This is a preprint of: "Hopf and zero-Hopf bifurcations on the Hindmarsh-Rose system", Claudio Aguinaldo Buzzi, Jaume Llibre, Joao Carlos Medrado, 1–8, 2015.
DOI: [10.1007/s11071-015-2429-y]

HOPF AND ZERO–HOPF BIFURCATIONS IN THE HINDMARSH–ROSE SYSTEM

CLAUDIO A. BUZZI [1], JAUME LLIBRE[2]
AND JOÃO C. MEDRADO[3]

Abstract. We prove the existence of the classical Hopf bifurcation and of the zero–Hopf bifurcation in the Hindmarsh–Rose system. For doing this some adequate change of parameters must be done in order that the computations become easier.

1. Introduction

These last years there was a big interest in studying the three–dimensional Hindmarsh–Rose polynomial ordinary differential system, see [4]. It appears as a reduction of the conductance based in the Hodgkin–Huxley model for neural spiking, see for more details [5]. This differential system can be written as:

$$
\begin{aligned}
\dot{x} &= y - x^3 + bx^2 + I - z,\\
\dot{y} &= 1 - 5x^2 - y,\\
\dot{z} &= \mu(s(x - x_0) - z),
\end{aligned}
\tag{1}
$$

where $b,\ I,\ \mu,\ s,\ x_0$ are parameters and the dot indicates derivative with respect to the time t.

2010 *Mathematics Subject Classification.* Primary 34C23, 34C25, 37G10.
Key words and phrases. Hindmarsh–Rose burster, Hopf bifurcation, zero–Hopf bifurcation, periodic solution.

The interest in system (1) basically comes from two main reasons. The first one is due to its simplicity since it is just a differential system in $\mathbb{R}^3$ with a polynomial nonlinearity containing only five parameters. And the second one because it captures the three main dynamical behaviors presented by real neurons: quiescence, tonic spiking and bursting. We can find in the literature many papers that investigate the dynamics presented by system (1), see for instance [1, 2, 3, 6, 7, 9, 10, 11, 12, 13, 15]. Among these amount of papers, none of them study the occurrence of a Hopf or a zero–Hopf bifurcation in this differential system.

In the present paper we consider some special choice of parameters that facilitates the study of the classical Hopf bifurcation and also of the zero–Hopf bifurcation. A *classical Hopf bifurcation* in $\mathbb{R}^3$ takes place in an equilibrium point with eigenvalues of the form $\pm\omega i$ and $\delta \neq 0$, while for a *zero–Hopf bifurcation* the eigenvalues are $\pm\omega i$ and 0. Here an equilibrium point with eigenvalues $\pm\omega i$ and 0 will be called a *zero–Hopf equilibrium.*

Our main results are the following.

Theorem 1. *The equilibrium point p_0 of the Hindmarsh–Rose system* (1) *given in* (4) *exhibits a zero–Hopf bifurcation for the choice of the parameters given in* (3) *and* (5)*, when $\varepsilon = \delta = 0$, and for $\varepsilon \neq 0$ sufficiently small the bifurcated periodic solution is of the form*

(2)
$$\begin{aligned} x(t,\varepsilon) = \;& \rho + \frac{\varepsilon}{2\omega^2}(-r_0(\omega-1)\sin(t\omega) + r_0(\omega+1)\cos(t\omega) - 2w_0) \\ & +\mathcal{O}(\varepsilon^2), \\ y(t,\varepsilon) = \;& 1 - 5\rho^2 + \frac{5\rho\varepsilon}{\omega^2}(-r_0\sin(t\omega) - r_0\cos(t\omega) + 2w_0) + \mathcal{O}(\varepsilon^2), \\ z(t,\varepsilon) = \;& s(\rho - x_0) + \frac{\varepsilon}{20\rho\omega^2}\Big(r_0(10\rho+\omega-1)\big(-10\rho+\omega^2+1\big)\sin(t\omega) \\ & -r_0(-10\rho+\omega+1)(-10\rho+\omega^2+1)\cos(t\omega) \\ & +2w_0\big((1-10\rho)^2+\omega^2\big)\Big) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Theorem 1 is proved in section 3. We note that the local stability of the periodic solution (2) is described at the end of section 3. Moreover, the proof of Theorem 1 is done using averaging theory, and the method used for proving Theorem 1 can be applied for studying the zero–Hopf bifurcation in other differential systems.

Theorem 2. *The equilibrium point p_0 of the Hindmarsh–Rose system* (1) *given in* (4) *exhibits a classical Hopf bifurcation for the choice of the parameters given in* (3) *and* (5)*, when $\varepsilon = 0$, $\delta \neq 0$ and*

$$\ell_1(p_0) = \frac{(\omega^2+1)R_1}{400\delta\rho^4\omega^3\,(\delta^2+\omega^2)\,(\delta^2+4\omega^2)\,R_2} \neq 0,$$

where R_1 and R_2 are given at the end of section 4. Moreover if $\ell_1(p_0) < 0$ then the Hopf bifurcation is supercritical, otherwise it is subcritical.

Theorem 2 is proved in section 4.

2. Preliminaries

The first step of our analysis is to change the parameters b and I to new parameters β and ρ in order to obtain a simplified expression of one of the equilibria. We consider the following change of parameters

$$\begin{aligned} b &= 5 + \frac{s}{\rho} - \frac{\beta}{\rho} + \rho, \\ I &= -1 - s x_0 + \beta\rho. \end{aligned} \tag{3}$$

Then it is easy to verify that

$$p_0 = (\rho, 1 - 5\rho^2, s(\rho - x_0)) \tag{4}$$

is an equilibrium of system (1).

Now, we choose the parameters μ, β and s in order that when $\varepsilon = 0$ the eigenvalues of the linear part of system (1) at the equilibrium point p_0 are δ, ωi and $-\omega i$. This is an scenario candidate to have a Hopf bifurcation. So, we get

(5)
$$
\begin{aligned}
\mu = \quad & -\frac{1+\delta-10\rho+(1+\delta)\omega^2}{10\rho}+\mu_1\varepsilon,\\
\beta = \quad & -\frac{1}{20\rho\,(1+\delta-10\rho+(1+\delta)\omega^2)}\big(-\delta^2\,(-1+10\rho-\omega^2)\,(1+\omega^2)\\
& -(-1+10\rho-\omega^2)\,(1+10\rho(-2+\rho(10+\rho))+\omega^2)\\
& +2\delta(1+5\rho(-4+\rho(20+\rho))+2\omega^2\\
& +5\rho(-4+(-10+\rho)\rho)\omega^2+\omega^4)\big)+\beta_1\varepsilon,\\
\mu s = \quad & -\frac{(-1+10\rho-\omega^2)\,((1+\delta-10\rho)^2+(1+\delta)^2\omega^2)}{100\rho^2}+s_1\varepsilon.
\end{aligned}
$$

Next step we translate the equilibrium point p_0 to the origin of coordinates doing the change of variables

$$
(x,y,z)=(\overline{x},\overline{y},\overline{z})+(\rho,1-5\rho^2,s(\rho-x_0)), \tag{6}
$$

and we obtain

$$
\begin{aligned}
\dot{\overline{x}} &= a_1\overline{x}+\overline{y}-\overline{z}+a_2\overline{x}^2-\overline{x}^3+\varepsilon a_3\overline{x}(2\rho+\overline{x}),\\
\dot{\overline{y}} &= -10\rho\overline{x}-\overline{y}-5\overline{x}^2,\\
\dot{\overline{z}} &= a_4\overline{x}+a_5\overline{z}+\varepsilon(s_1\overline{x}-\mu_1\overline{z}),
\end{aligned} \tag{7}
$$

where

$$
\begin{aligned}
a_1 = \quad & -\frac{1+\delta-20\rho-10\delta\rho+(1+\delta)\omega^2}{10\rho},\\
a_2 = \quad & -\frac{1+\delta-20\rho-10\delta\rho+30\rho^3+(1+\delta)\omega^2}{20\rho^2},\\
a_3 = \quad & \frac{1}{\rho\,(1+\delta-10\rho+(1+\delta)\omega^2)^2}\big(-10s_1\rho\,(1+\delta-10\rho+(1+\delta)\omega^2)\\
& +\mu_1\,(-1+10\rho-\omega^2)\,((1+\delta-10\rho)^2+(1+\delta)^2\omega^2)\big)-\frac{\beta_1}{\rho},\\
a_4 = \quad & -\frac{(-1+10\rho-\omega^2)\,((1+\delta-10\rho)^2+(1+\delta)^2\omega^2)}{100\rho^2},\\
a_5 = \quad & \frac{1+\delta-10\rho+(1+\delta)\omega^2}{10\rho}.
\end{aligned}
$$

In our approach we want to study the periodic solutions that born at the origin. So we perform the rescaling of variables $(\overline{x},\overline{y},\overline{z})=$

$(\varepsilon\overline{u}, \varepsilon\overline{v}, \varepsilon\overline{w})$. Then system (7) becomes

$$\begin{aligned} \dot{\overline{u}} &= a_1\overline{u} + \overline{v} - \overline{w} + \varepsilon(a_2\overline{u}^2 + 2\rho a_3\overline{u}) + \varepsilon^2(a_3\overline{u}^2 - \overline{u}^3), \\ \dot{\overline{v}} &= -10\rho\overline{u} - \overline{v} - 5\varepsilon\overline{u}^2, \\ \dot{\overline{w}} &= a_4\overline{u} + a_5\overline{w} + \varepsilon(s_1\overline{u} - \mu_1\overline{w}). \end{aligned} \tag{8}$$

Now we put the linear part of system (8) into its real Jordan normal form doing the change of variables

$$\begin{pmatrix} \overline{u} \\ \overline{v} \\ \overline{w} \end{pmatrix} = \frac{1}{10\rho} A \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \tag{9}$$

where the matrix A is

$$\begin{pmatrix} \dfrac{\delta + \Delta + \omega - \Delta\omega + \delta\omega^2 + \omega^3}{10\rho} & \dfrac{\delta + \Delta + \omega + \delta\omega - \omega^2}{10\rho} & 1 \\ \dfrac{\delta(1+\omega^2) + \Delta + (\Delta - 1)\omega - \omega^3}{10\rho} & \dfrac{\delta(1-\omega) + \Delta - \omega(1+\omega)}{10\rho} & 1 \\ \dfrac{(1+\delta)\Delta}{10\rho} & \dfrac{\Delta}{10\rho} & 1 \end{pmatrix},$$

where $\Delta = 1 - 10\rho + \omega^2$. So we obtain

$$\begin{aligned} \dot{u} &= -\omega v + \varepsilon P_1(u, v, w) + \mathcal{O}(\varepsilon^2), \\ \dot{v} &= \omega u + \varepsilon P_2(u, v, w) + \mathcal{O}(\varepsilon^2), \\ \dot{w} &= \delta w + \varepsilon P_3(u, v, w) + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{10}$$

where the $P_i(u, v, w) = b_{i1}u + b_{i2}v + b_{i3}w + b_{i4}u^2 + b_{i5}uv + b_{i6}uw + b_{i7}v^2 + b_{i8}vw + b_{i9}w^2$, for $i = 1, 2, 3$. The polynomials P_i are presented in the Appendix 5.3.

3. Proof of Theorem 1

In this section we analyze the case $\delta = 0$. It means that we are in the scenario of a zero–Hopf bifurcation. Our approach for obtaining the periodic orbits of the system bifurcating from the zero–Hopf equilibrium p_0 is through the averaging theory of first order (see Appendix 5.1).

First we pass system (10) to cylindrical coordinates using $u = r\cos\theta$, $v = r\sin\theta$ and $w = w$. We get

$$
\begin{aligned}
\dot{r} &= \varepsilon(\cos\theta P_1 + \sin\theta P_2) + \mathcal{O}(\varepsilon^2),\\
\dot{\theta} &= \omega + \varepsilon(\cos\theta P_2 - \sin\theta P_1)/r + \mathcal{O}(\varepsilon^2),\\
\dot{w} &= \varepsilon P_3 + \mathcal{O}(\varepsilon^2),
\end{aligned} \tag{11}
$$

where we are denoting $P_i(r\cos\theta, r\sin\theta, w)$ just by P_i, for $i = 1, 2, 3$.

Now we take θ as a new independent variable. So system (11) becomes

$$
\begin{aligned}
\frac{dr}{d\theta} &= \varepsilon\frac{\cos\theta P_1 + \sin\theta P_2}{\omega} + \mathcal{O}(\varepsilon^2),\\
\frac{dw}{d\theta} &= \varepsilon\frac{P_3}{\omega} + \mathcal{O}(\varepsilon^2).
\end{aligned} \tag{12}
$$

Here we are ready to apply the averaging theory of first order, presented in Appendix 5.1 to system (12). Like in (15) we must compute the integrals

$$
\begin{aligned}
g_1(r,w) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos\theta P_1 + \sin\theta P_2}{\omega} d\theta,\\
g_2(r,w) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{P_3}{\omega} d\theta.
\end{aligned}
$$

Performing the calculations we obtain

$$
\begin{aligned}
g_1(r,w) &= \frac{r}{2000\rho^4\omega^5\Delta}\left(B_0 w + B_1\right),\\
g_2(r,w) &= \frac{1}{8000\rho^4\omega^5}\left(C_0 r^2 + C_1 w^2 + C_2 w\right),
\end{aligned}
$$

where

$$\begin{aligned}
\Delta = &\quad (-1+10\rho-\omega^2)\,,\\
B_0 = &\quad \Delta\,(1-10\rho(3+\rho(-30+\rho(97+30\rho)))+2\omega^2\\
&\quad +10\rho(1+2\rho)(-2+3\rho(-2+5\rho))\omega^2+(1+10\rho)\omega^4)\,,\\
B_1 = &\quad 100\rho^3\omega^2\,(10s1\rho(1-10\rho+\omega^2+20\rho\omega^2+(1+30\rho)\omega^4)\\
&\quad +\mu_1(-(-1+10\rho)^3+2(1+200\rho^2(-1+5\rho))\omega^2\\
&\quad -2\beta_1\Delta(1+\omega^2+10\rho(-1+\omega^2))),\\
C_0 = &\quad \Delta\,(1+\omega^2)\,(1+10\rho(-2+\rho(10+3\rho))+\omega^2)\,,\\
C_1 = &\quad 4\Delta\,(1+10\rho(-2+\rho(10+3\rho))+\omega^2)\,,\\
C_2 = &\quad 800\rho^3\omega^2(10s_1\rho+\mu_1((1-10\rho)^2+\omega^2)-2\beta_1\Delta.
\end{aligned}$$

Consider

$$w_0 = -\frac{B_1}{B_0} \text{ and } r_0 = \sqrt{\frac{C_2B_1B_0 - C_1B_1^2}{C_0B_0^2}}.$$

If $\dfrac{C_2B_1B_0 - C_1B_1^2}{C_0} > 0$ then (r_0, w_0) is a singular point of the system $(\dot r, \dot w) = (g_1(r,w), g_2(r,w))$. According to Theorem 3, this solution (when it exists) provides a periodic solution $(r(\theta,\varepsilon), w(\theta,\varepsilon))$ of the differential system (12) such that $(r(0,\varepsilon), w(0,\varepsilon))$ tends to (r_0, w_0) when ε tends to 0.

Going back through the changes of variables and using the statement (a) of Theorem 3 we have

$$(r(\theta,\varepsilon), w(\theta,\varepsilon)) = (r_0, w_0) + \mathcal{O}(\varepsilon),$$

Then in cylindrical coordinates (r, θ, w), the periodic solution is

$$(r(t,\varepsilon), \theta(t,\varepsilon), w(t,\varepsilon)) = (r_0, \omega t, w_0) + \mathcal{O}(\varepsilon).$$

and, in the variables (u, v, w), the periodic solution becomes

$$(u(t,\varepsilon), v(t,\varepsilon), w(t,\varepsilon)) = (r_0\cos(\omega t), r_0\sin(\omega t), w_0) + \mathcal{O}(\varepsilon).$$

Now undo the linear change of variables (9) that transform system (8) into system (10), and we obtain the periodic solution

$$
\begin{aligned}
\overline{u}(t,\varepsilon) = &\ \frac{-r_0(\omega-1)\sin(t\omega)+r_0(\omega+1)\cos(t\omega)-2w_0}{2\omega^2}+\mathcal{O}(\varepsilon),\\
\overline{v}(t,\varepsilon) = &\ \frac{5\rho(-r_0\sin(t\omega)-r_0\cos(t\omega)+2w_0)}{\omega^2}+\mathcal{O}(\varepsilon),\\
\overline{w}(t,\varepsilon) = &\ \frac{1}{20\rho\omega^2}\left(r_0(-10\rho+\omega+1)\Delta\cos(t\omega)+2w_0\left((1-10\rho)^2+\omega^2\right)\right.\\
&\left. -r_0(10\rho+\omega-1)\Delta\sin(t\omega)\right)+\mathcal{O}(\varepsilon),
\end{aligned}
$$

of system (8).

Finally using that $(\overline{x},\overline{y},\overline{z})=(\varepsilon\overline{u},\varepsilon\overline{v},\varepsilon\overline{w})$ and (6) we get the periodic solution given in (2).

Note that the periodic solution (2) borns by a zero–Hopf bifurcation from the zero–Hopf equilibrium p_0, because when $\varepsilon\to 0$ this periodic solution tends to the equilibrium p_0. Moreover we know the kind of linear stability of this periodic solution. From Theorem 3 we need to know the eigenvalues of the Jacobian matrix of the map (g_1,g_2) evaluated at (r_0,w_0). If both eigenvalues have negative real part, this periodic orbit is an attractor. If both eigenvalues have positive real part, this periodic orbit is a repeller. If both eigenvalues are purely imaginary, then this periodic solution is linear stable. Finally if one eigenvalue has negative real part and the other has positive real part the periodic solution has an unstable and a stable invariant manifold formed by two cylinders.

This completes the proof of Theorem 1 and consequently the study of the zero–Hopf bifurcation for the Hindmarsh–Rose differential system at the mentioned equilibrium point. Now it remains to study the classical Hopf bifurcation.

4. Proof of Theorem 2

In what follows we perform the study of a classical Hopf bifurcation using a result that can be found in the book of Kuznetzov (see details on Appendix 5.2).

First of all we consider system (7), with $\varepsilon = 0$, given by

$$
\begin{aligned}
\dot{\overline{x}} &= a_1\overline{x} + \overline{y} - \overline{z} + a_2\overline{x}^2 - \overline{x}^3,\\
\dot{\overline{y}} &= -10\rho\overline{x} - \overline{y} - 5\overline{x}^2,\\
\dot{\overline{z}} &= a_4\overline{x} + a_5\overline{z},
\end{aligned}
\tag{13}
$$

The matrix of the linear part of (13) is

$$
A = \begin{pmatrix} a_1 & 1 & -1 \\ -10\rho & -1 & 0 \\ a_4 & 0 & a_5 \end{pmatrix},
$$

and its eigenvalues are δ, ωi and $-\omega i$. In order to prove that we have a Hopf bifurcation at the equilibrium point p_0 it remains to prove that the first Lyapunov coefficient $\ell_1(p_0)$ is different from zero. According to Theorem 4, to compute $\ell_1(p_0)$ we need not only the matrix A but also the bilinear and trilinear forms, B and C, associate to terms of second and third order of system (13), the inverse of matrix A and the inverse of the matrix $2\omega i Id - A$, where Id is the identity matrix of $\mathbb{R}^3$.

The bilinear form B evaluated at two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ is given by

$$
B(u, v) = (2a_2u_1v_1, -10u_1v_1, 0).
$$

And the trilinear form C evaluated at three vectors $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ is given by

$$
C(u, v, w) = (-6u_1v_1w_1, 0, 0).
$$

The inverse of the matrix A is

$$
A^{-1} = \frac{1}{100\delta\rho^2\omega^2} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},
$$

where

$$\begin{aligned}
a_{11} &= -10\rho(\delta\omega^2+\delta-10\rho+\omega^2+1),\\
a_{12} &= a_{11},\\
a_{13} &= -100\rho^2,\\
a_{21} &= 100\rho^2 a_{11} = 100\rho^2(\delta\omega^2+\delta-10\rho+\omega^2+1),\\
a_{22} &= 100\rho^2(\delta-10\rho+\omega^2+1),\\
a_{23} &= 1000\rho^3,\\
a_{31} &= -\Delta\left(\delta^2\omega^2+\delta^2-20\delta\rho+2\delta\omega^2+2\delta+100\rho^2-20\rho+\omega^2+1\right),\\
a_{32} &= a_{31},\\
a_{33} &= 10\rho(-10\delta\rho+\delta\omega^2+\delta+100\rho^2-20\rho+\omega^2+1),\\
\Delta &= 10\rho-\omega^2-1.
\end{aligned}$$

The inverse of the matrix $2\omega i\mathrm{Id}-A$ is

$$(2\omega i\mathrm{Id}-A)^{-1} = \frac{1}{300\rho^2\omega^2(\delta-2i\omega)}\begin{pmatrix} c_{11} & c_{12} & c_{13}\\ c_{21} & c_{22} & c_{23}\\ c_{31} & c_{32} & c_{33}\end{pmatrix},$$

where

$$\begin{aligned}
c_{11} &= -(2i\omega+1)c_{12} = 10\rho(2i\omega+1)(\delta\omega^2+\delta-20i\rho\omega-10\rho+\omega^2+1),\\
c_{12} &= -10\rho(\delta\omega^2+\delta-20i\rho\omega-10\rho+\omega^2+1),\\
c_{13} &= -100\rho^2(2i\omega+1),\\
c_{21} &= 100\rho^2(\delta\omega^2+\delta-20i\rho\omega-10\rho+\omega^2+1),\\
c_{22} &= 100\rho^2(-2i\delta\omega+\delta-10\rho-3\omega^2-2i\omega+1),\\
c_{23} &= 1000\rho^3,\\
c_{31} &= \Delta(2\omega-i)(-i\delta\omega-\delta+10\rho-i\omega-1)(\delta\omega+i\delta-10i\rho+\omega+i),\\
c_{32} &= -\Delta(-i\delta\omega-\delta+10\rho-i\omega-1)(i\delta\omega-\delta+10\rho+i\omega-1),\\
c_{33} &= 100\rho^2+(1+2i\omega)(1+\omega^2+20i\rho(i+\omega)-\delta\Delta).
\end{aligned}$$

Let $q=(q_1,q_2,q_3)$ be the eigenvector of A corresponding to the eigenvalue ωi, normalized so that $\overline{q}\cdot q=1$, where $\overline{q}$ is the conjugate vector of q. The expression of q is given by

$$q = \frac{1}{\sqrt{q_1\overline{q_1}+q_2\overline{q_2}+q_3\overline{q_3}}}(q_1,q_2,q_3),$$

where

$$\begin{aligned} q_1 &= 10\rho(\omega+i)(10i\rho+(1+\delta)(\omega-i)),\\ q_2 &= 100\rho^2(-i\delta\omega-\delta+10\rho-i\omega-1),\\ q_3 &= \Delta((1+\delta-10\rho)^2+(1+\delta)^2\omega^2), \end{aligned}$$

Let $p=(p_1,p_2,p_3)$ be the adjoint eigenvector such that $A^T p = -\omega i p$ and $\overline{p}\cdot q = 1$. The expression of p is given by

$$p = \frac{\sqrt{q_1\overline{q_1}+q_2\overline{q_2}+q_3\overline{q_3}}}{q_1\overline{p_1}+q_2\overline{p_2}+q_3\overline{p_3}}(p_1,p_2,p_3) = \frac{\sqrt{q\overline{q}}}{\overline{p}q}(p_1,p_2,p_3),$$

where

$$\begin{aligned} p_1 &= (\omega+i)(\delta\omega-i\delta+10i\rho+\omega-i),\\ p_2 &= i\delta\omega+\delta-10\rho+i\omega+1,\\ p_3 &= 10\rho. \end{aligned}$$

Applying the formula presented in Theorem 4 we obtain

$$\ell_1(p_0) = \frac{(\omega^2+1)R_1}{400\delta\rho^4\omega^3\left(\delta^2+\omega^2\right)\left(\delta^2+4\omega^2\right)R_2},$$

where R_1 is

$$2\delta^6\omega^{10} + 18\delta^5\omega^{10} + 60\delta^4\omega^{10} + 100\delta^3\omega^{10} + 90\delta^2\omega^{10} + 42\delta\omega^{10}+$$
$$40\delta^4\rho\omega^{10} + 200\delta^3\rho\omega^{10} + 360\delta^2\rho\omega^{10} + 280\delta\rho\omega^{10} + 80\rho\omega^{10} + 8\omega^{10}+$$
$$\delta^7\omega^8 + 13\delta^6\omega^8 + 82\delta^5\omega^8 + 250\delta^4\omega^8 + 2400\delta^3\rho^4\omega^8 + 9600\delta^2\rho^4\omega^8+$$
$$12000\delta\rho^4\omega^8 + 4800\rho^4\omega^8 + 405\delta^3\omega^8 + 120\delta^5\rho^3\omega^8 + 960\delta^4\rho^3\omega^8+$$
$$2640\delta^3\rho^3\omega^8 + 11360\delta^2\rho^3\omega^8 + 18040\delta\rho^3\omega^8 + 8480\rho^3\omega^8 + 361\delta^2\omega^8-$$
$$200\delta^4\rho^2\omega^8 - 1800\delta^3\rho^2\omega^8 - 5400\delta^2\rho^2\omega^8 - 6200\delta\rho^2\omega^8 - 2400\rho^2\omega^8+$$
$$168\delta\omega^8 - 60\delta^6\rho\omega^8 - 570\delta^5\rho\omega^8 - 2090\delta^4\rho\omega^8 - 3690\delta^3\rho\omega^8-$$
$$3330\delta^2\rho\omega^8 - 1460\delta\rho\omega^8 - 240\rho\omega^8 + 32\omega^8 + 4\delta^7\omega^6 + 36000\delta^2\rho^7\omega^6+$$
$$108000\delta\rho^7\omega^6 + 72000\rho^7\omega^6 + 32\delta^6\omega^6 + 1800\delta^4\rho^6\omega^6 + 12600\delta^3\rho^6\omega^6+$$
$$27000\delta^2\rho^6\omega^6 + 23400\delta\rho^6\omega^6 + 247200\rho^6\omega^6 + 148\delta^5\omega^6 - 42000\delta^3\rho^5\omega^6$$
$$-132000\delta^2\rho^5\omega^6 - 186000\delta\rho^5\omega^6 - 96000\rho^5\omega^6 + 400\delta^4\omega^6-$$
$$2400\delta^5\rho^4\omega^6 - 19800\delta^4\rho^4\omega^6 - 62400\delta^3\rho^4\omega^6 - 124600\delta^2\rho^4\omega^6-$$
$$249200\delta\rho^4\omega^6 - 169600\rho^4\omega^6 + 620\delta^3\omega^6 + 60\delta^6\rho^3\omega^6 + 600\delta^5\rho^3\omega^6-$$
$$1760\delta^4\rho^3\omega^6 - 33840\delta^3\rho^3\omega^6 - 54860\delta^2\rho^3\omega^6 - 5880\delta\rho^3\omega^6+$$
$$17440\rho^3\omega^6 + 544\delta^2\omega^6 + 700\delta^6\rho^2\omega^6 + 6900\delta^5\rho^2\omega^6 + 29600\delta^4\rho^2\omega^6+$$
$$65500\delta^3\rho^2\omega^6 + 73500\delta^2\rho^2\omega^6 + 38600\delta\rho^2\omega^6 + 7200\rho^2\omega^6 + 252\delta\omega^6-$$
$$20\delta^7\rho\omega^6 - 330\delta^6\rho\omega^6 - 2100\delta^5\rho\omega^6 - 6980\delta^4\rho\omega^6 - 12540\delta^3\rho\omega^6-$$
$$12210\delta^2\rho\omega^6 - 6060\delta\rho\omega^6 - 1200\rho\omega^6 + 48\omega^6 - 360000\delta^2\rho^8\omega^4-$$
$$720000\delta\rho^8\omega^4 - 720000\rho^8\omega^4 + 6\delta^7\omega^4 - 18000\delta^4\rho^7\omega^4 - 117000\delta^3\rho^7\omega^4$$
$$-333000\delta^2\rho^7\omega^4 - 1506000\delta\rho^7\omega^4 - 2472000\rho^7\omega^4 + 38\delta^6\omega^4+$$
$$900\delta^5\rho^6\omega^4 + 6300\delta^4\rho^6\omega^4 - 2100\delta^3\rho^6\omega^4 + 84900\delta^2\rho^6\omega^4-$$
$$193200\delta\rho^6\omega^4 + 254400\rho^6\omega^4 + 132\delta^5\omega^4 + 9000\delta^5\rho^5\omega^4 + 126000\delta^4\rho^5\omega^4$$
$$+489000\delta^3\rho^5\omega^4 + 336000\delta^2\rho^5\omega^4 - 244000\delta\rho^5\omega^4 + 192000\rho^5\omega^4+$$
$$300\delta^4\omega^4 - 600\delta^6\rho^4\omega^4 - 9600\delta^5\rho^4\omega^4 + 10200\delta^4\rho^4\omega^4 + 379600\delta^3\rho^4\omega^4$$
$$+1359600\delta^2\rho^4\omega^4 + 1405600\delta\rho^4\omega^4 + 366400\rho^4\omega^4 + 430\delta^3\omega^4-$$
$$2820\delta^6\rho^3\omega^4 - 37920\delta^5\rho^3\omega^4 - 190040\delta^4\rho^3\omega^4 - 551360\delta^3\rho^3\omega^4$$
$$-847740\delta^2\rho^3\omega^4 - 601880\delta\rho^3\omega^4 - 150560\rho^3\omega^4 + 366\delta^2\omega^4+$$
$$100\delta^7\rho^2\omega^4 + 2800\delta^6\rho^2\omega^4 + 19200\delta^5\rho^2\omega^4 + 68300\delta^4\rho^2\omega^4+$$
$$142300\delta^3\rho^2\omega^4 + 164700\delta^2\rho^2\omega^4 + 95800\delta\rho^2\omega^4 + 21600\rho^2\omega^4 + 168\delta\omega^4$$
$$-60\delta^7\rho\omega^4 - 630\delta^6\rho\omega^4 - 2880\delta^5\rho\omega^4 - 8000\delta^4\rho\omega^4 - 13480\delta^3\rho\omega^4-$$
$$13050\delta^2\rho\omega^4 - 6620\delta\rho\omega^4 - 1360\rho\omega^4 + 32\omega^4 + 360000\delta^3\rho^8\omega^2+$$
$$990000\delta^2\rho^8\omega^2 + 1800000\delta\rho^8\omega^2 + 720000\rho^8\omega^2 + 4\delta^7\omega^2 - 4$$
$$+5000\delta^4\rho^7\omega^2 + 438000\delta^3\rho^7\omega^2 + 2913000\delta^2\rho^7\omega^2 + 8586000\delta\rho^7\omega^2+$$
$$4656000\rho^7\omega^2 + 22\delta^6\omega^2 + 1800\delta^5\rho^6\omega^2 - 232800\delta^4\rho^6\omega^2-$$
$$1512000\delta^3\rho^6\omega^2 - 2021200\delta^2\rho^6\omega^2 + 4983400\delta\rho^6\omega^2 + 6087200\rho^6\omega^2+$$
$$58\delta^5\omega^2 + 30000\delta^5\rho^5\omega^2 - 84000\delta^4\rho^5\omega^2 - 1164000\delta^3\rho^5\omega^2-$$

$5296000\delta^2\rho^5\omega^2 - 9258000\delta\rho^5\omega^2 - 4512000\rho^5\omega^2 + 100\delta^4\omega^2-$
$1200\delta^6\rho^4\omega^2 + 68000\delta^5\rho^4\omega^2 + 529800\delta^4\rho^4\omega^2 + 1746000\delta^3\rho^4\omega^2+$
$3481400\delta^2\rho^4\omega^2 + 3426800\delta\rho^4\omega^2 + 1180800\rho^4\omega^2 + 120\delta^3\omega^2-$
$6820\delta^6\rho^3\omega^2 - 69160\delta^5\rho^3\omega^2 - 259960\delta^4\rho^3\omega^2 - 580640\delta^3\rho^3\omega^2-$
$801460\delta^2\rho^3\omega^2 - 577960\delta\rho^3\omega^2 - 159520\rho^3\omega^2 + 94\delta^2\omega^2 + 200\delta^7\rho^2\omega^2+$
$3500\delta^6\rho^2\omega^2 + 17700\delta^5\rho^2\omega^2 + 47000\delta^4\rho^2\omega^2 + 80900\delta^3\rho^2\omega^2+$
$87300\delta^2\rho^2\omega^2 + 51000\delta\rho^2\omega^2 + 12000\rho^2\omega^2 + 42\delta\omega^2 - 60\delta^7\rho\omega^2-$
$510\delta^6\rho\omega^2 - 1740\delta^5\rho\omega^2 - 3620\delta^4\rho\omega^2 - 5100\delta^3\rho\omega^2 - 4590\delta^2\rho\omega^2-$
$2300\delta\rho\omega^2 - 480\rho\omega^2 + 8\omega^2 + 180000\delta^3\rho^8 + 90000\delta^2\rho^8 + \delta^7-$
$27000\delta^4\rho^7 + 855000\delta^3\rho^7 + 582000\delta^2\rho^7 + 5\delta^6 + 900\delta^5\rho^6-$
$237300\delta^4\rho^6 + 462700\delta^3\rho^6 + 760900\delta^2\rho^6 + 10\delta^5 + 21000\delta^5\rho^5-$
$310000\delta^4\rho^5 - 995000\delta^3\rho^5 - 564000\delta^2\rho^5 + 10\delta^4 - 600\delta^6\rho^4+$
$55200\delta^5\rho^4 + 279800\delta^4\rho^4 + 381600\delta^3\rho^4 + 147600\delta^2\rho^4 + 5\delta^3-$
$3940\delta^6\rho^3 - 30760\delta^5\rho^3 - 72640\delta^4\rho^3 - 65760\delta^3\rho^3 - 19940\delta^2\rho^3 + \delta^2+$
$100\delta^7\rho^2 + 1400\delta^6\rho^2 + 5400\delta^5\rho^2 + 8500\delta^4\rho^2 + 5900\delta^3\rho^2 + 1500\delta^2\rho^2-$
$20\delta^7\rho - 150\delta^6\rho - 390\delta^5\rho - 470\delta^4\rho - 270\delta^3\rho - 60\delta^2\rho.$

and R^2 is

$100\delta^2\rho^2\omega^2 + 100\delta^2\rho^2 - 20\delta^2\rho\omega^4 - 40\delta^2\rho\omega^2 - 20\delta^2\rho + \delta^2\omega^6 + 3\delta^2\omega^4$
$+3\delta^2\omega^2 + \delta^2 - 2000\delta\rho^3 + 600\delta\rho^2\omega^2 + 600\delta\rho^2 - 60\delta\rho\omega^4 - 120\delta\rho\omega^2-$
$60\delta\rho + 2\delta\omega^6 + 6\delta\omega^4 + 6\delta\omega^2 + 2\delta + 20000\rho^4 - 2000\rho^3\omega^2 - 4000\rho^3+$
$100\rho^2\omega^4 + 800\rho^2\omega^2 + 700\rho^2 - 40\rho\omega^4 - 80\rho\omega^2 - 40\rho + \omega^6+$
$3\omega^4 + 3\omega^2 + 1.$

Now the proof of Theorem 2 follows directly from Theorem 4.

5. Three appendices

5.1. **Averaging theory of first order.** We consider the initial value problems

$$\dot{\mathbf{x}} = \varepsilon F_1(t,\mathbf{x}) + \varepsilon^2 F_2(t,\mathbf{x},\varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{14}$$

and

$$\dot{\mathbf{y}} = \varepsilon g(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0,$$

with $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x}_0$ in some open Ω of $\mathbb{R}^n$, $t \in [0, \infty)$, $\varepsilon \in (0, \varepsilon_0]$. We assume that $\mathbf{F_1}$ and $\mathbf{F_2}$ are periodic of period T in the variable t, and we set

$$g(\mathbf{y}) = \frac{1}{T} \int_0^T F_1(t, \mathbf{y}) dt. \tag{15}$$

Theorem 3. *Assume that F_1, $D_{\mathbf{x}} F_1$,$D_{\mathbf{xx}} F_1$ and $D_{\mathbf{x}} F_2$ are continuous and bounded by a constant independent of ε in $[0, \infty) \times \Omega \times (0, \varepsilon_0]$, and that $y(t) \in \Omega$ for $t \in [0, 1/\varepsilon]$. Then the following statements holds:*

(a) *For $t \in [0, 1/\varepsilon]$ we have $\mathbf{x}(t) - \mathbf{y}(t) = O(\varepsilon)$ as $\varepsilon \to 0$.*
(b) *If $p \neq 0$ is a singular point of system (15) and $det D_{\mathbf{y}} g(p) \neq 0$, then there exists a periodic solution $\phi(t, \varepsilon)$ of period T for system (14) which is close to p and such that $\phi(0, \varepsilon) - p = O(\varepsilon)$ as $\varepsilon \to 0$.*
(c) *The stability of the periodic solution $\phi(t, \varepsilon)$ is given by the stability of the singular point.*

We have used the notation $D_{\mathbf{x}} g$ for all the first derivatives of g, and $D_{\mathbf{xx}} g$ for all the second derivatives of g.

For a proof of Theorem 3 see [14].

5.2. **Classical Hopf bifurcation.** Assume that a system $\dot{x} = F(x)$ has an equilibrium point p_0. If its linearization at p_0 has a pair of conjugate purely imaginary eigenvalues and the others eigenvalues have non vanishing real part, then this is the setting for a classical Hopf bifurcation. We can expect to see a small–amplitude limit cycle branching from the equilibrium point p_0. It remains to compute the first Lyapunov coefficient $\ell_1(p_0)$ of the system near p_0. When $\ell_1(p_0) < 0$ the point p_0 is a weak focus of system restricted to the central manifold of p_0 and the limit cycle that emerges from p_0 is stable. In this case we say that the Hopf bifurcation is supercritical. When $\ell_1(p_0) > 0$ the point p_0 is also a weak focus of the system restricted to the central manifold of p_0 but the limit cycle that borns from p_0 is unstable. In this second case we say that the Hopf bifurcation is subcritical.

Here we use the following result presented on page 180 of the book [8] for computing $\ell_1(p_0)$.

Theorem 4. *Let $\dot{x} = F(x)$ be a differential system having p_0 as an equilibrium point. Consider the third order Taylor approximation of F around p_0 given by $F(x) = Ax + \frac{1}{2!}B(x,x) + \frac{1}{3!}C(x,x,x) + \mathcal{O}(|x|^4)$. Assume that A has a pair of purely imaginary eigenvalues $\pm\omega i$, and these eigenvalues are the only eigenvalues with real part equal to zero. Let q be the eigenvector of A corresponding to the eigenvalue ωi, normalized so that $\overline{q}\cdot q = 1$, where $\overline{q}$ is the conjugate vector of q. Let p be the adjoint eigenvector such that $A^T p = -\omega i p$ and $\overline{p}\cdot q = 1$. If Id denotes the identity matrix, then*

$$
\begin{aligned}
\ell_1(p_0) = \frac{1}{2\omega} Re(&\overline{p}\cdot C(q,q,\overline{q}) - 2\overline{p}\cdot B(q, A^{-1}B(q,\overline{q})) \\
&+\overline{p}\cdot B(\overline{q}, (2\omega i Id - A)^{-1}B(q,q))).
\end{aligned}
$$

5.3. **Expressions of the P_i's.** In this appendix we present the expressions of the quadratic polynomials P_i, for $i = 1,2,3$ in system (10).

$$
\begin{aligned}
P_1 = \;& \frac{1}{K_1}\big(Q_1(K_7 - K_5Q_1)(1 + 10\rho(-1+\omega) + \omega^2 + \delta(1+\omega^2)), \\
& +Q_2 - 100Q_1^2\rho^2(1 + \delta - 10\rho + \omega + \delta\omega)\big), \\
P_2 = \;& \frac{1}{K_1}\big(Q_1(K_7 - K_5Q_1)(1 + \omega^2 - 10\rho(1+\omega) + \delta(1+\omega^2))\big) \\
& +Q_2 + 100Q_1^2\rho^3 2(-1 + 10\rho + \delta(-1+\omega) + \omega), \\
P_3 = \;& \frac{1}{K_1}\big(K_7(1+\delta))(10\rho - 1 + \omega^2) - Q_2\big) \\
& +Q_1(Q_1(K_5(1+\delta) + 100\rho^2),
\end{aligned}
$$

where Q_i for $i = 1, 2$ are linear polynomials in the variables u, v and w and, the real numbers K_j for $j = 1 \dots 7$ are

$$
\begin{aligned}
Q_1 = &\quad (\delta(-1+\omega)+\omega+\omega^2)u+(\delta+\omega+\delta\omega-\omega^2)v-2(1+\delta)\omega w,\\
Q_2 = &\quad 400(10s_1\rho Q_1-(K_2u+K_3v+2K_4w)\mu_1)\omega(\delta^2+\omega^2)\rho^3,\\
K_1 = &\quad 800\rho^3\omega(\delta^2+\omega^2),\\
K_2 = &\quad (-1+10\rho-\omega^2)(\delta^2(-1+\omega)+\omega(1-10\rho+\omega)\\
&\quad +\delta(-1+10\rho+\omega(2+\omega))),\\
K_3 = &\quad (-1+10\rho-\omega^2)(\delta(1+\delta-10\rho)\\
&\quad +((1+\delta)^2-10\rho)\omega-(1+\delta)\omega^2),\\
K_4 = &\quad \omega((1+\delta-10\rho)^2+(1+\delta)^2\omega^2),\\
K_5 = &\quad (1+\delta-20\rho-10\delta\rho+30\rho^3+(1+\delta)\omega^2),\\
K_6 = &\quad (1+\delta-10\rho+(1+\delta)\omega^2)^2,\\
K_7 = &\quad K_1(-\beta_1+\frac{1}{K_6}\big(10s_1\rho(10\rho-1-\delta-(1+\delta)\omega^2)\\
&\quad +\mu_1(-1+10\rho-\omega^2)((1+\delta-10\rho)^2+(1+\delta)^2\omega^2)\big).
\end{aligned}
$$

Acknowledgments

The first author is partially supported by FAPESP grant 2013/2454-1, CAPES grant 88881.068462/2014-01 and EU Marie-Curie IRSES Brazilian-European partnership in Dynamical Systems (FP7-PEOPLE-2012-IRSES 318999 BREUDS).

The author is partially supported by a MINECO/FEDER grant MTM2008-03437, MINECO grant MTM2013-40998-P, an AGAUR grant number 2014SGR-568, an ICREA Academia, the grants FP7-PEOPLE-2012-IRSES 318999 and 316338, the MINECO/FEDER grant UNAB13-4E-1604, and a CAPES grant number 88881.030454/2013–01 from the program CSF–PVE.

The third author is partially supported by FAPEG, CNPq and CAPES.

References

[1] V.N. Belykh, I.V. Belykh, M. Colding–Jorgensen and E. Mosekilde, Homoclinic bifurcations leading to the emergence of bursting oscillations in cell

models, *The European Physical Journal E: Soft Matter and Biological Physics* **3** (2000), 205–219.

[2] J.M. González-.Miranda, Observation of a continuous interior crisis in the Hindmarsh–Rose neuron model, *Chaos* **13** (2003), 845–852.

[3] J.M. González–Miranda, Complex bifurcation structures in the Hindmarsh–Rose neuron model, *Int. J. Bifurcation and Chaos* **17** (2007), 3071–3083.

[4] J.L. Hindmarsh and R.M. Rose, A model of neuronal bursting using three coupled first order differential equations, *Proceedings of the Royal Society of London. Series B, Biological Sciences* **221** (1984), 87–102. **246** (2009), 541–551.

[5] A.L. Hodgkin and A.F. Huxley, A quantitative description of membrane current and its application to conduction and excitation in nerve, *J. Physiol. London* **117** (1952), 500–544.

[6] G. Innocenti and R. Genesio, On the dynamics of chaotic spiking–bursting transition in the Hindmarsh–Rose neuron, *Chaos* **19** (2009), 023124–8 pp.

[7] G. Innocenti, A. Morelli, R. Genesio and A. Torcini, Dynamical phases of the Hindmarsh–Rose neuronal model: Studies of the transition from bursting to spiking chaos, *Chaos* **17** (2007), 043128–11 pp.

[8] Y. Kuznetsov, *Elements of applied bifurcation theory*, Applied Mathematical Sciences, Vol. **112**, Springer–Verlag, New York, 2004.

[9] D. Linaro, A. Champneys, M. Desroches and M. Storace, Codimension–two homoclinic bifurcations underlying spike adding in the Hindmarsh–Rose burster, *SIAM J. Appl. Dyn. Syst.* **11** (2012), 939–962.

[10] A. Silnikov and M. Kolomiets, Methods of the qualitative theory for the Hindmarsh–Rose model: a case study. a tutorial, *Int. J. Bifurcation and Chaos* **18** (2008), 2141–2168.

[11] M. Storace, D. Linaro and E. de Lange, The Hindmarsh–Rose neuron model: bifurcation analysis and piecewise–linear approximations, *Chaos* **18** (2008), 033128–10 pp.

[12] D. Terman, Chaotic spikes arising from a model of bursting in excitable membranes, *SIAM J. Appl. Math.* **51** (1991), 1418–1450.

[13] D. Terman, The transition from bursting to continuous spiking in excitable membrane models, *J. Nonlinear Science* **2** (1992), 135–182.

[14] F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems*, Universitext, Springer, New York, 1996.

[15] X.J. Wang, Genesis of bursting oscillations in the Hindmarsh–Rose model and homoclinicity to a chaotic saddle, *Physica D* **62** (1993), 263–274.

[1] Departamento de Matemática, IBILCE–UNESP, Rua C. Colombo, 2265, CEP 15054–000 S. J. Rio Preto, São Paulo, Brazil

E-mail address: buzzi@ibilce.unesp.br

[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: jllibre@mat.uab.cat

[3] Instituto de Matemática e Estatística, IME–UFG, Campus Samambaia, CEP 13081–970 Goiânia, Goiás, Brazil

E-mail address: medrado@ufg.br