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THE HOPF BIFURCATION IN THE SHIMIZU-MORIOKA SYSTEM

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Abstract. We study the local Hopf bifurcations of codimension one and two which occur in the Shimizu-Morioka system. This system is a simplified model proposed for studying the dynamics of the well known Lorenz system for large Rayleigh numbers. We present an analytic study and their bifurcation diagrams of these kinds of Hopf bifurcation, showing the qualitative changes in the dynamics of its solutions for different values of the parameters.

1. Introduction

In this paper we study the local Hopf bifurcations of codimension one and two and the kind of stability of the Hopf periodic orbits in the dynamics of the Shimizu-Morioka system given by

$$\dot{x} = y, \qquad \dot{y} = x - \lambda y - xz, \qquad \dot{z} = -\alpha z + x^2, \tag{1}$$

where $(x, y, z) \in \mathbb{R}^3$ are the state variables, and α and λ are real parameters. System (1) is a simplified model proposed in [18] for studying the dynamics of the well known Lorenz system [9]. Later the system gained self-interest and several articles have appeared in the literature, dealing mainly with the chaotic behavior of the solutions and the emergence of strange attractor , see for instance [6, 17, 18, 20, 21, 22]. It was shown in [17] among other properties that system (1) presents Lorenz-like strange attractors, for example taking $\alpha = 0.45$ and $\lambda = 0.75$ (see Figure 1 of [13]).

In this note we perform an analytic bifurcation analysis of dynamical aspects of the solutions of system (1), when the parameters vary, aiming to give a contribution to the understanding of its complex behavior. Our approach permits a geometric synthesis of the bifurcation analysis, based on the algebraic expression

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and geometric location of the codimension 2 Hopf point leading to the bifurcation of periodic orbits.

The study presented here is close to those realized in some papers, which was performed in [12] (see also [3]). But our approach is different, mainly in the computations of the Lyapunov coefficients which are necessary to study the Hopf bifurcations. In [12] the authors study the system

$$\dot{x} = y - x, \qquad \dot{y} = \beta x - xz, \qquad \dot{z} = -\chi z + \eta x^2.$$

This system and system (1) are equivalent if $\beta = \lambda = 1$ and $\eta > 0$, taking $\alpha = \chi$ and doing the change of variables $(x, y, z) \mapsto (\sqrt{\eta}\, x, -\sqrt{\eta}\, x + \sqrt{\eta}\, y, z)$ in system (1), but when $\beta \neq \lambda$ or $\eta \leq 0$ these systems are not equivalent.

Our main result is the following one.

Theorem 1. *The following statements hold for system* (1)*:*

(a) *For* $\alpha = \frac{2-\lambda^2}{\lambda}$ *and* $\lambda \in (0, \sqrt{2})$ *system* (1) *has two non-hyperbolic singular points* Q_- *and* Q_+ *and, if* $h(\lambda) = 3\,\lambda^4 - 5\,\lambda^2 - 1 \neq 0$*, a one codimension Hopf bifurcation take place at these points, permitting the existence of limit cycles near them. These cycles on the central manifolds of* Q_- *and* Q_+ *are unstable if* $h(\lambda) < 0$ *and stable if* $h(\lambda) > 0$*.*

(b) *For* $\alpha = \frac{2-\lambda^2}{\lambda}$ *with* $\lambda \in (0, \sqrt{2})$ *and* $h(\lambda) = 0$ *a two codimension Hopf bifurcation take place at the points* Q_- *and* Q_+*, with the creation of two limit cycles, one unstable and the other stable on the central manifolds of* Q_- *and* Q_+ *.*

The paper is organized as follows. In section 2 through a linear analysis of system (1) we present a study of the bifurcations which occurs with its singular points. In section 3 we describe a method to compute the focus quantities, related to the stability of the limit cycles which appear in the Hopf bifurcations. In section 4 we present a brief review of the theory used to study codimension one and two Hopf bifurcations. These methods are used in Section 5 to prove statements (a) and (b) of Theorem 1. For some extensions of the Hopf bifurcation see [1].

2. Analysis of the singular points

The statement (a) and (b) of the next proposition are not new, in fact they are well know in the literature see for instance [13, 5].

Proposition 2. *The following statements hold for system* (1)*.*

(a) *For* $\alpha < 0$ *the origin of system* (1) *is the unique hyperbolic singular point. It is a saddle with a one-dimensional stable manifold and two-dimensional unstable manifold;*

(b) *For* $\alpha = 0$ *the z-axis of system* (1) *is filled of singular points. The origin becomes a non-hyperbolic singular point and a degenerate pitchfork bifurcation occurs on it. More precisely, for* $\alpha > 0$ *sufficiently small, this line of singular points disappear, the origin becomes a hyperbolic saddle with a two-dimensional stable manifold and an one-dimensional unstable manifold and two new singular points* Q_- *and* Q_+ *are created, they are symmetric with respect to the z-axis. These new equilibria are hyperbolic and asymptotically stable if* $\alpha > \frac{2-\lambda^2}{\lambda}$ *and* $\lambda > 0$*. For either* $\alpha = \frac{2-\lambda^2}{\lambda}$ *and* $\lambda \in (-\infty, -\sqrt{2})$ *or* $\alpha < \frac{2-\lambda^2}{\lambda}$ *and* $\lambda \in (-\infty, -\sqrt{2}) \cup (0, \sqrt{2})$*,* Q_- *and* Q_+ *are unstable singular points.*

Proof. For $\alpha < 0$ the origin $(0,0,0)$ is the unique singular point of system (1) and the eigenvalues of its linear part are

$$\sigma_0 = -\alpha, \qquad \sigma_\pm = \frac{-\lambda \pm \sqrt{\lambda^2+4}}{2}, \tag{2}$$

with eigenvectors given by $v_0 = (0,0,1)$, $v_\pm = (1,\sigma_\pm,0)$, respectively. As the eigenvalues are all reals and $\alpha < 0$, $\sigma_+\sigma_- < 0$, by the Invariant Manifold Theorem and the Hartman Theorem (see for instance [7]), the origin is a hyperbolic saddle with an one-dimensional stable manifold tangent to the line generated by v_- and a two-dimensional unstable manifold tangent to the plane generated by v_0 and v_+ for all λ. Note that for $\alpha < 0$ the solutions in the invariant z-axis go away from the origin.

If $\alpha = 0$ the invariant z-axis is filled by singular points of system (1). Then the origin is a non-isolated degenerate singular point. Moreover, the eigenvalues of the linear part of system (1) at this point are 0 and $\sigma_\pm$.

When the parameter α crosses the zero value the vector fields associated to system (1) cross this degenerate situation transversally. On the other words, for $\alpha > 0$ the z-axis filled of singular points which exists for $\alpha = 0$ disappears, and system (1) has only the singular points

$$Q_0 = (0,0,0), \qquad Q_\pm = (\pm\sqrt{\alpha},0,1).$$

The eigenvalues of the linear part of system (1) at Q_0 are given in (2) and we have $\sigma_0 < 0$ and $\sigma_- < 0$ and $\sigma_+ > 0$. Therefore Q_0 is a hyperbolic saddle with a two-dimensional stable manifold and an one-dimensional unstable manifold for all λ. Thus under the creation and subsequent elimination of the line of singular points when α crosses the zero value, the origin Q_0 of system (1) gains one dimension in the stable manifold and loses one dimension in the unstable one, as stated in statement (b) of the proposition.

Under the change of coordinates $(x,y,z) \mapsto (-x,-y,z)$, system (1) is invariant. Hence the kind of stability of the singular point Q_+ follows from the kind of stability of Q_-. The characteristic polynomial of the linear part of system (1) at Q_- is

$$p(\sigma) = -\sigma^3 - (\alpha+\lambda)\sigma^2 - \alpha\lambda\sigma - 2\alpha.$$

The rest of proof follows from the next proposition. □

Proposition 3. *Consider $\alpha > 0$. The singular point Q_- is asymptotically stable to system* (1) *if $\alpha > \frac{2-\lambda^2}{\lambda}$ and $\lambda > 0$, and unstable if either $\alpha = \frac{2-\lambda^2}{\lambda}$ and $\lambda \in (-\infty,-\sqrt{2})$ or $\alpha < \frac{2-\lambda^2}{\lambda}$ and $\lambda \in (-\infty,-\sqrt{2}) \cup (0,\sqrt{2})$.*

Proof. The proof follows easily from the Routh-Hurwitz stability criterion (see [14] page 58). □

The next proposition is a straightforward consequence of the relations between roots and coefficients of a polynomial in one variable.

Proposition 4. *Consider $\alpha > 0$. If $\alpha = \frac{2-\lambda^2}{\lambda}$ and $\lambda \in (0,\sqrt{2})$, then the linear part of system* (1) *at the singular point Q_- has one negative eigenvalue and two conugated pure imaginary eigenvalues.*

Following [12], the symmetric bifurcation which occurs when the parameter α crosses the zero value is called *degenerate pitchfork bifurcation*, due to the line of equilibria which exists for $\alpha = 0$, and it has already been observed in other systems which also present chaotic behavior (see for instance [16], page 4 and [12]).

3. Center Theorem and focus quantities

In this section we summarize the method described in [4] (see also [10, 11]) for studing the center problem on a center manifold for vector fields in $\mathbb{R}^3$. Let $X : U \to \mathbb{R}^3$ be a real analytic vector field, such that $DX(0)$ has two pure imaginary eigenvalues and one non-zero. By a linear change of variables and a possible rescaling of the time the system of differential equations $\dot{\mathbf{u}} = X(\mathbf{u})$ can be written as

$$\begin{aligned} \dot{u} &= -v + P(u,v,w) = \tilde{P}(u,v,w), \\ \dot{v} &= u + Q(u,v,w) = \tilde{Q}(u,v,w), \\ \dot{w} &= \beta w + R(u,v,w) = \tilde{R}(u,v,w), \end{aligned} \tag{3}$$

where β is a real non-zero number. We denote again by X this new vector field.

A non-constant C^1 function H from a neighborhood of the origin of $\mathbb{R}^3$ into $\mathbb{R}$ is a *local first integral* of system (3) if it is constant on the orbits of (3), i.e. H satisfies

$$XH = \tilde{P}\frac{\partial H}{\partial u} + \tilde{Q}\frac{\partial H}{\partial v} + \tilde{R}\frac{\partial H}{\partial w} \equiv 0, \tag{4}$$

in a neighborhood of the origin. A non-constant formal power series H in u, v and w is a *formal first integral* for system (3) if when $\tilde{P}$, $\tilde{Q}$, and $\tilde{R}$ are expanded in power series at the origin, every coefficient in the formal power series in (4) is zero. If w and $\dot{w}$ do not appear in system (3) the system is in $\mathbb{R}^2$, the singular point at the origin is either a focus (every trajectory near the origin spirals towards the origin, or every trajectory does so in reverse time) or a center (a punctured neighborhood is composed entirely of periodic orbits). The problem of distinguishing between these two cases is the center problem. It was solved by Poincaré and Lyapunov in terms of the non-existence or existence of a local first integral. A proof is given in [15].

From Theorem 5.1 page 152 of [7] we know that system (3) admits a local center manifold W^c_{loc} at the origin. The following theorem provides one the main tools for detecting a center on a center manifold . See [4] for a proof.

Theorem 5. *The following statements are equivalent.*

(a) *The origin is a center for* $X|_{W^c_{loc}}$.
(b) *There is a local analytic first integral at the origin for system* (3) *of the form* $H(u,v,w) = u^2 + v^2 + \cdots$ *(here the dots mean higher order terms).*
(c) *There is a formal first integral at the origin for system* (3) *of the form* $H(u,v,w) = u^2 + v^2 + \cdots$.

The *Lyapunov Center Theorem* correspond to the equivalence of statements (a) and (b); for a proof see also [2]. From this theorem we can restrict our attention to investigate the conditions for the existence or non-existence of a first integral of the form $H(u,v,w) = u^2 + v^2 + \cdots$, which is equivalent to determine necessary and sufficient conditions for the existence of a center or a focus on the local center manifold, respectively.

In what follows we consider that P, Q and R in (3) are polynomials. We start by introducing the complex variable $x = u + iv$. Therefore the first two equations in (3) are equivalent to the unique equation $\dot{x} = ix + \cdots$. Adding to this equation its complex conjugate, changing $\bar{x}$ (where as usual $\bar{x}$ denote the conjugate of x) by

y, thinking in y as an independent complex variable, and substituting w by z, we obtain the following complexification of system (3)

$$\begin{aligned} \dot{x} &= ix + \sum_{p+q+r=2}^{n} a_{pqr} x^p y^q z^r, \\ \dot{y} &= -iy + \sum_{p+q+r=2}^{n} b_{pqr} x^p y^q z^r, \\ \dot{z} &= \beta z + \sum_{p+q+r=2}^{n} c_{pqr} x^p y^q z^r, \end{aligned} \tag{5}$$

where $b_{qpr} = \bar{a}_{pqr}$ and the c_{pqr} are such that $\sum_{p+q+r=2}^{n} c_{pqr} x^p \bar{x}^q w^r$ is real for all $x \in \mathbb{C}$ and $w \in \mathbb{R}$. Again we denote by X the new vector field associated to system (5) on $\mathbb{C}^3$. Now the existence of a first integral $H(u, v, w) = u^2 + v^2 + \cdots$ for a system (3) is equivalent to the existence of a first integral of the form

$$H(x, y, z) = xy + \sum_{j+k+l=3} v_{jkl} x^j y^k z^l \tag{6}$$

for system (5).

By computing the coefficients of XH and equating them to zero we investigate the existence of a first integral H for a system (5) . When H has the form (6) we can calculated explicitly the coefficient $g_{k_1k_2k_3}$ of $x^{k1} y^{k2} z^{k3}$ in XH (see [4]). But when $(k_1, k_2, k_3) = (k, k, 0)$ for a positive integer k, we can solved in a unique way for $v_{k_1k_2k_3}$ the equation $g_{k_1k_2k_3} = 0$ in terms of the known quantities $v_{\alpha\beta\gamma}$ such that $\alpha + \beta + \lambda < k_1 + k_2 + k_3$. Hence if $g_{kk0} = 0$ for all $k \in \mathbb{N}$ a formal first integral H exists. When the coefficient g_{kk0} is non-zero an obstruction to the existence of the formal series H occurs. Such a coefficient is called the kth *focus quantity.*

The focus quantities $g_{110} = 0$ and g_{220} are determined in a unique way, but the others depend on the choices made for v_{kk0}, $k \in \mathbb{N}$, $k \geq 2$. Once such computations are made, H is determined and satisfies

$$XH(x, y, z) = g_{220}(xy)^2 + g_{330}(xy)^3 + \cdots .$$

It follows that if for one choice of the v_{kk0} at least one focus quantity is non-zero, the same is true for every other choice of the v_{kk0}. A sufficient and necessary condition for the existence of a center on the center manifold is to vanish all focus quantities, otherwise we have a focus (see [4]).

In rest of this work we denote the kth focus quantity g_{kk0} by ν_k.

4. Hopf bifurcation method

Let (θ, ρ) be polar coordinates on the local center manifold W^c_{loc}, such that $\rho = 0$ corresponds to the origin in cartesian coordinates. Consider system (3) restricted to its local center manifold and let $\Pi(\rho)$ the respective Poincaré first return map on a sufficiently short segment of the axis $\theta = 0$ starting at $\rho = 0$. By the kth *Lyapunov coefficient* we mean the coefficient l_k in the expansion of displacement map $\Pi(\rho) - \rho$, i.e.

$$\Pi(\rho) - \rho = l_1\rho + l_2\rho^2 + \cdots .$$

It follows by the proof of Theorem 6.2.3 of the page 261 of [15] that

$$l_1 = c_1\nu_1 \text{ and } l_k \,|_{l_1=\cdots=l_{k-1}=0} = \; c_k\nu_k \,|_{\nu_1=\cdots=\nu_{k-1}=0}, \tag{7}$$

where $c_1, \dots, c_k$ are positive constants.

A method to compute the Lyapunov coefficients can be found in the pages 177–181 of [7] or in [8, 12].

A singular point (x_0, μ_0) of a μ-parameter family of vector fields $X(x, \mu)$ in $\mathbb{R}^3$ is called a *Hopf point* if the Jacobian matrix $DX(x_0, \mu_0)$ has a real eigenvalue $\lambda_1 \neq 0$ and a pair of purely imaginary eigenvalues $\lambda_{2,3} = \pm i\omega_0$. There is a two-dimensional center manifold at a Hopf point and it is invariant by the flow of the system $\dot{x} = X(x, \mu)$, see page 152 of [7]. If varying the parameters the complex eigenvalues cross the imaginary axis with non-zero derivative, the Hopf point is called *transversal*, i.e. if μ is one-dimensional parameter then $\dfrac{d\xi}{d\mu}(\mu_0) \neq 0$ (where $\xi(\mu) \pm i\omega(\mu)$ are the conjugated complex eigenvalues of the linear part of $X(x, \mu)$ at singular point x_μ when $|\mu - \mu_0|$ is enough small). At a neighborhood of transversal Hopf point with $l_1(\mu_0) \neq 0$ the system $\dot{x} = X(x, \mu)$ restricted to a center manifold, is orbitally topologically equivalent to the following complex normal form

$$\dot{w} = (\xi + i\omega)w + \sigma w|w|^2,$$

where $w \in \mathbb{C}$, $\sigma = \text{sign } l_1(\mu_0) = \pm 1$, $l_1(\mu_0)$ the first Lyapunov coefficient at the Hopf point, and ξ, ω are real functions having derivatives of arbitrary higher order, which are continuations of 0 and ω_0, see page 98 of [7]. There is one family of stable (unstable) periodic orbits if $l_1 < 0$ ($l_1 > 0$) on the space of phases variables and parameters shrinking to a singular Hopf point.

A *Hopf point of codimension* 2 is a Hopf point where $l_1(\mu_0) = 0$ and $l_2(\mu_0) \neq 0$. It is called *transversal* if the manifolds $\xi(\mu) = 0$ ($\xi(\mu)$ is the real part of the conjugated complex eigenvalues) and $l_1(\mu)$ have transversal intersections, i.e. the map $\mu \mapsto (\xi(\mu), l_1(\mu))$ is regular at μ_0. The system $\dot{x} = X(x, \mu)$ restricted to a center manifold at a neighborhood of a transversal Hopf point of codimension 2 is orbitally topologically equivalent to

$$\dot{w} = (\xi + i\omega_0)w + \tau w|w|^2 + \sigma w|w|^4, \tag{8}$$

where ξ and τ are the unfolding parameters and $\sigma = \text{sign } l_2(\mu_0) = \pm 1$, see page 311 of [7]. The bifurcation diagram of system (8) on the space of parameters (ξ, τ) for $\sigma = 1$ is showed in Figure 1. Where the lines $H_1^{\pm} = \{\pm\tau > 0\}$ correspond to the Hopf bifurcation of codimension one with negative and with positive Lyapunov coefficient, respectively. Along these lines the singular point has eigenvalues $\lambda_{1,2} = \pm\omega_0 i$. Moreover the singular point is stable for $\xi < 0$ and unstable for $\xi > 0$. The first Lyapunov coefficient is $l_1(\xi, \tau) = \tau$. Therefore the point of the Hopf bifurcation of codimension two H_2 occurs when $\xi = \tau = 0$ and separates the two branches, H_1^- and H_1^+ of τ-axis. An unstable limit cycle bifurcates from the singular point if we cross H_1^+ from right to left, while a stable limit cycle appears if we cross H_1^- in the opposite direction. These limit cycles collide and disappear on the curve

$$T = \{(\xi, \tau) \ : \ 4\xi - \tau^2 = 0\},$$

corresponding to a nondegenerate fold bifurcation of the cycles. Along this curve the system has a semistable limit cycle of multiplicity one, see page 311 of [7].

The bifurcation diagrams for $\sigma = -1$ can be found in [7], page 313, and in [19] .

From (7) the Hopf method described above can be applied changing the Lyapunov coefficients by the focus quantities. Thus in rest of this paper we shall use the focus quantities in place of the Lyapunov coefficients to study Hopf bifurcations of codimension one and two.

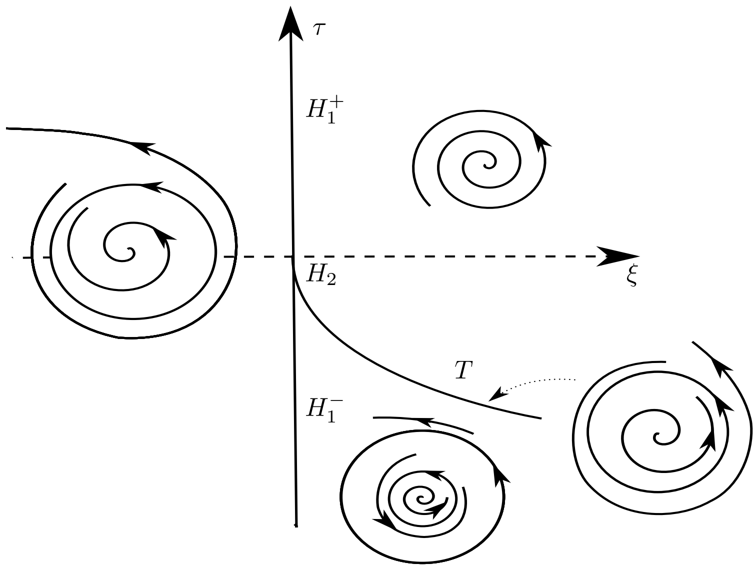

FIGURE 1. Diagram at the point H_2 of a two codimension Hopf bifurcation.

5. HOPF BIFURACTION IN THE SHIMIZU-MORIOKA SYSTEM

In this section we study the stability of the singular point Q_- of system (1) under the conditions $\alpha = \frac{2-\lambda^2}{\lambda}$ and $\lambda \in (0, \sqrt{2})$ given in Preposition 4, i.e. on the Hopf axis correspondent to the τ-axis of Figure 1. We prove the following theorem.

Theorem 6. *Consider the two-parameter family of differential equations* (1). *The first focus quantity at the point Q_- for parameter values satisfying $\alpha = \dfrac{2-\lambda^2}{\lambda}$ and $\lambda \in (0, \sqrt{2})$ is given by*

$$\nu_1(\lambda) = \frac{\lambda\sqrt{2-\lambda^2}\left(3\lambda^4 - 5\lambda^2 - 1\right)}{4\left(\lambda^4 - 2\lambda^2 - 4\right)\left(\lambda^4 - 2\lambda^2 - 1\right)}.$$

For $\lambda \in (0, \sqrt{2})$ such that $h(\lambda) = 3\,\lambda^4 - 5\,\lambda^2 - 1$ is different from zero, system (1) *has a transversal Hopf point at Q_- for $\alpha = \dfrac{2-\lambda^2}{\lambda}$.*

Now for the parameter values $\lambda_c = \sqrt{\dfrac{5+\sqrt{37}}{6}}$ and $\alpha = \dfrac{7-\sqrt{37}}{\sqrt{6(5+37)}}$ system (1) *has a transversal Hopf point of codimension 2 at Q_- which is unstable because $\nu_2 > 0$.*

Proof. For simplify the computations, we introduce the new parameters (β, ε) by

$$\begin{aligned}
\lambda &= \frac{-\varepsilon\left(\beta^2+\varepsilon^2+2\right) + \sqrt{\left(-\beta^2-\varepsilon^2+2\right)\left(\beta^4+\left(\beta^2+2\right)\varepsilon^2\right)}}{\beta^2+\varepsilon^2}, \\
\alpha &= -\varepsilon - \frac{\sqrt{\left(-\beta^2-\varepsilon^2+2\right)\left(\beta^4+\left(\beta^2+2\right)\varepsilon^2\right)}}{\beta^2+\varepsilon^2-2}.
\end{aligned}$$

The Jacobian determinant of this change of parameters in the point $(0,\beta)$ is

$$-\frac{2\beta^5\left(\beta^4-2\beta^2-4\right)}{\left(-\beta^4\left(\beta^2-2\right)\right)^{3/2}}.$$

Thus the change of parameters is well defined for $(\varepsilon,\beta)\in(-\delta,\delta)\times(0,\sqrt{2})$, with δ enough small. In this new parameters the linear part of system (1) at the singular point Q_- has a real eigenvalue and two conjugated complex given by $\varepsilon\pm i\beta$. Furthermore the conditions $\alpha=\frac{2-\lambda^2}{\lambda}$ and $\lambda\in(0,\sqrt{2})$ correspond to $\varepsilon=0$ and $\beta\in(0,\sqrt{2})$. Hence, in this case, we have

$$\alpha=\frac{\beta^2}{\sqrt{2-\beta^2}}\quad\text{and}\quad\lambda=\sqrt{2-\beta^2} \tag{9}$$

in system (1). Now, doing the change of coordinates $(x,y,z)\mapsto(\tilde{x}-\sqrt{\alpha},\tilde{y},\tilde{z}+1)$, the singular point Q_- is translated to the origin $(0,0,0)$. Now we shall write the linear part of system in the coordinates $(\tilde{x},\tilde{y},\tilde{z})$ at the origin in its real Jordan normal form. For this we introduce the variables (u,v,w) by

$$\begin{pmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{pmatrix}=\begin{pmatrix}-\dfrac{\beta}{2\sqrt[4]{2-\beta^2}} & \dfrac{1}{2}\sqrt[4]{2-\beta^2} & 2\left(2-\beta^2\right)^{5/4}\\ \frac{1}{2}\beta\sqrt[4]{2-\beta^2} & \frac{\beta^2}{2\sqrt[4]{2-\beta^2}} & -4\left(2-\beta^2\right)^{3/4}\\ 1 & 0 & 4\beta\sqrt{2-\beta^2}\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix},$$

and so system (1) becomes

$$\begin{aligned}
\dot{u} &= -v-\frac{\beta\sqrt{2-\beta^2}\left(6+\beta^2\right)u^2}{-16-8\,\beta^2+4\,\beta^4}-\frac{\left(-8+2\,\beta^2+\beta^4\right)vu}{-8-4\,\beta^2+2\,\beta^4}\\
&\quad+\frac{2\left(-2+\beta^2\right)\left(-8+4\,\beta^2+\beta^4\right)wu}{-4-2\,\beta^2+\beta^4}+\frac{\sqrt{2-\beta^2}\left(-4+\beta^4\right)v^2}{\beta(-16-8\,\beta^2+4\,\beta^4}) \\
&\quad-\frac{2\sqrt{2-\beta^2}\left(8-8\,\beta^2+\beta^6\right)wv}{\beta(-4-2\,\beta^2+\beta^4)}+\frac{4\left(2-\beta^2\right)^{5/2}\left(-4+4\,\beta^2+\beta^4\right)w^2}{\beta(-4-2\,\beta^2+\beta^4)},\\
\dot{v} &= u+\frac{\left(8+\beta^4\right)u^2}{16+8\,\beta^2-4\,\beta^4}+\frac{\sqrt{2-\beta^2}\left(4+\beta^4\right)vu}{\beta(-8-4\,\beta^2+2\,\beta^4)}\\
&\quad-\frac{2\sqrt{2-\beta^2}\left(-8+8\,\beta^2-2\,\beta^4+\beta^6\right)wu}{\beta(-4-2\,\beta^2+\beta^4)}+\frac{\beta^2\left(-2+\beta^2\right)v^2}{4(-4-2\,\beta^2+\beta^4)}\\
&\quad-\frac{2\left(-8+8\,\beta^2-4\,\beta^4+\beta^6\right)vw}{-4-2\,\beta^2+\beta^4}+\frac{4\left(-2+\beta^2\right)^2\left(8-2\,\beta^2+\beta^4\right)w^2}{-4-2\,\beta^2+\beta^4},\\
\dot{w} &= -\frac{2w}{\beta\sqrt{2-\beta^2}}+\frac{\left(-4+3\,\beta^2\right)u^2}{64-32\,\beta^4+8\,\beta^6}+\frac{\left(-1+\beta^2\right)vu}{2\beta\sqrt{2-\beta^2}\,(-4-2\,\beta^2+\beta^4)}\\
&\quad+\frac{\left(-4+8\,\beta^2-3\,\beta^4\right)wu}{\beta\sqrt{2-\beta^2}\,(-4-2\,\beta^2+\beta^4)}-\frac{6\left(-2+\beta^2\right)^2w^2}{-4-2\,\beta^2+\beta^4}\\
&\quad+\frac{v^2}{32+16\,\beta^2-8\,\beta^4}+\frac{2\left(-2+\beta^2\right)vw}{-4-2\,\beta^2+\beta^4}.
\end{aligned}$$

Note that the above system is in the form (3). Now we apply the method described in section 3.

Firstly we introduce the change of variables $(u, v, w) \mapsto (x, y, z) = (u + iv, u - iv, w)$ with inverse given by $(x, y, z) \mapsto (u, v, w) = \left(\frac{1}{2}(x + y), -\frac{i}{2}(x - y), z\right)$. Hence we obtain system (1) in the complex form (5). Aigain denote by X the vector field associated to this last system in complex form and let H be given by (6). We have that

$$XH(x, y, z) = \sum_{m\geq 2} H_m(x, y, z),$$

where, H_m are homogeneous polynomials of degree m in the variables (x, y, z). It is easy to see that $H_2 \equiv 0$. Denoting the coefficients of H_m by g_{jkl} with $j+k+l = m$, we can solve easily the equations $g_{jkl} = 0$ with $j + k + l = 3$ in terms of the coefficients $\nu_{\alpha\beta\gamma}$ of H such that $\alpha + \beta + \gamma \leq 3$. For instance the equation $g_{003} = 0$ is given by

$$-\frac{6\nu_{003}}{\beta\sqrt{2-\beta^2}} = 0,$$

which solution in terms of the coefficients of H is $\nu_{003} = 0$. Analogously, we can solve the equations $g_{jkl} = 0$ with $j + k + l = 4$ in terms of the coefficients $\nu_{\alpha\beta\gamma}$ of H with $\alpha + \beta + \gamma \leq 4$, except the equation $g_{220} = 0$, because this equation does not depend on the coefficients of H, only on the coefficients of X. Hence we have that the first focus quantity is $g_{220} = \nu_1$ given by

$$\nu_1 = \frac{\sqrt{2-\beta^2}\left(3\beta^4 - 7\beta^2 + 1\right)\beta}{4\left(\beta^4 - 2\beta^2 - 4\right)\left(\beta^4 - 2\beta^2 - 1\right)}. \tag{10}$$

Following the above ideas we obtain the second focus quantity $g_{330} = \nu_2$, i.e.

$$\begin{aligned}\nu_2 \;=\; & (\sqrt{2-\beta^2}\left(-162\beta^{22} + 2268\beta^{20} - 14289\beta^{18} + 47071\beta^{16} - 80155\beta^{14}\right.\\ & +63495\beta^{12} - 20967\beta^{10} + 16999\beta^8 - 23136\beta^6 + 9300\beta^4 + 9760\beta^2\\ & \left.-80\right))/(96\beta\left(\beta^4 - 2\beta^2 - 4\right)^3\left(\beta^4 - 2\beta^2 - 1\right)^2\left(9\beta^2\left(\beta^2 - 2\right) - 4\right)).\end{aligned}$$

Note that the polynomial $\beta^4 - 2\beta^2 - 4$ has only two real roots, $\pm\sqrt{1+\sqrt{5}}$, and so it has negative sign in $(-\sqrt{1+\sqrt{5}}, \sqrt{1+\sqrt{5}})$. Now the polynomial $\beta^4 - 2\beta^2 - 1$, also has only two real roots, $\pm\sqrt{1+\sqrt{2}}$, and so it has negative sign in $(-\sqrt{1+\sqrt{2}}, \sqrt{1+\sqrt{2}})$. Therefore the sign of the first focus quantity is determined by the sign of $h(\beta) = 3\beta^4 - 7\beta^2 + 1$, since we are considering $\beta \in (0, \sqrt{2})$ and so the denominator of (10) is positive. Observe that, for $\beta \in (0, \sqrt{2})$, the first focus quantity vanishes on $\beta_c = \sqrt{\frac{1}{6}\left(7 - \sqrt{37}\right)}$ and the second is deferent from zero, i.e. $\nu_1(\beta_c) = 0$ and $\nu_2(\beta_c) = \frac{\sqrt{274249\sqrt{37} - 591726}}{15568}$. Moreover $\nu_1 > 0$ for $\beta \in (0, \beta_c)$ and $\nu_1 < 0$ for $\beta \in (\beta_c, \sqrt{2})$.

Clearly, in the plane of parameters (ε, β), we have a transversal Hopf point in Q_- for $\varepsilon = 0$ and $\beta \in (0, \sqrt{2}) \setminus \{\beta_c\}$. Now as the map $(\varepsilon, \beta) \mapsto (\varepsilon, \nu_1(\beta))$ is regular in $(0, \beta_c)$, since

$$\frac{d\nu_1}{d\beta}(\beta_c) = -\frac{\sqrt{37\left(14066\sqrt{37} - 84205\right)}}{1946},$$

it follows that we have a transversal Hopf point of codimension 2 in Q_- for $\varepsilon = 0$ and $\beta = \beta_c$.

In the parameter λ, by (9), the function h becomes $h(\lambda) = 3\,\lambda^4 - 5\,\lambda^2 - 1 \neq 0$,

$$\nu_1(\lambda) = \frac{\lambda\sqrt{2-\lambda^2}\left(3\lambda^4 - 5\lambda^2 - 1\right)}{4\,(\lambda^4 - 2\lambda^2 - 4)\,(\lambda^4 - 2\lambda^2 - 1)}$$

and $\nu_1(\lambda)$ is zero in the $(0, \sqrt{2})$ only for the value $\lambda_c = \sqrt{\dfrac{5+\sqrt{37}}{6}}$. Moreover in this point, by (9), $\alpha = \dfrac{7-\sqrt{37}}{\sqrt{6(5+37)}}$. □

The same results stated in Theorem 1 are valid also for the point Q_+, due to the symmetry of the system under the change $(x, y, z) \mapsto (-x, -y, z)$. The statements (a) and (b) of Theorem 1 follow from the above results.

The bifurcation diagram on the space of parameters (λ, α) of system (1) on the neighborhood of the two codimension Hopf point $H_2 = \left(\sqrt{\dfrac{5+\sqrt{37}}{6}}, \dfrac{7-\sqrt{37}}{\sqrt{6(5+37)}}\right)$ is described in Figure 2, where by section 4 the curves $H_1^{\pm}$ and T correspond respectively with the curves of Figure 1. Note that

$$H_1^- \cup H_2 \cup H_1^+ = \left\{(\lambda, \alpha) \;:\; \alpha = \frac{2-\lambda^2}{\lambda},\ \lambda \in (0, \sqrt{2})\right\}.$$

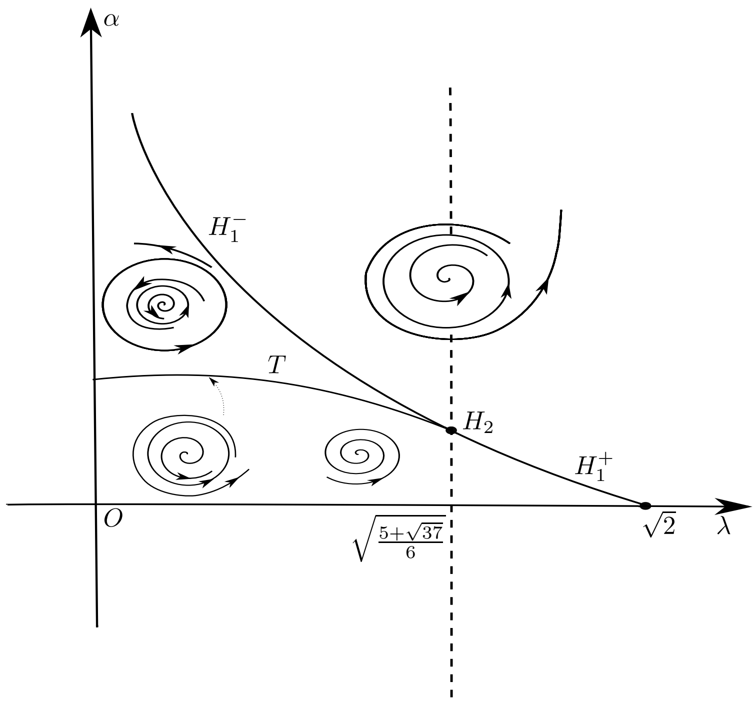

FIGURE 2. Diagram at the point H_2 of the two codimension Hopf bifurcation of system (1).

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