8) in the form

$$Y = y + g_1x^2 + g_2xy + g_3y^2 + g_4x^3 + g_5x^2y + g_6xy^2 + g_7y^3 + g_8x^4 + g_9x^3y + g_{10}x^2y^2 + g_{11}xy^3 + g_{12}y^4, \tag{19}$$

we obtain additionally condition 4) of the theorem, which is dual to condition 3) under involution (14).

To finish the proof of the theorem we show that each system whose coefficients satisfy one of conditions 1)–5) is linearizable. We can write out the linearizing transformation for each case explicitly.

When condition 1) of the theorem is fulfilled the system is written in the form

$$\dot{x} = -2b_{01}xy - a_{-12}y^2 - a_{-13}y^3, \quad \dot{y} = -y(1 - b_{01}y),$$

and it is linearizable by the substitution

$$X = (x - b_{01}xy - \frac{a_{-12}y^2}{3} - \frac{a_{-13}y^3}{4})(1 - b_{01}y), \quad Y = y/(1 - b_{01}y).$$

Note that if $3a_{-13} + 4a_{-12}b_{01} \neq 0$ then there is also a polynomial linearization of the second equation given by $Y = y + b_{01}y^2 + 12b_{01}^4xy^2/(3a_{-13} + 4a_{-12}b_{01}) + b_{01}^2y^3 + 3a_{-13}b_{01}^3y^4/(3a_{-13} + 4a_{-12}b_{01}$. The set of systems which are linearizable by this transformation is a proper subset of component 1). Its Zariski closure is exactly component 1).

In case 3) the system has the form

$$\dot{x} = x - 2b_{01}xy - a_{-12}y^2 + \frac{2}{3}b_{01}^2xy^2 + \frac{4}{9}a_{-12}b_{01}y^3, \quad \dot{y} = -y + b_{01}y^2 - \frac{2b_{01}^2y^3}{9}. \tag{20}$$

It is linearizable by the substitution

$$X = \frac{1}{27}(3 - 2b_{01}y)^2(-3x + 2b_{01}xy + a_{-12}y^2), \quad Y = \frac{3y(3 - b_{01}y)}{(3 - 2b_{01}y)^2}.$$

Cases 2) and 4) are dual to 1) and 3), respectively, under involution (14). Thus, there remains to consider case 5). If $b_{2,-1} = 0$ then the system is linearizable by

$$X = x - \frac{a_{-12}y^2}{3}, \quad Y = y,$$

[2]the routine is based on the algorithm of [12]

and if $b_{2,-1} \neq 0$ then the substitution

$$\begin{aligned} X =& x + a_{10}x^2 + \frac{6a_{10}^2 xy}{b_{2,-1}} + \frac{9a_{10}^3 y^2}{b_{2,-1}^2}, \\ Y =& y - \frac{b_{2,-1}x^2}{3} - 2a_{10}xy - \frac{3a_{10}^2 y^2}{b_{2,-1}}. \end{aligned}$$

linearizes the system. $\square$

We now compare our results with the results about real isochronous centers obtained in [6]. By the analogy with the real case we say that a linearization of system (4) is trivial if it is given by a polynomial transformation. By Theorem 2 all linearizations of real cubic Hamiltonian systems with a center at the origin are trivial. However it is not longer true for the case of complex Hamiltonian system (8) since, for instance, the linearizations of systems (13) and (20) are not trivial.

To further compare the condition of Theorem 2 with conditions of Theorem 4 we first note that the system (1) with Hamiltonian function (3) is written as

$$\begin{aligned} \dot{x} =& -2k_2k_3x - 2k_2k_4x^2 - 2k_2^2 y, \\ \dot{y} =& 2k_1^2 x + 2k_3^2 x + 6k_3k_4x^2 + 4k_4^2x^3 + 2k_2k_3y + 4k_2k_4xy. \end{aligned} \tag{21}$$

After a linear substitution which transforms the matrix of the linear approximation of (21) to the diagonal matrix and a rescaling of time we obtain from (21) the system

$$\begin{aligned} \dot{z}_1 =& ((k_1 - ik_3)^3 X(2k_1^2(k_1+ik_3)^2 + k_1k_2(k_1+ik_3)k_4X - 2k_2^2k_4^2X^2) + \\ & 2k_2(k_1-ik_3)^2(k_1+ik_3)k_4X(k_1^2 + ik_1k_3 + 3k_2k_4X)Y - \\ & 3k_2(k_1-ik_3)(k_1+ik_3)^2k_4(k_1^2+ik_1k_3+2k_2k_4X)Y^2 + \\ & 2k_2^2(k_1+ik_3)^3k_4^2Y^3)/((2k_1^2(k_1-ik_3)^3(k_1+ik_3)^2)), \\ \dot{z}_2 =& -(k_2(k_1-ik_3)^3k_4X^2(-3k_1(k_1+ik_3)+2k_2k_4X) + \\ & 2(k_1-ik_3)^2(k_1+ik_3)(k_1^2(k_1+ik_3)^2 + k_1k_2(k_1+ik_3)k_4X - 3k_2^2k_4^2X^2)Y + \\ & k_2(k_1-ik_3)(k_1+ik_3)^2k_4(k_1^2+ik_1k_3+6k_2k_4X)Y^2 - \\ & 2k_2^2(k_1+ik_3)^3k_4^2Y^3)/(2k_1^2(k_1-ik_3)^2(k_1+ik_3)^3). \end{aligned} \tag{22}$$

It is not difficult to see that system (22) is a Hamiltonian system with the Hamiltonian of the form (5). Denote the coefficient of $z_1^{q+1}z_2^s$ in the first equation of (5) by a_{qs} and the coefficient of $z_1^s z_2^{q+1}$ in the second equation of the system by b_{sq}. It is easily seen that

$$\begin{aligned} &(a_{10}, a_{01}, a_{12}, a_{20}, a_{11}, a_{02}, a_{-13}) = \Bigg(-\frac{k_2k_4}{2k_1^2 + 2ik_1k_3}, -\frac{k_2k_4}{k_1^2 - ik_1k_3}, \\ &\frac{3k_2k_4(k_1+ik_3)}{2k_1(k_1-ik_3)^2}, \frac{k_2^2k_4^2}{k_1^2(k_1+ik_3)^2}, -\frac{3k_2^2k_4^2}{k_1^4 + k_1^2k_3^2}, \frac{3k_2^2k_4^2}{k_1^2(k_1-ik_3)^2}, -\frac{k_2^2k_4^2(k_1+ik_3)}{k_1^2(k_1-ik_3)^3} \Bigg), \\ &(b_{01}, b_{10}, b_{2,-1}, b_{20}, b_{11}, b_{02}, b_{3,-1}) = \Bigg(-\frac{k_2k_4}{2k_1(k_1-ik_3)}, -\frac{k_2k_4}{k_1(k_1+ik_3)}, \\ &\frac{3k_2k_4(k_1-ik_3)}{2k_1(k_1+ik_3)^2}, \frac{3k_2^2k_4^2}{k_1^2(k_1+ik_3)^2}, -\frac{3k_2^2k_4^2}{k_1^2\,(k_1^2+k_3^2)}, \frac{k_2^2k_4^2}{k_1^2(k_1-ik_3)^2}, -\frac{k_2^2k_4^2(k_1-ik_3)}{k_1^2(k_1+ik_3)^3} \Bigg). \end{aligned}$$

We see from the above expressions that a_{qs} and b_{sq} are complex conjugate and satisfy condition (9). To eliminate k_1, k_2, k_3, k_4 from this system we subtract the right hand sides from the left hand sides, factor the expressions, take their numerators and then add to the obtaining set of polynomials the polynomials $1 - w_1k_1, 1 - w_2k_2, 1 - w_3(k_1^2 + k_3^2)$. Eliminating with `eliminate` of Singular the variables $k_1, k_2, k_3, k_4, w_1, w_2, w_3$ we obtain condition 5) of Theorem 4. This means, that in the space of parameters of system (8) the set corresponding to real isochronous Hamiltonian cubic systems is a subset of the set defined by equations 5) of Theorem 4.

Remark. Using the same approach we have also looked for linearizing transformations of system (8) in the form (16)-(19). In this case the computations yield only components 1), 2) and 5) of Theorem (4). It appears this is due to the fact that, as it is mentioned above, in case 1) almost for all points all points (when $3a_{-13}+4a_{-12}b_{01} \neq 0$) there is also a polynomial linearization of both equations (and similarly in case 2)), but in case 3) there is no polynomial linearization of one of equations.

To conclude the paper we can propose the following problem related to our study:

Open problem: *Is the set of systems given by conditions 1)–5) of Theorem 4 the set of all linearizable Hamiltonian cubic systems of the form* (8)?

We computed the set of Hamiltonian systems, where one of equations is linearizable by a polynomial of degree 5 and obtained the same conditions as given in Theorem 4 (because of computational complexity we did computations in the field $\mathbb{Z}_{32003}$). It indicates that most probably the set of systems given by conditions 1)–5) of Theorem 4 is the set of all linearizable cubic systems of the form (8) with at least one substitution being a polynomial one.

On the other side using with the approach of [22] we have tried to check if the variety of the union of the component 1)–5) of Theorem 4 is the same as the variety $V(\mathcal{L})$ of the ideal (15). Because of the computational complexity we were not able to check if these two varieties are equal in $\mathbb{C}^9$ (that is, computing in the field of characteristic zero) however some computations which we performed in the field of characteristic 32003 indicate that probably these varieties are not equal. Thus we expect that the answer to the question risen above is negative and there are some not trivial linearizable cubic Hamiltonian systems where the linearizing substitutions of both equations are not polynomial.

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