This is a preprint of: “Analytic integrability of Hamiltonian systems with exceptional potentials”, Jaume Llibre, Clàudia Valls, *Phys. Lett. A*, vol. 379, 2295–2299, 2015.
DOI: [http://dx.doi.org/10.1016/j.physleta.2015.07.034]

ANALYTIC INTEGRABILITY OF HAMILTONIAN SYSTEMS WITH EXCEPTIONAL POTENTIALS

JAUME LLIBRE[1] AND CLAUDIA VALLS[3]

ABSTRACT. We study the existence of analytic first integrals of the complex Hamiltonian systems of the form

$$H = \frac{1}{2}\sum_{i=1}^{2} p_i^2 + V_l(q_1, q_2)$$

with the homogeneous polynomial potential

$$V_l(q_1, q_2) = \alpha(q_2 - iq_1)^l (q_2 + iq_1)^{k-l}, \quad l = 0, \ldots, k, \quad \alpha \in \mathbb{C} \setminus \{0\}$$

of degree k called *exceptional potentials.* In Remark 2.1 of J. Math. Phys. **46** (2005), 062901, the authors state: *The exceptional potentials V_0, V_1, V_{k-1}, V_k and $V_{k/2}$ when k is even are integrable with a second polynomial first integral. However nothing is known about the integrability of the remaining exceptional potentials.*

Here we prove that the exceptional potentials with k even different from V_0, V_1, V_{k-1}, V_k and $V_{k/2}$, have no independent analytic first integral different from the Hamiltonian one.

Additionally in the cases V_2 and V_{k-2} with k either even or odd we show that they do not have rational first integrals independent of the Hamiltonian.

1. Introduction and statement of the main results

Ordinary differential equations in general and Hamiltonian systems in particular play a very important role in many branches of the applied sciences. The question whether a differential system admits a first integral is of fundamental importance as first integrals give conservation laws for the model and that enables to lower the dimension of the system. Moreover knowing a sufficient number of first integrals allows to solve the system explicitly. Until the end of the 19th century the majority of scientists thought that the equations of classical mechanics were integrable and finding the first integrals was mainly a problem of computation. In fact, now we know that the integrability is a non-generic phenomenon inside the class of Hamiltonian systems (see [3]), and in general it is very hard to determine whether a given Hamiltonian system is integrable or not.

2010 *Mathematics Subject Classification.* 37J35.
Key words and phrases. Hamiltonian system with 2–degrees of freedom, homogeneous potentials of degree k, exceptional potentials, integrability.

In this work we are concerned with the integrability of the natural Hamiltonian systems defined by a Hamiltonian function of the form

$$H = \frac{1}{2}\sum_{i=1}^{2} p_i^2 + V(q_1, q_2), \tag{1}$$

where $V(q_1, q_2) \in \mathbb{C}[q_1, q_2]$ is a homogeneous polynomial potential of degree k. As usual $\mathbb{C}[q_1, q_2]$ is the ring of polynomial functions over $\mathbb{C}$ in the variables q_1 and q_2. To be more precise we consider the following system of four differential equations

$$\dot{q}_i = p_i, \quad \dot{p}_i = -\frac{\partial V}{\partial q_i}, \qquad i = 1, 2. \tag{2}$$

Let $A = A(\mathbf{q}, \mathbf{p})$ and $B = B(\mathbf{q}, \mathbf{p})$ be two functions, where $\mathbf{p} = (p_1, p_2)$ and $\mathbf{q} = (q_1, q_2)$. We define the *Poisson bracket* of A and B as

$$\{A, B\} = \sum_{i=1}^{2} \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \right).$$

The functions A and B are *in involution* if $\{A, B\} = 0$. A non–constant function $F = F(\mathbf{q}, \mathbf{p})$ is a *first integral* for the Hamiltonian system (2) if it is in involution with the Hamiltonian function H, i.e. $\{H, F\} = 0$. Since the Poisson bracket is antisymmetric it follows that H itself is always a first integral. A 2–degree of freedom Hamiltonian system (2) is *completely* or *Liouville integrable* if it has 2 functionally independent first integrals H and F. As usual H and F are *functionally independent* if their gradients are linearly independent at all points of $\mathbb{C}^4$ except perhaps in a zero Lebesgue measure set.

Let $\mathrm{PO}_2(\mathbb{C})$ denote the group of 2×2 complex matrices A such that $AA^T = \alpha\,\mathrm{Id}$, where Id is the 2×2 identity matrix and $\alpha \in \mathbb{C} \setminus \{0\}$. The potentials $V_1(\mathbf{q})$ and $V_2(\mathbf{q})$ are *equivalent* if there exists a matrix $A \in \mathrm{PO}_2(\mathbb{C})$ such that $V_1(\mathbf{q}) = V_2(A\mathbf{q})$. Therefore we divide all potentials into equivalent classes. In what follows a potential means a class of equivalent potentials in the above sense. This definition of equivalent potentials is motivated by the following simple observation (for a proof see [1]). Let V_1 and V_2 be two equivalent potentials. If the Hamiltonian system (2) with the potential V_1 is integrable, then it is also integrable with the potential V_2.

It was shown in [2] that among all equivalent potentials one can always choose a representative V such that the polynomial V has one root in an arbitrary point of $\mathbb{CP}^1 \setminus \{[1 : +i], [1 : -i]\}$. This is always possible except for cases when all linear factors of V have the form $q_2 \pm iq_1$, that is, if the potential V is of the form

$$V = V_l = \alpha(q_2 - iq_1)^l (q_2 + iq_1)^{k-l}, \quad l = 0, \ldots, k, \quad \alpha \in \mathbb{C} \setminus \{0\}.$$

These potentials are called *exceptional*. It was proved in [1] that the exceptional potentials V_0, V_1, V_{k-1}, V_k and $V_{k/2}$ when k is even are integrable . It is easy to find that for these exceptional potentials the additional polynomial first integral is:

$$I_0 = p_1 - ip_2, \quad I_1 = k(p_1 - ip_2)^2 - 4\alpha(q_2 + iq_1)^k,$$
$$I_{k-1} = k(p_1 + ip_2)^2 - 4\alpha(q_2 - iq_1)^k, \quad I_k = p_1 + ip_2,$$

and when k is even

$$I_{k/2} = q_2 p_1 - q_1 p_2.$$

It is also claimed in [2] and [1] that nothing is known about the integrability of the remaining exceptional potentials. In this paper we focus on these remaining exceptional potentials. We restrict to the potentials V_l with $l = 2, \ldots, k/2 - 1, k/2 + 1, \ldots, k - 2$ and k even. Note that if $k \leq 4$ all the exceptional potentials are integrable with polynomial first integrals. So we will focus on the case $k \geq 5$.

System (2) becomes

$$\begin{aligned}
\dot{q}_1 &= p_1,\\
\dot{q}_2 &= p_2,\\
\dot{p}_1 &= \alpha i[(2l - k)q_2 + ik_1 q_1](q_2 - iq_1)^{l-1}(q_2 + iq_1)^{k-l-1},\\
\dot{p}_2 &= -\alpha i[(2l - k)q_1 - ikq_2](q_2 - iq_1)^{l-1}(q_2 + iq_1)^{k-l-1}.
\end{aligned} \tag{3}$$

Our main results are the following.

Theorem 1. *System* (3) *with* $k \geq 6$ *even and* $l = 2, \ldots, k/2 - 1, k/2 + 1, \ldots, k - 2$ *does not admit an additional analytic first integral independent of the Hamiltonian one.*

The proof of Theorem 1 is given in section 3. We state the following conjecture.

Conjecture 1. *System* (3) *with* $k \geq 5$ *odd and* $l = 2, \ldots, k-2$ *does not admit an additional analytic first integral independent of the Hamiltonian one.*

In the case in which $l = 2$ or $l = k - 2$ with k being either even or odd, we can also prove with different techniques the non–existence of rational first integrals.

Theorem 2. *System* (3) *with* $l = 2$ *or* $l = k-2$ *and* $k \geq 5$ *does not admit an additional rational first integral independent of the Hamiltonian one.*

The proof of Theorem 2 is given in section 2.

2. Weight-homogeneous polynomial differential system and proof of Theorem 2

We consider polynomial differential system of the form

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \mathbf{P}(\mathbf{x}), \quad \mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4 \tag{4}$$

with $\mathbf{P}(\mathbf{x}) = (P_1(\mathbf{x}), P_2(\mathbf{x}), P_3(\mathbf{x}), P_4(\mathbf{x})$ and $P_i \in \mathbb{C}[x_1, x_2, x_3, x_4]$ for $i = 1, 2, 3, 4$. As usual $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of positive integers, real and complex numbers, respectively; and $\mathbb{C}[x_1, x_2, x_3, x_4]$ denotes the polynomial ring over $\mathbb{C}$ in the variables x_1, x_2, x_3, x_4. Here t can be real or complex.

We say that system (4) is *weight–homogeneous* if there exist $\mathbf{s} = (s_1, s_2, s_3, s_4) \in \mathbb{N}^4$ and $d \in \mathbb{N}$ such that for arbitrary $a \in \mathbb{R}^+ = \{a \in \mathbb{R}, a > 0\}$ we have

$$P_i(a^{s_1}x_1, a^{s_2}x_2, a^{s_3}x_3, a^{s_4}x_4) = a^{s_i-1+d}P_i(x_1, x_2, x_3, x_4),$$

for $i = 1, 2, 3, 4$. We call $\mathbf{s} = (s_1, s_2, s_3, s_4)$ as the *weight exponent* of system (4) and d as *weight degree* with respect to the weight exponent $\mathbf{s}$.

We say that a polynomial $F(x_1, x_2, x_3, x_4)$ is a *weight-homogeneous polynomial with weight exponent* $\mathbf{s} = (s_1, s_2, s_3, s_4)$ *and weight degree* d if

$$F(a^{s_1}x_1, a^{s_2}x_2, a^{s_3}x_3, a^{s_4}x_4) = a^d F(x_1, x_2, x_3, x_4),$$

for all $a > 0$.

The following well–known proposition (easy to prove) reduces the study of the existence of analytic first integrals of a weight–homogeneous polynomial differential system (4) to the study of the existence of a weight-homogeneous polynomial first integrals.

Proposition 3. *Let H be an analytic function and let $H = \sum_i H_i$ be its decomposition into weight–homogeneous polynomials of weight degree i with respect to the weight exponent $\mathbf{s}$. Then H is an analytic first integral of the weight-homogeneous polynomial differential system* (4) *with weight exponent* $\mathbf{s}$ *if and only if each weight-homogeneous part H_i is a first integral of system* (4) *for all i.*

We introduce the change of variables

$$x_1 = p_1 + ip_2, \quad x_2 = p_1 - ip_2, \quad y_1 = q_1 + iq_2, \quad y_2 = q_1 - iq_2.$$

In these new variables system (3) becomes

$$\begin{aligned} \dot{x}_1 &= 2\alpha(l-k)(-1)^l i^k y_1^l y_2^{k-l-1}, \\ \dot{x}_2 &= -2\alpha l(-1)^l i^k y_1^{l-1} y_2^{k-l}, \\ \dot{y}_1 &= x_1, \\ \dot{y}_2 &= x_2. \end{aligned} \tag{5}$$

Note that system (5) is Hamiltonian with Hamiltonian

$$H = \frac{x_1 x_2}{2} + \alpha(-1)^l i^k y_1^l y_2^{k-l}, \quad \alpha \in \mathbb{C} \setminus \{0\}.$$

It is easy to check that system (5) is a weight–homogeneous polynomial differential system with weight exponent $s = (s_1, s_2, s_3, s_4)$ where

$$s_1 = l(k+1)+1, \quad s_2 = (l-1)(k+1), \quad s_3 = k+l+1, \quad s_4 = l-1 \tag{6}$$

and weight degree $d = 1 + k(l-1)$. Note that since $l \geq 2$ we have that $s_1, s_2, s_3, s_4, d \in \mathbb{N}$.

From Proposition 3 and the observation above it follows that for proving the existence of non–existence of analytic first integrals of system (5) it is sufficient to show the existence or non–existence of weight-homogeneous polynomial first integrals with weight exponents given in (6).

We recall that in the case in which k is even, we can be more precise and it is clear that system (5) is a weight-homogeneous polynomial differential system with weight exponent $(1, 1, k/2, k/2)$ and weight degree $d = k/2$.

Proof of Theorem 2. Instead of Proving Theorem 2 we will prove the following theorem which is equivalent to Theorem 2.

Theorem 4. *System* (5) *with* $l = 2$ *or with* $l = k-2$ *does not admit an additional rational first integral.*

We will only prove the case $l = k-2$ because the proof of the case $l = 2$ is exactly the same interchanging the roles of y_1 with y_2 and of x_1 with x_2. The proof follows directly from the following theorem which is Theorem 2.4 in [2].

Theorem 5. *The Hamiltonian system* (1) *with potential* $V = q_2^2 \tilde{V}(q_1, q_2)$ *where* $\deg \tilde{V} = k-2$ *and* $\tilde{V}(q_1, 0) \neq 0$*, does not admit an additional rational first integral.*

We note that in the case $l = k-2$ we have

$$V(y_1, y_2) = \alpha y_2^2 y_1^{k-2} = \alpha y_2^2 \tilde{V}(y_1, y_2), \tag{7}$$

where $\deg \tilde{V} = k-2$, $\tilde{V}(y_1, y_2) = \tilde{V}(y_1, 0) = \alpha y_1^{k-2}$. It follows directly from Theorem 5 that the Hamiltonian system with potential in (7) does not admit an additional rational first integral independent of the Hamiltonian one. This concludes the proof. □

3. Proof of Theorem 1

In this section we will prove the following equivalent result to Theorem 1.

Theorem 6. *System* (5) *with* $k \geq 6$ *even and* $l = 2, \ldots, \frac{k}{2}-1, \frac{k}{2}+1, \ldots k-2$ *does not admit an additional analytic first integral.*

We first observe that we only need to prove Theorem 6 for the cases $l = 2, \ldots, \frac{k}{2}-1$, because the proof of the cases $l = \frac{k}{2}+1, \ldots, k-2$ is exactly the same interchanging the roles of x_1 with x_2, and y_1 with y_2.

Before going into the technicalities of the proof of Theorem 6, we would like to highlight the main idea behind the proof. First we shall restrict system (5) to the zero level of the first integral H, which is a polynomial function. The restriction to this level set gives rise to a nontrivial rational first integral $\bar{F}$ of the restricted system. To be more precise, $\bar{F}(y_1, y_2, x_1)$ is a polynomial in the variables y_1, y_2, x_1 and x_1^{-1}. So, it can be written in the following form:

$$\bar{F} = \sum_{j=-\ell}^{m} g_j(y_1, y_2) x_1^j.$$

We recall again that system (5) is a weight–homogeneous polynomial differential system with weight exponent $(1, 1, k/2, k/2)$ and weight degree $d = k/2$. From section 3 it follows that for proving Theorem 6 it is sufficient to show that this system has no weight-homogeneous polynomial first integrals with weight exponent $(1, 1, k/2, k/2)$.

Let $F = F(y_1, y_2, x_1, x_2) \in \mathbb{C}[y_1, y_2, x_1, x_2]$ be a weight-homogeneous polynomial first integral of system (5) with weight exponent $(1, 1, k/2, k/2)$ and weight degree $d = \frac{k}{2}n$ with $n \geq 1$. We can express it as

$$F = \sum_{l_1+l_2+\frac{k}{2}l_3+\frac{k}{2}l_4=\frac{k}{2}n} F_{l_1,l_2,l_3,l_4} y_1^{l_1} y_2^{l_2} x_1^{l_3} x_2^{l_4}.$$

The function F cannot depend only on y_1 and y_2. Indeed, if $F = F(y_1, y_2)$ then from (5) we get

$$x_1 \frac{\partial F}{\partial y_1} + x_2 \frac{\partial F}{\partial y_2} = 0,$$

and consequently F is a constant. So F depends on x_1 or x_2, and thus $n \geq 2$.

We study the first integral F on the level set $H = 0$ by eliminating, for example x_2 as follows:

$$x_2 = -\frac{2\alpha(-1)^{l+k/2} y_1^l y_2^{k-l}}{x_1}.$$

Thus, we end up with the following system:

$$\begin{aligned} \dot{y}_1 &= x_1, \\ \dot{y}_2 &= -\frac{2\alpha(-1)^{l+k/2} y_1^l y_2^{k-l}}{x_1}, \\ \dot{x}_1 &= -2\alpha(-1)^{l+k/2}(k-l) y_1^l y_2^{k-l-1}. \end{aligned} \tag{8}$$

Note that the restriction of the polynomial first integral F to the level set $H = 0$ can be written as

$$\begin{aligned} \bar{F} &= \sum_{l_1+l_2+\frac{k}{2}l_3+\frac{k}{2}l_4=\frac{k}{2}n} \bar{F}_{l_1,l_2,l_3} y_1^{l_1} y_2^{l_2} x_1^{l_3} \Big(-\frac{2\alpha(-1)^{l+k/2} y_1^l y_2^{k-l}}{x_1} \Big)^{l_4} \\ &= \sum_{j=-n}^{n} \bar{F}_j(y_1, y_2) x_1^j \end{aligned} \tag{9}$$

where each $\bar{F}_j(y_1, y_2)$ is a homogeneous polynomial of weight degree $M := \frac{k}{2}(n-j)$. Indeed, the degree of $\bar{F}_j(y_1, y_2)$ is

$$l_1 + l_2 + l l_4 + (k-l)l_4 = l_1 + l_2 + k l_4.$$

using that $l_1 + l_2 = n - \frac{k}{2}(l_3 + l_4)$ and $l_3 - l_4 = j$ we can rewrite the above expression as

$$\frac{k}{2}n - \frac{k}{2}(l_3 + l_4) + k l_4 = \frac{k}{2}n - \frac{k}{2}l_3 + \frac{k}{2}l_4 = \frac{k}{2}n + \frac{k}{2}(l_4 - l_3) = \frac{k}{2}(n-j).$$

Note that system (8) is completely integrable with the first integrals

$$H_1 = \frac{y_2^{k-l}}{x_1} \quad \text{and} \quad H_2 = \frac{y_1^{l+1}}{l+1} + \frac{(-1)^{l+k/2} x_1^2}{2\alpha(k-l+1) y_2^{k-l-1}}.$$

Hence

$$\begin{aligned} \bar{F} &= \sum_{j_1, j_2} F_{j_1, j_2} \Big(\frac{y_1^{l+1}}{l+1} + \frac{(-1)^{l+k/2} x_1^2}{2\alpha(k-l+1) y_2^{k-l-1}} \Big)^{j_2} \\ &= \sum_{j_1, j_2} \sum_{m=0}^{j_2} \tilde{F}_{j_1, j_2, m} \frac{y_2^{j_1(k-l) - m(k-l-1)} y_1^{(l+1)(j_2-m)}}{x_1^{j_1 - 2m}}. \end{aligned}$$

where

$$\tilde{F}_{j_1, j_2, m} = F_{j_1, j_2} \binom{j_2}{m} \frac{(-1)^{m(l+k/2)}}{(l+1)^{j_2-m}(2\alpha(k-l+1))^m}.$$

Using that $\bar{F}$ must satisfy (9), we must have that for any $m = 0, \ldots, j_2$,

$$(j_2 - m)(l+1) - m(k-l-1) + j_1(k-l) = \frac{k}{2}n + \frac{k}{2}(j_1 - 2m)$$

that is

$$j_2(l+1) + j_1(k-l) = \frac{k}{2}n + \frac{k}{2}j_1$$

which yields

$$j_2 = \frac{1}{l+1}\Big(\frac{k}{2}(n-j_1) + l j_1\Big).$$

Note that $\bar{F}$ must be a polynomial in the variables y_1 and y_2. Thus

$$j_1(k-l) - \frac{k-l-1}{l+1}\Big(\frac{k}{2}(n-j_1) + l j_1\Big) \geq 0.$$

This implies that

$$j_1(k-l+1) \geq (k-l-1)n \quad \text{that is} \quad j_1 \geq \frac{k-l-1}{k-l+1}n.$$

Moreover using again (9) we have that $j_1 \leq n$. Therefore,

$$\bar{F} = \sum_{(k-l-1)n/(k-l+1) \leq j_1 \leq n} \beta_{j_1, n, k, l} H_1^{j_1} H_2^{(k(n-j_1) + 2 l j_1)/(2(l+1))},$$

with $\beta_{j_1, n, k, l} \in \mathbb{C}$.

To conclude the proof of Theorem 1 it is sufficient to show that $\bar{F} = 0$. Indeed if $\bar{F} = 0$ then any weight homogenous polynomial first integral with weight exponent $(1,1,k/2,k/2)$ and weight degree $d = kn/2$ restricted to $H = 0$ is zero and thus system (5) cannot have a weight homogenous polynomial first integral F with weight exponent $(1,1,k/2,k/2)$ and weight degree $d = kn/2$ independent with H since otherwise when restricted to $H = 0$ this first integral would not be zero.

To show that $\bar{F} = 0$, we write

$$F = F(y_1, y_2, x_1, x_2) = \bar{F} + \Big(x_2 + \frac{2\alpha(-1)^{l+k/2} y_1^l y_2^{k-l}}{x_1}\Big) G,$$

where G is a weight-homogenous polynomial with weight exponent $(1,1,k/2,k/2)$ and weight degree $d = kn/2$. Imposing that F is a first integral of system (5), after simplifying by $x_2 + \frac{2\alpha(-1)^{l+k/2} y_1^l y_2^{k-l}}{x_1}$ and then setting $x_2 = -\frac{2\alpha(-1)^{l+k/2} y_1^l y_2^{k-l}}{x_1}$ we get

$$\begin{aligned}
&\sum_{(k-l-1)n/(k-l+1)\le j_1 \le n} \frac{\beta_{j_1,n,k,l} j_1 (k-l)}{y_2} H_1^{j_1} H_2^{(k(n-j_1)+2lj_1)/(2(l+1))} \\
&\quad + \sum_{(k-l-1)n/(k-l+1)\le j_1 \le n} \frac{\beta_{j_1,n,k,l}(k(n-j_1)+2lj_1)(-1)^{l+k/2}(1-k+l)x_1^2 y_2^{l-k}}{4\alpha(l+1)(1+k-l)} \\
&\quad \cdot H_1^{j_1} H_2^{(k(n-j_1)+2lj_1)/(2(l+1))-1} + \frac{2(k-l)(-1)^{l+k/2} y_1^l y_2^{k-l}\alpha}{x_1 y_2} G \\
&\quad + x_1 \frac{\partial G}{\partial y_1} - \frac{2\alpha(-1)^{l+k/2} y_1^l y_2^{k-l}}{x_1}\frac{\partial G}{\partial y_2} - 2\alpha(-1)^{l+k/2}(k-l) y_1^l y_2^{k-l-1} \frac{\partial G}{\partial x_1} \\
&= 0.
\end{aligned}$$

Solving this partial differential equation we obtain

$$\begin{aligned}
G = {}& \frac{(-1)^{l+k/2+1}(k-l)}{2\alpha y_1^l y_2} \sum_{(k-l-1)n/(k-l+1)\le j_1 \le n} j_1 \beta_{j_1,n,k,l} H_1^{j_1} H_2^{(k(n-j_1)+2lj_1)/(2(l+1))} \\
& + \frac{(-1)^{l+k/2}(1-k+l)^2 x_1^5}{8(1+k-l)\alpha^2 y_1^l (2\alpha y_1^{l+1}(k-l+1) y_2^{k-l-1} + (-1)^{l+k/2}(l+1)x_1^2)} \\
& \sum_{(k-l-1)n/(k-l+1)\le j_1 \le n} \beta_{j_1,n,k,l}(k(n-j_1)+2lj_1) H_1^{j_1} H_2^{(k(n-j_1)+2lj_1)/(2(l+1))} \\
& + x_1 K(H_1, H_2),
\end{aligned}$$

where K is any function in the variables H_1 and H_2. Since G must be a polynomial in the variables y_1 and y_2 and $k(n-j_1) + 2lj_1 \neq 0$ due to the fact that $j_1 \le n$, we must have $\beta_{j_1,n,k,l} = 0$, i.e. $\bar{F} = 0$. This concludes the proof of the theorem.

Acknowledgements

The first author is partially supported by a MINECO/FEDER grant MTM2008-03437 and MTM2013-40998-P, an AGAUR grant number 2014SGR-568, an ICREA Academia, the grants FP7-PEOPLE-2012-IRSES 318999 and 316338, grant UNAB13-4E-1604.

The second author is supported by Portuguese national funds through FCT - Fundação para a Ciência e a Tecnologia: project PEst-OE/EEI/LA0009/2013 (CAMGSD).

References

[1] J. Hietarinta, *Direct methods for the search of the second invariant*, Phys. Rep. **147** (1987), 87–154.

[2] A.J. Maciejewski and M. Przybylska, *Darboux points and integrability of Hamiltonian systems with homogeneous polynomial potential*, J. Math. Phys. **46** (2005), no. 6, 062901.

[3] K. Meyer, *Generic Hamiltonian dynamical systems are neither integrable nor ergodic*, Memoirs of the American Mathematical Society. **144**, American Mathematical Society, Providence, R.I., 1974.

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: jllibre@mat.uab.cat

[2] Departamento de Matemática, Instituto Superior Técnico, Universidade Técnica de Lisboa, Av. Rovisco Pais 1049-001, Lisboa, Portugal

E-mail address: cvalls@math.ist.utl.pt